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APPLIED MATHEMATICS-I

Er. M.K. Kanyal

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Applied Mathematics-I

As per UBTER, Uttarakhand and AICTE Syllabus

for

First Semester Diploma Students

by

Er. M.K. Kanyal

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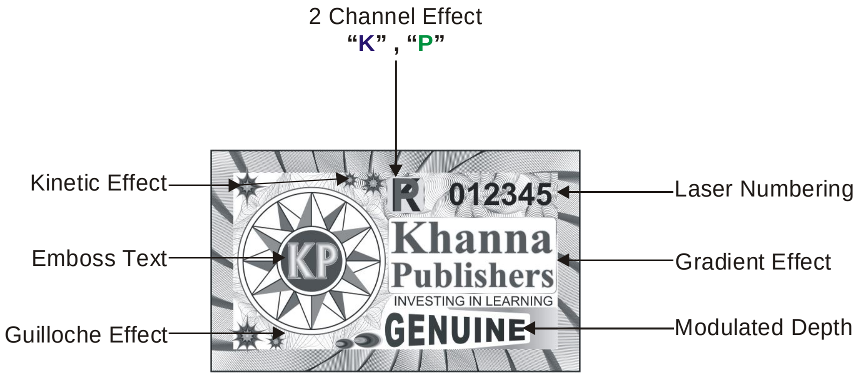

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Preface

It give us a great pleasure in presenting the new edition for Diploma level students of UBTER, Uttarakhand as per AICTE syllabus, for their **First Semester Diploma Course**. The aim of writing this book is to be provided the students a clear understanding of the basic concepts.

The subject matter of the book is arranged and presented such a way that the entire syllabus has been covered in the articles form in same sequence.

The book is divided into ***Four Units*** and ***Seventeen Chapters***. A large number of solved examples, exercises with answers given at the end of each chapter. This should be of a great help to understand the relevant concept and in developing problems solving skills.

Inspite of our best efforts, it is possible that some errors might have crept in. We shall acknowledge with gratitude, if any such error is brought in our notice.

Also, any suggestions and comments from students and teachers for improvement of the book are welcome.

We hope the book will be found useful by the readers.

—Publishers.

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Unit—I

Algebra

1.1. VALUE OF $^{n}p_{r}$ AND $^{n}c_{r}$ (WITHOUT PROOF)

• Factorial Notation

Let us consider n be a positive integer, the continued product of first n natural number is called factorial n, be denoted by n ! or $\lfloor n$.

Do you know?

0 ! = 1

Thus,

$$n\,! = \lfloor n = n\,(n-1)\,(n-2)\,...\,3\,.\,2\,.\,1$$
$$= n\,(n-1)\,(n-2)\,(n-3)\,...\,3\,.\,2\,.\,1$$
$$= n\,[(n-1)\,(n-2)\,(n-3)]\,...\,3\,.\,2\,.\,1]$$
$$= n\,[(n-1)\,!]$$

Theorem:

(i) $n\,(n-1)\,(n-2)\,...\,(n-r+1) = \dfrac{n\,!}{(n-r)\,!}$

$n\,(n-1)\,(n-2)\,...\,(n-r+1)$

$$= n\,(n-1)\,(n-2)\,...\,(n-r+1)$$

Multiplying numerator and denominator by $(n-r)$!

$$= \frac{n\,!}{(n-r)\,!}$$

For example, $(2n)! = (2n)\,(2n-1)\,(2n-2)\,(2n-3)\,...\,4\,.\,3\,.\,2\,.\,1$

$$= [(2n)\,(2n-2)\,(2n-4)\,...\,4\,...\,2]\,[(2n-1)\,(2n-3)\,...\,3\,...\,1]$$
$$= 2^{n} \times [n\,(n-1)\,(n-2)\,...\,2\,.\,1] \times [(2n-1)\,(2n-3)\,...\,3\,...1]$$
$$= 2^{n}\,.\,(n\,!)\,.\,[1\,.\,3\,.\,5\,...\,(2n-1)]$$

Example 1. *Prove that inequality,*

$$(n!)^2 \le n^n\,.n\,! < (2n)\,!\text{ for all positive integer } n.$$

Sol. $(2n)\,! = (1\,.\,2.\,3\,...\,n)\,(n+1)\,(n+2)\,...\,(2n-1)\,(2n) > (n\,!)\,n^{n}$

[∵ $(n + r) > n$ far $r = 1, 2\,...\,n$]

$(n\,!)\,.\,n^{n} < (2n)\,!$ ...(i)

$(n\,!)^{2} = (1\,.\,2\,.\,3\,...\,n)\,(n\,!) \le n^{n}\,.\,(n\,!)$ ...(ii)

[∵ $r \le h$ for each $r = 1, 2\,..\,n$]

From (*i*) and (*ii*), we get

$$(n\,!)^2 \leq n^n \,.\, n\,! < (2n)\,!$$

Example 2. *LCM [4 ! 5 ! 6 !]*

Sol. We have

$$4\,! = (4 \times 3 \times 2 \times 1) = 2^3 \times 3$$
$$5\,! = (5 \times 4 \times 3 \times 2 \times 1) = 2^3 \times 3 \times 5$$
$$6\,! = (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 2^4 \times 3^2 \times 5$$

$\therefore$ LCM [4 !, 5 !, 6 !] = $(2^4 \times 3^2 \times 5) = 720$.

Example 3. *Find the value of* $\dfrac{8!}{4!\,5!}$

Sol. Given, $\dfrac{8!}{4!\,5!}$

First, we have 8!

$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

Now consider 4!5!

First $4! = 4 \times 3 \times 2 \times 1 = 24$

and $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$\therefore$ $4!\,5! = 2880$

Hence, $\dfrac{8!}{4!\,5!} = \dfrac{40320}{2880} = 14$

Example 4. *Show that (n ! + 1) is not divisible by any natural number between 2 and n*

Sol. $n\,! = n\,(n-1)\,(n-2)\,...\,3\,.\,2\,.\,1$

n! is divisible by every natural number between 2 and n.

If (n ! + 1) is divided by any natural number between 2 and n, leaves 1 as remainder.

So, (n ! + 1) is not divisible by any natural number between 2 and n.

Example 5. *There are 4 routes between Allahabad and Ghaziabad. In how many different ways can a man go from Allahabad to Ghaziabad and return, if for returning.*

(i) any of the routes is taken

(ii) the same route is taken

(iii) the same route is not taken?

Sol. (*i*) If the man may take route for going from Allahabad to Ghaziabad.

When he return by any routes, in 4 different ways.

Total number of ways on going to Ghaziabad and returning back to Allahabad = 4 × 4 = 16

(*ii*) Now, 4 ways of going to Ghaziabad and in this case only 1 way of returning

$\therefore$ total number of ways = 4 × 1 = 4 (*iii*) 4 ways of going to Ghaziabad,

3 ways of returning back to Ghaziabad

Example 6. *How many words (with or without meaning) of three distinct letters of the English alphabet are there?*

Sol. There are 26 letters of the English alphabet

There are 25 ways of filling up the second place.

∴ the required number of words

$$= 26 \times 25 \times 24 = 15600.$$

Example 7. *How many numbers are there between 100 and 1000 such that 7 is the units place.*

Sol. Hundred's place can be filled with any of the digits from 1 to 9.

So, the hundred's place can be filled with any of the digits from 1 to 9.

The units place can be filled only 1 way.

∴ Required numbers = 9 × 10 × 1 = 90.

1.1.1. ^np_r (Permutations)

The different arrangement which can be made out of a given number of things by taking sum or all 0 at a time, are called permutations.

The number of all permutations of n dissimilar things taken r at a time is given by

$$^np_r = n(n-1)(n-2) \ldots (n-r+1)$$

$$= \frac{n!}{(n-r)!}.$$

Example 8. *Calculate :*

(*i*) $^8p_8$ (*ii*) $^{75}p_2$ (*iii*) $^{12}p_4.$

Sol. We have (*i*) $^8p_8 = 8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$

(*ii*) $$^{75}p_2 = \frac{75!}{(75-2)!} = \frac{75!}{73!}$$

$$= \frac{75 \times 74 \times 73!}{73!} = 5550$$

(*iii*) $$^{12}p_4 = \frac{12!}{(12-4!)} = \frac{12 \times 11 \times 10 \times 9 \times 8!}{8!}$$

$$= 11880.$$

Example 9. *Prove that* (*i*) $^np_n = {}^np_{n-1}$

(*ii*) $^np_r = n \,.\, {}^{n-1}p_{r-1}$

(*iii*) $^np_r = {}^{n-1}p_r + r.\, {}^{n-1}p_{r-1}.$

Sol. We have (*i*) $$^np_{n-1} = \frac{n!}{[n-(n-1)]!} = \frac{n!}{1!} = n! = {}^np_n$$

(*ii*) $$^np_r = \frac{n!}{(n-r)!} = \frac{n(n-1)!}{[(n-1)-(r-1)]!} = n \,.\, {}^{n-1}p_{r-1}$$

(*iii*) $$^{n-1}p_r + r\,.\,^{p-1}p_{r-1} = \left\{\frac{(n-1)\,!}{(n-1-r)\,!} + r\,.\,\frac{(n-1)!}{[(n-1)-(r-1)]\,!}\right\}$$

$$= \left\{\frac{(n-1)\,!}{(n-1-r)\,!} + r\,.\,\frac{(n-1)!}{[(n-r)\,!}\right\}$$

$$= \left\{\frac{(n-1)\,!}{(n-r-1)\,!} + r\,.\,\frac{(n-1)!}{[(n-r)\,(n-r-1)\,!}\right\}$$

$$= \frac{(n-1)\,!}{(n-r-1)\,!}\left\{1+\frac{r}{(n-r)}\right\} = \frac{n\,.\,(n-1)\,!}{(n-r)\,(n-r-1)\,!}$$

$$= n\,!/(n-r)\,! = {}^np_r.$$

Example 10. *In how many ways can the letters of the word 'PENCIL' be arranged so that N is always next to E.*

Sol. Putting E and N together and considering them as one letter, we have to arrange 5 letters at 5 places.

This is $^5p_5 = 5! = 120$ ways.

Example 11. *(i) How many words can be formed by using all the letters of the word 'ALLAHABAD'.*

(ii) How many of these words will not contain both L together?

Sol. (*i*) There are 9 letters out of which '*A*' is repeated 4 times, '*L*' is repeated 2 times, and the rest are all different.

$\therefore$ No. of words $= \dfrac{9\,!}{4\,!\,2!} = 7560$

(*ii*) When both '*L*' are together, we can letter. So, we will have to arranged 8 letters out of which '*A*' is repeated 4 times.

No. of arrangement with both '*L*' together $= \dfrac{8\,!}{4\,!} = 1680$

So the no. of arrangement with both '*L*' not occurring together = 7560 – 1680 = 5880.

Example 12. *How many 6 rings of different types had in 4 fingers.*

Sol. First ring can be worn in any of 4 fingers, there are 4 ways of wearing it.

Thus, rings may be worn in 4 ways.

$\therefore$ number of ways $= 4^6 = 4096$.

Theorem 1. *The number of circular permutations of n different objects is (n – 1) !.*

Proof. Fixing the position of an object can be done in *n* ways, as the position of anyone of them may be fixed.

So, each circular permutation corresponds to *n* linear permutations, depending upon where from we start.

There are *n* ! linear permutations, it follows that there are $\dfrac{n\,!}{n}$, *i.e.*, (*n* – 1) ! circular permutations.

Theorem 2. *The number of ways in which n persons can be seated round a table is (n – 1)!.*

Proof. Let us consider fix the position of one person and then arrange the remaining (*n* – 1) persons in all possible ways. Clearly, this can be done in (*n* – 1) ! ways.

So, the required number of ways = (*n* – 1) !.

Theorem 3. *Show that the number of ways in which n different beads can be arranged to form a necklace is 1/2 (n – 1) !.*

Proof. Fixing the position of one bead, the remaining $(n-1)$ beads can be arranged in $(n-1)$! ways.

In case of arranging the beads, there is no distinction between the clockwise and the anticlockwise arrangements.

Thus, the required number of ways = 1/2 $(n-1)$!.

1.1.2. ^nC_r (Combinations)

Each different groups which can be formed by taking some or all of a number of objects of their arrangements is called a combination.

- If number of all combinations of n distinct objects, taken or at a time, is given by

$$^nC_r = \frac{n!}{(r!) \times (n-r)!}$$

- If $0 \le r \le n$, then $^nC_r = {}^nC_{n-r}$
- If $1 \le r \le n$, then $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- If $1 \le r \le n$, then $n \times {}^{n-1}C_{r-1} = (n-r+1) \times {}^nC_{r-1}$
- If n and r are positive integers such that $1 \le r \le n$, then $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$
- If $1 \le r \le n$, then $^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+n}$
- If $^nC_p = {}^nC_q \;\Rightarrow\; p = q$ or $p + q = n.$

1.1.3. Binomial Theorem (without proof) for Positive Integral Index

$$(x+y)^n = {}^nC_0 + x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + \ldots + {}^nC_{n-1}\, xy^{n-1} + {}^nC_n\, y^n$$

Here, $(x + y)^0 = 1$

$$(x+y)^1 = (x+y) = {}^1C_0\, x + {}^1C_1\, y$$
$$(x+y)^2 = x^2 + 2xy + y^2 = {}^2C_0\, x^2 + {}^2C_1\, xy + {}^2C_2\, y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = {}^3C_0\, x^3 + {}^3C_1\, x^2y + {}^3C_2\, xy^2 + {}^3C_3\, y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$= {}^4C_0\, x^4 + {}^4C_1\, x^3y + {}^4C_2\, x^2y^2 + {}^4C_3\, xy^3 + {}^4C_4\, y^4 \text{ and so on.}$$

Pascal's Triangle

The pascal's triangle showing the coefficients of various terms in a binomical expansion.

Index	*Coefficients*
0	1
1	1 ∇ 1
2	1 ∇ 2 ∇ 1
3	1 ∇ 3 ∇ 3 ∇ 1
4	1 ∇ 4 ∇ 6 ∇ 4 ∇ 1
5	1 ∇ 5 ∇ 10 ∇ 10 ∇ 5 ∇ 1
6	1 ∇ 6 ∇ 15 ∇ 20 ∇ 15 ∇ 6 ∇ 1
7	1 ∇ 7 ∇ 21 ∇ 35 ∇ 35 ∇ 21 ∇ 7 ∇ 1
8	1 8 28 56 70 56 28 8 1

We observe that:

- every row starts with 1 and ends with 1;
- every coefficient (except the first and the last) in a row is the sun of two coefficients in the preceeding row, one just before it and the other just after it.

Look at the following pattern:

For $n = 6$, we have the coefficients ${}^6C_0, {}^6C_1, {}^6C_2, {}^6C_3, {}^6C_4, {}^6C_5, {}^6C_6$

For $n = 7$, we have the coefficients ${}^7C_0, {}^7C_1, {}^7C_2, {}^7C_3, {}^7C_4, {}^7C_5, {}^7C_6, {}^7C_7$,

For $n = 8$, we have the coefficients ${}^8C_0, {}^8C_1, {}^8C_2, {}^8C_3, {}^8C_4, {}^8C_5, {}^8C_6, {}^8C_7, {}^8C_8$ and so on.

1.1.3.1. *Binomial Theorem for Expansion Terms*

Theorem 4. *If x and y are real numbers then for all* $n \in N$,

$$(x + y)^n = {}^nC_0x^n + {}^nC_1 x^{n-1} y + {}^nC_2x^{n-2} y^2 + ... + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n,$$

i.e.,

$$(x + y)^n = \sum_{r=0}^{n} C_r \;.\; x^{n-r} y^r.$$

Proof. We shall prove the theorem by using the principle of mathematical induction.

Let P (n) be the statement:

$$(x + y)^n = {}^nC_0x^n + {}^nC_1 x^{n-1} y + {}^nC_2x^{n-2} y^2 + ... + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n.$$

Clearly, the statement P (1) is

$$(x + y)^1 = {}^1C_0 x^1 + {}^1C_1 x^0y = (x + y).$$

This shows that P (1) is true.

Let P (m) be true. Then,

$$(x + y)^m = {}^mC_0x^m + {}^mC_1 x^{m-1} y + {}^mC_2x^{m-2} y^2 + ... {}^mC_{m-1} + {}^mC_my^m \qquad ...(i)$$

Multiplying both sides of (i) by ($x + y$), we get

$$\begin{aligned}(x + y)^{m+1} &= {}^mC_0x^{m+1} + {}^mC_0x^my + {}^mC_1x^my \\ &\quad + {}^mC_1x^{m-1} y^2 + {}^mC_2x^{m-1} y^2 + {}^mC_2x^{m-2} y^3 + ... \\ &\quad + {}^mC_{m-1}x^2y^{m-1} + {}^mC_{m-1} xy^m + {}^mC_mxy^m + {}^mC_m{}^{m+1} \\ &= {}^mC_0x^{m+1} [{}^mC_0 + {}^mC_1] x^my + [{}^mC_1 + {}^nC_2] x^{m-1} y^2 + ... \\ &\quad + [{}^mC_{m-1} + {}^mC_m] xy^m + {}^mC_m y^{m+1} \\ &= {}^{m+1}C_0x^{m+1} + {}^{m+1}C_1x^my + {}^{m+1} C_2x^{m-1} y^2 + ... \\ &\quad + {}^{m+1}C_mxy^m + {}^{m+1} C_{m+1} y^{m+1}\end{aligned}$$

$$[\because {}^{m+1}C_0 = 1 = {}^mC_0;\ {}^mC_m = 1 = {}^{m+1}C_{m+1} \text{ and } {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r]$$

This shows that P (m + 1) is true, whenever P(m) is true.

Thus, by the principle of mathematical induction, the theorem for all $n \in$ N.

Remark. ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ are known as binomial coefficients.

1.1.3.2. *General Term*

The expansion of $(x + y)^n$ has (n + 1) terms. Since ${}^nC_r = {}^nC_{n-r}$, we have ${}^nC_0 = {}^nC_n$; ${}^nC_1 = {}^nC_{n-1}$, and so on.

Thus, the coefficient of the terms equidistant from the beginning and the end in a binomial expansion are equal.

General term in the expansion of $(x + y)^n$ and general term = (r + 1)th term,

$$t_{r+1} = {}^nC_rx^{n-r} y^r.$$

1.1.3.3. *Middle Term*

The middle terms in the expansion of $(x + y)^n$ *:*

We know that the expansion of $(x + y)$ has $(n + 1)$ terms.

(*i*) *If n is even, the middle term* = $\left(\frac{n}{2}+1\right)$ *th term.*

For example, in the expansion of $(x + 2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{\text{th}}$ *i.e.* 5^{th} term.

(*ii*) *If n is odd, the middle term*

$$\frac{1}{2}\ (n + 1)th\ term\ and\ \left\{\frac{1}{2}(n+1)+1\right\} th\ term.$$

For example, is the expansion of $(2x - y)^7$, the middle term is $\left(\frac{7+1}{2}\right)^{\text{th}} = 4^{\text{th}}$ and $\left(\frac{7\ +\ 1}{2}\right)^{\text{th}}$ *i.e.*, 5^{th} terns.

pth term from the end in $(x + y)^n$

= $(n + 1 - p + 1)$th term from the beginning.

= $(n - p + 2)$th term from the beginning.

Deductions

Replacing x by 1 and y by x in binomial expansion of $(x + y)^n$, we have

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n.$$

In this expansion, $(r + 1)$th term is the general term, given by

$$t_{r+1} = {}^nC_rx^r.$$

Replacing y by $-y$ in the binomial expansion of $(x + y)^n$, we get

$$(x - y)^n = {}^nC_0x^n - {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ...$$

$$\Rightarrow \qquad (x - y)^n = \sum_{r=0}^{n} (-1)^n\ .\ {}^nC_rx^{n-r}\ y^r$$

$$\Rightarrow \qquad t_{r+1} = (-1)^r\ .\ {}^nC_rx^{n-r}\ y^r.$$

In this expansion, $t_{r+q} = (-1)^r\ .\ {}^nC_rx^{n-r}y^r.$

$$(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + ... + (-1)^n\ .\ {}^nC_nx^n.$$

$$\Rightarrow \qquad (1 - x)^n = \sum_{r=0}^{n} (-1)^n\ .\ {}^nC_rx^r$$

Hence, in the expression of $(1 - x)^n$, we have $t_{r+1} = (-1)^r\ .\ {}^nC_rx^r$.

Example 13. *Expand* $(x^2 + 2y)^5$ *by the binomial theorem.*

Sol. Using the binomial theorem, we have

$$(x^2 + 2y)^5 = {}^5C_0\ .\ (x^2)^5 + {}^5C_1\ .\ (x^2)^4\ (2y) + {}^5C_2\ .\ (x^2)^3,\ (2y)^2$$
$$+ {}^5C_3\ .\ (x^2)^2\ (2y)^3\ .\ {}^5C_4\ .\ x^2\ (2y)^4 + {}^5C_5\ .\ (2y)^5$$
$$= x^{10} + 5x^8\ (2y) + 10x^6\ (4y^2) + 10x^4\ (8y^3) + 5x^2\ (16y^4) + 32y^5$$
$$= x^{10} + 10x^8y + 40x^6y^2 + 80x^4y^3 + 32y^5$$

Example 14. *Expand* $\left(2x - \frac{3}{y}\right)^5$ *by the binomial theorem.*

Sol. Using the binomial theorem, we have

$$\left(2x - \frac{3}{y}\right)^5 = {}^5C_0 \,.\, (2x)^5 - {}^5C_1 \,.\, (2x)^4 \,.\, \left(\frac{3}{y}\right) + {}^5C_2 \,.\, (2x)^3 \,.\, \left(\frac{3}{y}\right)^2$$

$$- {}^5C_3 \,.\, (2x)^2 \left(\frac{3}{y}\right)^3 + {}^5C_4 \,.\, (2x) \left(\frac{3}{y}\right)^4 - {}^5C_5 \,.\, \left(\frac{3}{y}\right)^5$$

$$= 32x^5 - \left(5 \times 16x^4 \times \frac{3}{y}\right) + \left(10 \times 8x^3 \times \frac{9}{y^2}\right) - \left(10 \times 4x^2 \times \frac{27}{y^3}\right)$$

$$+ \left(5 \times 2x \times \frac{81}{y^4}\right) - \frac{243}{y^5}$$

$$= 32x^5 - \frac{240x^4}{y} + \frac{720x^3}{y^2} - \frac{1080x^2}{y^3} + \frac{810x}{y^4} - \frac{243}{y^5}.$$

Example 15. *Using the binomial theorem, expand* $[(x + y)^5 + (x - y)^5]$ *and hence find the value of* $[(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5]$.

Sol. We have, $(x + y)^5 + (x - y)^5$

$$= [{}^5C_0x^5 + {}^5C_1x^4y + {}^5C_2x^3y^2 + {}^5C_3x^2y^3 + {}^5C_4xy^4 + {}^5C_5y^5]$$
$$+[{}^5C_0x^5 - {}^5C_1x^4y + {}^5C_2x^3y^2 - {}^5C_3x^2y^3 + {}^5C_4xy^4 - {}^5C_5y^5]$$
$$= 2\,[{}^5C_0x^5 + {}^5C_2x^3y^2 + {}^5C_4\,xy^4]$$
$$= 2\,[x^5 + 10x^3y^2 + 5xy^4]$$

Putting, $x = \sqrt{2}$ and $y = 1$

$$(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5 = 2[\,(\sqrt{2}\,) + 10\,(\sqrt{2}\,)^3 \,.\, 1^2 + 5\,(\sqrt{2}\,)\,1^4]$$

$$= 2\,[4\sqrt{2} + 20\sqrt{2} + 5\sqrt{2}\,] = 58\sqrt{2}\,.$$

Coefficient of *x* and Term Independent of *x* in Binomial Theorem

The coefficients of each expansion are the entries in Row n of Pascal's Triangle. So, the coefficient of each term r of the expansion of $(x + y)^n$ is given by $C(n,\ r - 1)$. The exponents of x descend, starting with n, and the exponents of y ascend, starting with 0, so the rth term of the expansion of $(x + y)^2$ contains $x_{n-(r-1)_{y_{r-1}}}$

In the expansion of $\left(x + \frac{1}{x}\right)^{2n}$, where, $n \neq 0$, middle term is $\left(\frac{2n+1+1}{2}\right)^{\text{th}} \cdot (n + 1)^{\text{th}}$ term as $2n$ is even.

This term is called the term independent of x or the constant term.

Example 16. Expand $\left(x^2 + \frac{3}{x}\right)^4$, $x \neq 0$.

Sol. $\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0\,(x^2)^4 + {}^4C_1\,(x^2)^3\left(\frac{3}{x}\right) + {}^4C_2\,(x^2)^2\left(\frac{3}{x}\right)^2 + {}^4C_3\,(x^2)\left(\frac{3}{x}\right)^3 + {}^4C_1\left(\frac{3}{x}\right)^4$

$= x^8 + 4x^6 \cdot \frac{3}{x} + 6x^4 \cdot \frac{9}{x^2} + 4x^2 \cdot \frac{27}{x^3} + \frac{81}{x^4}$

$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}$.

Example 17. *Expand* $(98)^5$.

Sol. $(98)^5 = (100 - 2)^5$

$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4 \cdot 2 + {}^5C_2\,(100)^3 2^2 - {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 - {}^5C_3\,(2)^5$

$= 10000000000 - 500000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000 \times 8 \times 5 \times 100 \times 16 - 32$

$= 9039207968$.

Example 18. *Find the 10th term of* $(2x^2 + 1/x)^{12}$.

Sol. We know that $(a + b)^n$

$(r + 1)^{th}$ term, $t_{r+1} = {}^nC_r = {}^nC_r\, a^{n-r}\, b^r$

$\therefore$ 10^{th} term, $t_{10} = t_{9+1} = {}^{12}C_9\,(2x^2)^{12-9} \cdot \left(\frac{1}{x}\right)^9$ $\left[\because a = 2x^2, b = \frac{1}{x}, n = 12, r = 9\right]$

$= -{}^9C_4\left(\frac{4x}{5}\right)^4\left(\frac{5}{2x}\right)^5$ $[\because {}^9C_5 = {}^9C_4]$

$= -\left[\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}\frac{2^8\,x^4}{5^4} \cdot \frac{5^5}{2^5\,x^5}\right]$

$= -\frac{5040}{x}$.

Example 19. *Find the* 5^{th} *term from the end in the expansion* $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^7$.

Sol. pth term from the end = $(n - p + 2)^{th}$ term from the beginning

$\therefore$ In the expression of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^n$, we have

5^{th} term from the end $= (9 - 5 + 2)^{th}$ term $= 6^{th}$ term $= t_6 = t_{5+1}$

$= (-1)^5 \cdot {}^9C_5\left(\frac{x^3}{2}\right)^{9-5} \cdot \left(\frac{2}{x^2}\right)^5 = -{}^9C_4 \cdot \left(\frac{x^3}{2}\right)^4\left(\frac{2}{x^2}\right)^5$

$= -\left(\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times \frac{x^{12}}{16} \cdot \frac{32}{x^{10}}\right) = -252x^2$.

Example 20. *Find the middle term in the expansion of* $\left(x - \frac{1}{2y}\right)^{10}$.

Sol. Total number of terms

Middle term = 6^{th} term = t_{5+1}

$$= (-1)^5 \cdot {}^{10}C_5\, x^{(10-5)} \cdot \left(\frac{1}{2y}\right)^5 = \frac{-63x^5}{8y^5}.$$

Example 21. *Find the coefficients of* x^{32} *and* x^{-17} *in the expansion of* $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

Sol. Let the term containing x^{32} be the $(r + 1)^{th}$ term

$$t_{r+1} = (-1)^r\, {}^{15}C_r\, (x^4)^{15-r} \left(\frac{1}{x^3}\right)^r$$

$$= (-1)^r\, {}^{15}C_r\, x^{(60-7r)} \quad ...(i)$$

Putting $60 - 7r = 32$, we get $r = 4$.

∴ $(4 + 1)^{th}$, *i.e.*, the 5^{th} term contains x^{32}

Putting $r = 4$ in (*i*), we get

$$t^5 = (-1)^4 \cdot {}^{15}C_4\, x^{32} = \left(\frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1}\right) x^{32} = 1365x^{32}.$$

∴ coefficient of x^{32} in the given expansion is 1365.

Again, let the term containing x^{-17} be the $(s + 1)^{th}$ term.

We have, $$t_{s+1} = (-1)^s\, {}^{15}C_s\, (x^4)^{(15-s)} \left(\frac{1}{x^3}\right)^5$$

$$= (-1)^s\, {}^{15}C_s\, x^{(60-7s)} \quad ...(ii)$$

Putting $60 - 7s = -17$, we get $s = 11$.

Hence, $(11 + 1)$th, *i.e.*, 12th term contains x^{-17}.

Putting $s = 11$ in (*ii*), we get

$$t_{12} = (-1)^{11}\, {}^{15}C_{17}\, x^{-17}$$

$$= -{}^{15}C_4\, x^{-17} = \left(\frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1}\right) x^{-17}$$

$$= -1365x^{-17}$$

Thus, the coefficient of x^{-17} in the given expansion is –1365.

Example 22. *Find the term independent of x in the expansion of*

(*i*) $\left(x^2 + \frac{1}{x}\right)^9$ (*ii*) $\left(2x + \frac{1}{x}\right)^{10}$.

Sol. (*i*) Let the $(r + 1)$th term be independent of x.

In the expansion of $\left(x^2 + \frac{1}{x}\right)^9$, we have

$$t_{r+1} = {}^9C_r \cdot (x^2)^{9-r} \cdot \left(\frac{1}{x}\right)^r = {}^9C_r\, x^{18-3r} \quad ...(i)$$

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