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***Published by* :**

Romesh Chander Khanna & Vineet Khanna
for **KHANNA PUBLISHERS**
2-B, Nath Market, Nai Sarak,
Delhi-110006 (India)

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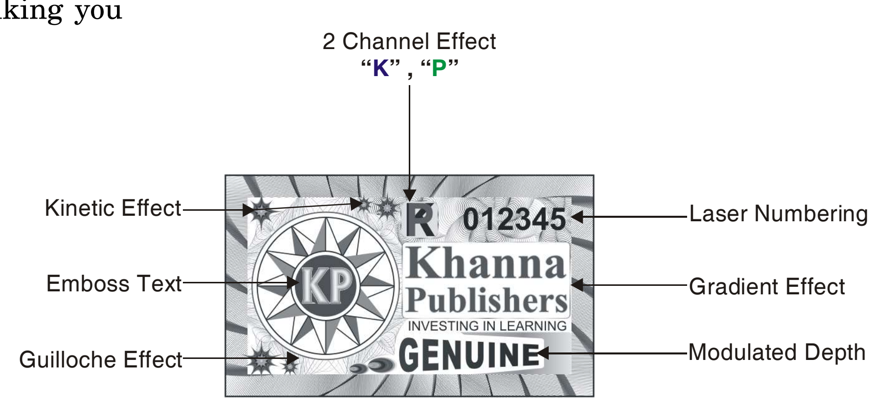

ISBN No. 978-93-92549-07-6

First Edition : 2021

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Unit-I

Chapter 1

Analysis

FIELD PROPERTIES

The real number system is of all a set $\{a, b, c, ...\}$ on which the operations of addition and multiplication are defined so that every pair of real numbers has a unique sum and product, both real numbers, with the following properties.

(i) Commutative laws: $a + b = b + a$ and $ab = ba$.

(ii) Associative laws: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

(iii) Distributive law: $a(b + c) = ab + ac$.

(iv) There are distinct real numbers 0 and 1, $a + 0 = a$ and $a1 = a$ for all a.

(v) For each a there is a real number $-a \ni a + (-a) = 0$, and if $a \neq 0$, there is a real number $1/a \ni a(1/a) = 1$.

THE ORDER RELATION

The real number system is ordered by the relation <, which has the following properties.

(i) For each pair of real numbers a and b, exactly one of the following is true: $a = b$ or $a < b$ or $b < a$.

(ii) Transitive: If $a < b$ and $b < c$, then $a < c$.

(iii) If $a < b$, then $a + c < b + c$ for any c and if $0 < c$, then $ac < bc$.

- **The Triangle Inequality:** If a and b are any two real numbers, then $|a + b| \leq |a| + |b|$.
- If a and b are any two real numbers, then $|a - b| \geq ||a| - |b||$ and $|a + b| \geq ||a| - |b||$.

SUPREMUM OF A SET

A set K of real numbers is bounded above if there is a real number $b \ni x \leq b$ whenever $x \in K$. In this case b is an upper bound of K.

If β is an upper bound of K, but no number less than β then β is a supremum of K, $\beta = \sup K$.

- **The Archimedean Property:** If a and b are positive number, then $na > b$ for some integer n.

Infimum of a Set

A set K of real numbers is bounded below if there is a real number $a \ni x \geq a$ whenever $x \in K$. In this case a is a lower bound of K.

If α is a lower bound of K, but no number greater than α, then α is an infimum of K, $\alpha = \inf K$.

- A set K is said to be bounded if it is bounded above as well as bounded below.

Open Set and Closed Set

If x_0 is a real number and $\varepsilon > 0$, then the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is an ε-neighborhood of x_0. If a set K contains an ε-neighborhood of x_0, then K is a neighborhood of x_0, and x_0 is an interior point of K.

The set of interior points of K is the interior of K, denoted by K^0. If every point of K is an interior point, then K is open.

- A set K is closed if K^c is open.

Deleted Neighborhood

A deleted neighborhood of a point x_0 is a set that contains every point of some neighborhood of x_0 except for x_0 itself.

- The union of open sets is open
- The intersection of closed sets is closed

Limit Point, Boundary Point, Isolated Point

Let K be a subset of R. Then

(i) x_0 is a limit point of K if every deleted neighborhood of x_0 contains a point of K.

(ii) x_0 is a boundary point of K if every neighborhood of x_0 contains at least one point in K and one not in K. The set of boundary points of K is the boundary of K, denoted by ∂K. The closure of K is denoted by $\overline{K}$, $\overline{K} = K \cup \partial K$

(iii) x_0 is an isolated point of K, if $x_0 \in K$ and there is a neighborhood of x_0 that contains no other point of K.

(iv) x_0 is an exterior to K, if x_0 is in the interior of K^c. The collection of such points is the exterior of K.

- *A* set is closed iff it contains all its limit points.

Dense and Perfect Sets

- A subset K of R is said to be dense in R if every point of R is a limit point of K.
- A set K is said to be dense in itself if every point of K is a limit point of K.
- A subset K of R is said to be a perfect set if it is closed and dense in itself.

Open Coverings

A collection A of open sets is an open covering of a set K if every point in K is contained in a set A belonging to A, (i.e), if $K \subset \cup \{A \mid A \in \text{A}\}$.

- **Heine–Borel Theorem:** If A is an open covering of a closed and bounded subset K of the real line, then K has an open covering $\widetilde{A}$ consisting of finitely many open sets belonging to A.
- **Bolzano–Weierstrass Theorem:** Every bounded infinite set of real numbers has at least one limit point.

Countable and Uncountable Sets

An infinite set K is said to be countably infinite or denumerable or enumerable if it is equivalent to the set N of Natural numbers. Otherwise it is uncountable.

- Countable union of countable set is countable.
- Every subset of a countable set is countable.
- The set of all rational numbers is countable.

Limit

$f(x)$ approaches the limit l as x approaches x_0 and write $\lim_{x \to x_0} f(x) = l$

- If $\lim_{x \to x_0} f(x)$ exists, then it is unique.(i.e) if $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} f(x) = l_2$, then $l_1 = l_2$.
- If $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$ then

$$\lim_{x \to x_0} (f+g)(x) = l_1 + l_2,$$

$$\lim_{x \to x_0} (f-g)(x) = l_1 - l_2$$

$$\lim_{x \to x_0} (fg)(x) = l_1 l_2$$

- and if $l_2 \neq 0$ $\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{l_1}{l_2}$.

Continuous

(i) A function g is continuous at x_0, if g is defined on an open interval (a, b) containing x_0 and $\lim_{x \to x_0} g(x) = g(x_0)$

(ii) A function g is continuous from the left at x_0, if g is defined on an open interval (a, x_0) and $g(x_{0^-}) = g(x_0)$

(iii) A function g is continuous from the right at x_0, if g is defined on an open interval (x_0, b) and $g(x_{0^+}) = g(x_0)$

Piecewise Continuous

A function g is piecewise continuous on $[a, b]$ if

(i) $g(x_{0^+})$ exists for all x_0 in $[a, b)$

(ii) $g(x_{0^-})$ exists for all x_0 in $(a,b]$

(iii) $g(x_{0^+}) = g(x_{0^-}) = g(x_0)$ for all but finitely many points x_0 in (a, b).

If (iii) fails to hold at some x_0 in (a,b), g has a jump discontinuity at x_0. Also, g has a jump discontinuity at a if $g(a+) \neq g(a)$ or at b if $g(b-) \neq g(b)$

- If f and g are continuous on a set K, then so are $f+g$, $f-g$ and fg.

  f/g is continuous at each x_0 in $K \ni g(x_0) \neq 0$.

Removable Discontinuities

- Let g be defined on a deleted neighborhood of x_0 and discontinuous at x_0. A function g has a removable discontinuity at x_0 if $\lim_{x \to x_0} g(x)$ exists.

Uniform Continuity

A function g is uniformly continuous on a subset K of its domain if, $\forall\, \varepsilon > 0$, there is a $\delta > 0 \ni |g(x) - g(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$ and $x, x_0 \in K$.

- If g is continuous on a closed and bounded interval $[a, b]$, then g is uniformly continuous on $[a, b]$.
- If g is continuous on a set K, then g is uniformly continuous on any finite closed interval contained in K.
- If g is monotonic and non constant on $[a, b]$, then g is continuous on $[a,b]$ iff its range $R_g = \{g(x) | x \in [a, b]\}$ is the closed interval with endpoints $g(a)$ and $g(b)$.
- If g is continuous on a finite closed interval $[a,b]$, then g is bounded on $[a, b]$.
- **Intermediate Value Theorem:** Suppose that f is continuous on $[a, b]$, $f(a) \neq f(b)$, and μ is between $f(a)$ and $f(b)$. Then $f(c) = \mu$ for some c in (a, b).

Derivative

A function f is differentiable at an interior point x_0 of its domain if the difference quotient

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, x \neq x_0 \text{ limit exists}$$

- If f is differentiable at x_0, then f is continuous at x_0.
- If f and g are differentiable at x_0, then so are $f+g$, $f-g$ and fg with

  (i) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

  (ii) $(f-g)'(x_0) = f'(x_0) - g'(x_0)$

  (iii) $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

  The quotient f/g is differentiable at x_0 if $g(x_0) \neq 0$, with

  (iv) $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{[g(x_0)]^2}$

- **The Chain Rule:** Suppose that g is differentiable at x_0 and f is differentiable at $g(x_0)$. Then the composite function $h = f \circ g$, defined by $h(x) = f(g(x))$, is differentiable at x_0, with

$$h'(x_0) = f'(g(x_0))\, g'(x_0).$$

RIGHT- AND LEFT-HAND DERIVATIVES

If f is defined on $[x_0, b)$ the right-hand derivative of f at x_0 is defined to be

$$f'_+(x_0) = \lim_{x \to x_{0+}} \frac{f(x) - f(x_0)}{x - x_0} \text{ the limit exists,}$$

If f is defined on $(a, x_0]$ the left-hand derivative of f at x_0 is defined to be

$$f_-'(x_0) = \lim_{x \to x_{0-}} \frac{f(x) - f(x_0)}{x - x_0} \text{ the limit exists.}$$

DIFFERENTIABLE ON THE CLOSED INTERVAL

(i) f is differentiable on the closed interval $[a,b]$ if f is differentiable on the open interval (a, b) and $f_+'(a)$ and $f_-'(b)$ both exist.

(ii) f is continuously differentiable on $[a, b]$ if f is differentiable on $[a, b]$, f' is continuous on (a, b), $f_+'(a) = f'(a+)$ and $f_-'(b) = f'(b-)$ both exist.

- **Rolle's Theorem:** Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a,b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some c in the open interval (a, b).
- **Intermediate Value Theorem for Derivatives:** Suppose that f is differentiable on $[a, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c) = \mu$ for some c in (a, b).
- **Generalized Mean Value Theorem:** If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then $[g(b) - g(a)]\, f'(c) = [f(b) - f(a)]\, g'(c)$ for some $c \in (a, b)$.
- **Mean Value Theorem:** If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a,b), then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some $c \in (a, b)$.
- If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).
- If f' exists and does not change sign on (a, b), then f is monotonic on (a, b), increasing, non decreasing, decreasing, or non increasing as $f'(x) > 0$, $f'(x) \geq 0$, $f'(x) < 0$, or $f'(x) \leq 0$, respectively, for all $x \in (a, b)$.
- If $|f'(x)| \leq M$, $a < x < b$, then $|f(x) - f(x_0)| \leq M|x - x_0|$, $x, x_0 \in (a, b)$.
- **L'Hospital's Rule:** Suppose that f and g are differentiable and g' has no zeros on (a, b). Let $\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = 0$ or $\lim_{x \to b-} f(x) = \pm\infty$ and $\lim_{x \to b-} g(x) = \pm\infty$, and suppose that $\lim_{x \to b-} \dfrac{f'(x)}{g'(x)} = l$ (finite or $\pm\infty$). Then $\lim_{x \to b-} \dfrac{f(x)}{g(x)} = l$.

SEQUENCE

- A function $S : N \to R$ is known as real sequence and its denoted by $\{S_n\}$.

BOUNDED SEQUENCES

A sequence $\{S_n\}$ is bounded above if there is a real number $b \ni S_n \le b$ for all n, bounded below if there is a real number $a \ni S_n \ge a$ for all n, bounded if there is a real number $r \ni |S_n| \le r$ for all n.

CONVERGENCE OF A SEQUENCE

A sequence $\{S_n\}$ converges to a limit l if $\forall\ \varepsilon > 0$ there is an integer $m \ni |S_n - l| < \varepsilon$ if $n \ge m$.

$\{S_n\}$ is convergent if $\lim\limits_{n\to\infty} S_n = l$.

A sequence that does not converge is diverges, or is divergent.

- Every convergent sequence is bounded.
- The limit of a convergent sequence is unique.
- Every bounded sequence has a limit point.
- The set of the limit points of a bounded sequence has the greatest and the least members.
- A necessary and sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.
- A sequence $\{S_n\}$ is said to be a cauchy sequence if $\forall \varepsilon > 0 \exists\ m \in N \ni |S_{n+p} - S_n| < \varepsilon$ $\forall\ n \ge m \forall\ p \ge 1$
- **Sandwich theorem**: If $\{a_n\}$ and $\{b_n\}$ are two sequences converging to same limit l then $\{c_n\}$ is a sequence $\ni a_n \le c_n \le b_n\ \forall\ n \ge m, m \in N$, then limiting value of $c_n = l$
- **Cauchy's first theorem on limits**: If $\lim\limits_{n\to\infty} a_n = l$ then $\lim\limits_{n\to\infty}\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) = l$
- If a sequence $\{a_n\}$ of positive terms converges to a positive limit l, then $\lim\limits_{n\to\infty}\left(a_1 a_2 \ldots a_n\right)^{1/n} = l$
- **Cesaro's theorem**: If the sequences $\{a_n\}$ and $\{b_n\}$ converge to finite limit a and b respectively, then $\lim\limits_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n} = ab$
- **Cauchy's second theorem on limits:** If $\{a_n\}$ is a positive term sequence if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists then so does $\lim\limits_{n\to\infty} (a_n)^{1/n}$, and the two limits are equal.

A sequence $\{u_n\}$ is decreasing if $u_n \ge u_{n+1}$ for all n, or increasing if $u_n \le u_{n+1}$ for all n. A monotonic sequence is a sequence that is either increasing or decreasing. If $u_n > u_{n+1}$ for all n, then $\{u_n\}$ is decreasing, while if $u_n < u_{n+1}$ for all n, $\{u_n\}$ is increasing.

- (i) If $\{u_n\}$ is increasing, then $\lim\limits_{n\to\infty} u_n = \sup\{u_n\}$.

  (ii) If $\{u_n\}$ is decreasing, then $\lim\limits_{n\to\infty} u_n = \inf\{u_n\}$.

Subsequence of a Sequence

A sequence $\{x_k\}$ is a subsequence of a sequence $\{x_n\}$ if $x_k = x_{n_k}$, $k \geq 0$ where $\{x_{n_k}\}$ is an increasing infinite sequence of integers in the domain of $\{x_n\}$.

- If $\lim_{n\to\infty} x_n = x \quad (-\infty \leq x \leq \infty)$, then $\lim_{k\to\infty} x_{n_k} = x$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.
- If $\{x_n\}$ is monotonic and has a subsequence $\{x_{n_k}\} \ni \lim_{k\to\infty} x_{n_k} = x \quad \{-\infty \leq x \leq \infty\}$, then $\lim_{n\to\infty} x_n = x$.
- A point $\bar{x}$ is a limit point of a set S iff there is a sequence $\{x_n\}$ of points in $S \ni x_n \neq \bar{x}$ for $n \geq 1$, and $\lim_{n\to\infty} x_n = \bar{x}$.
- If $\{x_n\}$ is bounded, then $\{x_n\}$ has a convergent subsequence.

Series

- A necessary condition for convergence of an infinite series $\sum u_n$ converges, then $\lim_{n\to\infty} u_n = 0$
- **Cauchy's Convergence Criterion for Series:** A series $\sum u_n$ converges iff for every $\varepsilon > 0$ there is an integer $m \ni |u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon \ \forall n \geq m$ and $p \geq 1$.
- If a series $\sum u_n$ of positive monotonic decreasing terms converges then not only $u_n \to 0$ but also $nu_n \to 0$ as $n \to \infty$.
- The positive term geometric series $1 + r + r^2 + \ldots$ converges for $r < 1$ and diverges for $r \geq 1$.
- A positive term series $\sum \frac{1}{n^p}$ is converges iff $p > 1$
- **The Comparison Test**: If $0 \leq u_n \leq v_n \ \forall n \geq m, m \in N$, then

  (i) $\sum u_n$ is convergent if $\sum v_n$ is convergent

  (ii) $\sum v_n$ is divergent if $\sum u_n$ is divergent
- Suppose that $u_n \geq 0$ and $v_n > 0$ for $n \geq m$, and $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ where, $0 < l < \infty$. Then $\sum u_n$ and $\sum v_n$ converge or diverge together.
- **Cauchy's Root Test** : If $u_n \geq 0$ for $n \geq m, m \in N$ such that $\lim_{n\to\infty} (u_n)^{1/n} = l$ then the series

  (i) converges if $l < 1$, (ii) diverges if $l > 1$, (iii) test fails if $l = 1$
- **D'Alembert's Ratio Test:** If $u_n \geq 0$ for $n \geq m, m \in N$ such that $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = l$ then the series

  (i) converges if $l < 1$, (ii) diverges if $l > 1$, (iii) test fails if $l = 1$
- **Raabe's Test:** If $u_n \geq 0$ for $n \geq m, m \in N$ such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ then the series

  (i) converges if $l > 1$, (ii) diverges if $l < 1$, (iii) test fails if $l = 1$

- **Logarithmic Test:** If $u_n \geq 0$ for $n \geq m, m \in N$ such that $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = l$ then the series (i) converges if $l > 1$, (ii) diverges if $l < 1$, (iii) test fails if $l = 1$
- **The Integral Test:** Let $u_n = f(n),\ n \geq a,$ where f is positive, non increasing, and locally integrable on $[a, \infty)$. Then $\sum u_n$ and $\int_a^\infty f(x)\,dx$ converges or diverge together.
- **Alternating Series Test** : The series $\sum (-1)^n u_n$ converges if $0 \leq u_{n+1} \leq u_n$ and $\lim_{n\to\infty} u_n = 0.$
- **Absolute Convergence:** A series $\sum u_n$ converges absolutely if $\sum |u_n|$ is convergent.
- **Abel's Test:** If $u_{n+1} \leq u_n$ for $n \geq N,\ \lim_{n\to\infty} u_n = 0$ and $\sum u_n$ is a convergent series, then the series $\sum u_n b_n$ is also converges.
- **Dirichlet's Test:** If b_n is a positive, monotonic decreasing function with limit zero, and if for the series $\sum u_n$, the sequence $\{S_n\}$ of partial sums is bounded then the series $\sum u_n b_n$ is convergent.

SEQUENCE AND SERIES OF FUNCTIONS

Suppose that $\{f_n\}$ is a sequence of functions, defined on an interval I and the sequence of values $\{f_n(x)\}$ converges for each x in I. $\{f_n\}$ converges pointwise on I to the limit function f is defined by

$$f(x) = \lim_{n\to\infty} f_n(x), x \in S.$$

- If the series $\sum f_n$ converges for every point $x \in I$ and $f(x) = \sum_{n=0}^{\infty} f_n(x),\ \forall x \in [a, b]$ the function f is called the sum of the series $\sum f_n$ on $[a,b]$.

UNIFORM CONVERGENCE

A sequence $\{f_n\}$ is said to converge uniformly on an interval $[a,b]$ to a function f if for any $\varepsilon > 0$ and for all $x \in [a, b]$ there is an integer N (independent of x but depend on ε) such that for all $x \in [a, b]$, $|f_n(x) - f(x)| < \varepsilon,\ \forall n \geq N.$

- Uniform convergence implies pointwise convergence. Converse not true.
- **Cauchy's Uniform Convergence Criterion:** A sequence of functions $\{f_n\}$ converges uniformly on $[a,b]$ iff for each $\varepsilon > 0$ there is an integer N such that $|f_{n+p}(x) - f_n(x)| < \varepsilon$ $\forall n \geq N, p \geq 1.$
- Let $\{f_n\}$ be a sequence of functions, such that $\lim_{n\to\infty} f_n(x) = f(x), x \in [a, b]$ and let $M_n = \operatorname{Sup}_{x\in[a,b]} |f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on $[a,b]$ iff $M_n \to 0$ as $n \to \infty$.
- **Weierstrass's Test :** The series $\sum f_n$ converges uniformly on $[a,b]$ if there exist a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a, b]$, $|f_n(x)| \leq M_n,\ \forall n,$
- If $\{f_n\}$ converges uniformly to f on $[a,b]$ and each f_n is continuous on $[a,b]$ then so is f. (*i.e.*,) a uniform limit of continuous functions is continuous.

- **Abel's Test:** If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of x in the interval $[a,b]$ and $b_n(x)$ is bounded for all values of n and x concerned, and if the series $\sum u_n(x)$ is uniformly convergent on $[a,b]$, then so also is the series $\sum b_n(x)u_n(x)$
- **Dirichlet's Test:** If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of x in $[a,b]$ and $b_n(x)$ tends uniformly to zero for $a \le x \le b$, and if there is a number $M>0$ independent of x and n such that for all values of x in $[a,b]$, $\left|\sum_{r=1}^{n} u_r(x)\right| \le M, \forall n$ then the series $\sum b_n(x)u_n(x)$ is uniformly convergent on $[a,b]$

MONOTONIC FUNCTIONS

A function f is non decreasing on an interval I if

$$f(x) \le f(y) \text{ whenever } x \text{ and } y \text{ are in } I \text{ if } x < y \qquad ...(1)$$

or non-increasing on I if

$$f(x) \ge f(y) \text{ whenever } x \text{ and } y \text{ are in } I \text{ and } x < y \qquad ...(2)$$

In either case, f is monotonic on I. If $\le$ can be replaced by $<$ in (1), f is increasing on I. If $\ge$ can be replaced by $>$ in (2), f is decreasing on I. In either of these two cases, f is strictly monotonic on I.

- Suppose that g is monotonic on (a, b) and define $\alpha = \inf_{a<x<b} f(x)$ and $\beta = \sup_{a<x<b} f(x)$.

(i) If g is non decreasing, then $g(a+) = \alpha$ and $g(b-) = \beta$.

(ii) If g is non increasing, then $g(a+) = \beta$ and $g(b-) = \alpha$.

- A function $f(x)$ is said to be convex in $[x_1, x_2]$ if $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2)$, $0 \le \lambda \le 1$ in interval $[x_1, x_2]$ and $f''(x) \ge 0$.
- A function $f(x)$ is said to be concave in $[x_1, x_2]$ if $f(\lambda x_1 + (1-\lambda)x_2) \ge \lambda f(x_1) + (1-\lambda)f(x_2)$, $0 \le \lambda \le 1$ in interval $[x_1, x_2]$ and $f''(x) \le 0$.
- A point x is said to be fixed point of the curve if $f(x) = x$.

FUNCTIONS OF BOUNDED VARIATION

- A finite set P of points, $x_0, x_1, x_2, \ldots, x_n$ where $a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b$ is called a partition of the interval $[a, b]$.
- Corresponding to partition P, $|f(x_i) - f(x_{i-1})|$ is known as length of the i^{th} sub interval and $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ is known as the variation of the function in $[a, b]$.
- Supremum of the sum is known as variation or total variation of f on $[a, b]$. (i.e) $V(f, a, b) = \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.
- The function f is said to be bounded variation on $[a, b]$ iff its total variation $V(f, a, b)$ is finite.

- A bounded monotonic function is a function of bounded variation.
- The sum (difference) of two functions of bounded variation is also of bounded variation.
- The variation function of a function f of bounded variation is continuous iff f is a continuous function.

Riemann Integrable

- Corresponding to the partition P, Upper sum and Lower sum are defined as follows;
  $U(P,f)=\sum_{i=1}^{n} M_i \Delta x_i$ and $L(P,f)=\sum_{i=1}^{n} m_i \Delta x_i$ respectively, where M and m denotes supremum and infimum of $f(x)$ in i^{th} sub interval.
- The upper and lower integral on $[a, b]$ are defined as $\overline{\int_a^b} fdx = \inf U$ and $\underline{\int_a^b} fdx = \sup L.$
- $\overline{\int_a^b} fdx = \underline{\int_a^b} fdx = \int_a^b fdx$, we say that f is Riemann integrable on $[a, b]$.
- A necessary and sufficient condition for the integrability of a bounded function f is that $\lim\{U(P,f)-L(P,f)\}=0$ when the norm of the partition P tends to 0
- If f is bounded on $[a, b]$, the oscillation of f on $[a, b]$ is defined by
  $$W_f[a,b] = \sup_{a \le x, x' \le b} |f(x)-f(x')|$$
- A bounded function f is integrable on $[a, b]$ iff $\forall \varepsilon > 0 \, \exists$ a partition $P \ni U(P,f)-L(P,f) < \varepsilon$.
- If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.
- If f is monotonic on $[a, b]$, then f is integrable on $[a, b]$.
- A bounded function f having a finite number of points of discontinuity on $[a, b]$ is integrable on $[a, b]$.
- If f and g are integrable on $[a, b]$, then so is $f+g$
- If f is integrable on $[a, b]$ and m is a constant, then mf is integrable on $[a, b]$.
- If f and g are integrable on $[a, b]$ and $f(x) \le g(x)$ for $a \le x \le b$, then $\int_a^b f(x)dx \le \int_a^b g(x)\, dx.$
- If f is integrable on $[a, b]$, then so is $|f|$.
- If f and g are integrable on $[a, b]$, then so is the product fg.
- **Fundamental Theorem of Calculus:** If f is continuous on $[a, b]$, then f has an antiderivative on $[a, b]$. If F is any anti derivative of f on $[a, b]$, then $\int_a^b f(x)\, dx = F(b)-F(a).$

Improper Integrals

- If f is locally integrable on $[a, b)$, define $\int_a^b f(x)\, dx = \lim_{c \to b-} \int_a^c f(x)\, dx$ provided that the limit exists (finite).

OBJECTIVE COMPETITIVE MATHEMATICS

For IIT-JAM (Joint Admission Test for MSc) from IITs, CSIR-UGC NET/SET (JRF & LS), PhD Entrance, UPSC Examination and Useful for Other Competitions

About the Book:

The text begins with supportive theory and high degree of standard questions with solved examples followed by detailed explanations of the objective type questions.

This book will also come in handy for IIT-JAM, CSIR-UGC NET/SET (JRF & LS), PhD Entrance, UPSC Examination and other Competitions.

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