1

Statics of a Particle

1.1. Force and its Characteristics

Force is that which produces or tends to produce change in the state of rest or of uniform motion in a straight line, of a body.

Let a horizontal force P be applied to a body placed on a rough horizontal plane. When P is a small, the body does not move. When P is increased, the body will start moving in a straight line if the line of action of P passes through the centre of gravity (c.g.) of the body : there will be motion of translation as well as of rotation if the line of action of P does not pass through the c.g. of the body.

Thus we see that the effect of a force depends on three characteristics – (1) magnitude, (2) direction, (3) position or line of action. The complete effect of a force can be found only if we know all these three characteristics.

If we draw a straight line parallel to the line of action of the force, whose length is proportional to the magnitude of the force, the line is said to represent the force in *magnitude and direction.* Thus let the force P be 15 kilograms acting in the north-east direction. Let 1 cm length represent 5 kilograms. Then a straight line AB of length 3 cm drawn in the north-east direction will represent the force P in direction and magnitude. An arrow is placed on the line with the arrow-head pointing north-east to show the sense of the force, *i.e.* the force is acting from A towards B. The force represented by the line AB is written as $\overrightarrow{AB}$.

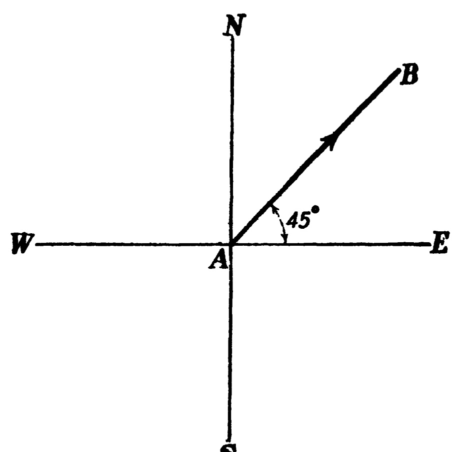


Fig. 1.1-A

If the line AB is drawn through the point at which P acts, then AB is said to represent P in *direction, magnitude* and *position*, or in short, AB represents P completely.

Any quantity which possesses magnitude as well as direction is called a *vector quantity.* Some examples of vector quantities are force velocity and acceleration. Any vector quantity can be represented by means of a straight line which is called a *vector*. Thus the force P is a vector quantity which is represented by the vector $\overrightarrow{AB}$.

Vectors are also denoted by a single letter, like **P** or **a**, etc. The magnitude of the vector **a** is represented by $|\mathbf{a}|$ or a.

A *free vector* can be moved any-where in space, provided it retains its direction and magnitude unchanged.

A vector which can be applied at any point of its line of action is called a *sliding vector.*

A *bound* or *fired* vector passes through a fixed point.

1.2. (*i*) **M.K.S. and C.G.S. Units**

The units of mass, distance and time are called fundamental units. All other units are known as derived units.

In the M.K.S. system, mass is measured in kilograms, distance in metres and time in seconds. The abbreviations for them are respectively kg, m and sec.

The weight of a body is the force of attraction exerted on it by the earth. If the mass of a body is m, then its weight is $m \times g$, where g is the acceleration due to gravity. The value of g on the earth's surface is approximately 9.81 metres per sec per sec which is written as 9.81 m/sec^2. Sometimes we take g = 9.8 m/sec^2.

If the mass of a body is m kg then its weight is $m \times 9.81$ Newtons. Since weight is a force, the unit of force is also Newton. The abbreviation for Newton is N.

Now a force of mg Newtons is equal to the weight of a body of mass m kg. We say that

$$mg \text{ Newtons} = \text{weight of } m \text{ kg}$$

or $$= m \text{ kg-wt}$$

or $$= m \text{ kgf.}$$

Thus if a force in Newtons is divided by $g = 9.81$, then the force is obtained in kg-wt. or kgf. Kg-wt or kgf is called the gravitational unit of force. In the gravitational system, mass and weight are equal numerically. Newton is called the absolute unit of force.

In the C.G.S. system mass is measured in grams (gm), length in centimetres (cm) and time in seconds. Weight and force are measured in dynes. Thus if the mass of a body is m gm, then its weight is mg dynes, where $g = 981$ cm/sec^2. The gravitational unit of force and weight, is gmf.

Pressure is force per unit area. It is measured in kgf/cm^2, or kgf/square millimetre (mm^2), or N/cm^2, or N/mm^2 and so on.

When the point of application of a force **F** moves through a distance x along its line of action, work is said to be done and its magnitude is F . x. When F is measured in kgf and x in metres, the work done = F . x. kgf-m. When F is measured in Newtons and x in metres, the work done = F . x N-m = F . x Joules. One Newton-metre is called one Joule.

The unit for energy is the same as for work.

In the F.P.S. system, the three fundamental units are the pound (lb), the foot (ft) and the sec. Force is measured in poundals, which are obtained by multiplying lb by $g = 32.2$ ft/sec^2. The gravitational unit is lb-wt.

1.2. ***(ii)*** **S.I. Units**

The following are the base units in this system as required in this book :

Quantity	*Name*	*Abbreviation or symbol*
Length	Metre	m
Mass	Kilogram	kg
Time	Second	s

S.I. derived units

Quantity	*Name*	*Symbol*
Force	Newton	N
Moment	Newton-metre	N-m
Work and energy	Joule	J
Power	Watt	W
Area	Square metre	m^2
Velocity	Metre per second	m/s or ms^{-1}
Acceleration	Metre per second per second	m/s^2 or ms^{-2}
Density	Kilogram per cubic metre	kg/m^3
Pressure and stress	Pascal	Pa

One Newton is that force which acting on a mass of one kilogram will produce in it an acceleration of one metre per second per second.

One Joule is the work done when the point of application of a force of one Newton is displaced through one metre along its line of action. One Joule is also called one Newton-metre.

$$1 \text{ J} = 1 \text{ N-m}.$$

Power is rate of doing work A rate of one Joule per second is called one Watt.

$$1 \text{ W} = 1 \text{ J/s}$$

Pressure, and also stress, is force per unit area. A pressure of one Newton per square metre is called one Pascal.

$$1 \text{ Pa} = 1 \text{ N/m}^2$$

S.I. prefixes for multiples

Factor	*Prefix name*	*Symbol*
10	deca	*da*
10^2	hecto	*h*
10^3	kilo	*k*
10^6	Mega	M
10^9	Giga	G
10^{12}	Tera	T

S.I. prefixes for sub-multiples

Factor	*Prefix name*	*Symbol*
10^{-1}	deci	*d*
10^{-2}	centi	*c*
10^{-3}	milli	*m*
10^{-6}	micro	μ
10^{-9}	nano	*n*
10^{-12}	pico	*P*

The use of the prefixes hecto, deca, deci and centi is not recommended.

Compound prefixes should be avoided. For example, 25000 kg should be written as 25×10^3 kg or 25×10^6 g or 25 Mg (Mega gram), but not as 25 *k* kg.

The following symbols are written in Roman type, lower case :

m, kg, s.

The following symbols, which have been derived from the names of persons, are written in Roman type, capital letters :

N, J, W, Pa.

The prefixes M, G, T are written in Roman type, capital letters.

The prefix k for kilo is written in Roman type, lower case.

The following prefixes are written in italics, lower case :

h *da*, *d*, *c*, μ, *n*, *p*

Correct way of writing S.I. units

The S.I unit names and symbols do not change in plural. The symbols are not followed by a full stop. No space should be left between prefix symbols and unit symbols. Compound prefixes are not used.

Some examples are given below.

Quantity	*Correct*	*Incorrect*
Force	250 Newton or 250 N	250 Newtons
Length	10 m 15 mm	10 m. 15 mm.
Length	20 metre or 20 m	20 metres
Volume	20 m^3	20 cu m
Volume	50 ml	50 cc
Velocity	5 m/s or 5 m s^{-1}	5 m/sec
Velocity	40 km/h	40 kmph
Acceleration	0.5 m/s^2 or 0.5 ms^{-2}	0.5 m/sec^2
Force	10 kN	10 k N or 10 k-N
Mass	2000 kg or 2×10^3 kg	2 k kg

Alternative unit for pressure. As already mentioned, the unit for pressure and stress is Pascal (Pa). For convenience, the unit N/mm^2 is also used.

Since 1000 mm = 1 m, and 1 Pa = 1 N/m^2, it follows that

$$1\ N/mm^2 = 1000^2\ N/m^2 = 10^6\ N/m^2$$
$$= 10^6\ Pa = 1\ MPa \text{ (one Mega Pascal)}$$
$$1\ kN/mm^2 = 10^6\ kN/m^2 = 10^9\ N/m^2 = 10^9\ Pa$$
$$= 1\ GPa \text{ (one Giga Pascal)}$$

1.3. Resultant and Components

If the combined effect of several forces P_1, P_2, P_3, ... acting on a body is the same as that of a single force R, then R is called the *resultant of* P_1, P_2, P_3 ..., and the forces P_1, P_2, P_3 ...... are called the *components* of R.

1.4. Law of Parallelogram of Forces

The resultant of two forces acting at a point can be found by the application of the law of parallelogram of forces, which is stated below.

If two forces, acting at a point O, be represented in direction and magnitude by straight lines OA and OB, and the parallelogram OACB be completed, then their resultant acts through O and is represented in magnitude and direction by the diagonal OC of the parallelogram which passes through O.

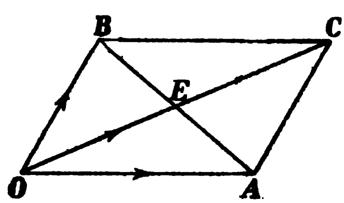


Fig. 1.4-A

Cor. Let the diagonals of the parallelogram intersect at E. Then E is the middle point of each AB and OC.

The resultant of forces $\vec{OA}$ and $\vec{OB} = \vec{OC} = 2\,\vec{OE}$.

Hence the resultant of forces represented in direction and magnitude by $\vec{OA}$ and $\vec{OB}$ is represented by $2\,\vec{OE}$, where E is the middle point of AB.

1.5. Resultant of Two Forces Acting at a Point

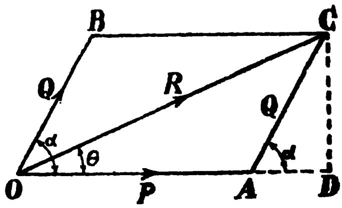


Fig. 1.5-A

Let two forces P and Q, acting at O, be represented in direction and magnitude by the sides OA and OB respectively of the parallelogram $OACB$. Then OC represents their resultant R.

Let $\angle AOB = \alpha$, $\angle AOC = \theta$.

Draw CD perpendicular to OA.

Since AC is equal and parallel to OB, we get $AC = Q$.

Also $\quad \angle CAD = \angle AOB = \alpha$.

$\therefore \quad AD = Q\cos\alpha, \qquad DC = Q\sin\alpha$

$$OC^2 = OD^2 + DC^2 = (OA + AD)^2 + DC^2$$

$$R^2 = (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$$

$$= P^2 + 2PQ\cos\alpha + Q^2\cos^2\alpha + Q^2\sin^2\alpha$$

$$= P^2 + Q^2 + 2PQ\cos\alpha \qquad \text{...(1)}$$

Equation (1) gives the magnitude of R.

$$\tan\theta = \frac{DC}{OD} = \frac{Q\sin\alpha}{P + Q\cos\alpha} \qquad \text{...(2)}$$

Equation (2) gives the direction of R.

Particular Cases

(*i*) Let $\alpha = 90°$, *i.e.*, let the forces act at right angles. Then parallelogram $OACB$ becomes a rectangle.

From (1) and (2), or directly,

$$R^2 = P^2 + Q^2,$$

$$\tan\theta = \frac{P}{Q}.$$

(*ii*) Let $\qquad P = Q$.

Then from (1), $R^2 = P^2 + P^2 + 2P^2\cos\alpha = 2P^2(1 + \cos\alpha)$

$$= 4P^2\cos^2\frac{\alpha}{2}$$

$$\mathbf{R = 2P\cos\frac{\alpha}{2}}$$

From (2), $\qquad \tan\theta = \dfrac{P\sin\alpha}{P + P\cos\alpha} = \dfrac{\sin\alpha}{1 + \cos\alpha}$

$$= \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2 \frac{\alpha}{2}} = \tan \frac{\alpha}{2}$$

$$\therefore \qquad \theta = \frac{\alpha}{2}$$

i.e. the resultant bisects the angle between the forces, a result which is quite obvious also from first principles.

Note. It is easy to see that the greatest resultant of two force P and Q is $P + Q$ when the two forces act in the same line and same sense and that the least resultant is $P—Q$ when they have the same line of action and opposite senses.

If two forces acting at a point are in equilibrium they must be equal in magnitude, have the same line of action and opposite senses.

1.6. Resolution of Forces

Finding the components of a given force in two given directions is called *resolution.*

Let the given force be R, and let it be required to find its components in directions making angles α and β with its line of action.

Let OC represent R in magnitude and direction and let the lines OX and OY make angles α and β respectively with OC. Through C, draw CA parallel to OY meeting OX at A, and CB parallel to OX, meeting OY at B. Then OA and OB represent the components of R along OX and OY respectively.

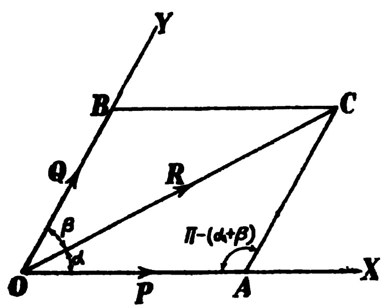


Fig. 1.6 A-1

Let $\qquad \vec{OA} = P, \qquad \vec{OB} = Q.$

$$\angle OCA = \angle BOC \text{ (alternate angles)} = \beta$$

$$\therefore \qquad \angle OAC = \pi - (\alpha + \beta)$$

In Δ OAC, by trigonometry,

$$\frac{OA}{\sin \beta} = \frac{AC}{\sin \alpha} = \frac{OC}{\sin [\pi - (\alpha + \beta)]}$$

$$\therefore \qquad \frac{P}{\sin \beta} = \frac{Q}{\sin \alpha} = \frac{R}{\sin (\alpha + \beta)}$$

for AC is equal to OB which is proportional to Q.

Hence $\qquad \mathbf{P} = \dfrac{\mathbf{R} \sin \beta}{\sin (\alpha + \beta)}$

$$Q = \frac{R \sin \alpha}{\sin (\alpha + \beta)}.$$

Particular case. Let *OX* and *OY* be at right angles. Then *OACB* becomes a rectangle and

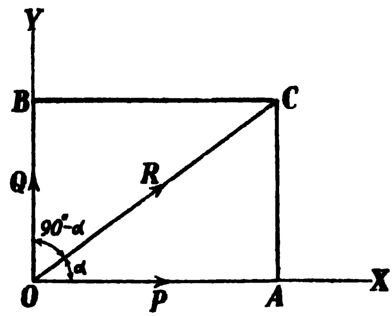


Fig. 1.6 A-2

$$\alpha + \beta = 90°$$

or $$\beta = 90° - \alpha$$

$$\therefore \quad \frac{P}{R} = \cos \alpha$$

or $$P = R \cos \alpha$$

$$\frac{Q}{R} = \cos (90° - \alpha)$$

$$= \sin \alpha$$

or $$Q = R \sin \alpha$$

When the components *P* and *Q* are at right angles, they are called the **resolved parts** of *R*.

We see that the resolved part of *R* in a direction inclined at angle α to *R* (*i.e.* along *OX*)

$$= P = R \cos \alpha.$$

This result is important. *To find the resolved part of a force in a given direction, multiply the force with the cosine of the angle between the line of action of the force and the given direction.*

In the application of this rule, care must be exercised in the measurement of the angle between the line of action of the force and the given direction. Let *R* be the force and *X ' X* the given direction Let *O* be their point of intersection. *The positive direction of R is that in which the arrow-head points away from O.* If it is required to find the resolved part of *R* along *OX*, then multiply *R* with cos *XOA*, where *OA* is the positive direction of *R*. Let $\angle XOA = \alpha$. Then resolved part of *R* along *OX* = *R* cos α.

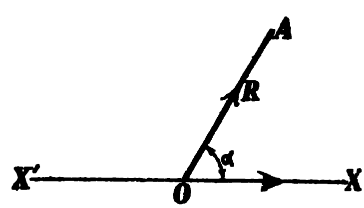


Fig. 1.7 A-3

Clearly, angle $X'OA = 180° - \alpha$. Hence the resolved part of *R* along $OX' = R \cos (180° - \alpha) = -R \cos \alpha$. We can also find the resolved part of *R* along *OX'* by first finding the resolved part of *R* along *OX* and then reversing its sign. Thus the resolved part of *R* along *OX* is $R \cos \alpha$; reversing the sign, we see that the resolved part of *R* along *OX'* is $-R \cos \alpha$. This method is often convenient, since the use of obtuse angle is avoided.

Consider the case shown in Fig. 1.6 A-4.

The angle between R and OX is not XOA. Produce AO to A' ; *then* OA' *is the positive direction of* R and the angle between R and OX is XOA'. Hence the resolved part of R along OX is

$$R \cos XOA' = R \cos \alpha,$$

and the resolved part of R along OX' is

$$R \cos (180° - \alpha) = -R \cos \alpha.$$

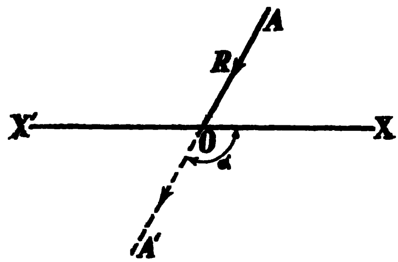


Fig. 1.6 A-4

Let it be required to find the resolved parts of a force F_1 along OX and OY (see Fig. 1.6 A-5). Produce AO to A'. Then OA' is the positive direction of F_1. Clearly $\angle X'OA' = 60°$.

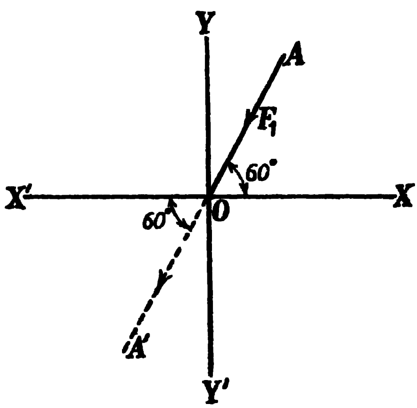


Fig. 1.6 A-5

$\therefore$ Resolved part of F_1 along $OX' = F_1 \cos 60° = \dfrac{F_1}{2}$.

$\therefore$ Resolved part of F_1 along $OX = -\dfrac{F_1}{2}$.

$$\angle A'OY' = 30°.$$

$\therefore$ Resolved part of F_1 along OY'

$$= F_1 \cos 30°$$

$$= \frac{F_1 \sqrt{3}}{2}.$$

$\therefore$ Resolved part of F_1 along

$OX \quad = -\dfrac{F_1 \sqrt{3}}{2}$.

Next, let us find the resolved parts of F_2 along OX and OY (*see* Fig. 1.6 A-6).

$$\angle AOX' = 180° - 120° = 60°$$

$\therefore$ Resolved part of F_2 along

$OX' = F_2 \cos 60°$

$$= \frac{F_2}{2}$$

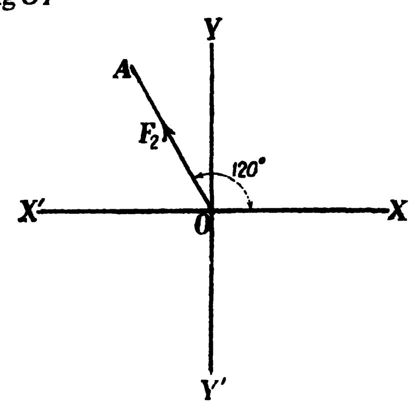


Fig. 1.6 A-6

$\therefore$ Resolved part of F_2 along OX

$$= -\frac{F_2}{2}$$

(which is also equal to $F_2 \cos 120°$)

$$\angle AOY = 120° - 90° = 30°$$

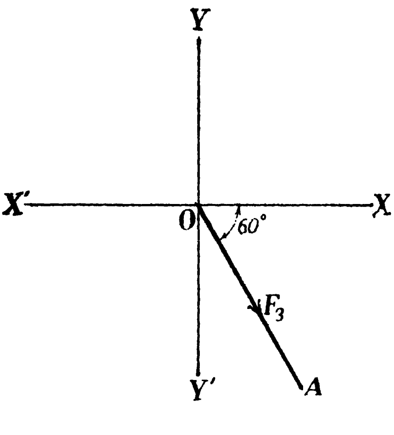


Fig. 1.6 A-7

$\therefore$ Resolved part of F_2 along OY

$$= F_2 \cos 30°$$

$$= \frac{F_2 \sqrt{3}}{2}$$

(which is also equal to $F_2 \sin 120°$).

Similarly resolved part of F_3 along OX (*see* Fig. 1.6 A-7)

$$= \frac{F_3}{2}$$

and resolved part of F_3 along OY

$$= -\frac{F_3 \sqrt{3}}{2}.$$

From the above discussion, we derive the following rule which may be found helpful.

If the positive direction of a force F makes with a line OX an angle θ, *and OY is perpendicular to OX, the angles being measured in the same sense, the resolved parts of F along OX and OY are respectively F cos* θ *and F sin* θ.

Cor. The resolved part of a force F in its own direction is F, and the resolved part in a perpendicula direction is zero.

Ex. 1. *The resultant o two forces is 8 N and its direction is inclined at 60° to one of the forces whose magnitude is 4 N. Find the magnitude and direction of the other force.*

Sol. Let $\vec{OA} = 4$ N, $\vec{OC} = 8$ N

Let $\vec{OB} = P$ be the other force inclined at $\angle \alpha$ to OC.

Clearly $\angle OCA = \alpha$,

$$\vec{AC} = P.$$

Also $\angle OAC = 180° - (60° + \alpha)$

From ΔOCA,

$$\frac{4}{\sin \alpha} = \frac{P}{\sin 60°} = \frac{8}{\sin [180° - (60° + \alpha)]}.$$

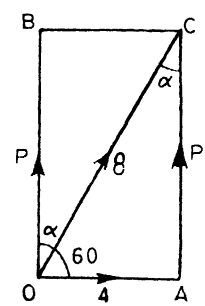


Fig. 1.6 E-1

Taking the first and third members,

$$\frac{4}{\sin\alpha} = \frac{8}{\sin(60^\circ + \alpha)}$$

$$2\sin\alpha = \sin(60^\circ + \alpha)$$

$$= \sin 60^\circ \cos\alpha + \cos 60^\circ \sin\alpha$$

$$= \frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha$$

$$3\sin\alpha = \sqrt{3}\cos\alpha,\ \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\therefore \qquad \alpha = 30^\circ.$$

Taking the first and second members and carrying the value of α, we get

$$\frac{4}{\sin 30^\circ} = \frac{P}{\sin 60^\circ}$$

$$P = \frac{4\sin 60^\circ}{\sin 30^\circ} = 4\sqrt{3}\text{ N}$$

Hence the other force is $4\sqrt{3}$N at right angles to the force of 4N.

Ex. 2. *Find a point within a quadrilateral such that, if it be acted on by forces represented by the lines joining it to the angular points of the quadrilateral, it will be in equilibrium.*

Sol. Let $ABCD$ be the quadrilateral and E, F the middle points of the sides AB, CD respectively. Let P be the point such that the forces $\vec{PA}$, $\vec{PB}$, $\vec{PC}$, $\vec{PD}$, are in equilibrium.

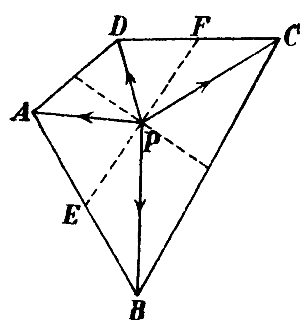


Fig. 1.6 E-2

Now $\vec{PA} + \vec{PB} = 2\,\vec{PE}$ [Art 1.4 Cor.]

$\vec{PC} + \vec{PD} = 2\,\vec{PF}$

$\vec{PA} + \vec{PB} + \vec{PC} + \vec{PD} = 2[\vec{PE} + \vec{PF}]$

The system will be in equilibrium

if $\qquad \vec{PE} + \vec{PF} = 0,$

or $\qquad \vec{PE} = -\vec{PF}$

i.e. the forces $\vec{PE}$ and $\vec{PF}$ should be equal and opposite ; in other words, these force should have the same line of action and $PE = PF$ numerically.

$\therefore$ P must be the middle point of the line joining EF.

Similarly, we can show that P must be the middle point of the line joining the middle points of AD and BC.

Hence P is the point of intersection of the lines joining the middle points of opposite sides of the quadrilateral.

PROBLEMS

1. A loaded wagon is at rest on a railway line and is pulled by a horizontal force of 200 N at an angle 50° to the railway line. What is the force tending to urge the wagon forwards ? [**Ans.** 128.56 N]
2. Find the resultant of two forces, 13 N and 11 N acting at an angle whose tangent is 12/5. [**Ans.** 20 N inclined at $\tan^{-1}$ 33/56 to the force of 13 N]
3. Find the components of a force of 100 N in directions inclined to it at 30° and 40° on opposite sides. [**Ans.** 68.41 N, 53.2 N]
4. Find a horizontal force and a force inclined at an angle of 60° with the vertical whose resultant shall be a given vertical force F. [**Ans.** $F\sqrt{3}$ and $2F$]
5. The resultant of two forces P and Q is at right angles to P. Show that the angle between the two forces is $\cos^{-1}\left(-\frac{P}{Q}\right)$.
6. Two forces equal to $2P$ and P respectively act on a particle ; if the first be doubled and the second increased by 120 N the direction of the resultant is unaltered ; find the value of P. [**Ans.** 120 N]
7. Show that the system of forces represented by the lines joining any point to the angular points of a triangle is equivalent to the system represented by straight lines drawn from the same point to the middle points of the sides of the triangle.
8. Find a point within a triangle such that, if it be acted on by forces represented by lines joining it to the angular points of the triangle, it will be in equilibrium. [**Ans.** The centroid of the triangle]
9. A boat B is in the middle of a canal 100 m wide, and is pulled through two ropes BA (150 m long) and BC (100 metres long) by two men on the banks. The pull in BC = 1500 N. Find the pull Q in BA so that the boat moves parallel to the banks. Find also the resultant pull on the boat.

   [**Ans.** Q = 2250 N Resultant force = 3420 N]

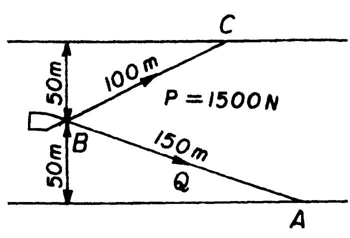


Fig. 1.6 P-9

10. Convert (*i*) 6.1 tonne into kg.
   (*ii*) 10 MN into N
   (*iii*) 15 MPa into N/mm^2
   (*iv*) 1 GPa into kN/mm^2

   [**Ans.** (*i*) 6100 kg (*ii*) 10×10^6 N (*iii*) 15 N/mm^2 (*iv*) 1 kN/mm^2]

11. Write the following as per S.I. units.

(*i*) The volume of this bottle is 750 cc

(*ii*) Draw a line of length 15 MM.

(*iii*) A room is 5 m wide × 6 m long. Its area is 30 sq m

(*iv*) 1 mN = 10^6 N.

(*v*) The mass of this stone is 50 kgs.

(*vi*) The acceleration of the car is 1 m/sec^2.

(*vii*) This pencil is 60 m-m. in length.

(*viii*) Strain energy in the bar = 100 m-N.

[**Ans.** (*i*) The volume of this bottle is 750 ml.

(*ii*) Draw a line of length 15 mm.

(*iii*) A room is 5 m wide × 6 m long. Its area is 30 m^2

(*iv*) 1 MN = 10^6 N

(*v*) The mass of this stone is 50 kg.

(*vi*) The acceleration of the car is 1 m/s^2 (or 1 ms^{-2})

(*vii*) This pencil is 60 mm in length.

(*viii*) Strain energy in the bar = 100 N-m (or 100 J)].

1.7. Extension of the Law of parallelogram of Forces

We have seen that the resultant of forces represented by *OA* and *OB* is represented by the diagonal *OC* of the parallelogram *OACB*. Similarly, the resultant of forces represented by *l.OA* and *l.OB* is represented by *l.OC*. In other words, the law of parallelogram of forces enables us to find the resultant of two forces represented by *OA* and *OB* on the *same* scale. When the scales are not the same the resultant can be found by the following theorem :

The resultan of two forces, acting at a point O in directions OA and OB and represented in magnitude by λ.OA and μ.OB is represented by (λ + μ). *OC, where C is a point in AB such that λ.AC = μ.CB.*

Complete the parallelograms *ODAC* and *OCBE*.

By the low of parallelogram of forces,

$$\lambda.\ \vec{OA} = \lambda\vec{OD} + \lambda\vec{OC}$$

$$\mu.\ \vec{OB} = \mu.\vec{OE} + \mu.\vec{OC}$$

$$\therefore\quad \lambda.\vec{OA} + \mu.\vec{OB}$$

$$= (\lambda + \mu).\vec{OC} + \lambda.\vec{OD} + \mu.\vec{OE}$$

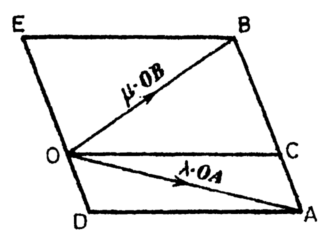


Fig. 1.7 A

But $\lambda.\vec{OD} + \mu.\vec{OE} = \lambda.\vec{CA} + \mu.\vec{CB} = -\lambda.\vec{AC} + \mu.\vec{CB} = 0$

$$\therefore\quad \lambda.\vec{OA} + \mu.\vec{OB} = (\lambda + \mu).\ \vec{OC}.$$

Cor. If $\lambda = \mu = 1$, we get

$$\vec{OA} + \vec{OB} = 2.\vec{OC}$$

where C is the middle point of AB.

Ex. 1. *Show that if A, B, C, P are any four points, the resultant of the three forces acting at P and represented by PA, PB and PC is represented by 3 $\vec{PG}$, where G is the centroid of the triangle ABC.*

Sol. Let D be the middle point of BC. Then

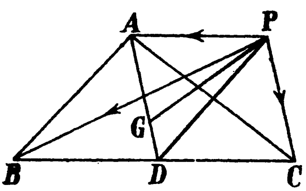


Fig. 1.7 E-1

$$\vec{PB} + \vec{PC} = 2.\vec{PD}$$

On AD take a point G such that

$$1.AG = 2.GD$$

Then, by Art. 1.7,

$$1.\vec{PA} + 2.\vec{PD} = (1 + 2).\vec{PG} = 3.\vec{PG}$$

The resultant of $\vec{PA}, \vec{PB}, \vec{PC}$ is $3\vec{PG}$. But AD is a median of the ΔABC and G divides AD in the ratio 2 : 1, so that G is the centroid of the triangle. Hence the preposition is proved.

Ex. 2. *Two forces P, Q and their resultan R act at a point O. If their directions meet a transversal in L, M, N respectively, prove that*

$$\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}. \qquad \textit{(Roorkee)}$$

Sol. The force P along OL is the same as $\frac{P}{OL} . OL$ along OL, and the force Q along OM is the same as $\frac{Q}{OM}$. OM along OM.

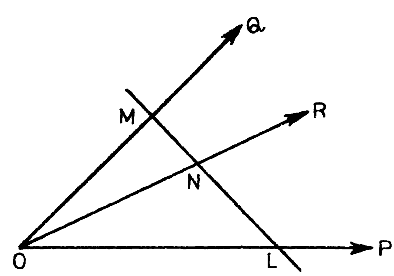


Fig. 1.7 E-2

$\therefore$ Resultant of P and Q.

$= $ Resultant of $\frac{P}{OL} . OL$ along OL and

$\frac{Q}{OM} . OM$ along OM

$= \left(\frac{P}{OL} + \frac{Q}{OM}\right) ON$ along ON

$= R$ (given)

$$\therefore \qquad \frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}.$$

1.8. Theorem of Resolved Parts

The algebraic sum of the resolved parts of two forces in a given direction is equal to the resolved part of their resultant in the same direction.

Let the two forces be P and Q represented by OA and OB respectively. Complete the parallelogram $OACB$. Then OC represents the resultant R of P and Q.

Let OX be the given direction. Draw AL, BM, CN perpendiculars to OX and AK perpendicular to CN.

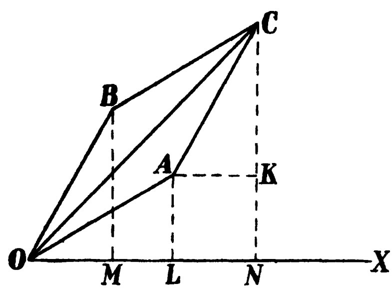


Fig. 1.8 A

Since BO is parallel to CA, and BM is parallel to CK,

$$\angle OBM = \angle ACK$$

In the Δs BOM, ACK

$$\angle OBM = \angle ACK$$

$$\angle OMB = \angle AKC$$

(each being a rt. angle)

$OB = AC$ (being opposite sides of the parallelogram $OACB$)

$\therefore$ The triangles are congruent.

$$\therefore \quad OM = AK = LN$$

$$\therefore \quad OL + OM = OL + LN = ON$$

But OL, OM, ON represent respectively the resolved parts of P, Q, R along OX.

Hence the theorem is proved.

Cor. The theorem can be easily extended to any number of concurrent coplanar forces.

Let three forces P_1, P_2, P_3 act at a point O and let OX be the given direction. Let $R(P_1)$ denote the resolved part of P_1 along OX and so on.

Let the resultant of P_1 and P_2 be R_1, and that of R_1 and P_3 be R_2, so that R_2 is the resultant of P_1, P_2, P_3.

Then applying the theorem to P_1 and P_2,

$$R(P_1) + R(P_2) = R(R_1) \qquad \text{...(1)}$$

Again applying the theorem to R_1 and P_3,

$$R(R_1) + R(P_3) = R(R_2) \qquad \text{...(2)}$$

Adding equations (1) and (2),

$$R(P_1) + R(P_2) + R(P_3) = R(R_2)$$

i.e. sum of the resolved parts of P_1, P_2, P_3 along OX is equal to the resolved part of their resultant in that direction.

Thus the theorem is true for three forces. Proceeding thus we can prove that it is generally true.

Note. "Concurrent" means meeting at a point. "Coplanar" means lying in the same plane.

1.9. Resultant of any Number of Concurrent Coplanar Forces

By applying Art. 1.8, we can find the resultant of any number of forces acting on a particle in the same plane.

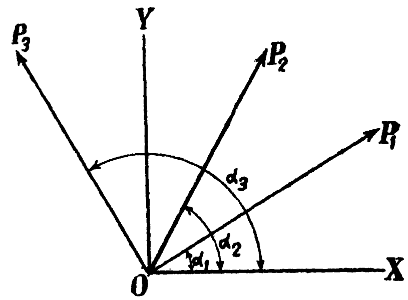


Fig. 1.9-A

Let the forces be P_1, P_2, P_3 ...... acting at O. Let OX be any convenient direction and OY a direction perpendicular to OX.

Let the forces make angles $\alpha_1, \alpha_2, \alpha_3$ ...... with OX. Let R be their resultant inclined at $\angle\theta$ to OX.

By Art. 1.8, resolved part of R along OX.

= Sum of the resolved parts of P_1, P_2, P_3 ..... along OX.

$$\therefore \quad R\cos\theta = P_1\cos\alpha_1 + P_2\cos\alpha_2 + P_3\cos\alpha_3 + \ldots\ldots$$
$$= X, \text{ say} \qquad \ldots(1)$$

Similarly, resolving along OY,

$$R\sin\theta = P_1\sin\alpha_1 + P_2\sin\alpha_2 + P_3\sin\alpha_3 + \ldots..$$
$$= Y, \text{ say} \qquad \ldots(2)$$

Squaring and adding (1) and (2)

$$R^2 = X^2 + Y^2 \qquad \ldots(3)$$

Dividing (2) by (1),

$$\tan\theta = Y/X \qquad \ldots(4)$$

Equation (3) gives the magnitude of R, and equation (4) its direction.

1.10. If the Forces in Art. 1.9 are in equilibrium, then

$$R = 0$$
$$\therefore \quad X^2 + Y^2 = 0.$$

Now X^2 and Y^2 are positive quantities and their sum cannot be zero unless each of them is zero.

$$\therefore \quad X = 0, \qquad Y = 0.$$

Hence if any number of forces acting at a point are in equilibrium the algebraic sum of their resolved parts in any two perpendicular directions are separately zero.

Conversely, if the sum of resolved parts in each of two directions at right angles is zero, the forces are in equilibrium.

Ex. 1. *A particle O is acted on by the following forces :*

(*i*) *20 N inclined 30° to north of east.*

(*ii*) *25 N towards the north.*

(*iii*) *30 N towards north west.*

(*iv*) *35 N inclined 40° to south of west.*

Find the resultant. (Bombay)

Sol. The system of forces is shown in Fig. 1.10 E-1.

Let the resultant be R inclined at $\angle\ \theta$ to the north of east.

Let X = sum of the resolved parts of the forces along OE.

Y = sum of the resolved parts of the forces along ON.

Resolved part of 20 N along $OE = 20 \cos 30°$

$= 17.32$ N.

Resolved part of 25 N along $OE = 0$

(Resolved part of a force in a direction perpendicular to its own is zero).

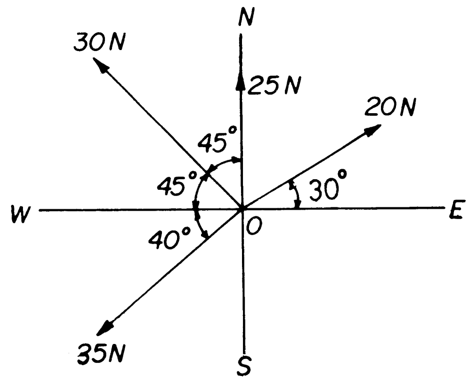


Fig. 1.10 E-1

Resolved part of 30 N along $OW = 30 \cos 45°$

$= 21.21$ N

" " " " $OE = -21.21$ N

[Or thus : 30 N makes $\angle 135°$ with OE. Hence its resolved parts along $OE = 30 \cos 135° = -21.21$ N)

Resolved part of 35 N along $OW = 30 \cos 40°$

$= 26.81$ N

" " " " $OE = -26.81$ N

$$X = 17.32 + 0 - 21.21 - 26.81 = -30.7$$

i.e. $$R \cos \theta = -30.7 \quad \text{...(1)}$$

Resolved part of 20 N along $ON = 20 \sin 30° = 10$ N.

[Or thus : Angle between 20 N and ON is 60°. Hence resolved part of 20 N along $ON = 20 \cos 60° = 10$ N.]

Resolved part of 25 N along $ON = 25$ N.

[Resolved part of a force in it own direction is equal to the force itself.]

Resolved part of 30 N along ON

$= 30 \cos 45°$ or $30 \sin 135° = 21.21$ N.

Resolved part of 35 N along OS

$= 35 \sin 40° = 22.50$ N

$\therefore$ Resolved part of 35 N along $ON = -22.50$ N

$$\therefore \quad Y = 10 + 25 + 21.21 - 22.50 = 33.71 \text{ N}$$

i.e. $$R \sin \theta = 33.71 \quad \text{...(2)}$$

By squaring and adding (1) and (2)

$$R^2 = (30.7)^2 + (33.71)^2 = 2738.80$$

$$R = \mathbf{45.60\ N.}$$

Dividing (2) by (1)

$$\tan\theta = -\frac{33.71}{30.7} = -1.098.$$

Now from (1) and (2) we see that cos θ is negative while sin θ is positive. Hence θ lies between 90° and 180°. The angle whose tangent is 1.098 is 47° 42′.

$$\theta = 180° - 47°\ 42' = \mathbf{132°\ 18'}$$

Note. To find the value of θ, first determine the quadrant by looking at signs of sin θ and cos θ. Then take the *numerical value* of tan θ and find the acute angle whose tangent is equal to that quantity. Let this angle be α.

If θ lies in the 1st quadrant, θ = α.

If θ lies in the 2nd quadrant, θ = 180° – α.

If θ lies in the 3rd quadrant,θ = 180° + α.

If θ lies in the 4th quadrant, θ = 360° – α.

Ex. 2. *Forces of 2, 3, 4, 5, 6, kN act at an angular point of a regular hexagon towards the other angular points taken in order ; find their resultant.*

(Nagpur)

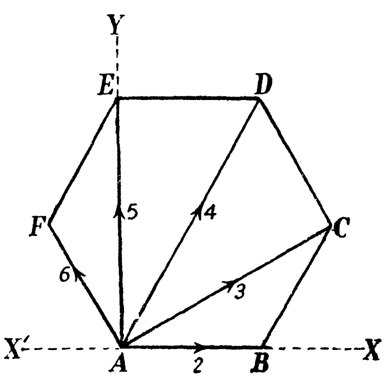


Fig. 1.10 E-2

Sol. *ABCDEF* is the hexagon. We shall resolve the forces along *AB* or *XX* and *AE* or *AY*.

Let *R* be the resultant inclined at ∠ θ with *AB*.

Each of the angles *BAC, CAD, DAE, EAF* is 30°.

Resolved part of 2 kN along *AB* = 2 kN.

Resolved part of 3 kN along $AB = 3\cos 30° = \frac{3\sqrt{3}}{2}$ kN

Resolved part of 4 kN along *AB* = 4 cos 60° = 2 kN.

Resolved part of 5 kN along *AB* = 0

„ „ 6 „ = – 6 cos *FAX′*

= – 6 cos 60° = – 3 kN

∴ Sum of resolved parts of the forces along *AB*

$$= 2 + \frac{3\sqrt{3}}{2} + 2 + 0 - 3 = 1 + \frac{3\sqrt{3}}{2}\ \text{kN}$$

Resolved part of 2 kN along AE $= 0$

" " 3 " " " $= 3 \sin 30° = 3/2$ kN

" " 4 " " " $= 4 \sin 60° = 2\sqrt{3}$ kN

" " 5 " " " $= 5$ kN

" " 6 " " " $= 6 \cos FAE = 6 \cos 30°$

$= 3\sqrt{3}$ kN.

$\therefore$ Sum of the resolved parts of the forces along AE

$$= 0 + \frac{3}{2} + 2\sqrt{3} + 5 + 3\sqrt{3} = \frac{13}{2} + 5\sqrt{3} \text{ kN.}$$

$$\therefore \quad R \cos \theta = 1 + \frac{3\sqrt{3}}{2} = 3.60$$

$$R \sin \theta = \frac{13}{2} + 5\sqrt{3} = 15.16.$$

$$R = \sqrt{(3.6)^2 + (15.16)^2} = \mathbf{15.58 \text{ kN}}$$

$$\theta = \tan^{-1} \frac{15.16}{3.60} = \tan^{-1} 4.211 = \mathbf{76° \ 39'}$$

Hence the resultant is 15.58 kN inclined at 76° 39′ to the first force.

Ex. 3. *ABCDEF is a regular hexagon. Forces 4, X, 8, Y, 6 N act along AB, CA, AE, AD, FA respectively. Find the values of X and Y in order that the system may be in equilibrium.*

Sol. Produce BA to B', CA to C', FA to F'.

Resolved part of force of 4 N along $AB = 4$ N.

AC' is the positive direction of the force X, and $\angle B'AC' = 30°$.

Resolved part of X along AB'

$$= X \cos 30° = \frac{X\sqrt{3}}{2}.$$

$\therefore$ Resolved part of X along AB

$$= -\frac{X\sqrt{3}}{2}.$$

Resolved part of Y along AB

$$= Y \cos 60° = \frac{Y}{2}.$$

Resolved part of force of 8 N along $AB = 0$.

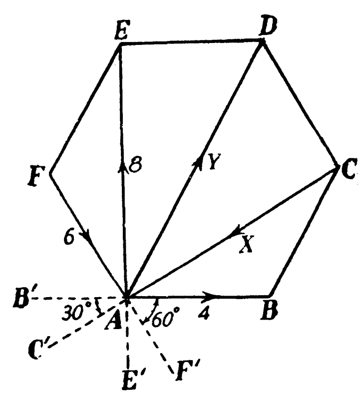


Fig. 1.10 E-3

AF' is the positive direction of the force of 6 N and $\angle BAF' = 60°$.

$\therefore$ Resolved part of 6 N along $AB = 6 \cos 60° = 3$ N.

The sum of the resolved parts of the forces along AB must be zero since the system is in equilibrium.

$$\therefore \qquad 4 - \frac{X\sqrt{3}}{2} + \frac{Y}{2} + 3 = 0$$

or $$X\sqrt{3} - Y = 14 \qquad ...(1)$$

Produce EA to E'.

Resolved part of 4 N along $AE = 0$

Resolved part of X along $AE' = X \sin 30° = \frac{X}{2}$.

$\therefore$ Resolved part of X along $AE = -\frac{X}{2}$.

Resolved part of Y along $AE = Y \cos 30° = \frac{Y\sqrt{3}}{2}$.

Resolved part of 8 N along AE = 8 N

Resolved part of 6 N along $AE' = 6 \sin 60°$

$= 3\sqrt{3}$ N

$\therefore$ Resolved part of 6 N along $AE = -3\sqrt{3}$ N.

The sum of the resolved parts of the forces along AE must be zero.

$$\therefore \qquad -\frac{X}{2} + \frac{Y\sqrt{3}}{2} + 8 - 3\sqrt{3} = 0$$

or $$X - Y\sqrt{3} = 16 - 6\sqrt{3} \qquad ...(2)$$

Solving (1) and (2)

X = 9.32 N, Y = 2.144 N.

Ex. 4. *A string of length l is fastened to two points A, B at the same level at a distance 'a' apart. A ring of weight W can slide on the string, and a horizontal force X is applied to it such that it is in equilibrium vertically beneath B. Prove that* $X = \frac{aW}{l}$ *and that the tension of the string is* $\frac{W(l^2 + a^2)}{2l^2}$.

Sol. Let C be the ring, vertically beneath B, and let $BC = x$, $\angle ACB = \theta$.

Then $$AC = l - x$$

$$AB^2 + BC^2 = AC^2$$

$$a^2 + x^2 = (l - x)^2 = l^2 + x^2 - 2lx$$

$$2lx = l^2 - a^2, \quad \text{or} \quad x = \frac{l^2 - a^2}{2l}$$

$$l - x = l - \frac{l^2 - a^2}{2l} = \frac{l^2 + a^2}{2l}$$

$$\sin \theta = \frac{a}{l - x} = \frac{2al}{l^2 + a^2}$$

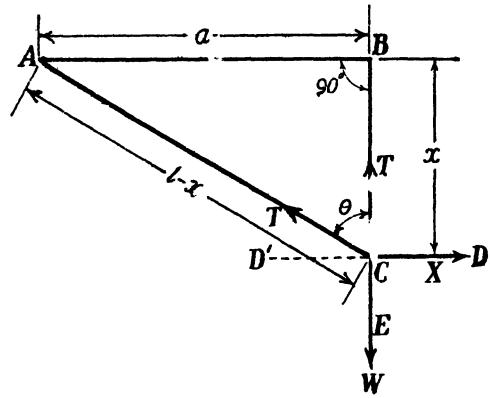


Fig. 1.10 E-4

$$\cos\theta = \frac{x}{l-x} = \frac{l^2-a^2}{l^2+a^2}.$$

Since the ring C slides on the string, the tension in BC = tension in AC = T (say).

Tne forces acting on C are

T along CB,

T along CA,

X along the horizontal $D'CD$,

W, the weight of the ring acting vertically downwards.

These forces are in equilibrium.

The sum of the resolved parts of the forces in the vertical direction must be zero.

$$T + T\cos\theta = W, \text{ or } T\left(1 + \frac{l^2-a^2}{l^2+a^2}\right) = W$$

$$T = \frac{W(l^2+a^2)}{2l^2}.$$

Similarly, resolving horizontally,

$$X = T\sin\theta = \frac{W(l^2+a^2)}{2l^2} \cdot \frac{2al}{l^2+a^2} = \frac{aW}{l}.$$

Ex. 5. *Four smooth pegs A, B, C, D are fixed in a vertical plane so that they form the four highest corners of a regular hexagon with the side BC horizontal. A loop is thrown over the pegs supporting a weight W, the loop being of such a length that angles formed by it at the lower pegs are right angles. Find the tension of the string and the pressures on the pegs.*

Sol. $ABCDE$ is the string, with weight W resting at E. Angles BAE, CDE are right angles. As the pegs are smooth, E is symmetrically situated with respect to A and D.

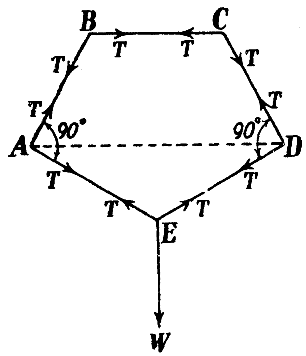


Fig. 1.10 E-5

$\angle BAD = \frac{1}{2}$ of the angle of a regular hexagon = 60°.

$\therefore \quad \angle DAE = 90° - 60° = 30°$

Similarly, $\angle ADE = 30°$

$\therefore \quad \angle AED = 120°$.

Since the pegs are smooth, the tension in the string is the same at every point. Let this tension be T.

Resultant of tensions T each in EA and ED

$$= 2T \cos \frac{AED}{2}$$

$$= 2T \cos 60° = T.$$

This must be equal to W (since E is in equilibrium).

$$\therefore \qquad \mathbf{T = W}$$

Pressure at A = Resultant of tensions T each in AE and AB acting at right angles

$$= 2T \cos \frac{90°}{2} = T\sqrt{2} = W\sqrt{2}.$$

Similarly, pressure at $D = W\sqrt{2}$.

Pressure at B = Resultant of tensions T each in BA and BC acting at 120°

$$= 2T \cos \frac{120}{2} = T = W.$$

Similarly pressure at $C = W$.

Note. At every point of a string, tension is equal and opposite. The direction of the tension at any point must be marked so as to show the effect of the tension at the point. Consider the equilibrium of E. If the string ED be cut, E will fall down, showing that E is being pulled towards D by the string. Hence at E, the direction of the tension is from E to D in ED. Similarly the direction of T in EA is E to A for the *equilibrium of* E.

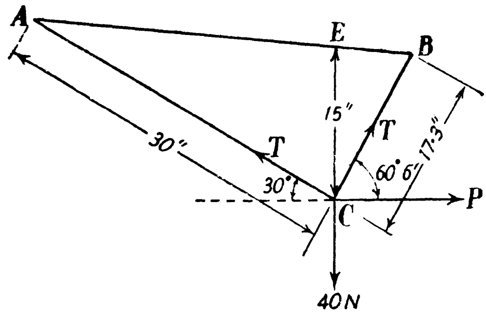
Next consider the peg A. The tensions in AE and AB are pulling at A, tending to move it towards the right. Hence at A, the directions of the tensions are from A to E and A to B.

Similarly, when considering the equilibrium of B, the tensions are to be marked from B to C and B to A.

Ex. 6. *A string ACB of length 47.3 cm is tied to two points A and B at the same level. A smooth ring of weight 40 N which can freely slide along the string is at C, 30 cm away from A along the string and pulled by a horizontal force P. If point C is 15 cm below the level of AB, determine the magnitude of P.*

Sol. As the ring is smooth, tension in AC

$= \text{tension in } BC = T \text{ (say)}$



Draw CE perpendicular to AB. Then $CE = 15$ cm

$$\sin B = \frac{15}{17.3}$$
$$= 0.867$$
$$B = 60°\ 6'$$
$$\cos B = 0.495$$
$$\sin A = \frac{15}{30} = \frac{1}{2}$$
$$A = 30°$$
$$\cos A = 0.866.$$

Angle between CB and P

$$= \angle B = 60°\ 6'.$$

Angle between CA and P produced backwards

$$= \angle A = 30°.$$

The ring at C is in equilibrium under the action of the forces P acting horizontally, T acting along CB, T acting along CA and the weight 40 N acting vertically.

Resolving vertically,

$$T \sin 60°\ 6' + T \sin 30 = 40$$
$$(0.867 + 0.5)T = 40$$
$$T = \frac{40}{1.367} = 29.26 \text{ N}.$$

Resolving horizontally,

$$T \cos 60°\ 6' + P = T \cos 30$$
$$P = (0.866 - 0.495)\ T$$
$$= 0.371 \times 29.26 = \mathbf{10.88\ N}.$$

PROBLEMS

1. Three forces 13, 10 and 5 MN act in one plane at a point, the angles between the direction of each pair being the same. Find the magnitude and direction of their resultant. [**Ans.** 7 MN inclined at 38° 12′ to the first force]

2. Forces P_1, P_2, P_3, P_4, act on a particle O at the centre of a square $ABCD$; P_1 and P_2 act along diagonals OA and OB and P_3 and P_4 perpendicular to the sides AB and BC. If

$$P_1, : P_2 : P_3 : P_4 :: 4 : 6 : 5 : 1$$

find the resultant in magnitude and direction.

[**Ans.** 12.31 making an angle of 78° 41′ with AB]

3. $ABCD$ is a square and E is the middle point of AB. Forces of 7, 8, 12, 5, 9 and 6 N act at a point in the directions AB, EC, BC, BD, CA, DE respectively. Find the magnitude and direction of the single force which will keep the particle at rest. [**Ans.** 11.49 N inclined to AB at 252°]

4. Eight points are taken on the circumference of a circle at equal distances, and from one of the points straight lines are drawn to the rest ; if these straight lines represent forces acting at a point, show that the direction of the resultant coincides with the diameter through that point and that its magnitude is four times that diameter.

[**Hint.** Let the points be A, B, C, D, E, F, G, H and let O be the centre of the circle. Clearly $\angle BAF = 90°$; hence BF is a diameter and O is its middle point. $\vec{AB} + \vec{AF} = 2\vec{AO} = \vec{AE}$. Similarly, $\vec{AC} + \vec{AG} = 2\vec{AO} = \vec{AE}$, $\vec{AD} + \vec{AH} = 2\vec{AO} = \vec{AE}$ and $\vec{AE} = \vec{AE}$. Adding these equations, we get the required result].

5. A string of length 310 mm has its extremities attached to two fixed points situated 250 mm apart in a horizontal line. If the string can bear any tension upto 36 N, find the greatest load that can be supported at a point of the string distant 240 mm from one extremity. [**Ans.** 37.5 N]

6. A string of length 310 mm has its extremities attached to two fixed points at the same level 250 mm apart. A small ring from which a weight of 9 N is suspended can slide on the string and is acted upon by a horizontal force of such a magnitude that in the position of equilibrium the ring is at a distance of 70 mm from the nearer end of the string. Show that the horizontal force is approximately 5 N and find the tension in the string. [**Ans.** 7(8/81) N]

1.11. Law of Triangle of Forces

If three forces acting upon a particle be represented in magnitude and direction by the sides of a triangle, taken in order, they will be in equilibrium.

Let forces P, Q, R acting at O be represented in direction and magnitude by the sides of a triangle ABC taken in order.

Complete the parallelogram $BCAD$. Then BD, being equal and parallel to CA represents Q.

By the low of parallelogram of forces, the resultant of $\vec{BC}$ and $\vec{BD}$ = $\vec{BA} = -\vec{AB} = -R$, *i.e.* the resultant of P and $Q = -R$. Hence the resultant of P, Q and $R = -R + R = 0$.

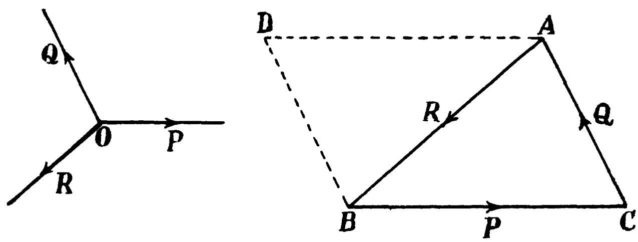


Fig. 1.11-A

Hence the system is in equilibrium.

Cor. The resultant of forces $\vec{BC}$ and $\vec{CA}$ = resultant of $\vec{BC}$ and $\vec{BD}$ = $\vec{BA}$.

Hence if two forces are represented in direction and magnitude by two sides of a triangle, taken in the same order, their resultant is represented by the third side taken in the opposite order.

1.12. Converse of the Law of Triangle of Forces

If three forces acting at a point be in equilibrium, they can be represented in magnitude and direction by the sides of any triangle which is drawn so as to have its sides respectively parallel to the directions of the forces.

Let the forces P, Q, R acting at O, be in equilibrium.

Cut off OA and OB respectively from the lines of action of P and Q to represent these forces.

Complete the parallelogram $OACB$ and join OC.

Then OC represents the resultant of P and Q. But the resultant of P and Q is equal and opposite to R (since P, Q R are in equilibrium). Hence OC represents a force equal and opposite to R and, therefore CO represents R.

Since AC is equal and parallel to OB, it represents Q.

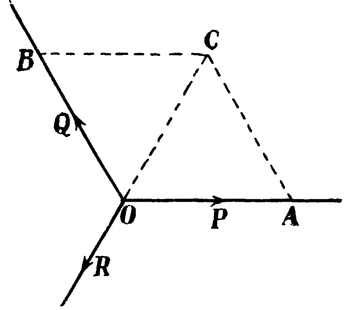


Fig. 1.12-A

In the triangle OAC, OA represents P, AC represents Q and OC represents R. Hence the sides of the triangle OAC, taken in order, represent the forces P, Q, R.

Any other triangle whose sides are respectively parallel to those of the triangle OAC, will have its sides proportional to the sides of triangle OAC and, therefore, proportional to P, Q, R. Hence the theorem is proved.

1.13. Lami's Theorem

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.

Let P, Q, R be the three forces acting at O along the lines OL, OM, ON respectively and let $\angle MON = \alpha$, $\angle NOL = \beta$, $\angle LOM = \gamma$.

Construct a triangle ABC whose sides are respectively parallel to OL, OM, ON. Then by the converse of the triangle of forces,

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB} \qquad ...(1)$$

Produce BC to X. Now CX is parallel to OL and CA is a parallel to OM.

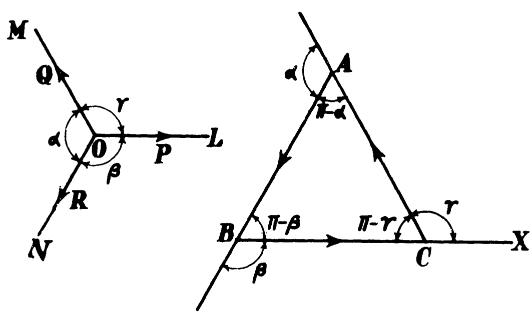


Fig. 1.13-A

$\therefore \qquad \angle ACX = \angle MOL = \gamma$

$\therefore \qquad \angle ACB = \pi - \gamma$

Similarly, $\quad \angle CAB = \pi - \alpha, \quad \angle ABC = \pi - \beta.$

By trigonometry,

$$\frac{BC}{\sin(\pi - \alpha)} = \frac{CA}{\sin(\pi - \beta)} = \frac{AB}{\sin(\pi - \gamma)}$$

or

$$\frac{BC}{\sin \alpha} = \frac{CA}{\sin \beta} = \frac{AB}{\sin \gamma} \qquad ...(2)$$

From (1) and (2),

$$\frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma}.$$

Note. In applying Lami's theorem, either the angle between the forces or its supplement, may be taken, since $\sin \alpha = \sin(180° - \alpha)$.

1.14. Tension and Compression

When a string, whose weight is negligible, is pulled with a force P, then at every point of the string equal and opposite forces P, act. Each of these forces is called the tension of the string. *It must be remembered that tensions at every point of a string are equal and opposite.* The arrow showing the direction of tension in any part of the string is to be marked so as to show the effect of the tension on the point whose equilibrium is under consideration Let AB be a string whose end A is fixed to a nail and let a weight W be attached to B. Evidently the string is pulling at A ; hence if the equilibrium of A is being considered, the arrow showing the tension T in the part adjacent to A points downwards. Again the string is clearly preventing the weight at B from falling downs *i.e.* the string is pulling B upwards. Hence the arrow near B points upwards.

String under tension
Fig. 1.14 A-1

We will assume that when a light string passes round a smooth peg or pulley, its tension remains unchanged.

When a bar is pulled at its ends by equal and opposite forces, it is in tension and the above remarks apply to it. A bar under tension is called a **tie.**

When a bar is pushed at both ends by equal and opposite forces, it is said to under **compression.**

Let AB be a bar subjected to compressive forces P at A and B. At every point of AB equal and opposite forces P' (where $P' = P$) act. P' acting at A must

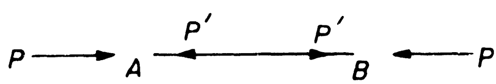


Fig. 1.14 A-2. Rod under compression.

balance the external force P ; hence when considering the equilibrium of A, the arrow showing P' must point towards A. So when considering the equilibrium of B, the arrow points towards B.

A bar under compression is called a **strut.**

1.15. Action and Reaction

When two bodies are in contact, each exerts a force on the other. One of these forces is called *action,* and the other is called *reaction.* Action and reaction are equal and opposite and when bodies are smooth, they are normal to the surfaces in contact.

Fig. 1.22 shows a block resting on a plane surface. The reaction R of the plane surface on the block is perpendicular to the surface.

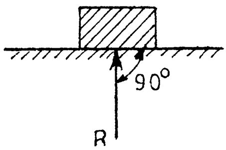


Fig. 1.22

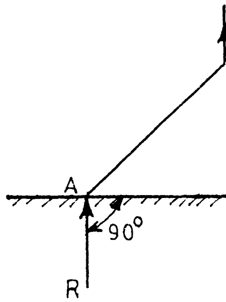


Fig. 1.23

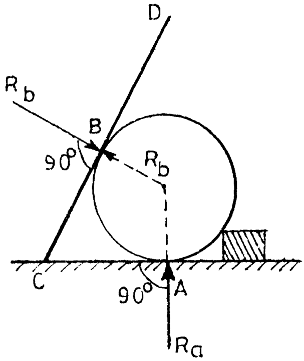


Fig. 1.24

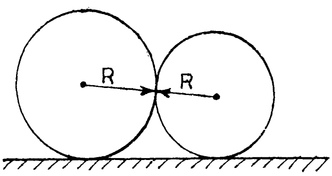


Fig. 1.25

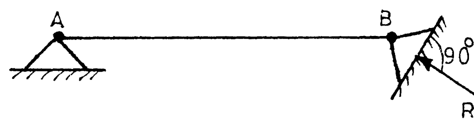


Fig. 1.26

Fig. 1.23 shows a bar AB with end A resting on a plane surface. The reaction R of the surface on the bar AB at A is perpendicular to the surface.

Fig. 1.24 shows a sphere in contact with a plane surface at A and with a bar at B. At A, the reaction R_a is normal to the sphere and its line of action passes through the centre of the sphere. At B, the bar exerts a force R_b, on the sphere. R_b is perpendicular to CD and its line of action passes through the centre of the sphere. The sphere exerts an equal and opposite force on the bar.

Fig. 1.25 shows two spheres in contact, each exerting a force R on the other whose lines of action pass through the centres.

Fig. 1.26 shows a bar AB hinged at A and resting on rollers at B on an inclined plane. The reaction R of the inclined plane on the bar is perpendicular to the plane.

A diagram showing the forces acting on a body, together with reactions at the supports, but not showing the supports, is called a *free-body diagram (FBD)*.

A body when so isolated from its supports is called a *free-body*.

Ex. 1. *Forces of 7, 15, 13 N acting on a particle are in equilibrium. Find the angle between the first two forces.*

Sol. Let the angle between forces of 7 N and 15 N be α.

Construct a triangle ABC whose sides are 7, 15 and 13 units respectively.

Then $\quad \angle BCA = 180° - \alpha$ (as in Art. 1.13)

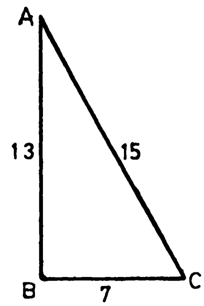


Fig. 1.15 E-1

By trigonometry,

$$\cos(180° - \alpha) = \frac{7^2 + 15^2 - 13^2}{2 \times 7 \times 15}$$

$$-\cos\alpha = \frac{1}{2} \text{ or } \alpha = \mathbf{120°}.$$

Ex. 2. *An electric light fixture weighing 15 N hangs from a point C, by two strings AC and BC. AC is inclined at 60° to the horizontal and BC at 45°*

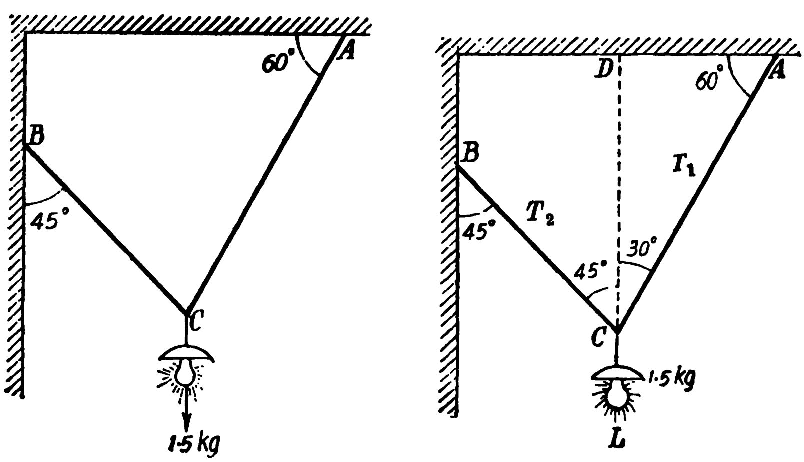


Fig. 1.15 E-2

to the vertical as shown in Fig. 1.15 E-2. Using Lami's theorem or otherwise, determine the forces in the strings AC and BC. (*A.M.I.E. Summer 1975*)

Sol. Produce the line of action of the weight of the light fixture L to meet the horizontal through A at D.

Clearly, $\angle ACD = 30°$

$\angle BCD = 45°$

$\angle ACL = 180° - 30°$

$\angle BCL = 180° - 45°$

Let T_1, T_2 be the tensions in AC and BC respectively.

Applying Lami's theorem,

$$\frac{T_1}{\sin BCL} = \frac{T_2}{\sin ACL} = \frac{15}{\sin ACB}$$

or
$$\frac{T_1}{\sin(180° - 45°)} = \frac{T_2}{\sin(180° - 30°)} = \frac{15}{\sin(30° + 45°)}$$

or
$$\frac{T_1}{\sin 45°} = \frac{T_2}{\sin 30°} = \frac{15}{\sin 75°}$$

$$\therefore \qquad T_1 = \frac{15 \times \sin 45°}{\sin 75°} = 10.98 \text{ N}$$

$$\therefore \qquad T_2 = \frac{15 \times \sin 30°}{\sin 75°} = 7.76 \text{ N}.$$

Ex. 3. *A fine light string ABCDE whose extremity A is fixed has weights w and w_1 attached to it at B and C and passes round a smooth peg at D carrying a weight of 40 N at the free end E. If in the position of equilibrium, BC is horizontal and AB, CD make angles of 150° and 120° respectively with BC, find*

(1) the tension in the portions AB, BC, CD, DE of the string :

(2) the values of the weights w and w_1 ;

(3) the pressure on the peg at D.

Sol. Let the tensions be T_1, T_2, T_3, T_4 as shown in Fig. 1.15 E-3.

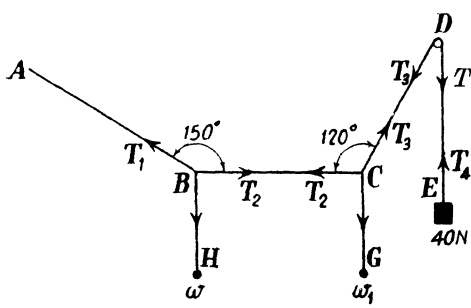


Fig. 1.15 E-3

E is in equilibrium under the vertical forces T_4 and 40 N.

$\therefore \quad T_4 =$ **40 N.**

As the peg is smooth,. tension in DC = tension in DE.

$T_3 = T_4$

= **40 N.**

Let G be a point on the line of action of weight w_1.

Then $\qquad \angle DCG = 360° - (\angle BCD + \angle BCG)$

$= 360° - (120° + 90°) = 150°$

Applying Lami's theorem to forces at C,

$$\frac{w_1}{\sin 120°} = \frac{T_2}{\sin 150°} = \frac{T_3}{\sin 90°} = 40 \qquad [\because \; T_3 = 40]$$

$$\therefore \qquad w_1 = 40 \sin 120° = 20\sqrt{3} = \mathbf{34.64 \text{ N}}.$$

$$T_2 = 40 \sin 150° = \mathbf{20 \text{ N}}.$$

Let H be a point on the line of action of weight w.
Then $\angle ABH = 360° - (150° + 90°) = 120°$.

Applying Lami's theorem to forces at B,

$$\frac{T_1}{\sin 90°} = \frac{w}{\sin 150°} = \frac{T_2}{\sin 120°} = \frac{40}{\sqrt{3}} \qquad [\because \; T_2 = 20]$$

$$\therefore \qquad T_1 = \frac{40}{\sqrt{3}} = \mathbf{23.09\ N}$$

$$w = \frac{40}{\sqrt{3}} \times \frac{1}{2} = \mathbf{11.55\ N}$$

CG and DE are parallel.

$$\therefore \qquad \angle DCG + \angle CDE = 180°$$

$$150° + \angle CDE = 180°$$

$$\angle CDE = 30°$$

Tension in each of DC and DE is 40 N. The pressure on D is the resultant of these tensions acting as 30°.

$$\therefore \quad \text{Pressure on } D = 2 \times 40 \times \cos\frac{30}{2} = 80 \times 0.966$$

$$= \mathbf{77.28\ N.}$$

Ex. 4. *Two equal lengths of tubing, of weight 2W each, are placed on two racks so that each rack supports half the weight of the tubing. Neglecting friction at all surfaces, determine the reactions exerted by the racks at A, B and C when $\alpha = 45°$.*

Find the least value of α for which equilibrium is possible.

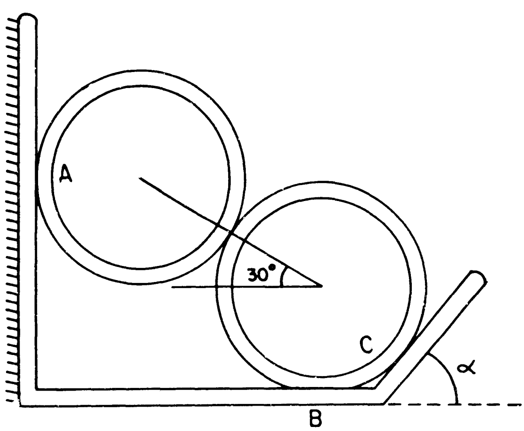


Fig. 1.15 E-4.1

Sol. The *FBD* of the upper, tubing is shown in Fig. 1.15 E-4.2. R_A is the reaction at A, S is the force exerted upon the upper tubing by the lower tubing; it is inclined to the horizontal at 30°.

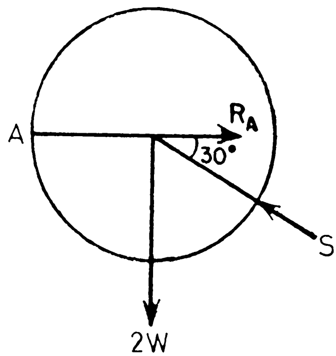


Fig. 1.15 E-4.2

Resolving vertically, we see that

$S \sin 30 = 2W$ and $S = 4W$

$\therefore \quad R_A = S \cos 30 = 4W \times \sqrt{3}/2$

$= 2W\sqrt{3}$. **Ans.**

The *FBD* of the lower tubing is shown in Fig. 1.15 E-4.3. R_C, the reaction at C, is perpendicular to the inclined surface and hence it makes angle α with the vertical.

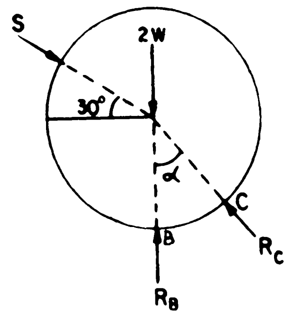


Fig. 1.15 E-4.3

Resolving horizontally, we get

$R_C \sin \alpha = S \cos 30° = 4W \times \sqrt{3}/2$

$R_C = 2W\sqrt{3}/\sin \alpha$

If $\alpha = 45°$, $R_C = 2W\sqrt{6}$. **Ans.**

Resolving vertically, we have

$R_B + R_C \cos \alpha = S \sin 30 + 2W$

$R_B = S \sin 30 + 2W - R_C \cos \alpha$

$$= 2W + 2W - \frac{2W\sqrt{3}}{\sin\alpha} \times \cos \alpha$$

$= 4W - 2W\sqrt{3} \cot \alpha.$

If $\alpha = 45°$, $R_B = 4W - 2W\sqrt{3}$. **Ans.**

R_A, R_B, R_C are the total reactions exerted by the two racks.

As α decreases, the tendency of the lower tubing to ascend the inclined surface increases. When it is about to ascend, $R_B = 0$.

$\therefore$ The minimum value of α for equilibrium is given by

$$R_B = 4W - 2W\sqrt{3} \cot \alpha = 0$$

$$\tan \alpha = \sqrt{3}/2, \quad \alpha = 40.9°. \quad \textbf{Ans.}$$

Ex. 5. *Three bars hinged at A and D and pinned at B and C as shown in Fig. 1.15 E-5.1, form a four link mechanism. Determine the value of P which will prevent motion. Neglect friction.* (M.S.U.R. College, Hyderabad)

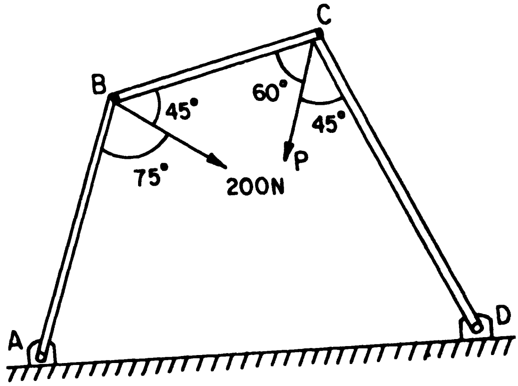


Fig. 1.15 E-5.1

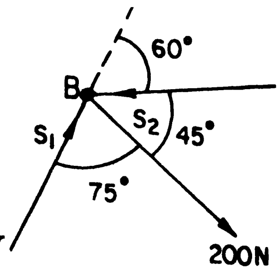


Fig. 1.15 E-5.2

Sol. Consider the equilibrium of pin B.

Let S_1 = force exerted by AB on B.

S_2 = force exerted by BC on B.

Fig. 1.15 E-5.2 is the free-body diagram of pin B. BC makes with AB produced an angle of $180° - (75° + 45°) = 60°$.

Resolving perpendicular to S_1, we get

$$S_2 \sin 60° = 200 \sin 75°$$

$$S_2 = 200 \sin 75°/\sin 60°.$$

Consider the equilibrium of pin C. Assume that the force exerted by DC on C is S_3.

Angle between DC produced and BC is $180° - (60° + 45°) = 75°$.

Fig. 1.15 E-5.3 is the free-body diagram of C. Resolving perpendicular to S_2, we get

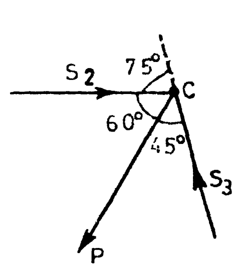


Fig. 1.15 F-5.3

$$P \sin 45° = S_2 \sin 75°$$

$$P = \frac{\sin 75°}{\sin 45°} \times \frac{200 \sin 75°}{\sin 60°}$$

$= \mathbf{304.7\ N.\ Ans.}$

Ex. 6. *Cords are looped around a small spacer separating two cylinders each weighing 400 N and pass over frictionless pulleys to weights of 200 N and 600 N. Determine the angle θ and the normal reaction R between the cylinders and the smooth surface inclined 15° to the horizontal as shown in Fig. 1.15 E-6.1.*

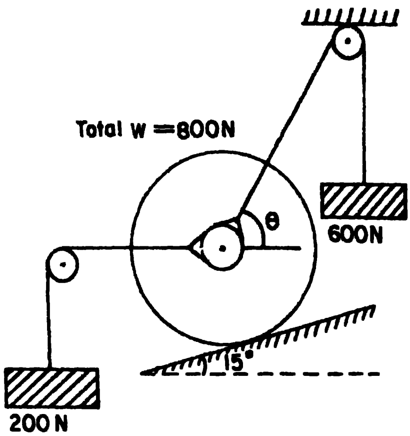


Fig. 1.15 E-6.1

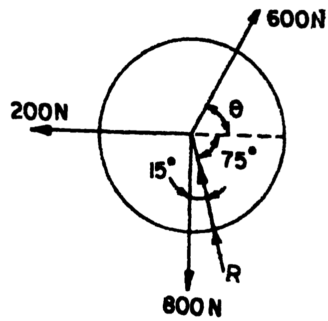


Fig. 1.15 E-6.2

Sol. When a cord passes over a smooth pulley, its tension remains unchanged. Accordingly, the *FBD* of the cylinders shall be as shown in Fig. 1.15 E-6.2.

Resolving the forces perpendicular to R, we write

$$600 \sin (75° + \theta) = 800 \sin 15° + 200 \sin 75°.$$

This equation gives

$$\sin (75° + \theta) = \frac{400.3}{600} = \sin 41.85°$$

$$\therefore \qquad 75° + \theta = 180° - (41.85°) = 138.15°$$

$$\theta = \mathbf{63.2°. \ Ans.}$$

Resolving vertically, we get

$$R \cos 15° + 600 \sin \theta = 800$$

$$R \cos 15° = 800 - 600 \sin 63.2° = 264.4$$

$$R = \mathbf{274 \ N. \ Ans.}$$

Ex. 7. *Three cylinders are piled in a rectangular ditch as shown in Fig. 1.15 E-7.1. Neglecting friction, determine the reaction between cylinder A and the vertical wall.*

Sol. B is at a distance of 120 mm from the right wall and C is at a distance of 100 mm from the left wall.

Hence the horizontal distance between B and C

$$= 360 - (100 + 120) = 140 \text{ mm}.$$

$$BC = 120 + 100 = 220 \text{ mm}.$$

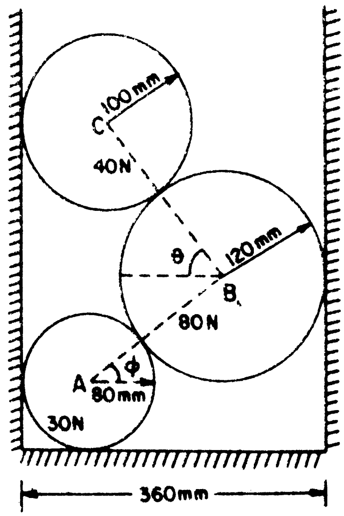


Fig. 1.15 E-7.1

θ, the inclination of BC to horizontal is given by

$$\cos\theta = \frac{140}{220} \qquad \theta = 50.5^\circ$$

Similarly, ϕ, the inclination of AB to horizontal

$$= \cos^{-1}\frac{160}{200} = 36.9^\circ$$

R_1, the force exerted by B on C, will act along the line BC. The free-body diagram of cylinder C is shown in Fig. 1.15 E-7.2. The force S_1 exerted by the wall on C is horizontal.

By Lami's theorem

$$\frac{R_1}{\sin 90^\circ} = \frac{S_1}{\sin(90 - 50.5)} = \frac{40}{\sin 50.5^\circ}$$

$$R_1 = 51.8\,N \qquad S_1 = 33\,N.$$

Fig. 1.15 E-7.3 is the FBD of cylinder B. R_2 is the action of A on B inclined at 36.9° to the horizontal. S_2 in the force exerted by the wall on B.

Resolving vertically, we write

$$R_2 \sin 36.9^\circ = 80 + 51.8 \sin 50.5^\circ$$

$$R_2 = 200\,N.$$

Resolving horizontally, we obtain

$$S_2 = 51.8\cos 50.5^\circ + 200\cos 36.9^\circ$$

$$= 193\text{ N}.$$

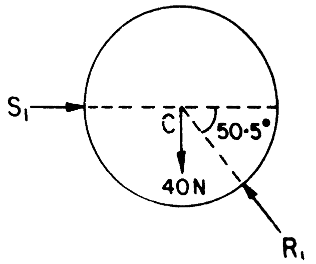


Fig. 1.15 E-7.2

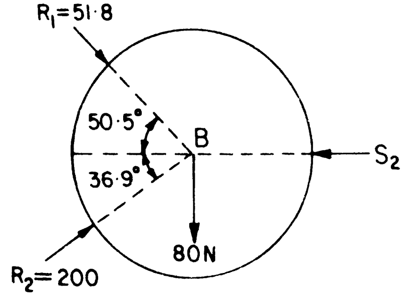


Fig. 1.15 E-7.3

Considering the three cylinders to be a free-body (Fig. 1.15 E-7.4),

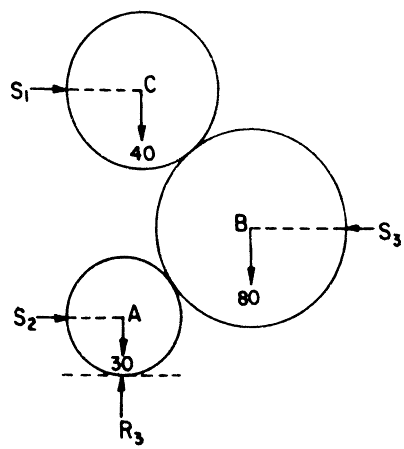


Fig. 1.15 E-7.4

we see that S_3 = reaction of the wall on A

$= S_2 - S_1 = 193 - 33 =$ **160 N. Ans.**

2nd method. From Fig. 1.15 E-7.4, we see by resolving vertically, that

$$R_3 = 40 + 80 + 30 = 150 \text{ N}$$

The free-body diagram of cylinder A is shown in Fig. 1.15 E-7.5.

Resolving vertically and horizontally, we get

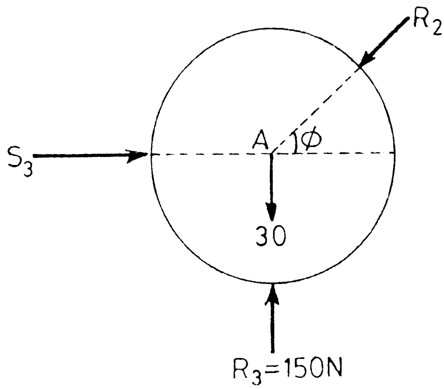
$$R_2 \sin \phi = 150 - 30 = 120$$

$$R_2 \cos \phi = S_3$$

whence $S_3/120 = \cot \phi = 4/3$

$S_3 =$ **160 N. Ans.**

In this method it is not necessary to calculate S_1, S_2, R_1, R_2.



$$\cos\phi = \frac{160}{200} = \frac{4}{5}$$

$$\cot\phi = \frac{4}{3}$$

Fig. 1.15 E-7.5

Ex. 8. *A rope AB, 9 m long, is connected at A and B to two points on the same level, 8 m apart. A load of 300 N is suspended from a point C on the rope, 3 m from A. What load connected to a point D on the rope, 2 m from B, will be necessary to keep the portion CD level ?* (Madras)

Sol. Let the load to be attached to D be W N.

Let E be the middle point of AB, so that

$$AE = EB = 4 \text{ m}.$$

EB and CD are equal and parallel. Hence $ECDB$ is a parallelogram.

$\therefore \quad EC = BD = 2$ m.

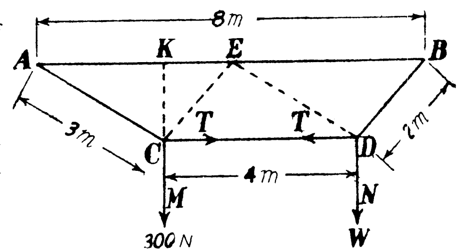


Fig. 1.15 E-8

In the triangle ACE,

$$\cos A = \frac{AE^2 + AC^2 - CE^2}{2AE\,AC} = \frac{4^2 + 3^2 - 2^2}{2 \times 4 \times 3} = \frac{7}{8}.$$

$$\sin A = \sqrt{1 - \left(\frac{7}{8}\right)^2} = \frac{\sqrt{15}}{8}$$

Let the vertical CM through C meet AE at K. Then

$$\angle ACM = \angle CKA + \angle CAK = 90° + A$$

$$\sin ACM = \sin(90° + A) = \cos A = \frac{7}{8}$$

$$\angle ACD = 180° - \angle BAC \qquad [\because \ AB \text{ is parallel to } CD]$$

$$\sin ACD = \sin BAC = \frac{\sqrt{15}}{8}$$

If T be the tension in CD, we have by Lami's theorem,

$$\frac{T}{\sin ACM} = \frac{300}{\sin ACD}$$

$$T = \frac{300 \times \sin ACM}{\sin ACD} = \frac{300 \times 7 \times 8}{8 \times \sqrt{15}} = \frac{2100}{\sqrt{15}} \text{ N.}$$

AE is equal and parallel to CD. Hence $AEDC$ is a parallelogram

$$\therefore \qquad ED = AC = 3 \text{ m}$$

In the triangle EBD,

$$\cos B = \frac{BE^2 + BD^2 - ED^2}{2BE.BD} = \frac{4^2 + 2^2 - 3^2}{2 \times 4 \times 2} = \frac{11}{16}$$

Let DN be the vertical through D. It is easy to see that

$$\angle BDN = 90^\circ + \angle EBD$$

$$\sin BDN = \cos EBD = 11/16.$$

$$\angle CDB = 180^\circ - \angle EBD$$

$$\sin CDB = \sin EBD = \sqrt{1 - \left(\frac{11}{16}\right)^2} = \frac{3\sqrt{15}}{16}$$

Applying Lami's theorem to the equilibrium of D

$$\frac{W}{\sin BDC} = \frac{T}{\sin BDN}$$

$$W = \frac{T \sin BDC}{\sin BDN} = \frac{2100}{\sqrt{15}} \times \frac{3\sqrt{15}}{16} \times \frac{16}{11} = \mathbf{572.73 \text{ N.}}$$

Ex. 9. *A heavy spherical ball of weight W rests in a V-shaped trough, whose sides are inclined at angles α and β to the horizontal. Find the pressure on each side. If a second equal ball be placed on the side of inclination α, so as to rest above the first, find the pressure of the lower ball on the side β.*

(Kerala)

Sol. Let XY be the horizontal and OA, OB the sides of the trough inclined at angles α, β to the horizontal.

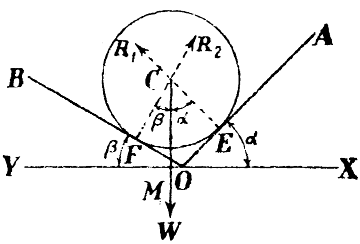


Fig. 1.15 E-9.1

Let C be the centre of the ball and E, F the points of contact with the sides of the trough.

Let R_1 and R_2 be the reactions at E and F, passing through the centre C.

Let CM be vertical through C. Then CM is the line of action of the weight W of the ball.

Now EC is perpendicular to OA and CM is perpendicular to OX.

$$\therefore \qquad \angle ECM = \angle XOA = \alpha.$$

Similarly, $\qquad \angle FCM = \angle YOB = \beta.$

$\therefore$ The angle between the positive directions of R_1 and $W = 180° - \alpha$
The angle between the positive directions of R_2 and $W = 180° - \beta$.
The angle between the positive directions of R_1 and $R_2 = \alpha + \beta$.
By Lami's theorem,

$$\frac{R_1}{\sin(180° - \beta)} = \frac{R_2}{\sin(180° - \alpha)} = \frac{W}{\sin(\alpha + \beta)}$$

$$R_1 = \frac{\mathbf{W} \sin \beta}{\sin(\alpha + \beta)}, R_2 = \frac{\mathbf{W} \sin \alpha}{\sin(\alpha + \beta)}.$$

Next let a second ball be placed on the side α. E_1 is the point of contact of the 2nd ball with the plane.

The two balls may be assumed to form a single body which is in equilibrium under the action of the following forces :

(1) The weights W each of the balls acting vertically through the centres O_1 and O_2 of the balls.

(2) Reactions R_1 and R_3 at E and E_1 acting at right angles to the side α.

(3) Reaction R_2 at F acting at right angles to side β.

The line O_1O_2 is parallel to the side α, since the balls are equal. R_1, R_3 are perpendicular to O_1O_2.

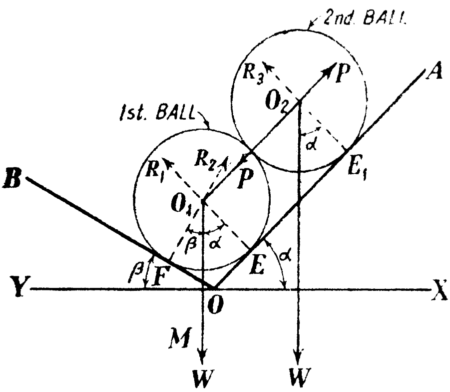


Fig. 1.15 E-9.2

The angle between R_1 and R_2 is $\alpha + \beta$. Hence R_2 is inclined to O_1O_2 at angle $90° - (\alpha + \beta)$.

Resolving along O_1O_2,

$$R_2 \cos\{90° - (\alpha + \beta)\} = W \sin\alpha + W \sin\alpha$$

$$\therefore \qquad R_2 = (2W \sin\alpha)/\sin(\alpha + \beta).$$

Ex. 10. *A string ACB of length l hangs between two vertical walls as shown in Fig. 1.15 E-10.1. Along this string a small pulley C, from which is suspended a load P, can roll without friction. In the particular case, where l = 2a = 4b, what configuration of equilibrium will the string assume ?*

Sol. Let the vertical through C meet the horizontals through A and B in M and N [See Fig. 1.15 E-10.2].

As the pulley C is smooth, tensions in BC and CA are equal and the resultant of these tensions will bisect $\angle ACB$. But this resultant should have the same line of action as P ; hence the line of action of P namely the vertical CNM, will bisect $\angle ACB$.

Let $\qquad \angle ACN = \angle NCB = \theta, BC = x$

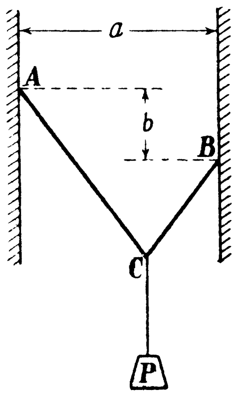


Fig. 1.15 E-10.1

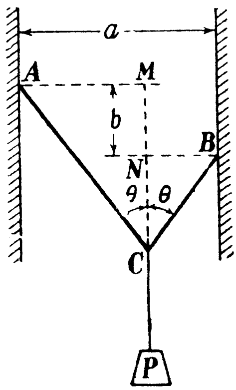


Fig. 1.15 E-10.2

Then $AC = l - x = 4b - x$

$$AM = AC \sin\theta = (4b - x)\sin\theta$$

$$BN = BC \sin\theta = x \sin\theta$$

$$a = AM + BN = (4b - x)\sin\theta + x\sin\theta = 4b\sin\theta.$$

or $$2b = 4b\sin\theta \qquad [\because\ 2a = 4b, \text{ or } a = 2b]$$

$$\sin\theta = \tfrac{1}{2},\ i.e.\ \theta = 30^\circ$$

$$NM = CM - CN = AC\cos\theta - BC\cos\theta$$

or $$b = (4b - x)\cos\theta - x\cos\theta$$

$$= 2b\sqrt{3} - x\sqrt{3} \text{ whence } x = \frac{2\sqrt{3} - 1}{\sqrt{3}}\,b.$$

Hence in the position of equilibrium, $\angle ACB = 60^\circ$ and $BC = \dfrac{2\sqrt{3} - 1}{\sqrt{3}}\,b.$

PROBLEMS

1. A heavy box weighing 2000 N is supported by two wire ropes, one end of each of which is tied to the same point of the box. The other end of one rope, which is inclined to the vertical at 45°, is attached to a staple in a wall, while the other end of the second rope, which is inclined to the vertical at 30°, is fixed to a hook in the ceiling. Find the tensions in the ropes. [**Ans.** 1035 N, 1464 N]
2. Two masses, each equal to 100 kg are attached to the extremities of a string which passes over two smooth pegs in the same horizontal line and distant 1 m apart. If a mass of 4 kg be attached to the string half-way between the pegs, find the depth to which the string will descend. [**Ans.** 10 mm]
3. A body is free to slide on a smooth vertical circular wire and is connected by a string, equal in length to the radius of the circle, to the highest point of the circle ; find the tension of the string and the reaction of the circle. [**Ans.** Each is equal to the weight of the body]

4. A body of weight 200 N is suspended by two strings 70 and 240 mm long, their ends being fastened to the extremities of a rod of length 250 mm. If the rod be so held that the body hangs immediately below its middle point, find the tensions of the strings. [**Ans.** 56 N and 192 N]

5. Three weightless strings AC, BC and AB are knotted together to form an isosceles triangle whose vertex is C. If a weight W be suspended from C and the whole be supported with AB horizontal, by two forces bisecting the angles at A and B, find the tension of the string AB. [**Ans.** $W/(2\cos C/2)$]

6. Two smooth spheres, each of radius 200 mm and weight 200 N, rest in a horizontal channel having vertical walls, the distance between which is 720 mm. Find the pressures at E, F and G.

[**Ans.** R_E = 267 N, R_F = 400 N, R_G = 267 N]

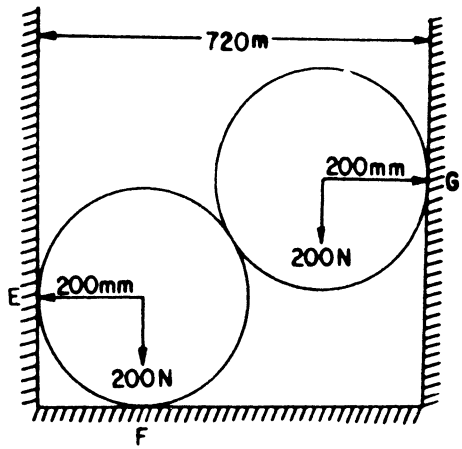


Fig. 1.15 P-6

7. Three cylinders weighing 100 N each and 160 mm in diameter are placed in a channel rectangular in section as shown in Fig. P-7. What is the pressure

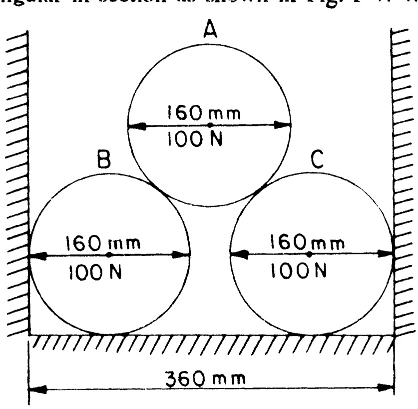


Fig. 1.15 P-7

exerted by A on B ? What is the pressure exerted by the lower two cylinders on the channel base and walls at the contact points ?

[**Ans.** Pressure of A on B = 64 N
Pressure of B on the base = 150 N
Pressure of B on the wall = Pressure of C on the wall = 40 N]

8. Three equal rings of weight W each rest on a smooth vertical circular wire at the corners of an equilateral triangle of which one side is vertical, the uppermost being connected with the other two by means of strings. Find their tensions.

[**Ans.** Tension in the vertical string W
Tension in the other string = $2W$]

9. Two equal spheres of 30 mm radius, are in equilibrium within a smooth spherical cup of 90 mm radius. Show that the action between the cup and one sphere is double that between the two spheres. *(Madras)*

10. A solid sphere of weight W rests upon two parallel bars which are in the same horizontal plane, the distance between bars being equal to the radius of the sphere ; find the reaction of each bar. [**Ans.** $W/\sqrt{3}$]

11. Two rollers of weights P and Q are connected by a flexible string DE and rest on two mutually, perpendicular planes AB and BC as shown in Fig. 1.15 P-11.

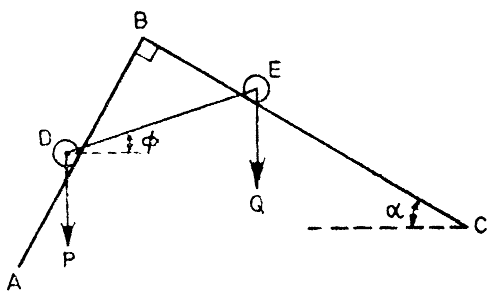


Fig. 1.15 P-11

Find the tension S in the string and angle ϕ that it makes with the horizontal when the system is in equilibrium, given P = 600 N, Q = 1000 N, $\alpha = 30°$. Assume that the string is inextensible and passes freely through slots in the smooth inclined planes AB and BC. *(Roorkee)*

[**Ans.** S = 721 N, ϕ = 16°6′]

12. Two identical rollers, each of weight Q = 1 kN are supported by an inclined plane and a vertical wall as shown in Fig. 1.15 P-12. Assuming smooth surfaces, find the reactions at A, B and C.

[**Ans.** R_a = 866 N ; R_b = 1440 N ; R_c = 1150 N]

13. A weight Q is suspended from point B of a cord ABC, the ends of which are pulled by equal weights P overhanging small pulleys A and C which are on the same level as shown in Fig. 1.15 P-13. Neglecting the radii of the pulleys, determine the sag BD if l = 12 m, P = 200 N, Q = 100 N. *(Nagpur)*

[**Ans.** BD = 1.55 m]

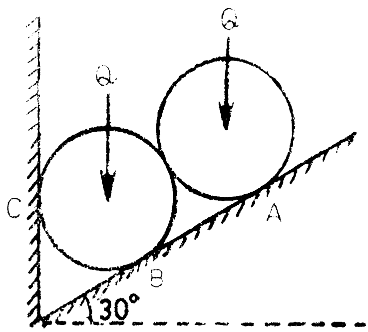


Fig. 1.15 P-12

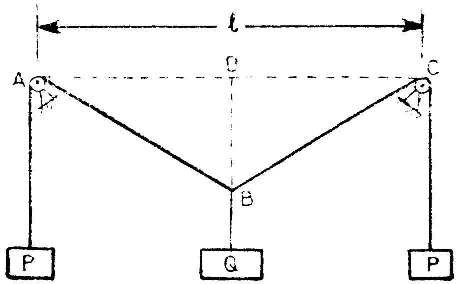


Fig. 1.15 P-13

14. Three bars in one plane, hinged at their ends as shown in Fig. 1.14 P-14 are submitted to the action of a force P = 100 N applied at B. Determine the magnitude of the force Q that will be necessary to apply at the hinge C in order to keep the system of bar in equilibrium in the position shown. (*Bangalore*)

[**Ans.** Q = 163 N]

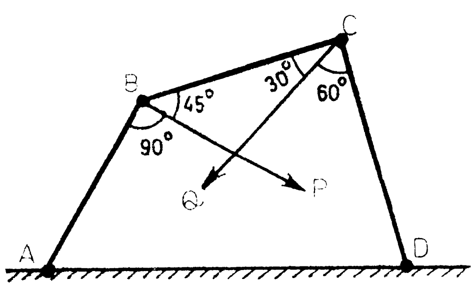


Fig. 1.15 P-14

1.16. Law of Polygon of Forces

If any number of forces acting at a point can be represented in direction and magnitude by the sides of a polygon taken in order, then the forces are in equilibrium.

Let the forces P_1, P_2, P_3, P_4, P_5 acting at O be represented in direction and magnitude by the sides AB, BC, CD, DE, EA of the polygon $ABCDE$. Join AC, AD.

Then
$$\vec{AB} + \vec{BC} = \vec{AC}$$
$$\vec{AC} + \vec{CD} = \vec{AD}$$
$$\vec{AD} + \vec{DE} = \vec{AE}$$
$$\vec{AE} + \vec{EA} = \vec{0}.$$

Hence the system is in equilibrium

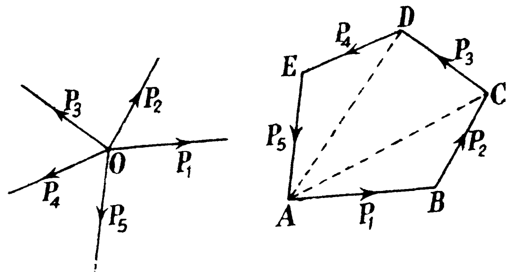


Fig. 1.16 A

Cor. It is clear that the resultant of the forces represented by AB, BC, CD, DE is represented by AE.

Note. The converse of the polygon of forces is not true. Thus suppose that the forces P_1, P_2, P_3, P_4, P_5 are in equilibrium. Then we can construct an infinite number of polygons whose sides are respectively parallel to the forces but their ratios are all different. ***However a polygon can be con: tructed whose sides are respectively parallel and proportional to the forces.***

1.17. Graphical Methods

The triangle of forces and the polygon of forces can be used to obtain graphically the resultant of forces acting at a point.

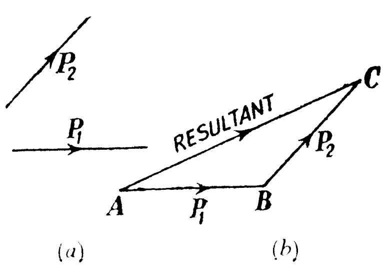
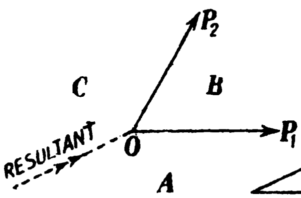


Fig. 1.17 A-1

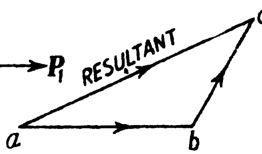
(*a*) Let the forces P_1 and P_2 act at a point. To find their resultant, draw a line AB parallel and proportional to P_1 on a suitably chosen scale ; then AB represents P_1 in direction and magnitude. Similarly draw BC to represent P_2. Join AC. Then AC represents the resultant of the forces in direction and magnitude. AC and the $\angle CAB$ can be easily measured.

(*b*) In Fig. 1.17 A-2 are shown two forces P_1, P_2 acting at O. The spaces about these forces have been named as A, B, C. The force P_1, which lies between spaces A and B is then named as AB. The force P_2 is named as BC, single it lies between the spaces B and C. This diagram showing the position of the forces is called a *space diagram*.

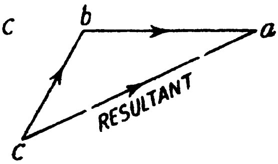
Draw a vector to represent P_1. [A vector is a straight line representing a force in magnitude and direction]. Since in the space diagram P_1 has been named as AB, the vector representing P_1 is named by the corresponding small letters ab. Similarly the vector representing P_2 or BC is named as bc. We thus obtain Fig. 1.17 A-3 representing the two forces. Join ac. Then ac represents the resultant. This diagram which shows the magnitudes and directions of the forces, is called a *vector diagram* or a *force diagram*.



Space diagram
Fig. 1.17 A-2



Vector diagram
Fig. 1.17 A-3



Vector diagram
Fig. 1.17 A-4

In the vector diagram, ab represents P_1. The sense of ab must be same as that of P_1. Thus the arrow-head should point from a towards b. Similarly, the sense of bc must be the same as that of P_2.

In the space diagram the forces have been named as AB and BC, while in the vector diagram they have been named as ab and bc.

We can also name the forces as BA and CB in the space diagram. Then they must be named as ba and cb in the vector diagram. The sense of ba must be the dame as that of P_1 and the sense of cb must be the same as that of P_2. Thus we obtain the vector diagram as shown in Fig. 1.17 A-4. The resultant is then represented by ca.

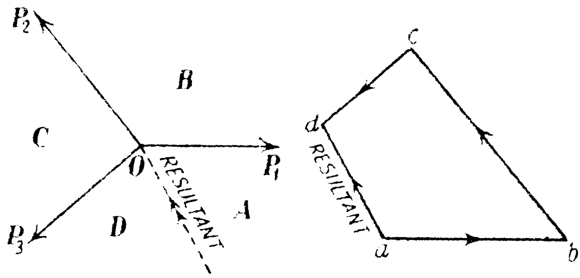
Thus we see that we can move either anti-clockwise or clockwise. But the movement must be uniform throughout ; *i.e.* either clockwise throughout or anti-clockwise throughout. If P_1 is named as AB, then P_2 must be named as BC and not as CB.

In the space diagram draw a line parallel to ac (shown dotted in Fig. 1.17 A-2). Then this line represents the direction and position of the resultant. If the forces P_1 and P_2 are named as AB and BC then the resultant is named as AC, *i.e.* if in naming the forces the movement is *anti-clockwise*, then in naming the resultant the movement is *clockwise*.

In the vector diagram, if the forces are ab and bc then the resultant is ac, *i.e. the line joining the first and the last points taken in the sense opposite to that of the forces.*

This method of naming the forces in the space and vector diagrams is called *Bow's notation*.

(*c*) Next, let it be required to find the resultant of three forces P_1, P_2, P_3 acting at O. Name the spaces as shown in Fig. 1.17 A-5 (*a*). Draw vectors *ab*, *bc*, *cd* to represent *AB*, *BC*, *CD*. Join *ad*. Then *ad* represents the resultant (Art. 1.16, Cor.) ; *ad* is the line joining the initial point of the first force and terminal point of the last force. The position and direction of the resultant have been shown in the space diagram by a dotted line with two arrow-heads which has been drawn parallel to *ad*. In the space diagram, the forces are *AB*, *BC*, *CD* and the resultant is *AD*.



(*a*) Space diagram (*b*) Vector diagram

Fig. 1.17 A-5

The force which applied at O will produce equilibrium, is called the *equilibriant* of the system. Evidently, the equilibriant is equal and opposite to the resultant and in the present case it is represented by *DA* in the space diagram, and by *da* in the vector diagram.

The method explained above is appicable to any number of forces.

(*d*) Finally, suppose that any number of forces say P_1, P_2, P_3, X, Y keep a particle O in equilibrium. If P_1, P_2, P_3 be known in direction and magnitude while X, Y be known in direction only, we can find the magnitudes of X, Y graphically.

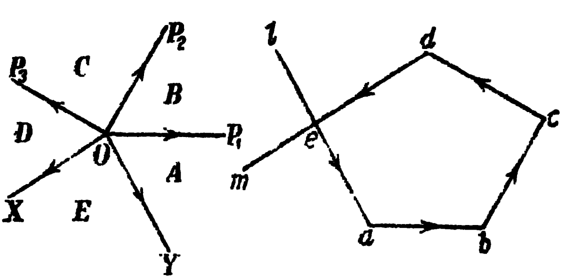
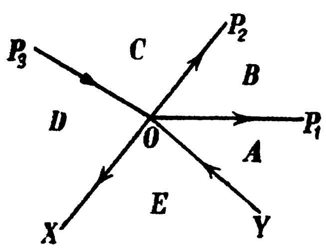


Fig. 1.17 A-6

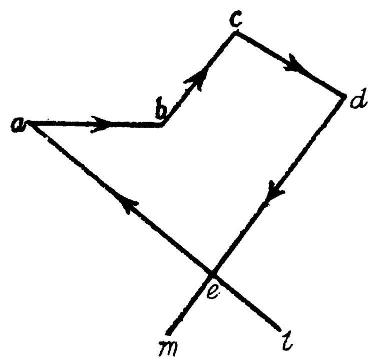
Name the spaces as A, B, C, D, E as shown in Fig. 1.17 A-6. Draw ab, bc, cd to represent AB, BC, CD. Through a, draw al parallel to AE. Through d, draw dm parallel to DE, meeting al in e. Then X is represented by de and Y is represented by ea.

It should be noted that not more than two of the forces should be unknown. To determine the forces X and Y, we must know their directions and magnitudes (position being already known, for they act at O), (*i.e.* in all four quantities should be known : if any two of these are known the remaining two can be determined by the method of this article.)

The method applies whether the forces converge or diverge. Suppose, for example, P_1, P_2 and X diverge from O while P_3 and Y converge towards O, as shown in the space diagram [Fig. 1.17 A-7 (*a*)]. The corresponding vector diagram is then as shown in Fig. 1.17 A-7 (*b*).



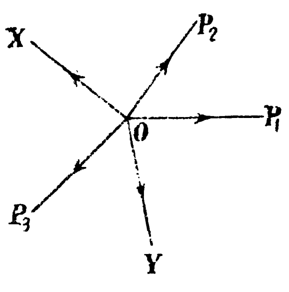
(*a*) Space diagram



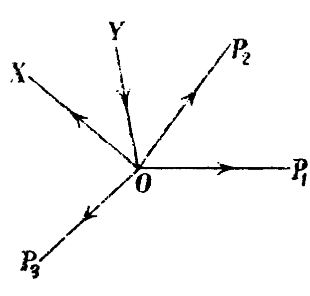
(*b*) Vector diagram

Fig. 1.17 A-7

The space diagram must be so drawn that the unknown forces X, Y occur together ; no other force should come in between them. Consider the



(*a*) Original space diagram

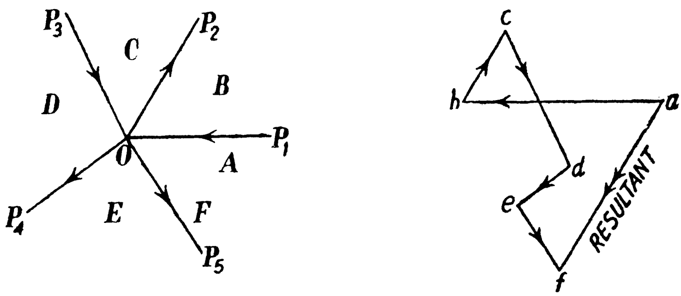


(*b*) Modified space diagram

Fig. 1.17 A S

space diagram [Fig. 1.17 A-8 (*a*)]. We note that P_3 falls between X and Y. The space diagram should be so modified that X and Y come together. The modified space diagram is shown in Fig. 1.17 A-8 (*b*).

(*e*) The lines in a vector diagram may intersect each other as shown in Fig. 1.17 A-9.



(*a*) Space diagram (*b*) Vector diagram

Fig. 1.17 A-9

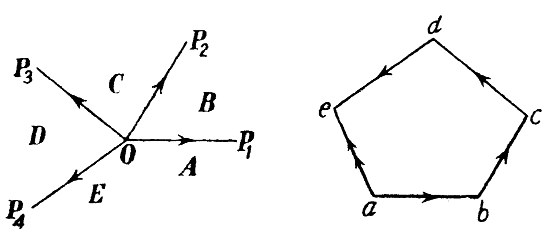
(*f*) It will be seen that the arrow-heads in the vector diagram follow the same order throughout for all the given forces, but the arrow-head on the resultant is in the opposite sense.

If, however, the equilibriant is required, then the sense of the arrow-head is to be the same as that of the arrows on the given forces.

1.18. Graphical Condition of Equilibrium (concurrent forces)

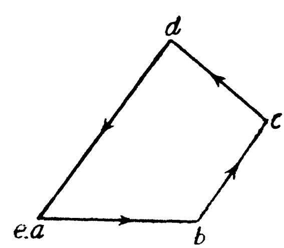
In Fig. 1.18 A-1, the resultant of forces *AB, BC, CD, DE* is represented by *ae* in the force polygon. If, however, *e* coincides with *a*, *i.e.* the last point falls on the first point as shown in Fig. 11.8 A-2, we obtain a closed polygon and *ae* = 0.

Then the resultant of the forces is zero.



(*a*) Space diagram (*b*) Force polygon

Fig. 1.18 A-1



Force polygon closes

Fig. 1.18 A-2

Hence a system of coplanar concurrent forces is in equilibrium if the force polygon closes.

Ex. 1. *The resultant of two forces, P and 100 N is 150 N inclined at 30° to the 100 N force. Find the magnitude and direction of P.*

Sol. Draw OA to represent the 100 N force on a suitable scale.

Draw OB to represent the resultant, making $\angle AOB = 30°$. Join AB.

Since $\vec{OA} + \vec{AB} = \vec{OB}$, it follows that AB represents P.

Produce OA to X. By measurements,

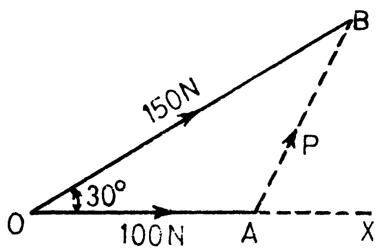


Fig. 1.18 E-1

P = **81N**, $\angle$**BAX = 68° 18′**

These results can be easily verified by calculation. By trigonometry,

$$AB^2 = OA^2 + OB^2 - 2OA.OB \cos AOB$$

$$P^2 = 100^2 + 150^2 - 2 \times 100 \times 150 \cos 30° = 652$$

$$P = 81 \text{ N}$$

$$\frac{\sin OAB}{OB} = \frac{\sin AOB}{AB}$$

$$\frac{\sin BAX}{150} = \frac{\sin 30°}{81} \qquad [\because \ \sin OAB = \sin BAX]$$

$$\angle BAX = 68° \ 18'.$$

Ex. 2. *Fig. 1.18 E-2.1 shows a rafter joint O in a roof truss. The rafters OA and OB are inclined to the horizontal at 30° each. The forces acting at O are*

(*i*) *a positive wind load of 500 kg acting normally to OA* ;

(*ii*) *a negative wind load of 300 kg normal to OB* :

(*iii*) *a vertical dead load of 1000 kg.*

Find the resultant force acting at O.

Sol. The rafters OA and OB are inclined to the horizontal at 30° and the wind loads are normal to them. Hence the wind loads are inclined to the vertical at 30°. The forces at O are shown in the space diagram [Fig 1.18 E-2.2 (*a*)].

The spaces have been named A', B', C', D'. The corresponding force polygon is $a'\ b'\ c'\ d'$ and $a'\ d'$ is the resultant.

By measurement, $a'\ d'$ = 1239 kg, inclination of $a'\ d'$ to the vertical = 18° 51′. The resultant is shown in the space diagram by a dotted line bearing two arrows.

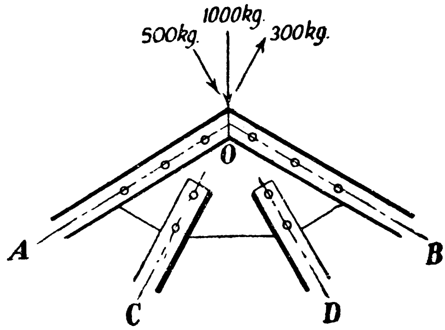
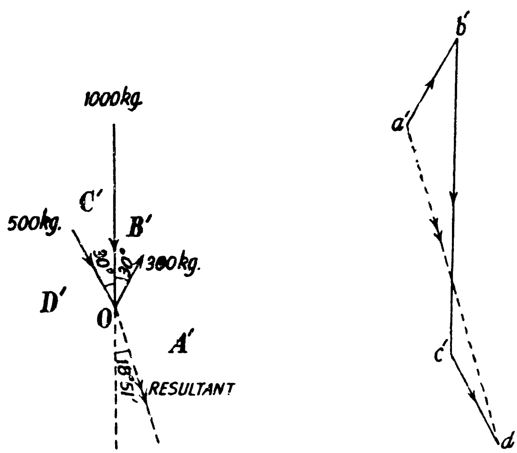


Fig. 1.18 E-2.1



(*a*) Space diagram. (*b*) Force polygon.

Fig. 1.18 E-2.2

Ex. 3. *The known forces acting at the apex joint of a steel roof truss are as shown in Fig. 1.18 E-3.1:*

(*i*) *a vertical dead load of 10 kN*

(*ii*) *a positive wind load of 5 kN normal to OA*

(*iii*) *a negative wind load of 3 kN normal to OB*

(*iv*) *a thrust of 20 kN in rafter OA*

(*v*) *a pull of 20 kN in the member C.*

Find the nature and magnitude of the forces in OB and the member D.

Sol. The rafters OA and OB are inclined to the horizontal at 30° while members C and D are inclined at 60° to the horizontal.

The forces acting at O are shown in the space diagram [Fig. 1.18 E-3.2 (*a*)]. The force in OD is marked X and that in OB is marked Y.

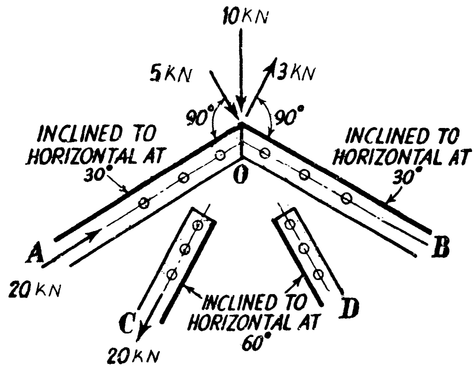


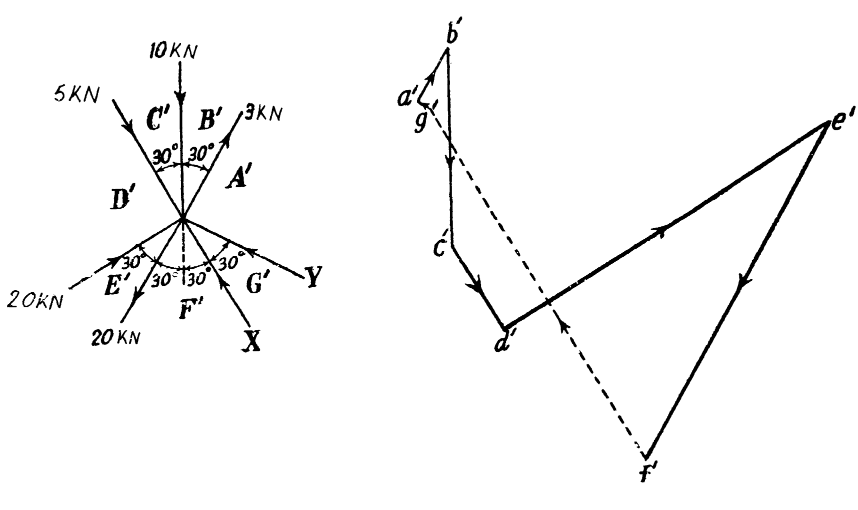
Fig. 1.18 E-3.1

The corresponding force diagram is shown in Fig. 1.18 E-3.2 (*b*). g' a' is parallel to Y and f' g' is parallel to X.

Moving in the anti-clockwise sense in the space diagram the forces X and Y are respectively named as F' G' and G' A'. Hence in the vector diagram they are represented by f' g' and g' a'. The arrows on X and Y are to be marked so as to be in the same sense as those on f' g' and g' a' respectively. On marking the arrows in this manner we see that both X and Y are compressive forces, *i.e.* member D and rafter OB exert a thrust at O.

By measurement, $X = f'g' = 21.7$ kN (thrust)

$Y = g'$ $a' = 560$ N (thrust)



(*a*) Space diagram (*b*) Force diagram

Fig. 1.18 E-3.2

Ex. 4. *In a simple jib crane, shown in Fig. 1.18 E-4, the vertical post is 5 metres, the tie is 4 metres and the jib is 7 metres. If the crane supports a load of 10 kN, find the magnitude and nature of forces in the jib and tie.*

Sol. The sides of the triangle ABC are parallel to the forces acting at C, and hence the forces in AB and BC are proportional to their lengths.

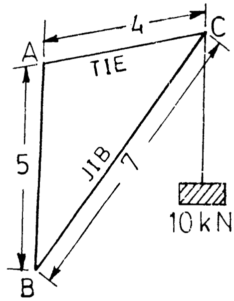


Fig. 1.18 E-4

The load at C being vertically downwards, it is represented by AB. Then taking the sides of the triangle in order, the other two forces are represented by BC and CA. Since a force represented by BC will exert a thrust at C, and that represented by CA will exert a pull at C, the force in BC is thrust while that in CA is pull.

Length of AB is 5 metres and it represents a force of 10 kN. Hence 1 metre represents a force of 2 kN.

Force in $BC = 7 \times 2$

$= \mathbf{14\ kN}$ (Compression).

Forces in $AC = 4 \times 2$

$= \mathbf{8\ kN}$ (Tension).

1.19. Jib Crane

The essentials of a jib crane are shown in Fig. 1.19 A (a). AD is a vertical post. AC is a rod called **jib** hinged at A so that it can turn round A in a vertical plane. It is supported by a chain or rod DC called **tie** which is attached to a point D in AD. At C there is a pulley over which passes a chain ECF ; the load W to be lifed is attached to the end F, while the end E is wound round a drum. The effort is applied at E to lift the load. Very often, CD is horizontal and EC coincides with it.

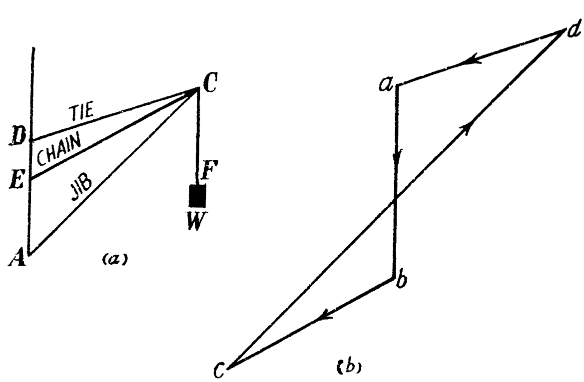


Fig. 1.19 A

Supposing the pulley at C to be smooth, the tension in EC = tension in $CF = W$.

Fig. 1.19 A (b) shows the vector diagram of the forces acting at C. The names are not according to Bow's rotation, ab represents W, bc represents the tension in EC which is also W ; cd has been drawn parallel to AC and ad parallel to DC.

The figures have been drawn with the following data :

The jib AC = 5 m, the tie CD = 3.6 m, AD = 2.4 m, AE = 1.6 m, W = 20 kN.

It will be found by measurement that

force in AC = 50 kN, thrust.

force in CD = 18 kN, tension.

Ex. 1. *In a jib crane, the height of the vertical post, as measured between its joints with the jib and the tie, is 2.1 m. The jib is 4.5 m and the tie is 3 m long. A load of 50 kN is suspended from a chain passing over a smooth pulley supported on the crane head and the chain is fixed to the post at a point 0.9 m above its junction with the jib. Find graphically the stresses in the jib and the tie. Find also graphically the stresses in the jib and the tie if (i) the supporting chain is taken parallel to the tie, and (ii) it is parallel to the jib.*

Sol. The crane and the forces acting at the crane head C are shown in Fig. 1.19 E-1 (a). Let the thrust in the jib AC be P kN, and let tension in the tie DC be T'. The tension T in the chain EC is 50 kN, being equal to the weight. The tension in the portion CM of the chain is also 50 kN. We can apply Bow's notation provided the two known forces in CM and CE are together, there being no unknown forces between them.

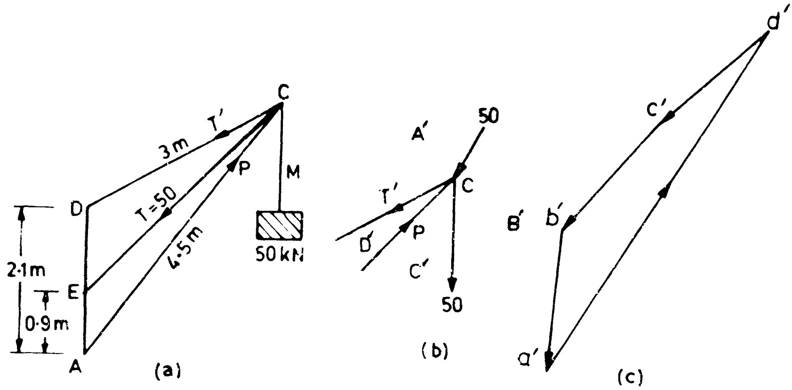


Fig. 1.19 E-1

The force $T = 50$ kN in CE tends to move the point C towards E. Produce EC towards the right hand side. Then T can be replaced by an equal force acting along this line and pushing at C. Hence the free-body diagrams of the crane head C can be drawn as shown in Fig. 1.19 E-1 (b) which may be taken as the space diagram. The spaces have been named as A', B', C' and D'. Fig. 1.19 E-1 (c) shows the vector polygon.

By measuring $c'\,d'$ and $a'\,d'$ we can find P and T. We get

$$P = 141.5 \text{ kN, thrust}$$

$$T = 55.5 \text{ kN, pull.}$$

(i) Let BE coincide with CD. Then the total force in the direction CD is $T + T' = 50 + T$. Hence at C, there are only three forces, *viz.*, 50 kN along CM, P along AC and $50 + T'$ along CD. Now the sides of the triangle ACD are parallel to these forces.

$$\therefore \qquad \frac{P}{AC} = \frac{50 + T'}{CD} = \frac{50}{DA}$$

or

$$\frac{P}{4.5} = \frac{50 + T'}{3} = \frac{50}{2.1}$$

whence $P = 107.14$ kN (compression)

$T = 21.43$ kN (tension)

(ii) Let CE coincide with AC. Then the forces acting at C are 50 kN along CM, $P - 50$ along AC, and T' along CD.

$$\therefore \qquad \frac{P - 50}{4.5} = \frac{T'}{3} = \frac{50}{2.1}$$

$P = 157.14$ kN (Compression)

$T' = 71.43$ kN (Tension).

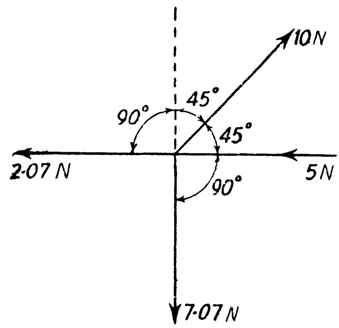
PROBLEMS

1. A string OA is attached to a fixed point O and carries a weight of 100 N at A. A horizontal force X is applied to the weight. If in the position of equilibrium OA is inclined to the veitical at 30° find graphically the magnitude of X and the tension of the string. [**Ans.** $X = 57.8$ N. Tension = 115.5 N].

2. Find the resultant of the following system of forces acting at a point :
   40 N towards the east.
   100 N towards 30° north of east.
   150 N towards north.
   200 N towards south-west. [**Ans.** 61 N inclined 75° to north of west]

3. Find the resultant of forces shown in Fig. 1.19 P-3.
   [**Ans.** The system is in equilibrium]

4. A and B are two points in the same horizontal line 1 m apart. OA and BO are two strings of lengths 0.7 m and 0.5 m carrying at O a weight of 200 N : a third string, attached to the weight at O, passes over a smooth peg C at the middle



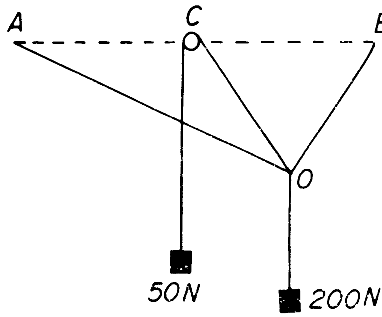
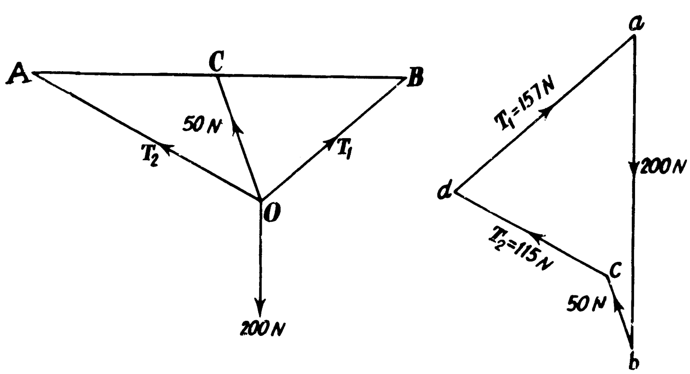


Fig. 1.19 P-3 Fig. 1.19 P-4.1

point of AB and carries a weight of 50 N at the other end as shown in Fig. 1.19 P-4.1. Find the tensions in the strings AO and BO.

[**Ans.** Tension in AO = 115 N
" " BO = 157 N]

The solution is shown in Fig. 1.19 P-4.2 (*a*) and (*b*).



(*a*) Space diagram (*b*) Vector diagram

Fig. 1.19 P-4.2

5. In the crane explained in Art. 1.19 the angle CDB = 45°, angle ACD = 15°. The chain EC coincides with DC. If W = 1000 N find the forces in BC and DC.

[**Ans.** Force in DC = 930 N tension,
Force in AC = 730 N thrust]