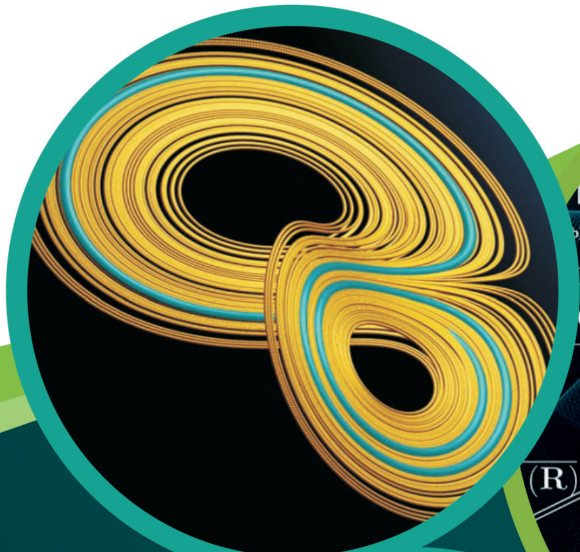
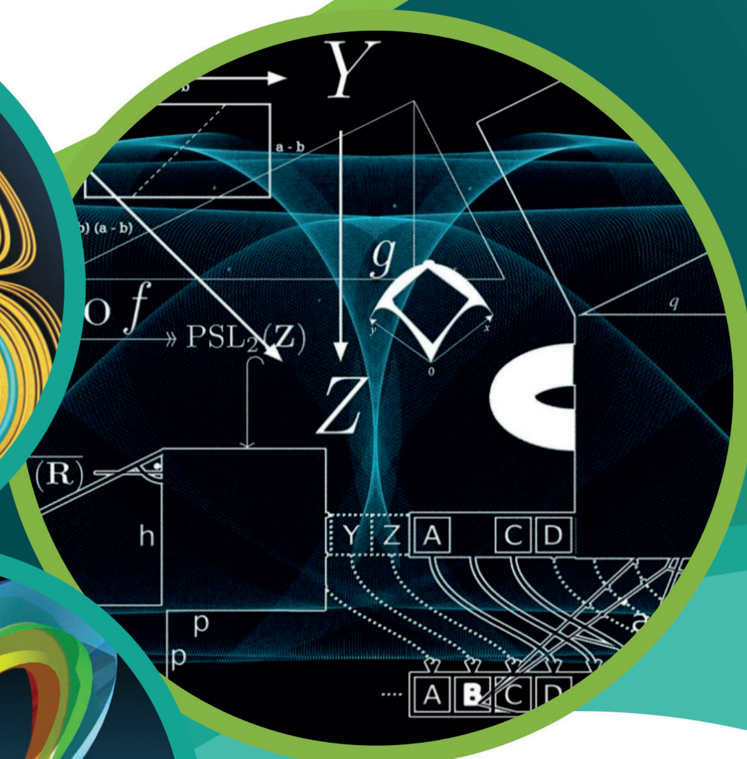
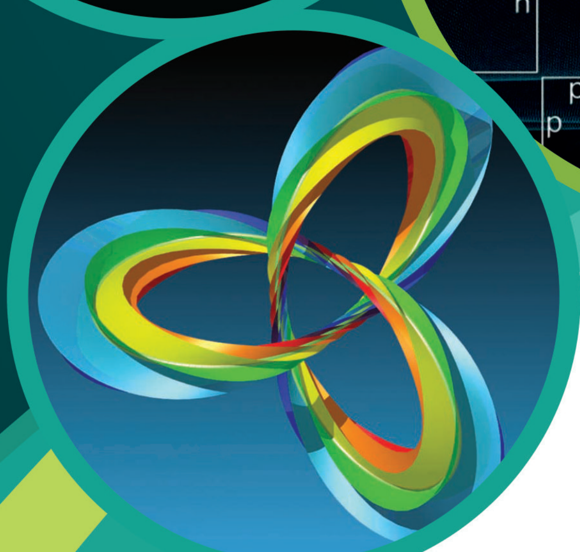
Fundamentals of Mathematical Analysis

Rashmi Bhardwaj
Ajaya Kumar Singh







KHANNA PUBLISHERS®
Investing in Learning®

FUNDAMENTALS OF MATHEMATICAL ANALYSIS

Rashmi Bhardwaj
Fellow of Institute of Mathematics and Applications, UK
Professor of Mathematics
Guru Gobind Singh Indraprastha University (GGSIPU), New Delhi (India)

Ajaya Kumar Singh
Reader (SS) in Mathematics,
Ekamra College, Bhubaneswar–751002

KHANNA PUBLISHERS®
Investing in Learning®
Operational Office :
4575/15, Onkar House, Opp. Happy School,
Ground Floor, Daryaganj, New Delhi 110 002
Phones : 011-45033819 • *Mob*. 09811541460
email : contactus@khannapublishers.in

Published by:

Romesh Chander Khanna & Vineet Khanna
for KHANNA PUBLISHERS
2-B, Nath Market, Nai Sarak
Delhi-110 006 (India)

Website : www.khannapublishers.in

© 1979 and onward

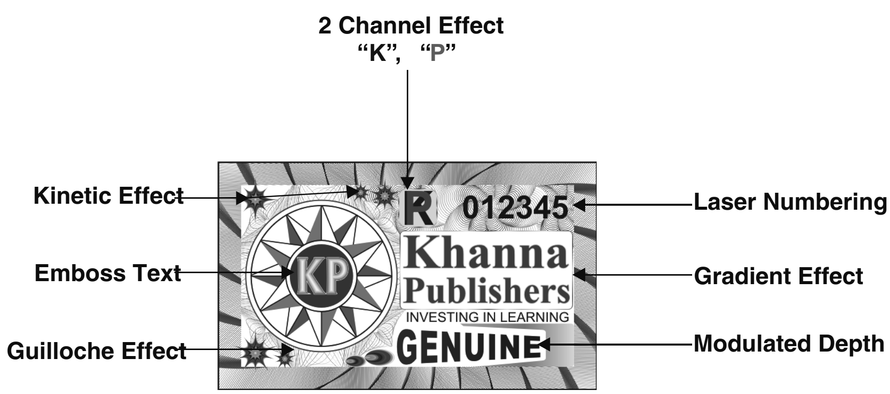
This book or part thereof cannot be translated or reproduced in any form without the written permission of the Authors and the Publishers. The right to translation, however, reserved with the author alone.

Copyright: Authors and Publishers Jointly

Hologram & Description

To all readers of our books, to prevent yourself from being defrauded by piraters, please make sure that there is an Hologram on the cover of our books with the below specifications. If you find any book without Hologram and Description, please mail us at **contactus@khannapublishers.in**

Thanking you



ISBN No.: 978-93-92549-36-6

***First Edition:* 2024**

Preface

The present text book 'Fundamentals of Mathematical Analysis' is specifically designed for under-graduate, post-graduate, Ph.D Scholar students of Indian and Foreign Universities, Autonomous and Degree Colleges with special emphasis being laid on B.A. & B.Sc (Mathematics) course as structured by UGC (CBCS) syllabus. The theory portion is done with utmost lucidity and systematic progression. The preliminary ideas and scope of the book is modified and detailed course contents are attached for better understanding of subsequent chapters are also given. The book contains twenty-one chapters.

First chapter (Chapter 1) contains Basic Concepts and Preliminaries. It gives ideas and background materials for the development of the ideas in the subsequent chapters. In addition Basic Features, preliminary ideas and some notations in Mathematical Reasoning, Set, Mathematical Induction, Relation, Function and it's properties, Real Sequences for the proper understanding and their importance background necessary for the study of Real Analysis. These are basic tools of Mathematics. All key discussions are reflected in this chapter. Set theory became foundation of all branches of Mathematics which introduced between 1873 and 1895 by a famous German Mathematician, George Cantor (1845–1918). Also we develop the concept of Relation and Function which are basic ideas and properties that underlie almost all mathematical and physical relationships between variable quantities, regardless of the form in which they are expressed. Function and Sequence play a central role in Calculus, Real Analysis and it's applications. The real or complex valued functions are frequently used in Real Analysis that is X be any set, The function $f: X \to \mathbb{R}$, $f : X \to \mathbb{C}$ are respectively. Such functions to define algebraic operations like addition and multiplication play an important role in the study of linear or Vector Space, Linear Transformations. The polish mathematician S. Banach (1892–1945) is introducing a Vector Spaces, Linear Transformations on Vector Spaces. Lastly Expected Short and long Possible Questions are also given.

Second chapter (Chapter 2) contains Real and Complex numbers. It gives ideas and background materials for the development of the ideas in the inadequacy of rationals. Rationals number system is inadequate for many purposes both as a field and as an ordered set. In addition Basic Features and some notations are used in this chapter. We try as far as possible to build our number system to include rationals and preserve all the essential properties of rationals (it's algebraic and Order properties). This extended system of numbers shall be called the real numbers and the collection of all real numbers shall be denoted by $\mathbb{R}$. We call an element of $\mathbb{R}$-$\mathbb{Q}$ an irrational number. A great Mathematician and analyst Richard Dedekind (1831–1916) constructed R out of the knowledge of $\mathbb{Q}$ by a method called as Dedekind section and proved that R is complete. In this chapter we discussed Absolute Value function on an ordered field F, Intervals and related theorems. Also discussed Completeness principles like bounded set, lub and glb property and concept of supremum and infimum. Most important relation Density and it's properties and related theorems. Lastly we discussed about decimal representation of real numbers for the proper understanding and their importance background necessary for the study of Real Analysis. These are basic tools of Mathematics. Real Number System became foundation of all branches of B.Sc, Master Degree in Mathematics and allied subjects. It is very essential for research work. Here we develop the Concept of Order properties that underlie almost all mathematical and physical relation.

Third chapter (Chapter 3) contains Cardinality of Sets, One-to-One correspondence and Bijection map. It gives for the development of the ideas in the finite and infinite set. In addition basic features and some notations are used in this chapter. Also discussed about Cardinality of a set, Equivalence between two sets and the concept of Infinite cardinal. Here the concept of Countability helps in higher studies. Also the most important reflection is Schrder–Bernstein property. In this chapter important theorems related with Countability and Uncountability. Lastly we discussed the set of Algebraic and Transcendental numbers and it's related theorems.

Fourth chapter (Chapter 4) contains Analytical (metric) properties of real line $\mathbb{R}$ or complex plane $\mathbb{C}$ and concepts of distance function or metric or distance between two elements on $\mathbb{R}$ or $\mathbb{C}$. We shall study the idea of neighbourhood of a point simply we write nbd, interior point of a set, open and closed sets, Limit points of a set of real numbers and Bolzano–Weierstrass theorem and existence of limit points of a set. Lastly we discussed definitions and notations, relations of closure, Interior Boundary (Edge) of a set and it's related theorems.

Fifth chapter (Chapter 5) contains sequences ($\mathbb{R}$ and $\mathbb{C}$) and their convergences. In this chapter we shall study a special class of functions whose domain is the set of natural numbers ($\mathbb{N}$) and range is the set of real numbers ($\mathbb{R}$) then sequence is called Real Sequence and when range is the set of complex numbers ($\mathbb{C}$) then sequence is called complex sequence. Next we discussed a related concept of boundedness of sequences. The most important concept of Analysis is that of Limit or Convergence which indicates that a sequence is convergent or divergent or null. Related theorems on limit and properties of limit function $f: \mathbb{C} \to \mathbb{R}$, c = set of all convergent sequences and c_0 = set of all null sequences are discussed. German mathematician Karl Weierstrass (1815–95) is one of the founder of Analysis which is formulated the completeness principle of $\mathbb{R}$ using monotonic sequence which has been proved using lub property of $\mathbb{R}$ (see Chapter 2). We can derive the completeness of $\mathbb{R}$ by assuming Weierstrass Completeness Principle. Similarly Cantor used the nested interval property to establish the completeness of $\mathbb{R}$. This chapter contains definition of subsequences, cluster point (Limit point). So we established a fundamental results (the Bolzano Weierstrass Theorem) about subsequence. The notable French mathematician A.L. Cauchy (1789–1857) introduced a concept of fundamental (Cauchy) sequence and established a Cauchy's completeness principle which turns out to be equivalent to completeness of $\mathbb{R}$. This principle plays in analysis. This chapter introduces recurrence relation, real number Euler Constant (γ) introduced by Leonard Euler (1707–83) and turns out to be finite and its approximate value is 0.577216..... It is surprising that no body yet knows whether Euler Constant is rational or irrational. Next it establishes Extended Real Number system which is related to the concept of limit. So we introduce two more concepts Limit Superior and Limit inferior and their properties and related theorems. Lastly we established the Complex Sequence. The most definitions and results on convergence of real sequences are meaningful for Complex Analysis but the concept of upper bound, Lower bound, monotonically, dgs to + ..., lim sup, lim inf depend upon order concepts therefore these are to be dropped in case of complex sequence. Even though $\mathbb{C}$ is not an ordered field but it is possible to define bounded subset of $\mathbb{C}$ by using the concept of modulus. This chapter gives a suggestion as to how to deal with completeness problem of $\mathbb{C}$ by definition and related theorems.

Sixth chapter (Chapter 6) contains Infinite Series (Real and Complex) and their convergences. In this chapter we shall discuss the techniques of testing the behaviour of the infinite series. Infinite Series of real or complex are well connected with Analysis. That is a series is the sum of the terms of a sequence. If $u_1, u_2, ..., u_n, ...$ is a sequence then the sum $u_1 + u_2 + ... + u_n + ...$ is called an Infinite Series and is denoted by $\sum_{n=1}^{\infty} u_n$ or simply notated by $\sum u_n$, where nth term (u_n) in an infinite series $\sum u_n$ is called the general term of the series. We discussed the techniques of testing the behaviour of the infinite series. Here this chapter emphasizes with Arithmetic Series, Geometric Series, Harmonic Series and Arithmetico Geometric Series. First of all more basic convergence tests for basic convergence tests for infinite series with non-negative terms are discussed and after that next section we discussed series that contain both positive and negative terms. If a series whose terms alternate between positive and negative called Alternating Series. That is either $u_1 - u_2 + u_3 - u_4 + ...$ form or $-u_1 + u_2 - u_3 + u_4 - ...$ form and discussed the convergence techniques by Leibnitz Test for Alternating Series.

Important part of this chapter is that a series $\sum u_n$ which has infinitely many positive and infinitely many negative terms or which has the complex terms. So we established a relation between Absolute Convergence and convergences by suitable definitions and theorems. Next we established Abel and Dirichlet's tests for arbitrary term series. Also Infinite Series Sum is effected by the rearrangement of terms where divergent series may become convergent by suitable examples by using Dirichlet's and Riemann Theorems in detail towards the end. But none of the tests is suitable for dealing with series that are not absolute convergence. For such series we require special tests like Abel's Summation formula and Dedekind's Test for conditional convergence test. Finally we have to say that the discussion about techniques of testing the behaviour of Sequence of Functions, Series of Functions, Power Series, Fourier Series, Integrals and Transforms, Complex integration, Power Series, Taylor Series and Laurent Series, and Concept of Metric Spaces (Four important C) in subsequent chapters.

Seventh chapter (Chapter 7) contains Limit of a function and it uses in different functions. In the preceding chapter we studied basic idea about neighbourhood (nbd) of a point and deleted nbd of a point, cluster point, limiting concepts, the limit of a sequence of real numbers. In this chapter we introduced the basic concepts and idea of a limit by mathematical notion of the limit of a function and limit theorems. Also we focused on computational methods and precise definitions. The development of calculus and Analysis was invented in the 1680's by English mathematician Sir Issac Newton (1642–1727) and German mathematician and philosopher Gottfried Wilhelm Leibnitz (1646–1716). Leibnitz introduced the term "Function" to indicate a quantity that depended on a variable and he invented "infinitesimally small" numbers as a way of handling the concept of a limit. We want to investigate the behaviour or change of $f: X \to \mathbb{R}$ where $X < \mathbb{R}$ in the nbd of the point $a \in \mathbb{R}$, where a may or may not belong to . The idea of the limit of a function f at a point a to be meaningful. It is necessary that f be defined at points near a. In this chapter, the discussions of limit relies on the use of sequences. The fundamental properties of limits of functions on intervals are discussed. Both mathematician formulated a notion of function and idea of quantities being "close to" one another. A precise definition of limit were taken by Karl Weierstrass (1815–97). He insisted the definition of limit is the one we use today. We discussed one sided limits of f at a. In particular, the existence of one sided limits can be

Contents

1

Basic Concepts and Preliminaries

1.1. INTRODUCTION

This chapter contains *Basic Concepts and Preliminaries*. It gives ideas and background materials for the development of the ideas in the subsequent chapters. In addition Basic Features and some notations in Mathematical Reasoning, Set, Mathematical Induction, Relation, Function and it's properties, Sequences for the proper understanding and their importance background necessary for the study of Real Analysis. These are basic tools of Mathematics. Set theory became foundation of all branches of Mathematics which introduced between 1873 and 1895 by a famous German Mathematician, George Cantor (1845–1918). Also we develop the concept of Relation and Function which are basic ideas and properties that underlie almost all mathematical and physical relationships between variable quantities, regardless of the form in which they are expressed. Function and Sequence play a central role in calculus, Real Analysis and it's applications. The real or complex valued functions are frequently used in Real Analysis that is X be any set. The function $f: X \to \mathbb{R}$, $f: X \to \mathbb{C}$ are respectively. Such functions to define algebraic operations like addition and multiplication play an important role in the study of linear or Vector Space, Linear Transformations. The polish mathematician S. Banach (1892–1945) is introducing a Vector Spaces, Linear Transformations on Vector Spaces. All key discussions are reflect in this chapter one by one.

1.2. MATHEMATICAL REASONING

Logic is a science of thought or a science of reasoning as expressed in language.

It is derived from the Greek word 'logika' which means a collection of rules used to draw valid conclusions. The word 'Statement' is the basic building blocks of logic. So we discuss the following important basic features and some notations:

1. **Statement/Proposition:** A proposition/statement is a declarative sentence which is either true or false, but not both.

   OR

   Any sentence which is either true or false but not both is called a proposition or a statement. The sentence must be a declarative sentence. For example,

   (*i*) The Sun rises in the East (T)
   (*ii*) Sita is a boy (F)
   (*iii*) $1 + 1 = 2$ (T)
   (*iv*) Today is Monday (T).

   For example, the following sentences are not statements:

   (*i*) What is your name?
   (*ii*) Wish you a happy birthday.
   (*iii*) How beautiful is that flower?

2. **Notation:** A proposition is normally/generally denoted by small roman letters like p, q, r, s, etc.
3. **Truth Value of Statement:** The truth or falsity of a statement is called as truth value of that statement. For example,

   (*i*) $p: 2 > 5$ (F) (*ii*) $q: 5 > 3$ (T).

   Note: If a statement is true, then it's truth value is True or T.

   If a statement is false, then it's truth value is False or F.
4. Interrogative sentences, exclamatory sentences, order sentences, variable in a sentence, words like wise, young, good, bad in a sentence are not statements. These are called FUZZY statements. For example,

   (*i*) What is your name?
   (*ii*) Oh! what a beautiful scene it is.
   (*iii*) Close the door.
   (*iv*) Rama is a good teacher.
   (*v*) $x < 5$.
5. **Open Statement:** A sentence has a variable but has a particular value is called a open statement. For example, $x + 3 = 5$.

1.2.1. Logical Operators Used in Logic

Logical Operators used in logic are as follows:

1. Negation/Not (~)
2. Conjunction/AND ($\wedge$)

3. Disjunction/or ($\vee$)
4. Exclusive or ($\underline{\vee}$ or $\oplus$)
5. Conditional/if ... then ($\rightarrow$)
6. Biconditional/iff ($\leftrightarrow$)
7. Implication ($\Rightarrow$) implies
8. Double Implification/Bi-implification ($\Leftrightarrow$)

1.2.1.1. *Negation/Not* (~)

Let p be a proposition then the negation of p is denoted by $\sim p$ or 'p'. It is read as 'not p' or 'It is not the case that p'.

Axiom: For any proposition p, if p is true, then $\sim p$ is false and if p is false, then $\sim p$ is true.

Truth Table: See below:

p	$\sim p$
T	F
F	T

For example,

$p \equiv$ Today is Friday.

$\sim p \equiv$ Today is not Friday.

1.2.1.2. *Conjunction/And* ($\wedge$)

Let p and q be propositions, then the conjunction of p and q is denoted by '$p \wedge q$'.

It is read as p and q.

Axiom: A conjunction $p \wedge q$ is true.

If both p and q are true, otherwise false.

Truth Table: See below:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

For example,

1. p: Raju went to the market.
2. q: Raju bought a notebook.
3. $p \wedge q$: Raju went to the market and bought a note book.

1.2.1.3. *Disjunction/OR* ($\vee$)

Let p and q be propositions, then the disjunction of p and q is denoted by $p \vee q$.

It is read as p or q.

Axiom: A disjunction $p \vee q$ is false.

If both p and q are false, otherwise true.

Truth Table: See below:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

For example,

1. p: Sunday is a holiday.
2. q: Thursday is a holiday.
3. $p \vee q$: Either Sunday is a holiday or Thursday is a holiday.

1.2.1.4. *Exclusive* ($\vee/\oplus$)

Let p and q be propositions.

Then the exclusive or of p and q is denoted by $p \oplus q$.

It is read as p exclusive or q.

Axiom: An exclusive or $p \oplus q$ is false when both p and q are true or false, otherwise true.

If both p and q are false, otherwise true.

Truth Table: See below:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1.2.1.5. *Conditional/if ... then* ($\rightarrow$)

Let p and q be propositions.

Then conditional of p and q is denoted by $p \rightarrow q$.

It is read as:

if p then q or p only if q

Axiom: A conditional $p \rightarrow q$ is false.

When p is true and q is false, otherwise true.

Truth Table: See below:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

For example,

1. p: Nita has ten rupees.
2. q: she will buy a bottle of jam.
3. $p \rightarrow q$: If Nita has ten rupees then she will buy a bottle of jam.

1.2.1.6. *Biconditional/iff* ($p \leftrightarrow q$)

Let p and q be prepositions.

Then biconditional of p and q is denoted by $p \leftrightarrow q$.

It is read as:

'p iff q'

'q iff p'

'q is necessary and sufficient for p'

'p is necessary and sufficient for q'.

Axiom: A biconditional $p \leftrightarrow q$ is true.

When both p and q are true or false, otherwise false.

Truth Table: See below:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

For example,

1. $p: 2$ is an even number.
2. $q: 4$ is an even number.
3. $p \leftrightarrow q: 2$ is an even number iff 4 is an even number.

1.2.1.7. *Implication* ($\Rightarrow$)

If a conditional $p \to q$ is a tautology, then we say that

p implies q *i.e.*, $p \Rightarrow q$.

1.2.1.8. *Double Implication/Bi-implication/Implies and is Implied by* ($\Leftrightarrow$)

If a biconditional $p \Leftrightarrow q$ is a tautology then we say that p implies and is implied by q *i.e.*, $p \Leftrightarrow q$.

Logical Connectives: In Section 1.2.1, the operator negation (~) / not is not a logical connective but other operators are also called logical connectives.

1.2.2. More About Connectives

1. Equivalent statement.
2. Compound statement/composite statement.

1.2.2.1. *Equivalent Statement*

Let p and q be propositions. Then the equivalent statement of p and q is denoted by '$p \Leftrightarrow q$' or '$p \equiv q$'.

It is read as 'p is equivalent to q' *i.e.*, p iff q.
i.e., p is necessary and sufficient for q.

Two statements are said to be equivalent if we have the same truth values.

1. $p \equiv \sim(\sim p)$
2. $p \vee q \equiv \sim(\sim p \wedge \sim q)$
3. $\sim(p \wedge q) \equiv \sim p \vee \sim q$
4. $\sim(p \vee q) \equiv \sim p \wedge \sim q$
5. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
6. $p \vee (q \wedge r) \equiv (p \vee q) \vee (p \vee r)$
7. $p \to (\sim q) \equiv \sim(p \wedge q)$

Truth Table:

p	$\sim p$	$\sim(p)$
T	F	T
F	T	F

1.2.2.2. *Compound Statement/Composite Statement*

More than one statement can be taken by using the connectives is called compound/composite statement. For example,

1. $p \wedge q$
2. $p\ (q \vee r)$
3. $p \to (p \to q)$
4. $p \to (p \to r)$
5. $\sim p \to (\sim p \wedge q)$
6. $p \vee (p \to r)$

Truth Table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

1.2.3. Conditional Statements

1. Converse Statement 2. Inverse Statement
3. Contrapositive Statement.

1.2.3.1. *Converse Statement*

Given $p \to q$

$\therefore$ Converse of $p \to q$ is '$q \to p$'

1.2.3.2. *Inverse Statement*

Given $p \to q$ and $p \to q$

$\therefore$ Inverse of $p \to q$ is '$\sim p \to \sim q$'.

1.2.3.3. *Contrapositive Statement*

Given $p \to q$

$\therefore$ Contrapositive of $p \to q$ is '$\sim q \to \sim p$'

Truth Table:

p	q	$\sim p$	$\sim q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	T	T	T	T

For example,

1. p: Two angles of a triangle are equal.
2. q: It is Isosceles triangle.
3. $p \to q$: If two angles of a triangle are equal then it is Isosceles.
4. $q \to p$: If the triangle is Isosceles then two angles are equal.
5. $\sim p \to \sim q$: If two angles of a triangle are not equal then it is not Isosceles.
6. $\sim q \to \sim p$: If the triangle is not Isosceles then the two angles are not equal.

1.2.4. Types of Proofs

1. Logically valid/Tautology (T).
2. Logically invalid/Contradiction/Fallacy (F).
3. Neither/Contingency (T, F).

1.2.4.1. *Logically Valid/Tautology* (*T*)

A compound statement which is always true is called as *Tautology*.

Examples of logically valid statements:

1. $p \vee (\sim p)$ (law of excluded middle)
2. $\sim (p \wedge \sim p)$ (law of contradiction)
3. $p \leftrightarrow \sim(\sim p)$ (law of double negation)

} (Classical laws of logic)

4. $p \wedge (p \to q) \to q$
5. $[p \to q) \wedge (p \to r)] \to [p \to q \wedge r]$
6. $[(p \to q) \wedge (q \to r)] \to [p \to r]$ (Principle of Syllogism)
7. $[\sim q \to \sim p] \leftrightarrow [p \to q]$ (Law of contrapositive/ Classical principle of reductio-ad-absurdum)

Truth Table of (1):

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

$\therefore$ $p \vee \sim p$ is a tautology.

1.2.4.2. *Logically Invalid/Contradiction/Fallacy* (*F*)

A compound statement which is always false, is called a *Fallacy*. For Example,

$p \wedge (\sim p)$

Truth Table:

p	$\sim p$	$p \wedge (\sim p)$
T	F	F
F	T	F

$p \wedge (\sim p)$ is a Fallacy.

1.2.4.3. *Neither/Contingency* (*T, F*)

A compound statement which is neither a tautology nor a Fallacy, is called as Neither/Contingency. For example,

1. $p \to q$
2. $\sim p \to \sim q$
3. $\sim q \to \sim p$
4. $\sim(p \vee q)$
5. $\sim p \wedge \sim q$

Truth Table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$\therefore$ $p \to q$ is a neither.

1.2.5. Conditional Theorems

1. Converse theorem
2. Inverse theorem
3. Contrapositive theorem/method of Contrapositive.

1.2.5.1. *Converse Theorem*

$(p \to q) \to (q \to p)$ is neither.

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

$\therefore$ Above statement is neither.

1.2.5.2. *Inverse Theorem*

$(p \to q)$ $(\sim p \to \sim q)$ is neither.

p	q	$\sim p$	$\sim q$	$p \to q$ (a)	$\sim p \to \sim q$ (b)	$a \to b$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

$\therefore$ Above statement is neither.

1.2.5.3. *Contrapositive Theorem/Method of Contrapositive*

1. $(p \to q) \to (\sim q \to \sim p)$ is tautology.

p	q	$\sim p$	$\sim q$	$p \to q$ (a)	$\sim q \to \sim p$ (b)	$a \leftrightarrow b$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	F	T	T

2. $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is tautology

p	q	$\sim p$	$\sim q$	$p \to q$ (a)	$\sim q \to \sim p$ (b)	$a \leftrightarrow b$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

1.2.6. Operations of Logical Compound Statements/ Precedence of Logical Operators

1. Bracket
2. (*i*) Precedence 1 ($\sim$)
   (*ii*) Precedence 2 ($\wedge$, $\vee$)
   (*iii*) Precedence 3 ($\to$, $\leftrightarrow$)

Example 1.1. Simplify $\sim p \wedge q \vee r$.

Answer: Step 1: $\sim p$, **Step 2:** $\sim p \wedge q = a$, **Step 3:** $a \vee r$

Example 1.2. Simplify $\sim p \to q \wedge r$.

Answer: Step 1: $\sim p$, **Step 2:** $q \wedge r = a$, **Step 3:** $\sim p \to a$.

1.2.7. Forms Used in Logic

1. English language form/English sentence/simple.
2. Logical expression form/Symbolic form.

1.2.7.1. *English Language Form/English Sentence/Simple*

For example, you can access the internet from campus only if you are a computer science major or you are not a freshman.

1.2.7.2. *Logical Expression Form/Symbolic Form*

- a: You can access the internet from campus
- c: You are a computer science major
- f: You are a fresh man.

$\therefore$ $a \to ((\vee (\sim f))$ is its logical expression form.

1.2.8. Logical Bit Operators : Bit = 0 and 1

Truth value = F and T

Here '0' represents 'F' and '1' represents 'T'.

Truth Table:

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
1	1	1	1	0
1	0	1	0	1
0	1	1	0	1
0	0	0	0	0

Example 1.3. Bitwise OR, Bitwise AND & Bitwise XOR: Find the bitwise OR, AND and XOR of the two bits 0110110110 and 1100011101

Solution:

0 1 1 0 1 1 0 1 1 0 → length of ten
1 1 0 0 0 1 1 1 0 1 → length of ten
1 1 1 0 1 1 1 1 1 1 → Bitwise OR
0 1 0 0 0 1 0 1 0 0 → Bitwise AND
1 0 1 0 1 0 1 0 1 1 → Bitwise XOR

1.3. THE ALGEBRA OF PROPOSITIONS/PROPERTIES OF PROPOSITION BASIC LOGICAL EQUIVALENT

1. **Idempotence:** (*i*) $p \vee p \Leftrightarrow p$, p is an idempotence statement, (*ii*) $p \wedge p \Leftrightarrow p$, p is an idempotence statement.
2. **Commutative:** (*i*) $p \vee q \Leftrightarrow q \vee p$, (*ii*) $p \wedge q \Leftrightarrow q \wedge p$.
3. **Associative:** (*i*) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$, (*ii*) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$.
4. **Distributive:** (*i*) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$, (*ii*) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.
5. **Double negation:** $\sim(\sim p) \Leftrightarrow p$
6. **De Morgan's laws:** (*i*) $\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$, (*ii*) $\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$
7. If 1 denotes tautology (T) and 0 denotes contradiction (F), then
   (*a*) (*i*) $p \vee 1 \Leftrightarrow 1$ (*ii*) $p \wedge 1 \Leftrightarrow p$
   (*b*) (*i*) $p \vee 0 \Leftrightarrow p$ (*ii*) $p \wedge 0 \Leftrightarrow 0$
   (*c*) (*i*) $p \vee (\sim p) \Leftrightarrow 1$ (*ii*) $p \wedge (\sim p) \Leftrightarrow 0$
   (*d*) (*i*) $\sim 1 \Leftrightarrow 0$ (*ii*) $\sim 0 \Leftrightarrow 1$
8. See below:
   (*i*) $(p \to q) \Leftrightarrow [(\sim q) \to (\sim p)]$
   (*ii*) $(p \to q) \Leftrightarrow [(\sim p) \vee q)]$
   (*iii*) $(p \leftrightarrow q) \Leftrightarrow [(p \to q) \wedge (q \to p)]$
9. **Identity:**
   (*i*) $p \wedge 1 \Leftrightarrow p$ (*ii*) $p \vee 0 \Leftrightarrow p$.
10. **Absorption law:**
    (*i*) $p \vee (p \wedge q) \Leftrightarrow p$ (*ii*) $p \wedge (p \vee q) \Leftrightarrow p$.
11. **Domination law:**
    (*i*) $p \vee 1 \Leftrightarrow 1$ (*ii*) $p \wedge 0 \Leftrightarrow 0$.

Theorem 1.3.1. Suppose A and B are logically equivalent statements involving variables $p_1, p_2, ..., p_n$ suppose that $C_1, C_2, ..., C_n$ are statements. If in A and B, we replace p_1 by C_1, p_2 by C_2 and so until we replace p_n by C_n. Then the resulting statement will still be logically equivalent.

Minterm and Disjunctive:

$x_1, x_2, ..., x_n$ be variables

if a compound statement is $a_1 \wedge a_2 \wedge ... \wedge a_n$, then a misterm based on the variables $x_1, x_2, ..., x_n$ of the form

$$a_1 \wedge a_2 \wedge ... \wedge a_n,$$

where each a_i is x_i.

A compound statement in $x_1, x_2, ..., x_n$, said to disjunctive normal form if it looks like $y_1 \vee y_2 \vee ... \vee y_m$, where the statement $y_1, y_2, ..., y_m$ are different minterms.

For example, $x_1 \wedge \sim x_2 \wedge \sim x_3$ is a minterm because x_1, x_2, x_3 are variables.

The compound statement $(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \sim x_2 \wedge \sim x_3)$ is in disjunctive normal form.

For example, $p \wedge (q \vee r)$ is not a minterm because it is not is the form of $a_1 \wedge a_2$.

The compound statement $(p \wedge (q \vee r)) \wedge ((p \wedge q) \vee (\sim q))$ is not in disjunctive normal form.

But $(p \wedge q \wedge r) \vee (p \wedge q \wedge (\sim r)) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r)$ is equivalent to $p \wedge (q \vee r)$.

$\therefore$ $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r)$ is in disjunctive normal form.

1.3.1. Logical Arguments

Argument: An argument is a finite collection of statements/premises (hypotheses) $A_1, A_2, ..., A_n$ followed by a statement B called *conclusion*.

Argument is of *two* types:

1. Premises 2. Conclusion

It can be written as

$$\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ \underline{A_n} \\ B \end{array}$$

is an elementary argument.

Logically Valid: An argument is valid/logically valid if when $A_1, A_2, ..., A_n$ are true then B is true.

Theorem 1.3.2. An argument with premises A_1, $A_2, ..., A_n$ and conclusion B is valid if the compound statement $A_1 \wedge A_2 \wedge ... \wedge A_n \to B$ is a tautology.

Proof: For the implication

$$A_1 \wedge A_2 \wedge ... \wedge A_n \to B$$

is a tautology, it is the case that whenever $A_1 \wedge A_2 ... \wedge A_n$ is true, then B is also true.

But $A_1 \wedge A_2 \wedge ... \wedge A_n$ is true, if each of $A_1, A_2, ..., A_n$ is true.

Hence, the theorem proof.

Theorem 1.3.3. Substitution: Let an argument with premises $A_1, A_2, ..., A_n$ and Conclusion B is valid and let $P_1, P_2, ..., P_m$ be variables.

If $P_1, P_2, ..., P_m$ are replaced by the statements $C_1, C_2, ..., C_m$, then the resulting argument is still valid.

1.3.2. Rules of Inference

1. **Modus Ponens/Law of Detachment** $\dfrac{\begin{array}{c} p \\ p \to q \end{array}}{q}$

2. **Modus Tollens:** $\dfrac{\begin{array}{c} p \to q \\ \sim q \end{array}}{\sim p}$

3. **Disjunctive Syllogism:** $\dfrac{\begin{array}{c} p \vee q \\ \sim p \end{array}}{q}$

4. **Chain Rule:** $\dfrac{\begin{array}{c} p \to q \\ q \to r \end{array}}{p \to r}$

5. **Resolution:** $\dfrac{\begin{array}{c} p \vee r \\ q \vee (\sim r) \end{array}}{p \vee q}$

1.4. SETS

1.4.1. Meaning of a Set

1. It is an undefined term.
2. It is a collection of well defined objects elements/membranes.
3. It is a collection of objects.
4. All elements can be taken within curly brackets alphabet *i.e.*, { }
5. It can be notated by capital letter *i.e.*, $A, B, C, D, E, F, ...$

   For example,

   (*i*) $A = \{1, 2, 3, 4, 5, ...\}$, (*ii*) $A = \{a, e, i, o, u\}$
   (*iii*) $A = \{*, \Delta, 0\}$ (*iv*) $A = \{*, ...\}$
   (*v*) $A = \{\text{Ram, Sita}\}$.

1.4.2. Notation of Sets

$\mathbb{N}$ = Set of natural numbers = {1, 2, 3, ...}

$\mathbb{Z}$ or I = Set of integers = {..., –2, –1, 0, 1, 2, ...}

$= Z^- \cup \{0\} \cup Z^+$

$\mathbb{Q}$ = Set of Rational numbers

$$= \left\{ \frac{p}{q} \middle| \; p, q \in Z, q \neq 0 \right\}$$

$$= \{..., -2, -1, 0, 1, 2, ... \frac{1}{2}, \frac{-5}{6}, 0.35, -1.4, ...\}$$

$\mathbb{R}$ = Set of Real numbers

$$= \left\{ ..., \frac{-5}{4}, \frac{-4}{3}, 0, \frac{1}{2}, \frac{3}{4}, 1, ... \right\} = Q \cup Q'$$

$\mathbb{C}$ = Set of complex numbers

$= \{x + iy \mid x, y \in Q, i^2 = -1\}$

$= \{..., -2 + 3i, 1, 0, -5, ...\}$

I^+ or $\mathbb{Z}^+$ = Set of positive integers = {0, 1, 2, ...}

W = Set of whole numbers = {0, 1, 2, ...}

$\mathbb{N}^*$ = Set of all natural numbers = {0, 1, 2, ...}

$= W$

$\mathbb{Z}^-$ or I^- = Set of negative integers = {..., –3, –2, –1}

$\mathbb{Q}'$ = Set of irrational numbers

$$= \left\{ \sqrt{2}, \sqrt{3}, ..., 2 + \sqrt{2}, \frac{2 + \sqrt{3}}{2 - \sqrt{2}} \right\}.$$

1.4.3. Description of a Set

1. Tabular/Enlist/Roster/Extension form.
2. Set builder/Rule/Set selector/intention/proposition way form.

1.4.3.1. *Tabular/Enlist/Roster/Extension Form*

For example,

$A = \{1, 2, 3\}$ (Tabular form)/Enlist

$= \{x \mid x \in N, x \leq 3\}$ (Set builder form)

$= \{x \mid x = 1 \text{ or } x = 2 \text{ or } x = 3\}$

(Proposition way form)

1.4.3.2. *Set Builder/Rule/Set Selector/Intention/Proposition Way Form*

For example,

$A = \{*, \Delta, 0\}$ Enlist

$= \{x \mid \; x = * \text{ or } x = \Delta \text{ or } x = 0\}$

(Proposition way form).

1.4.4. Types of Sets

1. Equal set
2. Empty set/Null/Void set
3. Equivalent set
4. Subset
5. Proper subset
6. Super set
7. Universal set
8. Comparable set
9. Power set.

1.4.4.1. *Equal Set*

Two sets A and B are equal, if:

1. $|A| = |B|$.
2. Each elements of A is equal to corresponding each element of B.

   For example,

   $A = \{1, 2\}$

and $B = \{x \mid \; (x - 1)(x - 2) = 0\} = \{1, 2\}$

$\therefore$ $A = B$.

1.4.4.2. *Empty Set/Null/Void Set*

1. A set is said to be empty set if it has no element.
2. It is denoted by

$\phi = \{ \}$ (Tabular form)

$= \{x \mid x \neq x\}$ (Set builder form)

$= \{x \mid x \neq p(x)\}$ (Proposition way form)

where $p(x)$ is a compound statement.

For example, $S = \{n \in N \mid n^2 + 1 = 0\}$, (Set builder form)

$= \{\,\}$, (Tabular form)

$= \phi$.

1.4.4.3. *Equivalent Set*

Two sets A and B are equivalent if

$$|A| = |B|$$

$\therefore$ We say $A \sim B$.

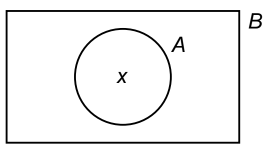
For example,

$A = \{1, 2, 3\} \Rightarrow |A| = 3,$

$B = \{2, 4, 6\} \Rightarrow |B| = 3$

$\therefore \quad A \sim B.$

1.4.4.4. *Subset*



(Venn diagram)

Fig. 1.1.

1. $A \subseteq B$ *i.e.*, $x \in A \Rightarrow x \in B$.
2. $\phi \subseteq$ every set *i.e.*, $\phi \subseteq A$, A is any set.
3. $A \subseteq A$, A is any set.
4. $\phi \subseteq \phi$.

Example 1.4. If $A = \{1, 2\}$, then find all subsets of A.

Solution:

$\phi \subseteq A$

$\{1\} \subseteq A$

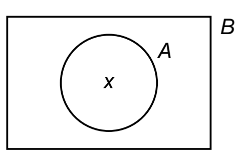
$\{2\} \subseteq A$

$\{1, 2\} \subseteq A$

$\therefore$ All subsets of A are ϕ, $\{1\}$, $\{2\}$, $\{1, 2\}$.

1.4.4.5. *Proper Subset*

1. $A \subset B$ or $A \subsetneq B$ *i.e.*, $x \in A \Rightarrow x \in B$



(Venn diagram)

Fig. 1.2.

2. If $A \subseteq B$ but $A \neq B$; then A is called a subset of B

For example, $N \subset Z \subset Q \subset R \subset \mathbb{C}$

or $N \subsetneq Z \subsetneq Q \subsetneq R \subsetneq \mathbb{C}$

Example 1.5. If $A = \{1, 2\}$, then find all proper subsets of A.

Solution:

$\phi \not\subset A$

$\{1\} \subset A$

$\{2\} \subset A$

Therefore, all proper subsets of A are $\{1\}$, $\{2\}$.

1.4.4.6. *Super Set*

If every element of A is an element of A, then B is a superset of A.

Mathematically,

$$B \supseteq A$$

i.e., B is a superset of A.

i.e., $\quad A \subseteq B$

For example,

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$

$\therefore \quad B \supseteq A.$

1.4.4.7. *Universal Set*

1. It is the highest set.
2. It has sub-set only.

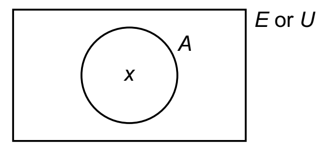


Fig. 1.3.

3. It is denoted by capital letter U or E.
4. Mathematically, if $A \subseteq E$, A is any set *i.e.*, $x \in A \Rightarrow x \in E$.

1.4.4.8. *Comparable Set*

Two sets A and B are called comparable sets if either $A \subseteq B$ or $B \subseteq A$.

Note: If $A \subseteq B$ and $B \subseteq A$, then $A = B$. This is called *Property of Extension*.

1.4.4.9. *Power Set*

The set of all possible subsets of that set is called the power set of a set.

Mathematically, if A is any set, then power set of A is written as

$$P(A) = \{B \mid B \subseteq A\}$$

i.e., $\quad B \in P(A) \Leftrightarrow B \subseteq A.$

Notes:

1. $A \subseteq A \cup B \Leftrightarrow A \in P(A \cup B)$
2. $B \subseteq A \cup B \Leftrightarrow B \in P(A \cup B)$
3. $A \cap B \subseteq A \Leftrightarrow (A \cap B) \in P(A)$
4. $A \cap B \subseteq B \Leftrightarrow (A \cap B) \in P(B)$
5. $\phi \subseteq A \Leftrightarrow \phi \in P(A)$.

Example 1.6. If $A = \{1, 2\}$, then find the following:

(*i*) All subsets of A (*ii*) All proper subsets of A

(*iii*) $P(A)$ (*iv*) $|P(A)|$.

Answer: (*i*) Subsets of A are ϕ, $\{1\}$, $\{2\}$, $\{1, 2\}$

(*ii*) Proper subset of A are $\{1\}$, $\{2\}$

(*iii*) $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

(*iv*) $|P(A)| = 4 = 2^2$.

Notes:

(*i*) If $|A| = n$, then $|P(A)| = 2^n$

(*ii*) If $|A| = 0$, then $|P(A)| = 2^0 = 1$.

For example, $A = \{\phi\}$

$\Rightarrow \quad |A| = 1$

$P(A) = \{\phi, \{\phi\}\} \Rightarrow |P(A)| = 2^1 = 2.$

For example, $A = \{\ \}$

$\Rightarrow \quad |A| = 0$

$P(A) = \{\phi\}$

$\Rightarrow \quad |P(A)| = 2^0 = 1.$

For example,

(*i*) If $A = \{a\}$, then $P(A) = \{\phi, \{a\}\}$

(*ii*) If $A = \{a, b\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

(*iii*) If $A = \{a, b, c\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$.

For example, True or False:

Let S denote the set $\{\{a\}, b, c\}$.

(*a*) $a \notin S$ (T) (*b*) $\{a\} \in S$ (T).

For example, True or False:

$\{\phi\} = \phi$ (T).

1.4.5. Operation on Sets

1. Union
2. Intersection
3. Set difference
4. Symmetric difference
5. Complement

1.4.5.1. *Union of Two Sets*

If A and B are two sets, then the union of two sets A and B is written as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

i.e., $\quad x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B.$

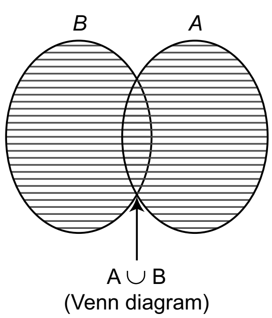


Fig. 1.4.

For example,

If $\quad A = \{a, b, c\}$

and $\quad B = \{a, x, y, b\}$

then $\quad A \cup B = \{a, b, c, x, y\}$ $A \cap B = \{a, b\}$.

Note: The union of n sets $A_1, A_2, ..., A_n$ is written as $A_1 \cup A_2 \cup A_3 \cup ... \cup A_n$ or $\bigcup_{i=1}^{n} A_n$.

1. $A \cup A = A$ (Law of idempotence)
2. $A \cup \phi = A$ (Law of identity)
3. $A \cup B = B \cup A$ (Commutative property)
4. $A \cup B = B$ iff $A \subseteq B$
5. $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative)
6. $A \subseteq (A \cup B)$
7. $B \subseteq (A \cup B)$.

1.4.5.2. *Intersection of Two Sets*

If A and B are two sets, then the intersection of two sets A and B written as follows.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

i.e., $\quad x \in (A \cap B) \Leftrightarrow x \in A \text{ and } x \in B.$

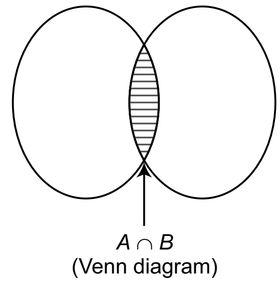


Fig. 1.5.

Notes:

1. $A \cap A = A$ (Law of idempotence)
2. $A \cap \phi = \phi$ (Law of identity)
3. $A \cap B = A$ iff $A \subseteq B$
4. $A \cap B = B \cap A$ (Commutative property)
5. $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative property)
6. $(A \cap B) \subseteq A$
7. $(A \cap B) \subseteq B$.

1.4.5.3. *Set Difference/Difference of Two Sets*

If A and B are two sets, then the difference of A and B is written as:

1. $A - B = \{x \mid x \in A \text{ and } x \notin B\} = A/B =$ only A
2. $B - A = \{x \mid x \in B \text{ and } x \notin A\} = B/A =$ only B

i.e., $\quad x \in A - B \Leftrightarrow x \in A, x \notin B$

$x \in B - A \Leftrightarrow x \in B, x \notin A$

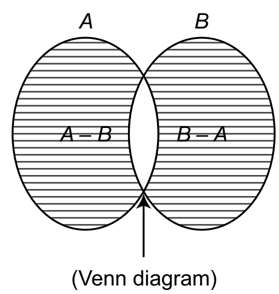


Fig. 1.6.

For example,

(*i*) $\{a, b, c\}/\{a, b\} = \{c\}$ (*ii*) $\{a, b, c\}/\{a, b\} = \{b, c\}$

(*iii*) $\{a, b, \phi\}/\phi = \{a, b, \phi\}$ (*iv*) $\{a, b, \phi\}/\{\phi\} = \{a, b\}$

For example,

$A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$

$A - B = \{1\}, B - A = \{4\}$

$\therefore \quad (A - B) \neq (B - A).$

1.4.5.4. *Symmetric Difference*

The union of the differences $A - B$ and $B - A$ are known as *symmetric difference*.

It is denoted by $A \,\Delta B/A \oplus B$

$$A \,\Delta B/A \oplus B = (A - B) \cup (B - A)$$

$$= (A \cup B) - (A \cap B)$$

$$= (\text{only } A) \cup (\text{only } B).$$

For example,

1. $\{a, b, c\} \Delta \{x, y, a\} = \{b, c, x, y\}$
2. $\{a, b, c\} \Delta \phi = \{a, b, c\}$
3. $\{a, b, c\} \Delta \{\phi\} = \{a, b, c, d\}$.

Notes:

1. $A \Delta A = \phi$
2. $A \Delta B = B \Delta A$ (Commutative)
3. $A \cup B = (A \Delta B) - (A \cap B)$
4. $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.

1.4.5.5. Complement of Set *A*

If A is a subset of an universal set then complement of set A is written as:

$$A' \text{ or } A^c = \{x \mid x \notin A\}$$

i.e., $x \in A^c \Leftrightarrow x \notin A$.

For example, if A = {Monday, Tuesday, Wednesday, Thursday, Friday}

Then A^c = {Saturday, Sunday}.

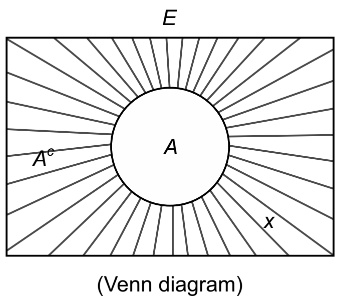


Fig. 1.7.

For example, if

$$A = \{x \in Z \mid x^2 > 0\}, E = Z, \text{ then } A^c = \{0\}.$$

Notes:

(*i*) $\phi' = E$ (*ii*) $E' = \phi$

(*iii*) $A \cup A' = E$ (*iv*) $A \cap A' = \phi$

(*v*) $(A')' = A$ (*vi*) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(*vii*) $A - B = A \cap B'$.

1.4.6. Algebra/Properties of Sets

1.4.6.1. *Closure*

If A, B are two sets then

1. $A \cup B$ is a set
2. $A \cap B$ is a set
3. $A - B$ is a set
4. $B - A$ is a set
5. $A \Delta B$ is a set
6. A^c is a set.

1.4.6.2. *Commutative*

1. $A \cup B = B \cup A$ 2. $A \cap B = B \cap A$
3. $A - B \neq B - A$ 4. $A \Delta B = B \Delta A$.

1.4.6.3. *Associative*

A, B and C are following *four* sets:

1. $A \cup (B \cup C) = (A \cup B) \cup C$
2. $A \cap (B \cap C) = (A \cap B) \cap C$
3. $A - (B - C) \neq (A - B) - C$
4. $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

1.4.6.4. *Idempotence*

1. $A \cup A = A$, A is idempotent.
2. $A \cap A = A$, A is idempotent.
3. $A - A = \phi$, A is not idempotent.
4. $A \Delta A = \phi$, A is not idempotent.

1.4.6.5. *Distributive*

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Notes:

1. $A \subseteq B \Leftrightarrow B' \subseteq A'$ 2. $A - B \Leftrightarrow A \cap B^c$
3. $B - A \Leftrightarrow B \cap A^c$ 4. $(A')' = A$
5. $A \cup B = B$ iff $A \subseteq B$ 6. $A \cap B = A$ iff $A \subseteq B$.

Theorem 1.4.1. (*i*) $A \cup B = B \cup A$

(*ii*) $A \Delta B = B \Delta A$

Proof: (*i*) $X \in A \cup B$

$\Leftrightarrow$ $x \in A$ or $x \in B$

$\Leftrightarrow$ $x \in B$ or $x \in A$

$\Leftrightarrow$ $x \in (B \cup A)$

$\therefore$ $A \cup B = B \cup A$

(*ii*) $X \in A \Delta B$

i.e., $x \in \{(A - B) \cup (B - A)\}$

i.e., $x \in (A - B)$ or $x \in (B - A)$

i.e., $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

i.e., $(x \in A$ and $x \in B')$ or $(x \in B$ and $x \in A')$

i.e., $x \in (A \cap B')$ or $x \in (B \cap A')$

i.e., $x \in (B' \cap A)$ or $x \in (A' \cap B)$

i.e., $(x \in B'$ and $x \in A)$ or $(x \in A'$ and $x \in B)$

i.e., $(x \notin B$ and $x \in A)$ or $(x \notin A$ and $x \in B)$

Theorem 1.4.2. DeMorgan's law:

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'.$$

Theorem 1.4.3. Another DeMorgan's law:

(*i*) $A - (B \cup C) = (A - B) \cap (A - C)$

(*ii*) $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof:

(*i*) $x \in A - (B \cup C)$

$\Leftrightarrow$ $x \in A$ and $x \notin (B \cup C)$

$\Leftrightarrow$ $x \in A$ and $(x \notin B$ and $x \notin C)$

$\Leftrightarrow$ $(x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$

$\Leftrightarrow$ $x \in (A - B)$ and $x \in (A - C)$

$\Leftrightarrow$ $x \in (A - B) \cap (A - C)$

$\therefore$ $A - (B \cup C) = (A - B) \cap (A - C)$ **(Proved)**.

(*ii*) $A - (B \cap C) = (A - B) \cup (A - C)$

$x \in A - (B \cap C)$

$\Leftrightarrow$ $x \in A$ and $x \notin (B \cap C)$

$\Leftrightarrow$ $x \in A$ and $(x \notin B$ or $x \notin C)$

$\Leftrightarrow$ $(x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$

$\Leftrightarrow$ $x \in (A - B)$ or $x \in (A - C)$

$\Leftrightarrow$ $x \in (A - B) \cup (A - C)$

$\therefore$ $A - (B \cap C) = (A - B) \cup (A - C)$ **(Proved)**.

Theorem 1.4.4. Generalised DeMorgan's law:

(*i*) $(A_1 \cup A_2 \cup ... \cup A_n)' = A_1' \cap A_2' \cap ... \cap A_n'$

(*ii*) $(A_1 \cap A_2 \cap ... \cap A_n)' = A_1' \cup A_2' \cup ... \cup A_n'$.

1.4.7. The Cartesian Product of Sets

If A and B are sets, then the Cartesian/Direct product of A and B is defined as:

$$A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

Note (1):

(*i*) $R \times R = \{(x, y) \mid x \in R, y \in R\}$

(*ii*) $A \times A = \{(x, y) \mid x \in A, y \in A\}$

(*iii*) $A \times B \times C = \{(x, y, z) \mid x \in A, y \in B, z \in C\}$

(*iv*) $A_1 \times A_2 \times ... \times A_n = \bigcap_{i=1}^{n} A_i = \{(x_1, ..., x_n) \mid x_i \in A_i, i = 0\,(1)\}$

(*v*) (*a*) $A \times A = A^2$, (*b*) $A \times A \times A = A^3$,

(*c*) $A \times A \times n\text{-times} \times A = A^n$

(*vi*) (*a*) $A \times \phi = \phi = \phi \times A$, (*b*) $\phi \times \phi = \phi$, (*c*) $A \times A \times \phi = \phi$

Note (2):

(*i*) If $|A| = m$ and $|B| = n$, then $|A \times B| = mn$

(*ii*) If $|P(A)| = 2^m$ and $|P(B)| = 2^n$, then $|P(A \times B)| = 2^{mn}$

(*iii*) If $|P(A)| = 2^m$ and $|P(B)| = 2^n$, then $|P(A)| \times |P(B)| = 2^m \times 2^n = 2^{m+n}$

(*iv*) If $|P(A \times B)| \neq |P(A)| \times |P(B)|$.

Note (3):

$(a, b) = (c, d)$ iff $a = c$ and $b = d$

i.e., elements of $A \times B$ are equal, iff they have the same first and second coordinates.

Note (4):

(*i*) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

(*ii*) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Example 1.7. If $A = [-4, 4]$ and $B = [0, 5]$, then $A/B = [-4, 0)$. What is B/A and A^c?

Solution: $A = [-4, 4], B = [0, 5]$

$A/B = [-4, 0)$

$A \cap B = [0, 4]$

$B/A = (4, 3]$

$A^c = E - A$

$= R - [-4, 4]$

$= (-\infty, 4) \cup (4, \infty)$.

For example,

$A = \{a, b\}$ and $B = \{x, y, z\}$

$\therefore$ $A \times B = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$

$B \times A = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}$.

Theorem 1.4.5. Let A, B and C be three sets prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Proof: $(x, y) \in \{A \times (B \cup C)\}$

$\Leftrightarrow$ $x \in A$ and $y \in (B \cup C)$

$\Leftrightarrow$ $x \in A$ and ($y \in B$ or $y \in C$)

$\Leftrightarrow$ ($x \in A$ and $y \in B$)

or ($x \in A$ and $y \in C$)

$\Leftrightarrow$ $(x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Leftrightarrow$ $(x, y) \in \{(A \times B) \cup (A \times C)\}$

$\therefore$ $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Theorem 1.4.6. Let A, B and C be three sets prove that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

What can you conclude about the sets $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$? Why?

Proof: $(x, y) \in \{(A \times B) \cup (A \times C)\}$

$\Leftrightarrow$ $(x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Leftrightarrow$ ($x \in A$ and $y \in B$)

or ($x \in A$ and $y \in C$)

$\Leftrightarrow$ $x \in A$ and $y \in$ (B or C)

$\Leftrightarrow$ $x \in A$ and $y \in (B \cup C)$

$\Leftrightarrow$ $(x, y) \in A \times (B \cup C)$

$\therefore$ $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ **(Proved)**.

From Theorem 1.4.5 and 1.4.6, we conclude that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Since $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

1.5. MATHEMATICAL INDUCTION

In order to establish the truth of a sequence of statement $P(n)$, $\forall$ positive integer n, we adopt method of mathematical induction as follows:

Step 1: We verify the truth of $P(1)$.

Step 2: For any positive integer k assume the truth of $P(k)$.

Step 3: Establish the truth of $P(k + 1)$.

Example 1.8. Prove that $1 + 2 + ... + n = \frac{n(n+1)}{2}$, for all positive integer n.

Solution: Let

$$P(n) = 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

Now $P(I): 1 = \frac{1(1+1)}{2}$

Since $\frac{1(1+1)}{2} = \frac{2}{2} = 1$, $P(1)$ is true.

Let $P(k)$ be true.

Thus $1 + 2 + ... + k$

$$= \frac{k(k+1)}{2}$$

Then $1 + 2 + ... + k + (k + 1)$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{(k+1)}{2}(k+2) = \frac{(k+1)\,(k+1+1)}{2}$$

This shows $P(k + 1)$ is true.

So by the method of induction $P(n)$ is true $\forall$ n.

Fundamentals of Mathematical Analysis

About the Book

This textbook is envisaged to provide coherent and compressive coverage of Fundamentals of Mathematical Analysis. It is intended to serve as a text in mathematical analysis for the B.A., B.Sc (Hons), Postgraduate, Ph.D and Research Scholar students of various universities. This book is very helpful for theoretical understanding as well as for research usage.

The main highlights of the book are:

- Basic Concepts and Preliminaries
- Numbers: Real and Complex, Cardinality of Sets and One-to-One Correspondence
- Analytical (Metric) Properties of $\mathbb{R}$ & $\mathbb{C}$, Sequences and Subsequences ($\mathbb{R}$ & $\mathbb{C}$) and Infinite Series ($\mathbb{R}$ & $\mathbb{C}$)
- Limits, Continuous Functions and Differentiable Functions
- The Riemann Integral and Darboux Integral, Bounded Variation and Rectification
- Improper Integral (The Generalised Riemann Integral) and Complex Integration
- Sequence of Functions, Series of Functions
- Power Series, Fourier Series, Integrals and Transforms, Taylor Series, Laurent Series
- Complex Numbers and Functions, Conformal Mapping and Analytic Functions
- Metric Spaces, Compactness and Connectedness

About the Authors

Prof. Rashmi Bhardwaj, fellow of Institute of Mathematics & Applications (FIMA, UK); Assessor, National Assessment and Accreditation Council (NAAC), UGC, Government of India, Fellow of Indian Society for Mathematical Modelling and Computer Simulation (FISMMACS), under IIT Kanpur, India and working as Professor of Mathematics and Head of Nonlinear Dynamics Research Lab, University School of Basic & Applied Sciences (USBAS), Guru Gobind Singh Indraprastha University (GGSIPU), New Delhi, India. Recipient of Government of India, India Meteorological Department (IMD) 27th Biennial MAUSM award. Visiting Associate of Inter University Center for Astronomy & Astrophysics (IUCAA) Pune, recipient of young scientist award of All India Council for Technical Education (AICTE), Government of India. Best researcher award of GGSIPU (3 consecutive years). Topper of University of Delhi with 100% marks in Integral Equation and Boundary Value Problem. Also, the topper of Zakir Husain College, University of Delhi in M.Sc (mathematics) and B.Sc (Hons.) (mathematics) examination.

She has about 28 years teaching experience in Mathematics for Ph.D, postgraduate and graduate level students. She supervised 16 candidates for Ph.D degree and six are presently working for Ph.D degree. She has published four patents, 10 books, 200 research papers published in Journals of International/National repute and conferences. Subject reviewer for many International Journals of repute. Life member of several international organizations in the sphere of mathematical sciences including Indian National Science Congress. Attended/delivered talks and Chaired Technical sessions at reputed academic conferences/workshops across the world established Nonlinear Dynamics Research Lab and Mathematics Lab in GGSIPU.

Dr. Ajaya Kumar Singh, Reader (SS) in Mathematics, Ekamra College, Bhubaneswar. He has more than 31 years teaching experience in Mathematical and Computer Science for undergraduate and postgraduate students. He served as Ph.D supervisor in Utkal University, Vani Vihar. He has published more than 10 research articles in Journal of National and International repute, Member of Odisha Mathematical Society and Subject Expert Committee in Mathematics of JEE, Directorate of Higher Secondary Education, Govt. of Odisha. His specialization in Mathematical Analysis and Computer Science. Also, he has competent in C++, R, Python and MATLAB and his credit 9 books published at State, National and International level.

KHANNA PUBLISHERS®
ISO 9001:2015
4575/15, Onkar House, Opp. Happy School,
Ground Floor, Daryaganj, New Delhi-110002
Phones: 011-45033819, 9811541460
E-mail: contactus@khannapublishers.in

Website:
www.khannapublishers.in

9 789392 549366