

# *Complex Numbers and Their Applications*

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## **1.1. Introduction**

There are equations such as

$$x^2 + 1 = 0, \quad x^2 - 4x + 13 = 0$$

which are not satisfied by any real number and this leads to the introduction of complex numbers. If we solve  $x^2 + 1 = 0$ , we get  $x^2 = -1$  or  $x = \pm\sqrt{-1}$ . The positive square root of  $-1$  is usually denoted by  $i$  and is read as 'iota'. Thus  $x = \pm i$ . Similarly on solving the equation

$$x^2 - 4x + 13 = 0,$$

$$\text{we get, } x = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

Here  $2 \pm 3i$  are complex numbers. A complex number is the sum of a real number and an imaginary number and it is of the form  $a + ib$ , where  $a$  and  $b$  are real numbers. If  $b = 0$ , the number is said to be wholly real and if  $a = 0$ , the number is said to be purely or wholly imaginary. It may be seen that  $i^2 = -1$ ,  $i^3 = -i$ ,  $i^4 = 1$ ,  $i^5 = i$  etc. We also write complex number  $a + ib$  as an ordered pair of real numbers  $(a, b)$ . Thus  $2 + 3i$  is written as  $(2, 3)$  and  $2 - 3i$  as  $(2, -3)$ .

If a complex number  $a + ib = 0$ , then  $a = 0$  and  $b = 0$ . If two complex numbers  $x + iy$  and  $a + ib$  are equal, i.e. if,  $x + iy = a + ib$ , then  $x = a$  and  $y = b$ . Thus  $x + iy = 3 + 4i \Rightarrow x = 3$  and  $y = 4$ .

Let

$$z_1 = x_1 + iy_1 = (x_1, y_1),$$

and

$$z_2 = x_2 + iy_2 = (x_2, y_2),$$

be any two complex numbers. Then the addition of  $z_1$  and  $z_2$  is defined as

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

Their difference as

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

Their product as

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2), \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \end{aligned}$$

Their quotient as

$$\begin{aligned}\frac{z_1}{z_2} &= \left( \frac{x_1+iy_1}{x_2+iy_2} \right) = \left( \frac{x_1+iy_1}{x_2+iy_2} \right) \times \left( \frac{x_2-iy_2}{x_2-iy_2} \right), (z_2 \neq 0) \\ &= \frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2} + i \frac{(y_1x_2-x_1y_2)}{x_2^2+y_2^2}.\end{aligned}$$

The numbers  $(x+iy)$  and  $(x-iy)$  are said to be complex conjugates with respect to each other. If we write  $z=(x+iy)$ , then  $\bar{z}=(x-iy)$ . Thus  $x=\frac{1}{2}(z+\bar{z})$  and  $iy=\frac{1}{2}(z-\bar{z})$ .

$$\text{Also } z \cdot \bar{z} = (x+iy)(x-iy) = x^2 + y^2.$$

### 1.2. Polar Form of a Complex Number

A complex number can always be put in the form  $r(\cos \theta + i \sin \theta)$ , where  $r$  and  $\theta$  are real.

Let  $z=x+iy=r(\cos \theta + i \sin \theta)$ . Equating real and imaginary parts, we have  $x=r \cos \theta$  and  $y=r \sin \theta$ . By squaring and adding, we get  $r^2=x^2+y^2$  or  $r=\sqrt{x^2+y^2}$ , taking the +ve sign only. Here  $r$  or  $\sqrt{x^2+y^2}$  is called the **modulus** of the complex number  $z=(x+iy)$ .

Further,  $\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta$  or  $\theta = \tan^{-1} \left( \frac{y}{x} \right)$ , where  $\theta$  is called **amplitude** or **argument** of the complex number  $z$ . The principal value of amplitude is that value of  $\theta$  which lies between  $-\pi$  and  $\pi$ . The general value of the amplitude is  $\theta + 2n\pi$ , where  $n$  is any integer. We also write  $r = \text{mod } z$  or  $r = |z|$ , read as mod  $z$  and  $\theta = \text{amp } z$  or  $\arg z$ .

It may be noted from above that

$$z \bar{z} = r^2.$$

### 1.3. Geometrical Representation

A complex number  $x+iy$  is defined as an ordered pair  $(x, y)$ . Therefore, it can be represented by the point  $P(x, y)$  with reference

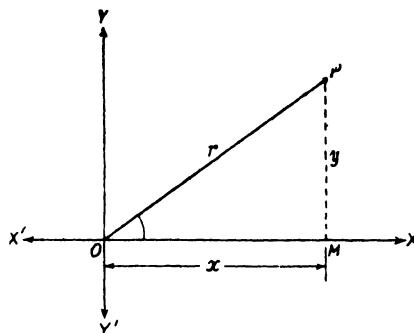


Fig. 1.1.

to co-ordinate axes  $X'OX$  and  $Y'CY$  in a plane. All points on  $X'OX$  represent real numbers and thus  $X'OX$  is called the real axis; all purely imaginary points,  $iy$ , lie on  $Y'CY$  which is called the imaginary axis.

From Fig. 1·1,

$$OP=r=\sqrt{(x^2+y^2)}=|z|,$$

and  $\angle POM=\theta=\arg z$ .

This type of representation of a complex number on  $XY$ -plane is called **Argand diagram**.

We can represent  $z_1 \pm z_2$ ,  $z_1 z_2$  and  $z_1/z_2$  on Argand diagram.

Let  $P_1$  and  $P_2$  represent numbers  $z_1=x_1+iy_1$  and  $z_2=x_2+iy_2$  respectively. Complete the parallelogram  $OP_1QP_2$ . It can easily be seen that the co-ordinates of  $Q$  are  $(x_1+x_2, y_1+y_2)$ . Thus  $Q$  represents the complex number

$$(x_1+x_2)+i(y_1+y_2)=(x_1+iy_1)+(x_2+iy_2)=z_1+z_2.$$

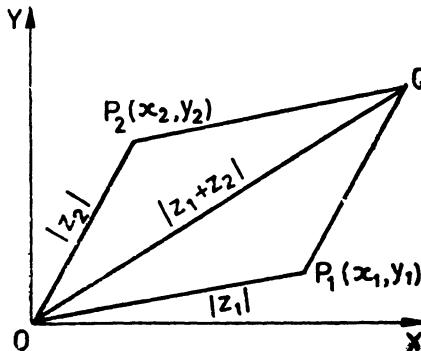


Fig. 1·2.

Now  
i.e.,  $|z_1+z_2| \leqslant |z_1| + |z_2|$ .

Thus the modulus of the sum of two complex quantities is less than or equal to the sum of their moduli.

Similarly, we can represent  $z_1 - z_2$ ,  $z_1 z_2$  and  $z_1/z_2$  on Argand diagram.

#### 1·4. Product and Quotient of Complex Numbers

Let  $z_1=r_1(\cos \theta_1+i \sin \theta_1)$ ,

and  $z_2=r_2(\cos \theta_2+i \sin \theta_2)$ ,

then, 
$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2), \\ &= r_1 r_2 \{(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ &\quad + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)\}, \\ &= r_1 r_2 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\}. \end{aligned}$$

Thus  $|z_1 z_2| = r_1 r_2 = |z_1| |z_2|$ ,

and  $\arg(z_1 z_2) = (\theta_1 + \theta_2) = \arg z_1 + \arg z_2$ .

This result is true for the product of any number of complex quantities. Thus

(i) the modulus of the product of any number of complex quantities is the product of their moduli.

(ii) the argument of the product of any number of complex quantities is the sum of their arguments. Further,

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}, \\ &= \frac{r_1}{r_2} \cdot \frac{(\cos \theta_1 + i \sin \theta_1)}{(\cos \theta_2 + i \sin \theta_2)} \times \frac{(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 - i \sin \theta_2)}, \\ &= \frac{r_1 \{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)\}}{\cos^2 \theta_2 + \sin^2 \theta_2}, \\ &= \frac{r_1}{r_2} \{ \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \}. \end{aligned}$$

$$\therefore \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|},$$

and  $\arg \left( \frac{z_1}{z_2} \right) = (\theta_1 - \theta_2) = \arg z_1 - \arg z_2.$

Thus

(i) the modulus of the quotient of two complex quantities is the quotient of their moduli.

(ii) the argument of the quotient of two complex quantities is the difference of their arguments.

**Example 1.** Express

$$(i) (1+i)(2+i)(3+i) \quad (ii) \frac{2+3i}{3+4i}$$

in the form  $A+iB.$

$$\text{Sol. } (1+i)(2+i)(3+i) = (2+3i+i^2)(3+i)$$

$$= (1+3i)(3+i) = 3+10i+3i^2 \quad [\because i^2 = -1] \\ = 10i = 0+10i.$$

$$\begin{aligned} (ii) \quad \frac{2+3i}{3+4i} &= \frac{(2+3i)}{(3+4i)} \times \frac{(3-4i)}{(3-4i)} \\ &= \frac{6-8i+9i-12i^2}{9-16i^2} = \frac{18+i}{9+16} \\ &= \frac{18+i}{25} = \frac{18}{25} + \frac{1}{25}i. \end{aligned}$$

**Example 2.** What locus is represented by  $|z-1|=2$ , on the Argand diagram.

**Sol.** We have  $|z-1|=2,$

$$\text{or} \quad |x+iy-1|=2 \quad \text{or} \quad |(x-1)+iy|=2,$$

$$\text{or} \quad (x-1)^2+y^2=4 \quad \text{or} \quad x^2+y^2-2x-3=0,$$

which is a circle with centre  $(1, 0)$  and radius 2. The centre will be on the real axis.

**Example 3.** Show that the points representing the numbers  $1+i$ ,  $-2+3i$ ,  $5i/3$  are collinear.

**Sol.** The given points are  $(1, 1)$ ,  $(-2, 3)$ ,  $(0, 5/3)$  as represented on the Argand diagram.

The equation of a line passing through  $(1, 1)$  and  $(-2, 3)$  is given by

$$y-1 = \frac{3-1}{-2-1}(x-1) \quad \text{or} \quad 2x+3y-5=0$$

Clearly the third point  $(0, 5/3)$  satisfies it and as such it also lies on the line. Thus the three given points are collinear.

**Example 4.** If the complex numbers  $z_1$ ,  $z_2$  and  $z_3$  be the vertices of an equilateral triangle and  $z_0$  be the circumcentre of the triangle. Then show that

$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2.$$

**Sol.** Let  $A(z_1)$ ,  $B(z_2)$  and  $C(z_3)$  be the vertices of the equilateral triangle and  $D(z_0)$  its circumcentre, on the Argand diagram. Also, let each side of the  $\triangle ABC$  be  $2a$ .

Draw a perpendicular  $BO$  from the vertex  $B$  on the side  $AC$ . Take  $O$  as origin,  $OC$  and  $OB$  as  $x$ -axis and  $y$ -axis respectively. Then

$$z_1 = OA = -a,$$

$$z_3 = OC = a.$$

Now  $z_2 = i \cdot OB$   
 $(\because$  The point  $B$  is on  $y$ -axis)

From  $\triangle OBC$ , we have

$$\frac{OB}{OC} = \tan 60^\circ = \sqrt{3}$$

$$\therefore OB = a\sqrt{3}.$$

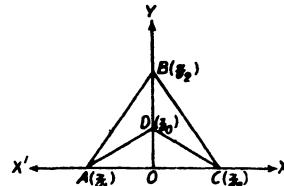
$$\text{or } z_2 = i \cdot a\sqrt{3}.$$

From  $\triangle ODC$ , we get

$$\frac{OD}{OC} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

$$\text{or } OD = \frac{a}{\sqrt{3}}.$$

$$\text{Now, } z_0 = i \cdot OD = i \cdot \frac{a}{\sqrt{3}}.$$



We have

$$\begin{aligned} z_1^2 + z_2^2 + z_3^2 &= (-a)^2 + (i a \sqrt{3})^2 + (a)^2 \\ &= -a^2 = 3 \cdot \left( \frac{i a}{\sqrt{3}} \right)^2 \end{aligned}$$

or  $z_1^2 + z_2^2 + z_3^2 = 3z_1^2$

**Example 5.** Prove that the area of the triangle whose vertices are the points  $z_1, z_2, z_3$  on the Argand diagram is

$$\sum \left\{ \frac{(z_2 - z_3) |z_1|^2}{4iz_1} \right\}.$$

**Sol.** Let  $z_1 = x_1 + iy_1$ ,  $z_2 = x_2 + iy_2$  and  $z_3 = x_3 + iy_3$ . Then the coordinates of the vertices of the triangle on the Argand diagram are  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  respectively.

Then the area of the triangle

$$\begin{aligned} \Delta &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \\ &= \frac{1}{2i} \begin{vmatrix} x_1 & iy_1 & 1 \\ x_2 & iy_2 & 1 \\ x_3 & iy_3 & 1 \end{vmatrix} \quad (\text{Multiplying } C_2 \text{ by } i) \\ &= \frac{1}{2i} \begin{vmatrix} x_1 & x_1 + iy_1 & 1 \\ x_2 & x_2 + iy_2 & 1 \\ x_3 & x_3 + iy_3 & 1 \end{vmatrix} \quad (\text{Adding } C_1 \text{ to } C_2) \\ &= \frac{1}{2i} \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} \quad (\because z_1 = x_1 + iy_1 \text{ and so on}) \\ &= \frac{1}{2i} \sum z_1(z_2 - z_3) \\ &= \frac{1}{2i} \sum \left( \frac{z_1 + \bar{z}_1}{2} \right) (z_2 - z_3) \\ &= \frac{1}{4i} \left[ \sum z_1(z_2 - z_3) + \sum \bar{z}_1(z_2 - z_3) \right]. \end{aligned}$$

Now  $\sum z_1(z_2 - z_3) = 0$

$$\begin{aligned} \therefore \Delta &= \frac{1}{4i} \sum \bar{z}_1(z_2 - z_3) \\ &= \frac{1}{4i} \sum \frac{\bar{z}_1(z_2 - z_3) \cdot z_1}{z_1} \\ &\quad (\text{Multiplying and dividing by } z_1) \\ &= \sum \frac{(z_1 \cdot \bar{z}_1)(z_2 - z_3)}{4iz_1} \\ &= \sum \frac{|z_1|^2 (z_2 - z_3)}{4iz_1} \\ &\quad (\because z_1 \bar{z}_1 = z_1^2) \end{aligned}$$

**EXERCISE 1 (a)**

1. Express the following in the form  $A+iB$

$$(a) (1+2i)(3-i),$$

$$(b) \left( \frac{4-5i}{2+3i} \right) \left( \frac{3+2i}{7+i} \right).$$

2. Show that the points representing the complex numbers  $-4+3i, 2-3i, -i$  are collinear.

3. If  $|z_1| = |z_2|$  and  $\arg z_1 + \arg z_2 = 0$ , show that  $z_1$  and  $z_2$  are conjugate complex numbers.

4. If  $z_1, z_2, z_3$  are vertices of an equilateral triangle, show that  $z_1^3 + z_2^3 + z_3^3 = z_1 z_2 + z_2 z_3 + z_3 z_1$ .

5. Show that if  $\arg \left\{ \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} \right\}$  is zero, the four points representing  $z_1, z_2, z_3, z_4$  on the Argand diagram lie on a circle or on a straight line.

6. Find the locus of  $z$  moving on the Argand diagram so that

$$(i) |z-z_1| = |z-z_3|$$

$$(ii) |z-z_1| + |z-z_3| = 2$$

$$(iii) |z-z_1| = 2|z-z_3|.$$

7. Interpret geometrically the following loci

$$(i) \left| \frac{z-i}{z+i} \right| > 2.$$

$$(ii) \operatorname{Arg} \left( \frac{z-1}{z+1} \right) = \frac{\pi}{2}.$$

8. Given that  $\frac{1}{\rho} = \frac{1}{Lp i} + cpi + \frac{1}{R}$ , express  $\rho$  in the form  $Ae^{i\theta}$ , giving the values of  $A$  and  $\theta$ .

### 1.5. De Moivre's Theorem

If  $n$  is an integer, positive or negative or a rational number, then the value or one of the values of  $(\cos \theta + i \sin \theta)^n$  is  $\cos n\theta + i \sin n\theta$ .

**Proof. Case I.** When  $n$  is a positive integer.

By actual multiplication, we have

$$\begin{aligned} (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) &= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ &\quad + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2), \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2). \dots (1) \end{aligned}$$

Multiplying both sides of (1) by  $(\cos \theta_3 + i \sin \theta_3)$ , we get

$$\begin{aligned} &(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)(\cos \theta_3 + i \sin \theta_3) \\ &= [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] (\cos \theta_3 + i \sin \theta_3), \\ &= [\cos(\theta_1 + \theta_2) \cos \theta_3 - \sin(\theta_1 + \theta_2) \sin \theta_3] \\ &\quad + i[\sin(\theta_1 + \theta_2) \cos \theta_3 + \cos(\theta_1 + \theta_2) \sin \theta_3], \\ &= \cos(\theta_1 + \theta_2 + \theta_3) + i \sin(\theta_1 + \theta_2 + \theta_3). \end{aligned}$$

Proceeding in this way, we have

$$\begin{aligned} (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)(\cos \theta_3 + i \sin \theta_3) \dots & \dots \\ (\cos \theta_n + i \sin \theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n) & \\ & + i \sin(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n) \end{aligned}$$

Putting  $\theta_1 = \theta_2 = \theta_3 = \dots = \theta_n = \theta$  on both sides, we get

$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ , which proves De-Moivre's theorem for a positive integer.

### Case II. When $n$ is a negative integer.

Let  $n = -m$ , where  $m$  is a positive integer. Then

$$\begin{aligned} (\cos \theta + i \sin \theta)^n &= (\cos \theta + i \sin \theta)^{-m} \\ &= \frac{1}{(\cos \theta + i \sin \theta)^m}, \\ &= \frac{1}{\cos m\theta + i \sin m\theta}, \quad [\text{By case I.}] \\ &= \frac{\cos m\theta - i \sin m\theta}{(\cos m\theta + i \sin m\theta)(\cos m\theta - i \sin m\theta)}, \\ &= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}, \\ &= \cos m\theta - i \sin m\theta, \\ &= \cos(-m)\theta + i \sin(-m)\theta, \\ &= \cos n\theta + i \sin n\theta. \quad [ \because -m = n ] \end{aligned}$$

$\therefore (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ , which prove De-Moivre's theorem for a negative integer.

### Case III. When $n$ is a rational number.

$$\text{Let } n = \frac{p}{q} \quad (q \neq 0)$$

$$\begin{aligned} \text{Now, } \left( \cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right)^q &= \cos \left( q \frac{\theta}{q} \right) + i \sin \left( q \frac{\theta}{q} \right), \\ &= \cos \theta + i \sin \theta. \end{aligned}$$

(By case I or II)

Taking  $q$ th root on both sides, we have

$$\cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \text{ as one of the values of } (\cos \theta + i \sin \theta)^{1/q}$$

$$\therefore (\cos \theta + i \sin \theta)^{1/q} = \left( \cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right).$$

Raising both sides to power  $p$ ,

$$\begin{aligned} (\cos \theta + i \sin \theta)^{p/q} &= \left( \cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right)^p, \\ &= \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q}. \end{aligned}$$

Putting  $\frac{p}{q} = n$ , we get

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

which proves De-Moivre's theorem for a rational number.

De-Moivre's theorem may also be used in the following forms.

$$(\cos \theta + i \sin \theta)^{-n} = \cos(-n)\theta + i \sin(-n)\theta = \cos n\theta - i \sin n\theta.$$

$$(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta.$$

$$(\cos \theta - i \sin \theta)^{-n} = \cos n\theta + i \sin n\theta.$$

**Example 1.** Simplify

$$\frac{(\cos 4\theta + i \sin 4\theta)^3 \cdot (\cos \theta - i \sin \theta)^{-4}}{(\cos \theta + i \sin \theta)^{-5} \cdot (\cos 2\theta + i \sin 2\theta)^{-6}}.$$

**Sol.** Now applying De-Moivre's theorem, we get

$$\begin{aligned} &= \frac{(\cos 12\theta + i \sin 12\theta)(\cos 4\theta + i \sin 4\theta)}{(\cos 5\theta - i \sin 5\theta)(\cos 12\theta - i \sin 12\theta)}, \\ &= \frac{(\cos \theta + i \sin \theta)^{12} \cdot (\cos \theta + i \sin \theta)^4}{(\cos \theta + i \sin \theta)^{-5} \cdot (\cos \theta + i \sin \theta)^{-12}}, \\ &= \frac{(\cos \theta + i \sin \theta)^{16}}{(\cos \theta + i \sin \theta)^{-17}} = (\cos \theta + i \sin \theta)^{16+17} \\ &= (\cos \theta + i \sin \theta)^{33} \\ &= \cos 33\theta + i \sin 33\theta. \end{aligned}$$

**Example 2.** Simplify  $\frac{(\cos \theta + i \sin \theta)^5}{(\sin \theta + i \cos \theta)^7}$ .

$$\begin{aligned} \text{Sol. } &\frac{(\cos \theta + i \sin \theta)^5}{(\sin \theta + i \cos \theta)^7} = (\cos \theta + i \sin \theta)^5 (\sin \theta + i \cos \theta)^{-7}, \\ &= (\cos \theta + i \sin \theta)^5 \cdot \left[ \cos \left\{ \frac{\pi}{2} - \theta \right\} \right. \\ &\quad \left. + i \sin \left\{ \frac{\pi}{2} - \theta \right\} \right]^{-7} \end{aligned}$$

By De-Moivre's theorem, we have

$$\begin{aligned} &(\cos 5\theta + i \sin 5\theta) \left[ \cos \left\{ \frac{-7\pi}{2} + 7\theta \right\} \right. \\ &\quad \left. + i \sin \left\{ \frac{-7\pi}{2} + 7\theta \right\} \right]^{-7} \\ &= \cos \left( 5\theta - \frac{7\pi}{2} + 7\theta \right) + i \sin \left( 5\theta - \frac{7\pi}{2} + 7\theta \right) \\ &= \cos \left( \frac{7\pi}{2} - 12\theta \right) - i \sin \left( \frac{7\pi}{2} - 12\theta \right) \\ &= -\sin 12\theta + i \cos 12\theta; \end{aligned}$$

**Example 3.** If  $a = \cos 2\alpha + i \sin 2\alpha$ ,  $b = \cos 2\beta + i \sin 2\beta$ ,  
 $c = \cos 2\gamma + i \sin 2\gamma$  and  $d = \cos 2\delta + i \sin 2\delta$ , prove that

$$(i) \sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2 \cos(\alpha + \beta - \gamma)$$

$$(ii) \sqrt{\frac{ab}{cd}} + \sqrt{\frac{cd}{ab}} = 2 \cos(\alpha + \beta - \gamma - \delta).$$

**Sol.** (i) We have  $\frac{ab}{c} = \frac{(\cos 2\alpha + i \sin 2\alpha)(\cos 2\beta + i \sin 2\beta)}{\cos 2\gamma + i \sin 2\gamma}$   
 $= \cos 2(\alpha + \beta - \gamma) + i \sin 2(\alpha + \beta - \gamma)$

$$\therefore \sqrt{\frac{ab}{c}} = [\cos 2(\alpha + \beta - \gamma) + i \sin 2(\alpha + \beta - \gamma)]^{1/2}$$
  
 $= \cos 2 \cdot \frac{1}{2}(\alpha + \beta - \gamma) + i \sin 2 \cdot \frac{1}{2}(\alpha + \beta - \gamma)$   
 $[By De Moivre's theorem]$   
 $= \cos(\alpha + \beta - \gamma) + i \sin(\alpha + \beta - \gamma) \dots (1)$

$$\therefore \sqrt{\frac{c}{ab}} = \left[ \frac{1}{\cos(\alpha + \beta - \gamma) + i \sin(\alpha + \beta - \gamma)} \right]$$
  
 $[By 1]$   
 $= [\cos(\alpha + \beta - \gamma) + i \sin(\alpha + \beta - \gamma)]^{-1}$   
 $= \cos(\alpha + \beta - \gamma) - i \sin(\alpha + \beta - \gamma) \dots (2)$   
 $[By De Moivre's theorem]$

From (1) and (2), we have

$$\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2 \cos(\alpha + \beta - \gamma)$$

(ii) This can be proved like part (i).

**Example 4.** Prove that  $\frac{(1 + \sin \phi + i \cos \phi)^n}{(1 + \sin \phi - i \cos \phi)^n}$

$$= \cos\left(\frac{n\pi}{2} - n\phi\right) + i \sin\left(\frac{n\pi}{2} - n\phi\right)$$

**Sol.** Let  $1 + \sin \phi = r \cos \theta$  and  $\cos \phi = r \sin \theta$

$$\therefore \tan \theta = \frac{r \sin \theta}{r \cos \theta} = \frac{\cos \phi}{1 + \sin \phi}$$

$$= \frac{\sin\left(\frac{\pi}{2} - \phi\right)}{1 + \cos\left(\frac{\pi}{2} - \phi\right)}$$

$$= \frac{2 \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \cos\left(\frac{\pi}{4} - \frac{\phi}{2}\right)}{2 \cos^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right)}$$

$$= \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right)$$

$$\tan \theta = \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right)$$

or

$$\theta = \frac{\pi}{4} - \frac{\phi}{2}$$

$$\begin{aligned} \text{L.H.S.} &= \frac{[r(\cos \theta + i \sin \theta)]^n}{[r(\cos \theta - i \sin \theta)]^n} = (\cos \theta + i \sin \theta)^n (\cos \theta - i \sin \theta)^{-n} \\ &= (\cos n\theta + i \sin n\theta)(\cos n\theta + i \sin n\theta) \\ &\quad [\text{By De-Moivre's theorem}] \\ &= \cos 2n\theta + i \sin 2n\theta \\ &= \cos 2n \left( \frac{\pi}{4} - \frac{\phi}{2} \right) + i \sin 2n \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \\ &\quad \left[ \because \theta = \frac{\pi}{4} - \frac{\phi}{2} \right] \\ &= \cos \left( \frac{n\pi}{2} - n\phi \right) + i \sin \left( \frac{n\pi}{2} - n\phi \right) = \text{R.H.S} \end{aligned}$$

**Example 5.** If  $\cos \alpha + 2 \cos \beta + 3 \cos \gamma = 0$

$= \sin \alpha + 2 \sin \beta + 3 \sin \gamma$ , Prove that

$$\cos 3\alpha + 8 \cos 3\beta + 27 \cos 3\gamma = 18 \cos(\alpha + \beta + \gamma),$$

and  $\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma = 18 \sin(\alpha + \beta + \gamma)$ .

**Sol.** Let  $a = \cos \alpha + i \sin \alpha$ ,  $b = 2 \cos \beta + 2i \sin \beta$ ,  
 $c = 3 \cos \gamma + 3i \sin \gamma$ .

$$\begin{aligned} \therefore a + b + c &= (\cos \alpha + i \sin \alpha) + (2 \cos \beta + 2i \sin \beta) \\ &\quad + (3 \cos \gamma + 3i \sin \gamma) \\ &= (\cos \alpha + 2 \cos \beta + 3 \cos \gamma) \\ &\quad + i(\sin \alpha + 2 \sin \beta + 3 \sin \gamma) = 0. \end{aligned}$$

Since  $a + b + c = 0$ ,

$$a^3 + b^3 + c^3 = 3abc.$$

$$\begin{aligned} \text{i.e. } &(\cos \alpha + i \sin \alpha)^3 + (2 \cos \beta + 2i \sin \beta)^3 + (3 \cos \gamma + 3i \sin \gamma)^3 \\ &- 3(\cos \alpha + i \sin \alpha)(2 \cos \beta + 2i \sin \beta)(3 \cos \gamma + 3i \sin \gamma) \\ \therefore &(\cos 3\alpha + i \sin 3\alpha) + (8 \cos 3\beta + 8i \sin 3\beta) \\ &+ (27 \cos 3\gamma + 27i \sin 3\gamma) \\ &- 18[\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma)] \end{aligned}$$

$$\begin{aligned} \text{or } &(\cos 3\alpha + 8 \cos 3\beta + 27 \cos 3\gamma) + i(\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma) \\ &= 18[\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma)] \end{aligned}$$

Equating real and imaginary parts, we get

$$\cos 3\alpha + 8 \cos 3\beta + 27 \cos 3\gamma = 18 \cos(\alpha + \beta + \gamma)$$

$$\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma = 18 \sin(\alpha + \beta + \gamma)$$

**Example 6.** Find general value of  $\theta$ , which satisfies the following equation,

$$(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \dots (\cos n\theta + i \sin n\theta) = 1.$$

$$\text{Sol. } \text{Here } (\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \dots (\cos n\theta + i \sin n\theta) = 1$$

$$\text{or } (\cos \theta + i \sin \theta)^1 \cdot (\cos \theta + i \sin \theta)^2 \dots (\cos \theta + i \sin \theta)^n \quad [\text{By De-Moivre's Theorem}]$$

$$= \cos 2r\pi + i \sin 2r\pi \quad [\because \cos 2r\pi + i \sin 2r\pi = 1]$$

$$\text{or } (\cos \theta + i \sin \theta)^{1+2+3+\dots+n} = \cos 2r\pi + i \sin 2r\pi$$

$$(\cos \theta + i \sin \theta)^{\frac{n(n+1)}{2}} = \cos 2r\pi + i \sin 2r\pi \quad \left[ \because \Sigma n = \frac{n(n+1)}{2} \right]$$

Applying De-Moivre's theorem, we get

$$\cos \frac{n(n+1)}{2} \theta + i \sin \frac{n(n+1)}{2} \theta = \cos 2r\pi + i \sin 2r\pi$$

Thus we have

$$\frac{n(n+1)}{2} \theta = 2r\pi$$

$$\theta = \frac{4r\pi}{n(n+1)}.$$

### 1.6. Roots of a Complex Number

We have seen that  $\cos \frac{\theta}{q} + i \sin \frac{\theta}{q}$  is one of the values of  $(\cos \theta + i \sin \theta)^{1/q}$ . Here we shall show that  $(\cos \theta + i \sin \theta)^{1/q}$  has  $q$  and only  $q$  different values.

Since  $\cos \theta$  and  $\sin \theta$  are periodic functions with period  $2n\pi$  ( $n$  being an integer).

$$\therefore \cos \theta + i \sin \theta = \cos(2n\pi + \theta) + i \sin(2n\pi + \theta)$$

$$\begin{aligned} \therefore (\cos \theta + i \sin \theta)^{1/q} &= [\cos(2n\pi + \theta) + i \sin(2n\pi + \theta)]^{1/q} \\ &= \cos\left(\frac{2n\pi + \theta}{q}\right) + i \sin\left(\frac{2n\pi + \theta}{q}\right) \dots (1) \end{aligned}$$

Giving  $n$  the values 0, 1, 2, 3, ...,  $(q-1)$  successively, we get following  $q$  values of  $(\cos \theta + i \sin \theta)^{1/q}$ .

$$\cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \quad (\text{for } n=0)$$

$$\cos \frac{2\pi + \theta}{q} + i \sin \frac{2\pi + \theta}{q} \quad (\text{for } n=1)$$

$$\cos \frac{4\pi + \theta}{q} + i \sin \frac{4\pi + \theta}{q} \quad (\text{for } n=2)$$

$$\begin{array}{ll} \cdots & \cdots \\ \cdots & \cdots \end{array}$$

$$\cos \frac{2(q-1)\pi + \theta}{q} + i \sin \frac{2(q-1)\pi + \theta}{q} \quad (\text{for } n=q-1)$$

Each of the above value is a root of  $(\cos \theta + i \sin \theta)^{1/q}$ .

**Note 1.** If we give to  $n$  values greater than  $q-1$ , no new roots are obtained but the same values will repeat. For example, when we put  $n=q$  in (1), we get  $(q+1)$ th root as

$$\begin{aligned} & \cos \left( \frac{2q\pi + \theta}{q} \right) + i \sin \left( \frac{2q\pi + \theta}{q} \right) \\ & = \cos \left( 2\pi + \frac{\theta}{q} \right) + i \sin \left( 2\pi + \frac{\theta}{q} \right) \\ & = \cos \frac{\theta}{q} + i \sin \frac{\theta}{q}, \end{aligned}$$

which is same as the value obtained by putting  $n=0$ .

**Note 2.** The roots of  $(\cos \theta + i \sin \theta)^{1/q}$  are in G.P., as shown below. The roots of  $(\cos \theta + i \sin \theta)^{1/q}$  can be written as

$$\cos \left( \frac{2n\pi + \theta}{q} \right) + i \sin \left( \frac{2n\pi + \theta}{q} \right) \quad \text{for } n=0, 1, 2, 3, \dots, q-1.$$

$$\begin{aligned} \text{Further, } & \cos \left( \frac{2n\pi + \theta}{q} \right) + i \sin \left( \frac{2n\pi + \theta}{q} \right) \\ & = \left( \cos \frac{2n\pi}{q} + i \sin \frac{2n\pi}{q} \right) \left( \cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right) \\ & = \left( \cos \frac{2\pi}{q} + i \sin \frac{2\pi}{q} \right)^n \left( \cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right) \\ & = \alpha^n \cdot \beta \quad \text{for } n=0, 1, 2, 3, \dots, q-1 \end{aligned}$$

$$\text{where } \alpha = \cos \frac{2\pi}{q} + i \sin \frac{2\pi}{q}$$

$$\text{and } \beta = \cos \frac{\theta}{q} + i \sin \frac{\theta}{q}.$$

Therefore, the roots of  $(\cos \theta + i \sin \theta)^{1/q}$  can be written as  $\beta, \alpha\beta, \alpha^2\beta, \dots, \alpha^{q-1}\beta$ .

Thus we see the roots are in G.P. with common ratio

$$\alpha = \cos \frac{2\pi}{q} + i \sin \frac{2\pi}{q}.$$

**Example 1.** Find all the values of  $(1)^{1/6}$ .

**Sol.**

$$\begin{aligned} 1 &= \cos 0 + i \sin 0 \\ &= \cos (2n\pi) + i \sin (2n\pi) \end{aligned}$$

$$\therefore (1)^{1/6} = [\cos(2n\pi) + i \sin(2n\pi)]^{1/6}$$

$$= \cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} \quad \dots(1)$$

where  $n=0, 1, 2, 3, 4, 5$  [By De-Moivre's theorem]

Putting  $n=0, 1, 2, 3, 4, 5$  successively in (1), we get the required roots as follows,

$$(i) \cos 0 + i \sin 0 = 1 \quad [\text{for } n=0]$$

$$(ii) \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \quad [\text{for } n=1]$$

$$(iii) \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \quad [\text{for } n=2]$$

$$(iv) \cos \pi + i \sin \pi = -1 \quad [\text{for } n=3]$$

$$(v) \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2} \quad [\text{for } n=4]$$

$$(vi) \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \quad [\text{for } n=5]$$

Thus the roots are  $1, -1, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ .

**Example 2.** Find all the values of  $(1+\sqrt{3}i)^{1/3}$ .

$$\begin{aligned} \text{Sol. Let } & 1 = r \cos \theta \\ & \sqrt{3} = r \sin \theta \end{aligned}$$

$$\therefore r = 2$$

$$\text{and } \theta = \frac{\pi}{3}$$

$$\begin{aligned} \therefore 1 + \sqrt{3}i &= 2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \\ &= 2 \left[ \cos \left( 2n\pi + \frac{\pi}{3} \right) + i \sin \left( 2n\pi + \frac{\pi}{3} \right) \right] \\ &= 2 \left[ \cos (6n+1) \frac{\pi}{3} + i \sin (6n+1) \frac{\pi}{3} \right] \end{aligned}$$

$$\begin{aligned} \therefore (1 + \sqrt{3}i)^{1/3} &= (2)^{1/3} \left[ \cos (6n+1) \frac{\pi}{3} + i \sin (6n+1) \frac{\pi}{3} \right]^{1/3} \\ &= (2)^{1/3} \left[ \cos (6n+1) \frac{\pi}{9} + i \sin (6n+1) \frac{\pi}{9} \right] \end{aligned}$$

for  $n=0, 1, 2$

Hence the required values are,

$$\begin{aligned} (2)^{1/3} \left( \cos \frac{\pi}{9} + i \sin \frac{\pi}{9} \right), (2)^{1/3} \left( \cos \frac{7\pi}{9} + i \sin \frac{7\pi}{9} \right), \\ (2)^{1/3} \left( \cos \frac{13\pi}{9} + i \sin \frac{13\pi}{9} \right) \end{aligned}$$

Thus the required values are

$$(2)^{1/3} \left( \cos \frac{r\pi}{9} + i \sin \frac{r\pi}{9} \right), \text{ for } r=1, 7, 13.$$

**Example 3.** Prove that the roots of the equation

$$x^{10} + 11x^5 - 1 = 0 \text{ are } \left( \frac{\pm \sqrt{5}-1}{2} \right) \left( \cos \frac{2n\pi}{5} + i \sin \frac{2n\pi}{5} \right).$$

where  $n=0, 1, 2, 3, 4$

**Sol.** Here  $x^{10} + 11x^5 - 1 = 0$  ... (i)

Put  $x^5 = t$  in (i),

$$\therefore t^2 + 11t - 1 = 0$$

$$t = \frac{-11 \pm \sqrt{125}}{2} = \frac{-11 \pm 5\sqrt{5}}{2}$$

$$\therefore x^5 = \frac{-11 \pm 5\sqrt{5}}{2} = \frac{-176 \pm 80\sqrt{5}}{32}$$

(Multiplying  $N^r$ , and  $D^r$ . by 16 so as to make the  $D^r$  fifth power of 2.)

$$\therefore x^5 = \left( \frac{-176 \pm 80\sqrt{5}}{32} \right) (\cos 2n\pi + i \sin 2n\pi)$$

$$\therefore x^5 = \left( \frac{\pm \sqrt{5}-1}{2} \right)^5 (\cos 2n\pi + i \sin 2n\pi)$$

$$\therefore x = \left( \frac{\pm \sqrt{5}-1}{2} \right) (\cos 2n\pi + i \sin 2n\pi)^{1/5}$$

$$\text{or } x = \left( \frac{\pm \sqrt{5}-1}{2} \right) \left( \cos \frac{2n\pi}{5} + i \sin \frac{2n\pi}{5} \right)$$

where  $n=0, 1, 2, 3, 4$ .

**Example 4.** Prove that  $\sqrt[n]{a+ib} + \sqrt[n]{a-ib}$  has  $n$  real values and find those of  $\sqrt[3]{1+\sqrt{3}i} + \sqrt[3]{1-\sqrt{3}i}$ .

**Sol.** Let  $a=r \cos \theta$  and  $b=r \sin \theta$

$$\therefore r=(a^2+b^2)^{1/2}$$

$$\text{and } \theta=\tan^{-1} \frac{b}{a}$$

$$\text{Now } (a+ib)^{1/n} + (a-ib)^{1/n} = [r(\cos \theta + i \sin \theta)]^{1/n} + [r(\cos \theta - i \sin \theta)]^{1/n}$$

$$= r^{1/n} [\cos (2k\pi + \theta) + i \sin (2k\pi + \theta)]^{1/n} + r^{1/n} [\cos (2k\pi + \theta) - i \sin (2k\pi + \theta)]^{1/n}$$

$$= r^{1/n} \left[ \cos \left( \frac{2k\pi + \theta}{n} \right) + i \sin \left( \frac{2k\pi + \theta}{n} \right) \right]$$

$$+ r^{1/n} \left[ \cos \left( \frac{2k\pi + \theta}{n} \right) - i \sin \left( \frac{2k\pi + \theta}{n} \right) \right]$$

[By De-Movire's theorem]

$$= 2r^{1/n} \cos\left(\frac{2k\pi + \theta}{n}\right)$$

for  $k = 0, 1, 2, \dots, n-1$ .

Thus there are  $n$  real values of the given expression.

Putting  $a=1$ ,  $b=\sqrt{3}$  and  $n=3$ ,

We have  $r=2$ ,  $\theta=\pi/3$ .

$$\therefore \sqrt[3]{1+\sqrt{3}i} + \sqrt[3]{1-\sqrt{3}i} = 2 \cdot 2^{1/3} \cdot \left[ \cos\left(\frac{2k\pi + \pi/3}{3}\right) \right]$$

for  $k = 0, 1, 2$

$$= 2^{4/3} \left[ \cos(6k+1) \frac{\pi}{9} \right]$$

Giving  $k$  the values 0, 1, 2 we get the required values as

$$2^{4/3} \cos \frac{\pi}{9}, 2^{4/3} \cos \frac{7\pi}{9}, 2^{4/3} \cos \frac{13\pi}{9}.$$

**Example 5.** Find the 7th root of unity and prove that the sum of their  $n$ th power always vanishes unless  $n$  be a multiple of 7,  $n$  being an integer, and then the sum is 7.

(Bhopal 1981)

**Sol.** Let  $1 = \cos 0 + i \sin 0$   
 $= \cos 2n\pi + i \sin 2n\pi$

$$\therefore (1)^{1/7} = (\cos 2n\pi + i \sin 2n\pi)^{1/7}$$

Applying De-Moivre's theorem, we have

$$(1)^{1/7} = \cos \frac{2n\pi}{7} + i \sin \frac{2n\pi}{7},$$

where,  $n = 0, 1, 2, 3, 4, 5, 6$ .

Therefore the roots are

$$\cos 0 + i \sin 0 = 1$$

$$\cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7} = \alpha \text{(say)}$$

$$\cos \frac{4\pi}{7} + i \sin \frac{4\pi}{7} = \alpha^2 \text{ and so on up to } \alpha^6$$

Let the sum of  $n$ th powers of the roots be  $S$ .

$$\begin{aligned} S &= 1 + \alpha^n + (\alpha^2)^n + \dots + (\alpha^6)^n \\ &= 1 + \alpha^n + (\alpha^n)^2 + \dots + (\alpha^n)^6 \\ &\quad [\text{A G.P. with common ratio } \alpha^n] \end{aligned}$$

$$S = \frac{1 - (\alpha^n)^7}{1 - \alpha^n} \quad \dots (1)$$

$$\left[ \text{Sum of G.P.} = \frac{a(1 - r^n)}{1 - r} \right]$$

When  $n$  is not a multiple of 7

$$(\alpha^n)^7 = \left( \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7} \right)^{7n}$$

$$= \cos \frac{2\pi \cdot 7n}{7} + i \sin \frac{2\pi \cdot 7n}{7}$$

[By De-Moivre's theorem]

$$= \cos 2n\pi + i \sin 2n\pi = 1$$

$$\therefore 1 - (\alpha^n)^7 = 1 - 1 = 0$$

$$\text{Also } 1 - \alpha^n = 1 - \left( \cos \frac{2n\pi}{7} + i \sin \frac{2n\pi}{7} \right) \neq 0$$

as  $n$  is not a multiple of 7.

$$\therefore S = 0 \text{ from (1).}$$

When  $n$  is a multiple of 7,  $n = 7m$  (say)

$$\therefore \alpha^n = \left( \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7} \right)^{7m} = \cos 2\pi m + i \sin 2\pi m = 1$$

Similarly  $\alpha^{2n}, \alpha^{4n}$  etc. each term is 1.

$$\begin{aligned} \therefore S &= 1 + \alpha^n + \alpha^{2n} + \dots + \alpha^{6n} \\ &= 1 + 1 + 1 + \dots + 1 = 7. \end{aligned}$$

**Example 6.** Solve the equation  $x^4 - x^3 + x^2 - x + 1 = 0$ .

**Sol.** To solve the given equation, multiply both sides by  $(x+1)$ .

$$\therefore (x+1)(x^4 - x^3 + x^2 - x + 1) = 0$$

$$\text{or } x^5 + 1 = 0$$

$$\begin{aligned} \text{or } x^5 &= -1 = \cos \pi \pm i \sin \pi \\ &= \cos (2n\pi + \pi) \pm i \sin (2n\pi + \pi) \end{aligned}$$

$$\therefore x = [\cos (2n+1)\pi \pm i \sin (2n+1)\pi]^{1/5} \text{ for } n = 0, 1, 2.$$

Applying De-Moivre's theorem, we get

$$x = \cos (2n+1) \frac{\pi}{5} \pm i \sin (2n+1) \frac{\pi}{5}$$

Giving  $n$  the values 0, 1, 2 the roots are

$$\cos \frac{\pi}{5} \pm i \sin \frac{\pi}{5}, \cos \frac{3\pi}{5} \pm i \sin \frac{3\pi}{5}, \cos \pi + i \sin \pi.$$

The root  $\cos \pi + i \sin \pi = -1$ , is due to multiplication by  $(x+1)$ .

Hence rejecting  $x = -1$ , the other four roots are the required roots.

**EXERCISE 1 (b)**

1. Simplify (i)  $\frac{(\cos 2\theta - i \sin 2\theta)^7 (\cos 3\theta + i \sin 3\theta)^{-5}}{(\cos 4\theta - i \sin 4\theta)^{-4} (\cos 6\theta + i \sin 6\theta)^6}$

(ii)  $\frac{(\cos \theta + i \sin \theta)^3}{(\sin \phi + i \cos \phi)^4}$

2. Show that  $\{(\cos \theta - \cos \phi) + i(\sin \theta - \sin \phi)\}^n$   
 $+ \{(\cos \theta - \cos \phi) - i(\sin \theta - \sin \phi)\}^n$   
 $= 2^{n+1} \sin^n \frac{\theta - \phi}{2} \cos n \left( \frac{\pi + \theta + \phi}{2} \right).$

3. If  $x = \cos \theta + i \sin \theta$ , show that

(i)  $x^m + \frac{1}{x^m} = 2 \cos m\theta$       (ii)  $x^m - \frac{1}{x^m} = 2i \sin m\theta$ .

4. Prove that

(i)  $(a+bi)^{\frac{m}{n}} + (a-bi)^{\frac{m}{n}} = 2(a^2+b^2)^{\frac{m}{2n}} \cos \left( \frac{m}{n} \tan^{-1} \frac{b}{a} \right)$

(ii)  $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos \frac{n\pi}{4}.$

5. (i)  $x_r = \cos \left( \frac{\pi}{2^r} \right) + i \sin \left( \frac{\pi}{2^r} \right)$ , show that

$x_1, x_2, x_3, \dots \infty = -1$ .

(ii) Prove that  $(1+\cos \theta + i \sin \theta)^n + (1+\cos \theta - i \sin \theta)^n$   
 $= 2^{n+1} \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2}.$

6. (i) If  $\alpha, \beta$  be the roots of  $x^2 - 2x + 4 = 0$ ,

prove that  $\alpha^n + \beta^n = 2^{n+1} \cos \frac{n\pi}{3}$

(ii) Show that the equation whose roots are the  $n$ th powers of the roots of  $x^2 - 2x \cos \theta + 1 = 0$  is given by  $x^2 - 2x \cos n\theta + 1 = 0$ .

7. If  $\sin \alpha + \sin \beta + \sin \gamma = 0 = \cos \alpha + \cos \beta + \cos \gamma$ , prove that

(i)  $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$

(ii)  $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$

(iii)  $\cos(\beta + \gamma) + \cos(\gamma + \alpha) + \cos(\alpha + \beta) = 0$   
 $= \sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0$

(iv)  $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

8. If  $(a_1 + ib_1)(a_2 + ib_2)(a_3 + ib_3) \dots (a_n + ib_n) = A + iB$ ,  
prove that

$$(i) \quad (a_1^2 + b_1^2)(a_2^2 + b_2^2) \dots (a_n^2 + b_n^2) = A^2 + B^2$$

$$(ii) \quad \tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \dots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$$

9. If  $n$  is a positive integer show that

$$(\sqrt{3+i})^n + (\sqrt{3-i})^n = 2^{n+1} \cos \frac{n\pi}{6}.$$

10. If  $x = \cos \theta + i \sin \theta$  and  $\sqrt{1-c^2} = nc - 1$ ,  
show that  $1+c \cos \theta = \frac{c}{2n} \left(1+nx\right) \left(1+\frac{n}{x}\right)$

11. Find the value of  $x$  such that

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta} = \frac{\sin n\theta}{\sin^n \theta}$$

where  $\alpha$  and  $\beta$  are roots of  $t^2 - 2t + 2 = 0$ .

12. Prove that if  $n$  is a positive integer and

$$(1+x)^n = p_0 + p_1x + p_2x^2 + \dots + p_nx^n$$

then (i)  $p_0 - p_2 + p_4 - \dots = 2^{n/2} \cos \frac{n\pi}{4}$ .

$$(ii) \quad p_0 + p_4 + p_8 + \dots = 2^{\frac{n-2}{2}} \cos \frac{n\pi}{4}.$$

13. Show that the sum of the infinite series

$$1 - \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} + \dots \text{ is } \frac{1}{2}(1 + \sqrt{2})^{1/2}.$$

14. Find all the values of

$$(i) \quad (16)^{1/4} \quad (ii) \quad (-1)^{1/5} \quad (iii) \quad (1-i\sqrt{3})^{1/4} \\ (iv) \quad (i)^{1/3} \quad (v) \quad (1+i)^{2/3}.$$

15. Find the continued product of the four values of

$$\left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^{3/4}$$

16. Show that if  $\alpha$  denotes any  $n$ th root of unity, then  
 $1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} = 0$

17. Solve the equation  $x^{12} - 1 = 0$  and find the roots that satisfy  
 $x^4 + x^3 + 1 = 0$ .

18. Solve (i)  $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$   
(ii)  $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$ .

19. Solve using De Moivre's theorem the equation  $(x-1)^n = x^n$  ( $n$  being a +ve integer) and show that its roots are given by

$$\frac{1}{2} \left\{ 1 + i \cot \left( \frac{r\pi}{n} \right) \right\}.$$

where  $r = 0, 1, 2, \dots, (n-1)$ .

20. Solve  $x^5=1$  by De Moivre's theorem and prove that the sum of  $n$ th powers of the roots of the equation,  $n$  being an integer not divisible by 5, is zero.

21. Express  $P = \frac{(\sqrt{3}-1)+i(\sqrt{3}+1)}{2\sqrt{2}}$

in the form  $r(\cos \theta + i \sin \theta)$  and derive all the value of  $P^{1/6}$ .

22. Show that the roots of the equation

$$(1+x)^{2n} + (1-x)^{2n} = 0$$

are given by

$$\pm i \tan\left(\frac{2r-1}{4n}\right)\pi$$

where  $r=1, 2, \dots, n$

1.7. Expansion of  $\cos n\theta$ ,  $\sin n\theta$  and  $\tan n\theta$ , in powers  $\cos \theta$ ,  $\sin \theta$  and  $\tan \theta$ , when  $n$  is a positive integer.

We have by De Moivre's theorem,  
 $\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$

$$\begin{aligned} &= \cos^n \theta + "C_1 \cos^{n-1} \theta \cdot (i \sin \theta) + "C_2 \cos^{n-2} \theta (i \sin \theta)^2 \\ &\quad + "C_3 \cos^{n-3} \theta (i \sin \theta)^3 + "C_4 \cos^{n-4} \theta (i \sin \theta)^4 \\ &\quad + \dots + "C_{n-1} \cos \theta (i \sin \theta)^{n-1} + (i \sin \theta)^n \end{aligned} \quad [\text{By Binomial theorem}]$$

$$\begin{aligned} &= (\cos^n \theta - "C_2 \cos^{n-2} \theta \sin^2 \theta + "C_4 \cos^{n-4} \theta \sin^4 \theta - \dots) \\ &\quad + i("C_1 \cos^{n-1} \theta \sin \theta - "C_3 \cos^{n-3} \theta \sin^3 \theta - \dots) \end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$\begin{aligned} \cos n\theta &= \cos^n \theta - "C_2 \cos^{n-2} \theta \sin^2 \theta + "C_4 \cos^{n-4} \theta \sin^4 \theta - \dots \quad (1) \\ \text{and } \sin n\theta &= "C_1 \cos^{n-1} \theta \sin \theta - "C_3 \cos^{n-3} \theta \sin^3 \theta \\ &\quad + "C_5 \cos^{n-5} \theta \sin^5 \theta - \dots \quad (2) \end{aligned}$$

Replacing  $\sin^2 \theta$  by  $1 - \cos^2 \theta$  in (1) and  $\cos^2 \theta$  by  $1 - \sin^2 \theta$  in (2), we get the required expansions of  $\cos n\theta$  and  $\sin n\theta$ .

Now, dividing (2) by (1), we get

$$\begin{aligned} \tan n\theta &= \frac{"C_1 \cos^{n-1} \theta \sin \theta - "C_3 \cos^{n-3} \theta \sin^3 \theta + "C_5 \cos^{n-5} \theta \sin^5 \theta - \dots}{\cos^n \theta - "C_2 \cos^{n-2} \theta \sin^2 \theta + "C_4 \cos^{n-4} \theta \sin^4 \theta - \dots} \\ &= \frac{"C_1 \tan \theta - "C_3 \tan^3 \theta + "C_5 \tan^5 \theta - \dots}{1 - "C_2 \tan^2 \theta + "C_4 \tan^4 \theta - \dots} \end{aligned} \quad [\text{Dividing } N' \text{ and } D' \text{ by } \cos^n \theta]$$

**Example 1.** Expand  $\cos 7\theta$  in powers of  $\cos \theta$  and  $\sin \theta$ .

**Sol.** We know  $\cos n\theta = \cos^n \theta - "C_2 \cos^{n-2} \theta \sin^2 \theta + "C_4 \cos^{n-4} \theta \sin^4 \theta - "C_6 \cos^{n-6} \theta \sin^6 \theta + \dots$

$$\therefore \cos 7\theta = \cos^7 \theta - \frac{7.6}{1.2} \cos^5 \theta \sin^2 \theta$$

$$+ \frac{7.6.5.4}{1.2.3.4} \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta$$

$$= \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta.$$

**Aliter**

$$(\cos \theta + i \sin \theta)^7 = \cos^7 \theta + {}^7C_1 \cos^6 \theta (i \sin \theta) + {}^7C_2 \cos^5 \theta (i \sin \theta)^2 + {}^7C_3 \cos^4 \theta (i \sin \theta)^3 + {}^7C_4 \cos^3 \theta (i \sin \theta)^4 + {}^7C_5 \cos^2 \theta (i \sin \theta)^5 + {}^7C_6 \cos \theta (i \sin \theta)^6 + {}^7C_7 (i \sin \theta)^7.$$

$$\therefore \cos 7\theta + i \sin 7\theta = (\cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta) + i(7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta).$$

Equating real parts on both sides, we get

$$\cos 7\theta = \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta.$$

**Example 2. Expand  $\sin 8\theta$ .**

$$\text{Sol. We know } \sin n\theta = {}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + {}^nC_5 \cos^{n-5} \theta \sin^5 \theta - {}^nC_7 \cos^{n-7} \theta \sin^7 \theta + \dots$$

$$\therefore \sin 8\theta = 8 \cos^7 \theta \sin \theta - 56 \cos^5 \theta \sin^3 \theta + 56 \cos^3 \theta \sin^5 \theta - 8 \cos \theta \sin^7 \theta.$$

**Example 3. Expand  $\tan 6\theta$  in powers of  $\tan \theta$ .**

$$\text{Sol. We know, } \tan n\theta = \frac{{}^nC_1 \tan \theta - {}^nC_3 \tan^3 \theta + {}^nC_5 \tan^5 \theta - \dots}{1 - {}^nC_2 \tan^2 \theta + {}^nC_4 \tan^4 \theta - \dots}$$

$$\therefore \tan 6\theta = \frac{{}^6C_1 \tan \theta - {}^6C_3 \tan^3 \theta + {}^6C_5 \tan^5 \theta}{1 - {}^6C_2 \tan^2 \theta + {}^6C_4 \tan^4 \theta - {}^6C_6 \tan^6 \theta} \\ = \frac{6 \tan \theta - 20 \tan^3 \theta + 6 \tan^5 \theta}{1 - 15 \tan^2 \theta + 15 \tan^4 \theta - \tan^6 \theta}.$$

**EXERCISE 1 (c)**

Prove that

$$1. \sin 7\theta = 7 \cos^6 \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta$$

$$2. \sin 6\theta = 6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta$$

$$3. 1 + \cos 10\theta = 2(16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta)^2$$

$$4. 1 - \cos 10\theta = 2(16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta)^2$$

$$5. \tan 9\theta = \frac{9 \tan \theta - 84 \tan^3 \theta + 126 \tan^5 \theta - 36 \tan^7 \theta + \tan^9 \theta}{1 - 36 \tan^2 \theta + 126 \tan^4 \theta - 84 \tan^6 \theta + 9 \tan^8 \theta}$$

- 1.8. Expansion of  $\cos^n \theta$  in terms of cosines of multiples of  $\theta$  when  $n$  is a positive integer,**

Let  $x = \cos \theta + i \sin \theta$

$$\therefore \frac{1}{x} = \cos \theta - i \sin \theta \quad \text{or} \quad x + \frac{1}{x} = 2 \cos \theta$$

$$\text{Hence, } (2 \cos \theta)^n = \left( x + \frac{1}{x} \right)^n$$

$$= x^n + {}^nC_1 x^{n-1} \cdot \frac{1}{x} + {}^nC_2 x^{n-2} \cdot \frac{1}{x^2} + \dots \\ + {}^nC_{n-2} x^2 \cdot \frac{1}{x^{n-2}} + {}^nC_{n-1} \cdot x \cdot \frac{1}{x^{n-1}} + \frac{1}{x^n}$$

Since  ${}^nC_r = {}^nC_{n-r}$ , we get

$$(2 \cos \theta)^n = x^n + {}^nC_1 x^{n-2} + {}^nC_2 x^{n-4} + \dots$$

$$\begin{aligned} &+ {}^nC_2 \left( \frac{1}{x^{n-4}} \right) + {}^nC_1 \left( \frac{1}{x^{n-2}} \right) + \frac{1}{x^n} \\ &= \left( x^n + \frac{1}{x^n} \right) + {}^nC_1 \left( x^{n-2} + \frac{1}{x^{n-2}} \right) \\ &\quad + {}^nC_2 \left( x^{n-4} + \frac{1}{x^{n-4}} \right) + \dots \end{aligned}$$

Now  $x^n + \frac{1}{x^n} = 2 \cos n\theta$ ,  $x^{n-2} + \frac{1}{x^{n-2}} = 2 \cos(n-2)\theta$  and so on. [By De Moivre's theorem]

$$\therefore 2^n (\cos \theta)^n = 2[\cos n\theta + {}^nC_1 \cos(n-2)\theta + {}^nC_2 \cos(n-4)\theta + \dots]$$

or  $2^{n-1} \cos^n \theta = \cos n\theta + {}^nC_1 \cos(n-2)\theta + {}^nC_2 \cos(n-4)\theta + \dots$

**Example 1.** Expand  $\cos^5 \theta$  in a series of cosines of multiples of  $\theta$ .

$$\begin{aligned} \text{Sol. We have } (2 \cos \theta)^5 &= \left( x + \frac{1}{x} \right)^5 \\ &= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5} \\ &= \left( x^5 + \frac{1}{x^5} \right) + 5 \left( x^3 + \frac{1}{x^3} \right) + 10 \left( x + \frac{1}{x} \right) \\ &= 2[\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta] \\ \therefore \cos^5 \theta &= \frac{1}{16} [\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta]. \end{aligned}$$

1.9. Expansion of  $\sin^n \theta$  in terms of sines of multiples of  $\theta$ , when  $n$  is a positive integer.

Let  $x = \cos \theta + i \sin \theta$

$$\therefore \frac{1}{x} = \cos \theta - i \sin \theta$$

$$x - \frac{1}{x} = 2i \sin \theta$$

$$\text{Now } (2i \sin \theta)^n = \left( x - \frac{1}{x} \right)^n$$

$$\begin{aligned} 2^n \cdot i^n \sin^n \theta &= x^n + {}^nC_1 x^{n-1} \cdot \left( -\frac{1}{x} \right) + {}^nC_2 x^{n-2} \left( -\frac{1}{x} \right)^2 \\ &\quad + \dots + {}^nC_{n-2} x^2 \cdot \left( -\frac{1}{x} \right)^{n-2} \\ &\quad + {}^nC_{n-1} x \cdot \left( -\frac{1}{x} \right)^{n-1} + \left( -\frac{1}{x} \right)^n. \end{aligned}$$

**Case I.** When  $n$  is even.

$$\begin{aligned} 2^n \cdot (-1)^{n/2} \cdot \sin^n \theta &= \left( x^n + \frac{1}{x^n} \right) - {}^n C_1 \left( x^{n-2} + \frac{1}{x^{n-2}} \right) \\ &\quad + {}^n C_2 \left( x^{n-4} + \frac{1}{x^{n-4}} \right) - \dots \dots \\ &\quad [\because {}^n C_r = {}^n C_{n-r}] \end{aligned}$$

Now  $x^n + \frac{1}{x^n} = 2 \cos n\theta$ ,  $x^{n-2} + \frac{1}{x^{n-2}} = 2 \cos (n-2)\theta$ , so on.

$$\therefore 2^n \cdot (-1)^{n/2} \sin^n \theta = 2 \cos n\theta - {}^n C_1 \cdot 2 \cos (n-2)\theta + {}^n C_2 \cdot 2 \cos (n-4)\theta - \dots$$

Dividing both sides by 2, we get

$$\text{or } 2^{n-1} \cdot (-1)^{n/2} \sin^n \theta = \cos n\theta - {}^n C_1 \cos (n-2)\theta + {}^n C_2 \cos (n-4)\theta - \dots \dots$$

**Case II.** When  $n$  is odd.

$\therefore (n-1)$  is an even integer

$$\therefore (2i \sin \theta)^n = \left( x - \frac{1}{x} \right)^n$$

$$\begin{aligned} \text{or } 2^n \cdot i^n \cdot \sin^n \theta &= x^n - {}^n C_1 x^{n-1} \cdot \frac{1}{x} + {}^n C_2 x^{n-2} \cdot \frac{1}{x^2} - \dots \\ &\quad - {}^n C_{n-2} x^2 \cdot \frac{1}{x^{n-1}} + {}^n C_{n-1} x \cdot \frac{1}{x^{n-2}} - \frac{1}{x^n} \\ \therefore 2^n i \cdot i^{n-1} \cdot \sin^n \theta &= \left( x^n - \frac{1}{x^n} \right) - {}^n C_1 \left( x^{n-2} - \frac{1}{x^{n-2}} \right) \\ &\quad + {}^n C_2 \left( x^{n-4} - \frac{1}{x^{n-4}} \right) + \dots \end{aligned}$$

$$\text{or } 2^n i \cdot (-1)^{\frac{n-1}{2}} \sin^n \theta = 2i \sin n\theta - {}^n C_1 \cdot 2i \sin (n-2)\theta + {}^n C_2 \cdot 2i \sin (n-4)\theta - \dots$$

Dividing both sides by  $2i$ , we get

$$2^{n-1} \cdot (-1)^{\frac{n-1}{2}} \sin^n \theta = \sin n\theta - {}^n C_1 \sin (n-2)\theta + {}^n C_2 \sin (n-4)\theta - \dots$$

**Example 1.** Show that

$$\sin^8 \theta = \frac{1}{128} [\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 2\theta + 35].$$

**Sol.** When  $n$  is even, we have

$$\begin{aligned} 2^{n-1} \cdot (-1)^{n/2} \cdot \sin^n \theta &= \cos n\theta - {}^n C_1 \cos (n-2)\theta \\ &\quad + {}^n C_2 \cos (n-4)\theta - {}^n C_3 \cos (n-6)\theta \\ &\quad + {}^n C_4 \cos (n-8)\theta + \dots \end{aligned}$$

Putting  $n=8$ , we get

$$2^7 \cdot (-1)^4 \sin^8 \theta = \cos 8\theta - {}^8 C_1 \cos 6\theta + {}^8 C_2 \cos 4\theta - {}^8 C_3 \cos 2\theta + {}^8 C_4 \cos 0\theta$$

$$128 \sin^8 \theta = [\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 2\theta + 35]$$

$$\therefore \sin^8 \theta = \frac{1}{128} [\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 2\theta + 35]$$

**Example 2.** Expand  $\sin^7 \theta \cos^4 \theta$  in series of sines of multiples of  $\theta$ .

**Sol.** Let  $x = \cos \theta + i \sin \theta$

$$\therefore x + \frac{1}{x} = 2 \cos \theta, \quad x - \frac{1}{x} = 2i \sin \theta$$

$$\begin{aligned}\therefore (2i \sin \theta)^7 \cdot (2 \cos \theta)^4 &= \left( x - \frac{1}{x} \right)^7 \left( x + \frac{1}{x} \right)^4 \\ &= \left( x - \frac{1}{x} \right)^3 \left[ \left( x - \frac{1}{x} \right) \left( x + \frac{1}{x} \right) \right]^4 \\ &= \left( x - \frac{1}{x} \right)^3 \cdot \left( x^2 - \frac{1}{x^2} \right)^4 \\ &= \left( x - \frac{1}{x} \right)^3 \cdot \left( x^8 - 4x^6 + 6 - \frac{4}{x^4} + \frac{1}{x^8} \right) \\ &= \left( x^3 - 3x + \frac{3}{x} - \frac{1}{x^3} \right) \left( x^8 - 4x^6 + 6 - \frac{4}{x^4} + \frac{1}{x^8} \right) \\ &= \left( x^{11} - \frac{1}{x^{11}} \right) - 3 \left( x^9 - \frac{1}{x^9} \right) - \left( x^3 - \frac{1}{x^3} \right) \\ &\quad - 11 \left( x^5 - \frac{1}{x^5} \right) - 6 \left( x^7 - \frac{1}{x^7} \right) \\ &\quad - 14 \left( x - \frac{1}{x} \right)\end{aligned}$$

$$\text{Now } x^n - \frac{1}{x^n} = 2i \sin n\theta$$

$$\begin{aligned}\therefore -2^{11} i \cdot \sin^7 \theta \cos^4 \theta &= 2i [\sin 11\theta - 3 \sin 9\theta - \sin 7\theta \\ &\quad - 11 \sin 5\theta - 6 \sin 3\theta - 14 \sin \theta] \\ \therefore \sin^7 \theta \cos^4 \theta &= -\frac{1}{2^{10}} [\sin 11\theta - 3 \sin 9\theta - \sin 7\theta \\ &\quad - 11 \sin 5\theta - 6 \sin 3\theta - 14 \sin \theta].\end{aligned}$$

### EXERCISE 1 (d)

Prove that

$$1. \cos^7 \theta = \frac{1}{2^6} [\cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta]$$

$$2. \sin^9 \theta = \frac{1}{256} [\sin 9\theta - 9 \sin 7\theta + 36 \sin 5\theta - 84 \sin 3\theta + 126 \sin \theta]$$

$$3. \sin^4 \theta \cos^2 \theta = \frac{1}{32} [\cos 6\theta - 2 \cos 4\theta - \cos 2\theta + 2]$$

$$4. \cos^5 \theta \sin^7 \theta = -\frac{1}{2^{11}} [\sin 12\theta - 2 \sin 10\theta - 4 \sin 8\theta + 10 \sin 6\theta + 5 \sin 4\theta - 20 \sin 2\theta].$$

### 1.10. Exponential Function of a Complex Variable

We know

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

Similarly we define an exponential function of a complex variable  $z = (x+iy)$ , as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots,$$

$$\text{or } e^{x+iy} = 1 + (x+iy) + \frac{1}{2!} (x+iy)^2 + \frac{1}{3!} (x+iy)^3 + \frac{1}{4!} (x+iy)^4 +$$

Putting  $x=0$ , we have

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{1}{2!} (iy)^2 + \frac{1}{3!} (iy)^3 + \frac{1}{4!} (iy)^4 + \dots \\ &= \left( 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \right) + i \left( y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right) \end{aligned} \quad .. (1)$$

$$\therefore e^{iy} = \cos y + i \sin y. \quad .. (1)$$

Changing  $i$  to  $-i$  in (1), we get

$$e^{-iy} = \cos y - i \sin y. \quad .. (2)$$

Adding (1) and (2), we get

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \quad .. (3)$$

Substracting (2) from (1) we get

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}. \quad .. (4)$$

The formulae (3) and (4) are known as **Euler's Formulae** for circular functions.

### Circular functions of a complex variable

If  $z$  is a complex variable, then we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

$$\text{and } \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

Evidently,  $\cos z + i \sin z = e^{iz}$

The values of  $\sin z$  and  $\cos z$  in terms of exponential functions are also called **Euler's Formulae**.

**Example 1.** Using Euler's formulae, prove that

- (i)  $\cos^2 \theta + \sin^2 \theta = 1$ ,
- (ii)  $\cos 2\theta = 2 \cos^2 \theta - 1$
- (iii)  $\sin 2\theta = 2 \sin \theta \cos \theta$ ,
- (iv)  $\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi$ .

**Sol.** (i) We know  $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\begin{aligned}\therefore \cos^2 \theta + \sin^2 \theta &= \frac{1}{4} (e^{i\theta} + e^{-i\theta})^2 + \frac{1}{4i^2} (e^{i\theta} - e^{-i\theta})^2 \\&= \frac{1}{4} [(e^{i\theta} + e^{-i\theta})^2 - (e^{i\theta} - e^{-i\theta})^2] \quad [\because i^2 = -1] \\&= \frac{1}{4} 4e^{i\theta} \cdot e^{-i\theta} = 1.\end{aligned}$$

$$\begin{aligned}(ii) \quad \cos 2\theta &= \frac{e^{2i\theta} + e^{-2i\theta}}{2} \\&= \frac{(e^{i\theta} + e^{-i\theta})^2 - 2e^{i\theta} \cdot e^{-i\theta}}{2} \\&= \frac{4 \cos^2 \theta - 2}{2} = 2 \cos^2 \theta - 1.\end{aligned}$$

$$\begin{aligned}(iii) \quad \sin 2\theta &= \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \\&= 2 \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right) \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) \\&= 2 \sin \theta \cos \theta.\end{aligned}$$

$$\begin{aligned}(iv) \quad e^{i(\theta \pm \phi)} &= e^{i\theta} \cdot e^{\pm i\phi} \\&= (\cos \theta + i \sin \theta)(\cos \phi \pm i \sin \phi)\end{aligned}$$

$$\begin{aligned}\cos(\theta \pm \phi) + i \sin(\theta \pm \phi) &= (\cos \theta \cos \phi \mp \sin \theta \sin \phi) \\&\quad + i (\sin \theta \cos \phi \pm \cos \theta \sin \phi)\end{aligned}$$

Equating imaginary parts on both sides, we get

$$\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi.$$

### 1.11. Period of $e^z$ , where $z$ is a Complex Variable

Let

$$z = x + iy$$

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y) \quad \dots(1)$$

$$\begin{aligned}\text{Also } e^{z+2n\pi i} &= e^x + i(2n\pi + y) = e^x [\cos(2n\pi + y) + i \sin(2n\pi + y)] \\&= e^x (\cos y + i \sin y) \quad \dots(2)\end{aligned}$$

From (1) and (2)

$$e^z = e^{z+2n\pi i}$$

Thus  $2\pi i$  is period of  $e^z$ .

### 1.12. Hyperbolic Functions.

We know trigonometric functions like  $\sin \theta$ ,  $\cos \theta$  etc. are connected with a circle. Similarly, there are functions of  $e^x$ , which are connected with a hyperbola and these are known as **Hyperbolic Functions**. These functions have a great similarity to trigonometric functions and are defined as under.

$$(i) \sinh x = \frac{e^x - e^{-x}}{2} \quad (v) \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$(ii) \cosh x = \frac{e^x + e^{-x}}{2} \quad (vi) \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$(iii) \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$(iv) \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

From (i) and (ii), we have

$$e^x = \cosh x + \sinh x$$

and

$$e^{-x} = \cosh x - \sinh x$$

The following table gives names and nomenclature of these hyperbolic functions.

Name	Abbreviation	Pronunciation
Hyperbolic sin of $x$	$\sinh x$	shin $x$
Hyperbolic cos of $x$	$\cosh x$	cosh $x$
Hyperbolic tan of $x$	$\tanh x$	than $x$
Hyperbolic sec of $x$	$\operatorname{sech} x$	shec $x$
Hyperbolic cosec of $x$	$\operatorname{cosech} x$	coshec $x$
Hyperbolic cot of $x$	$\coth x$	coth $x$

### 1.13 Relation Between Hyperbolic and Circular Functions

$$\begin{aligned} \cos ix &= \frac{1}{2} (e^{i(ix)} + e^{-i(ix)}) \\ &= \frac{1}{2} (e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x. \end{aligned}$$

$$\therefore \cos ix = \cosh x$$

$$\text{Similarly, } \sin ix = i \sinh x,$$

$$\text{and } \tan ix = i \tanh x.$$

$$\text{Also, } \cosh ix = \frac{1}{2}[e^{ix} + e^{-ix}] = \cos x$$

$$\sinh ix = \frac{1}{2}[e^{ix} - e^{-ix}] = i \sin x$$

$$\tanh ix = \frac{\sinh ix}{\cosh ix} = i \tan x \text{ and so on.}$$

### 1.14 Relations Between Hyperbolic Functions.

1.  $\cosh^2 x - \sinh^2 x = 1$
2.  $1 - \tanh^2 x = \operatorname{sech}^2 x$
3.  $\coth^2 x - 1 = \operatorname{cossec}^2 x$
4.  $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
5.  $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
6.  $\tanh(x \pm y) = \frac{\tanh x + \tanh y}{1 \pm \tanh x \tanh y}$
7.  $\cosh 2x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$
8.  $\sinh 2x = 2 \sinh x \cosh x$
9.  $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$
10.  $\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}$
11.  $\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}$
12.  $\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2}$
13.  $\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$

Proofs of some of the above results are given below.

$$(1) \text{ We know } \cosh x = \frac{e^x + e^{-x}}{2}$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\therefore \cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2, \\ = \frac{1}{4} \cdot 4 \cdot e^x \cdot e^{-x} = 1$$

$$\text{or } \cosh^2 x - \sinh^2 x = 1.$$

**Aliter.**  $\cos^2 x + \sin^2 x = 1$

Changing  $x$  to  $ix$ , we have

$$(\cos ix)^2 + (\sin ix)^2 = 1$$

$$\text{or } \cosh^2 x + (i \sinh x)^2 = 1 \\ \cosh^2 x - \sinh^2 x = 1.$$

$$(5) \quad \begin{aligned} \cosh(x+y) &= \frac{1}{2}(e^{x+y} + e^{-(x+y)}) \\ &= \frac{1}{2}(e^x \cdot e^y + e^{-x} \cdot e^{-y}) \\ &\quad \frac{1}{2}[(\cosh x + \sinh x)(\cosh y + \sinh y) \\ &\quad + (\cosh x - \sinh x)(\cosh y - \sinh y)] \\ &= \frac{1}{2}[2 \cosh x \cosh y + 2 \sinh x \sinh y] \\ &\quad (\text{On simplification}) \\ &= \cosh x \cosh y + \sinh x \sinh y \end{aligned}$$

$$\begin{aligned}
 (7) \quad \cosh 2x &= \frac{1}{2}(e^{2x} + e^{-2x}) \\
 &= \frac{1}{2}[(e^x + e^{-x})^2 - 2] \\
 &= \frac{1}{2}[4 \cosh^2 x - 2] \\
 &= 2 \cosh^2 x - 1 \\
 &= 2(1 + \sinh^2 x) - 1 \\
 &= 1 + 2 \sinh^2 x.
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad \sinh x + \sinh y &= \frac{1}{i} (\sin ix + \sin iy) \\
 &= \frac{1}{i} \left( 2 \sin \frac{ix+iy}{2} \cos \frac{ix-iy}{2} \right) \\
 &= \frac{2i}{i} \sinh \left( \frac{x+y}{2} \right) \cosh \left( \frac{x-y}{2} \right) \\
 &\quad [\because \sin ix = i \sinh x] \\
 &= 2 \sinh \left( \frac{x+y}{2} \right) \cosh \left( \frac{x-y}{2} \right).
 \end{aligned}$$

Similarly we can prove other formulae.

### 1.15. Series Expansions of $\sinh x$ and $\cosh x$

$$\begin{aligned}
 \text{We know } \sinh x &= \frac{e^x - e^{-x}}{2} \\
 &= \frac{1}{2} \left[ \left( 1 + x + \frac{x^3}{2!} + \frac{x^5}{3!} + \dots \right) \right. \\
 &\quad \left. - \left( 1 - x + \frac{x^3}{2!} - \frac{x^5}{3!} + \dots \right) \right] \\
 &= \frac{1}{2} \left[ 2 \left( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \right]
 \end{aligned}$$

$$\therefore \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\begin{aligned}
 \text{Now } \cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
 &= \frac{1}{2} \left[ \left( 1 + x + \frac{x^3}{2!} + \frac{x^5}{3!} + \dots \right) \right. \\
 &\quad \left. + \left( 1 - x + \frac{x^3}{2!} - \frac{x^5}{3!} + \dots \right) \right] \\
 &= \frac{1}{2} \left[ 2 \left( 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \right] \\
 \therefore \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots
 \end{aligned}$$

These expansions could easily be obtained by replacing  $x$  by  $ix$  in expansion of  $\sin x$  and  $\cos x$  respectively.

### 1.16. Periodicity of Hyperbolic Functions

Let  $n$  be an integer,

$$\begin{aligned}\sinh(\theta + 2n\pi i) &= \frac{e^{(0+2n\pi i)} - e^{-(0+2n\pi i)}}{2} \\ &= \frac{e^0 - e^{-\theta}}{2} = \sinh \theta \\ \left[ \because e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1 \right. \\ \left. e^{-2n\pi i} = \cos 2n\pi - i \sin 2n\pi = 1 \right]\end{aligned}$$

Thus period of  $\sinh \theta$  is  $2\pi i$ .

Similarly it can be shown that period of  $\cosh \theta$  is also  $2\pi i$ .

$$\begin{aligned}\text{Also } \sinh(\theta + \pi i) &= \frac{e^{(\theta+\pi i)} - e^{-(\theta+\pi i)}}{2} \\ &= -\frac{(e^\theta - e^{-\theta})}{2} \\ \left[ \because e^{\pi i} = \cos \pi + i \sin \pi = -1 \right. \\ \left. e^{-\pi i} = \cos \pi - i \sin \pi = -1 \right] \\ &= -\sinh \theta \\ \cosh(\theta + \pi i) &= \frac{e^{(\theta+\pi i)} + e^{-(\theta+\pi i)}}{2} \\ &= \frac{-e^\theta - e^{-\theta}}{2} \\ &= -\frac{(e^\theta + e^{-\theta})}{2} = -\cosh \theta \\ \therefore \tanh(\theta + \pi i) &= \frac{\sinh(\theta + \pi i)}{\cosh(\theta + \pi i)} \\ &= \frac{-\sinh \theta}{-\cosh \theta} = \tanh \theta\end{aligned}$$

Similarly  $\coth(\theta + \pi i) = \coth \theta$

Thus the period of  $\tanh \theta$  and  $\coth \theta$  is  $\pi i$ .

**Example 1.** Separate into real and imaginary parts.

- (iv)  $\sin(x+iy)$     (ii)  $\tan(x+iy)$     (iii)  $\sec(x+iy)$
- (iv)  $\sinh(x+iy)$     (v)  $e^{(x-iy)^2}$ .

**Sol.** (i)  $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$   
 $= \sin x \cosh y + i \cos x \sinh y$

$$(ii) \quad \tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$

Multiplying numerator and denominator on R.H.S. by  $2 \cos(x-iy)$ , we have

$$\tan(x+iy) = \frac{2 \sin(x+iy) \cos(x-iy)}{2 \cos(x+iy) \cos(x-iy)}$$

$$\begin{aligned}
 &= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} \\
 &= \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y} \\
 &= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i \frac{\sinh 2y}{\cos 2x + \cosh 2y}
 \end{aligned}$$

$$\begin{aligned}
 (iii) \sec(x+iy) &= \frac{1}{\cos(x+iy)} \\
 &= \frac{2 \cos(x-iy)}{2 \cos(x+y)\cos(x-iy)}
 \end{aligned}$$

[Multiplying  $N^r$  and  $D^r$  by  $2 \cos(x-iy)$ ]

$$\begin{aligned}
 &= \frac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cosh 2y} \\
 &= \frac{2 \cos x \cosh y}{\cos 2x + \cosh 2y} + i \frac{2 \sin x \sinh y}{\cos 2x + \cosh 2y}
 \end{aligned}$$

$$\begin{aligned}
 (iv) \sinh(x+iy) &= \frac{1}{i} \sin i(x+iy) \\
 &= \frac{1}{i} \sin(ix-y) = \frac{1}{i} [\sin ix \cos y - \cos ix \sin y] \\
 &= \frac{1}{i} [i \sinh x \cos y - \cosh x \sin y] \\
 &= \sinh x \cos y + i \cosh x \sin y \\
 &\quad \left[ \because -\frac{1}{i} = i \right]
 \end{aligned}$$

$$\begin{aligned}
 (v) e^{(x-iy)^2} &= e^{x^2 - y^2 - 2ixy} = e^{x^2 - y^2} \cdot e^{-2ixy} \\
 &= e^{x^2 - y^2} \cdot (\cos 2xy - i \sin 2xy)
 \end{aligned}$$

**Example 3.** If  $\sin(\alpha+i\beta)=x+iy$ , prove that

$$(i) \frac{x^2}{\cosh^2 \beta} + \frac{y^2}{\sinh^2 \beta} = 1$$

$$(ii) \frac{x^2}{\sin^2 \alpha} - \frac{y^2}{\cos^2 \alpha} = 1.$$

**Sol.** Here  $\sin(\alpha+i\beta)=x+iy$

$$\text{or } \sin \alpha \cosh \beta + i \cos \alpha \sinh \beta = x + iy$$

Equating real and imaginary parts, we have

$$x = \sin \alpha \cosh \beta,$$

$$y = \cos \alpha \sinh \beta$$

$$\text{or } \frac{x}{\cosh \beta} = \sin \alpha$$

$$\text{and } \frac{y}{\sinh \beta} = \cos \alpha$$

$$\therefore \frac{x^2}{\cosh^2 \beta} + \frac{y^2}{\sinh^2 \beta} = \sin^2 \alpha + \cos^2 \alpha$$

$$\text{or } \frac{x^2}{\cosh^2 \beta} + \frac{y^2}{\sinh^2 \beta} = 1$$

$$\text{Also } \frac{x}{\sin \alpha} = \cosh \beta$$

$$\text{and } \frac{y}{\cos \alpha} = \sinh \beta$$

$$\text{or } \frac{x^2}{\sin^2 \alpha} - \frac{y^2}{\cos^2 \alpha} = \cosh^2 \beta - \sinh^2 \beta$$

$$\therefore \frac{x^2}{\sin^2 \alpha} - \frac{y^2}{\cos^2 \alpha} = 1.$$

**Example 3.** If  $x+iy = \cosh(u+iv)$ , then prove that

$$(i) \quad \frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1$$

$$(ii) \quad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1.$$

$$\begin{aligned} \text{Sol. } \text{Here } x+iy &= \cosh(u+iv) = \cos(iu-v) \\ &= \cosh u \cos v + i \sinh u \sin v \end{aligned}$$

Equating real and imaginary parts, we have

$$x = \cosh u \cos v \quad \dots(1)$$

$$\text{and } y = \sinh u \sin v \quad \dots(2)$$

Eliminating  $v$  from (1) and (2), we have

$$\frac{x^2}{\cosh^2 u} = \cos^2 v \text{ and } \frac{y^2}{\sinh^2 u} = \sin^2 v$$

$$\therefore \frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v$$

$$\text{or } \frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1.$$

Again eliminating  $u$  from (1) and (2), we have

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = \cosh^2 u - \sinh^2 u$$

$$\therefore \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1.$$

**Example 4.** If  $\tan(\theta+i\phi)=\tan \alpha + i \sec \alpha$ , then show that

$$2\theta = n\pi + \frac{\pi}{2} + \alpha$$

$$e^{i\theta} = \pm \cot(\alpha/2)$$

$$\text{Sol. Here } \tan(\theta+i\phi) = \tan \alpha + i \sec \alpha$$

$$\therefore \tan(\theta-i\phi) = \tan \alpha - i \sec \alpha$$

$$\begin{aligned}
 \text{Now } \tan 2\theta &= \tan [(\theta + i\phi) + (\theta - i\phi)] \\
 &= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} \\
 &= \frac{(\tan \alpha + i \sec \alpha) + (\tan \alpha - i \sec \alpha)}{1 - (\tan \alpha + i \sec \alpha)(\tan \alpha - i \sec \alpha)} \\
 &= \frac{2 \tan \alpha}{1 - (\tan^2 \alpha + \sec^2 \alpha)} = -\frac{2 \tan \alpha}{2 \tan^2 \alpha} \\
 &= -\cot \alpha = \tan \left[ \left( n\pi + \frac{\pi}{2} \right) + \alpha \right]
 \end{aligned}$$

$$\therefore 2\theta = n\pi + \frac{\pi}{2} + \alpha$$

$$\begin{aligned}
 \tan 2i\phi &= \tan [(\theta + i\phi) - (\theta - i\phi)] \\
 &= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)} \\
 &= \frac{(\tan \alpha + i \sec \alpha) - (\tan \alpha - i \sec \alpha)}{1 + (\tan \alpha + i \sec \alpha)(\tan \alpha - i \sec \alpha)} \\
 &= \frac{2i \sec \alpha}{1 + (\tan^2 \alpha + \sec^2 \alpha)} = \frac{2i \sec \alpha}{2 \sec^2 \alpha} = i \cos \alpha
 \end{aligned}$$

$$\therefore i \tanh 2\phi = i \cos \alpha$$

$$\tanh 2\phi = \cos \alpha$$

$$\text{or } \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\cos \alpha}{1}$$

Applying componendo-dividendo, we have

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1 + \cos \alpha}{1 - \cos \alpha} = \frac{2 \cos^2 \alpha / 2}{2 \sin^2 \alpha / 2} = \cot^2 \frac{\alpha}{2}$$

$$\text{or } e^{4\phi} = \cot^2 \alpha / 2$$

$$\therefore e^{2\phi} = \pm \cot \alpha / 2.$$

**Example 5.** If  $\tan(x+i y) = \sin(u+iv)$ , then prove that

$$\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$

**Sol.** Here  $\tan(x+i y) = \sin(u+iv)$

$$\begin{aligned}
 \frac{\sin 2x}{\cos 2x + \cosh 2y} + \frac{i \sinh 2y}{\cos 2x + \cosh 2y} &= \sin u \cosh v \\
 &\quad + i \cos u \sinh v \\
 &\text{[see example 1 (ii)]}
 \end{aligned}$$

Equating real and imaginary parts on both sides, we have

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = \sin u \cosh v \quad \dots(1)$$

$$\text{and } \frac{\sinh 2y}{\cos 2x + \cosh 2y} = \cos u \sinh v \quad \dots(2)$$

Dividing (1) by (2), we get

$$\frac{\sin 2x}{\sinh 2y} = \frac{\sin u \cosh v}{\cos u \sinh v}$$

or  $\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}.$

**Example 6.**  $\sin(\theta + i\phi) = \tan \alpha + i \sec \alpha$ , show that  
 $\cos 2\theta \cosh 2\phi = 3$ .

**Sol.** Here  $\sin(\theta + i\phi) = \tan \alpha + i \sec \alpha$

$$\therefore \sin \theta \cosh \phi + i \cos \theta \sinh \phi = \tan \alpha + i \sec \alpha$$

Equating real and imaginary parts, we have

$$\sin \theta \cosh \phi = \tan \alpha \quad \dots(1)$$

and  $\cos \theta \sinh \phi = \sec \alpha \quad \dots(2)$

Squaring and subtracting, we have

$$\sin^2 \theta \cosh^2 \phi - \cos^2 \theta \sinh^2 \phi = \tan^2 \alpha - \sec^2 \alpha = -1$$

or  $\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cosh 2\phi}{2}\right) - \left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cosh 2\phi-1}{2}\right) = -1$

Simplifying, we have

$$\cos 2\theta \cosh 2\phi = 3.$$

### EXERCISE 1 (e)

1. Using Euler's formulae, prove that

$$(i) \sin^4 z + \cos^4 z = 1$$

$$(ii) \cos 2z = \cos^2 z - \sin^2 z$$

$$(iii) \sin 2z = 2 \sin z \cos z$$

2. Show that

$$(i) \{\sin(\alpha + \theta) - e^{i\alpha} \sin \theta\}^n = \sin^n \alpha \cdot e^{-n\theta i}$$

$$(ii) \{\sin(\alpha - \theta) + e^{\pm i\alpha} \sin \theta\}^n \\ = \sin^{n-1} \alpha \{\sin(\alpha - n\theta) + e^{\pm i\alpha} \sin n\theta\}$$

3. Prove that (i)  $\sinh 3\theta = 3 \sinh \theta + 4 \sinh^3 \theta$

$$(ii) \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$(iii) \tanh 3\theta = \frac{3 \tanh \theta + \tanh^3 \theta}{1 + 3 \tanh^2 \theta}.$$

4. Separate into real and imaginary parts

$$(i) \cos(\alpha + i\beta) \qquad (ii) \cot(\alpha - i\beta)$$

$$(iii) \operatorname{cosec}(\alpha + i\beta) \qquad (iv) \cosh(\alpha + i\beta)$$

$$(v) \tanh(\alpha + i\beta) \qquad (vi) e^{\cosh(x+iy)}$$

5. If  $\tan(\alpha+i\beta)=x+iy$ , prove that

$$x^2+y^2+2x \cot 2\alpha=1$$

and  $x^2+y^2-2y \coth 2\beta=-1$ .

6. If  $\tan(\theta+i\phi)=\sin(x+iy)$ , show that

$$\coth y \sinh 2\phi=\cot x \sin 2\theta.$$

7. If  $\sin(\theta+i\phi)=\rho(\cos \alpha+i \sin \alpha)$ , prove that

$$\rho^2=\frac{1}{2}(\cosh 2\phi - \cos 2\theta)$$

and  $\tan \alpha=\tanh \phi \cot \theta$ .

8. If  $u=\log \tan(\pi/4+\theta/2)$ , prove that

$$(i) \quad \tanh \frac{u}{2} = \tan \frac{1}{2} \theta$$

$$(ii) \quad \theta = -i \log \tan \left( \frac{\pi}{4} + \frac{iu}{2} \right).$$

9. If  $\tan(\theta+i\phi)=\cos \alpha+i \sin \alpha$ , prove that

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4}$$

and  $\phi = \frac{1}{2} \log \tan \left( \frac{\pi}{4} + \frac{\alpha}{2} \right)$ .

10. If  $\tan(x+iy)=\theta+i\phi$ , prove that

$$\theta^2 + \phi^2 = \frac{\cosh^2 y - \cos^2 x}{\cosh^2 y - \sin^2 x}.$$

11. If  $\cos(\theta+i\phi)=R(\cos \alpha+i \sin \alpha)$ , prove that

$$\phi = \frac{1}{2} \log \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

12. If  $\log \sin(\theta+i\phi)=\alpha+i\beta$ , prove that

$$(i) \quad 2 \cos 2\theta = e^{i\beta} + e^{-i\beta} - 4e^{i\alpha}.$$

$$(ii) \quad \cos(\theta-\beta) = e^{i\beta} \cos(\theta+\beta).$$

13. If  $\tan(x+i\beta)=i$ , where  $\alpha$  and  $\beta$  are real, prove that  $\alpha$  is indeterminate and  $\beta$  is infinite.

14. If  $\sin(\theta+i\phi)=e^{i\alpha}$ , prove that

$$\cos^2 \theta = \sinh^2 \phi = \pm \sin \alpha.$$

15. If  $x=2 \cos \theta \cosh \phi$ ,  $y=2 \sin \theta \sinh \phi$ , prove that

$$\sec(\theta+i\phi) + \sec(\theta-i\phi) = \frac{4x}{x^2+y^2}.$$

### 1.17. Inverse Hyperbolic Functions

If  $\sinh z = w$ , then  $z = \sinh^{-1} w$  is known as the inverse hyperbolic sin of  $w$  and pronounced as *shin inverse w*. Similarly other inverse hyperbolic functions are  $\cosh^{-1} w$ ,  $\tanh^{-1} w$  etc.

The inverse hyperbolic functions are many-valued functions but we shall consider only their principal values.

We shall express inverse hyperbolic functions in the logarithmic forms.

$$\text{Let } \sinh^{-1} w = z$$

$$\text{then } w = \sinh z = \frac{e^z - e^{-z}}{2}$$

$$\text{or } 2w = e^z - \frac{1}{e^{-z}}$$

$$\text{or } e^{2z} - 2we^z - 1 = 0$$

This equation is quadratic in  $e^z$ :

$$\therefore e^z = \frac{2w \pm \sqrt{4w^2 + 4}}{2}$$

As  $e^z$  is always positive, we shall take positive sign only

$$\therefore e^z = w + \sqrt{w^2 + 1}$$

$$\text{or } z = \log(w + \sqrt{w^2 + 1})$$

$$\text{or } \sinh^{-1} w = \log(w + \sqrt{w^2 + 1})$$

Similarly we can show,

$$\cosh^{-1} w = \log(w + \sqrt{w^2 - 1})$$

Further, let  $\tanh^{-1} w = z$

$$\therefore w = \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\text{or } \frac{w}{1-w} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

Applying componendo-dividendo, we get

$$\frac{1+w}{1-w} = \frac{2e^z}{2e^{-z}} = e^{2z}$$

$$\therefore 2z = \log \frac{1+w}{1-w}.$$

$$\text{or } z = \frac{1}{2} \log \frac{1+w}{1-w}$$

$$\text{or } \tanh^{-1} w = \frac{1}{2} \log \frac{1+w}{1-w}.$$

### 1.18. General values of Inverse Hyperbolic Functions

The general values of  $\sinh^{-1} x$ ,  $\cosh^{-1} x$  and  $\tanh^{-1} x$  are denoted by  $\text{Sinh}^{-1} x$ ,  $\text{Cosh}^{-1} x$  and  $\text{Tanh}^{-1} x$  respectively.

These general values are

$$\text{Sinh}^{-1} x = 2n\pi i + (-1)^n \log \{x + \sqrt{x^2 + 1}\}$$

$$\text{Cosh}^{-1} x = 2n\pi i \pm \log \{x + \sqrt{x^2 - 1}\}$$

$$\text{Tanh}^{-1} x = n\pi i + \frac{1}{2} \log \left( \frac{1+x}{1-x} \right).$$

**Example 1.** Separate  $\sin^{-1}(\alpha+i\beta)$  into real and imaginary parts.

**Sol.** Let  $\sin^{-1}(\alpha+i\beta)=x+iy$

$$\therefore \alpha+i\beta=\sin(x+iy)=\sin x \cosh y + i \cos x \sinh y$$

Equating real and imaginary parts, we get

$$\alpha = \sin x \cosh y \quad \dots(i)$$

$$\beta = \cos x \sinh y \quad \dots(ii)$$

$$\therefore \frac{\alpha^2}{\sin^2 x} - \frac{\beta^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1$$

$$\text{or} \quad \alpha^2 \cos^2 x - \beta^2 \sin^2 x = \sin^2 x \cos^2 x$$

$$\alpha^2(1-\sin^2 x) - \beta^2 \sin^2 x = \sin^2 x (1-\sin^2 x)$$

$$\text{or} \quad \sin^4 x - (\alpha^2 + \beta^2 + 1) \sin^2 x + \alpha^2 = 0$$

This equation is quadratic in  $\sin^2 x$ ,

$$\therefore \sin^2 x = \frac{1}{2} [(\alpha^2 + \beta^2 + 1) \pm \sqrt{(\alpha^2 + \beta^2 + 1)^2 - 4\alpha^2}]$$

$$x = \sin^{-1} [\pm \sqrt{\frac{1}{2}(\alpha^2 + \beta^2 + 1)} \pm \sqrt{(\alpha^2 + \beta^2 + 1)^2 - 4\alpha^2}]$$

Similarly eliminating  $x$  from (i) and (ii), we have

$$\frac{\alpha^2}{\cosh^2 y} + \frac{\beta^2}{\sinh^2 y} = 1$$

$$\text{or} \quad \alpha^2 \sinh^2 y + \beta^2 \cosh^2 y = \cosh^2 y \sinh^2 y$$

$$\alpha^2 \sinh^2 y + \beta^2 (1 + \sinh^2 y) = (1 + \sinh^2 y) \sinh^2 y$$

$$\text{or} \quad \sinh^4 y - (\alpha^2 + \beta^2 - 1) \sinh^2 y - \beta^2 = 0$$

$$\sinh^2 y = \frac{1}{2}[(\alpha^2 + \beta^2 - 1) \pm \sqrt{(\alpha^2 + \beta^2 - 1)^2 + 4\beta^2}]$$

$$\therefore y = \sinh^{-1} [\pm \sqrt{\frac{1}{2}(\alpha^2 + \beta^2 - 1)} \pm \sqrt{(\alpha^2 + \beta^2 - 1)^2 + 4\beta^2}].$$

**Example 2.** Separate into real and imaginary parts

$$\tan^{-1}(\cos \theta + i \sin \theta).$$

**Sol.** Let  $\tan^{-1}(\cos \theta + i \sin \theta) = A + iB$

$$\therefore \cos \theta + i \sin \theta = \tan(A + iB) \quad \dots(1)$$

$$\text{and} \quad \cos \theta - i \sin \theta = \tan(A - iB) \quad \dots(2)$$

$$\begin{aligned}
 \text{Now } \tan 2A &= \tan \{(A+iB)+(A-iB)\} \\
 &= \frac{\tan(A+iB) + \tan(A-iB)}{1 - \tan(A+iB) \tan(A-iB)} \\
 &= \frac{(\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta)}{1 - (\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)} \\
 &\quad [\text{from (1) and (2)}] \\
 &= \frac{2 \cos \theta}{1 - (\cos^2 \theta + \sin^2 \theta)} \\
 &= \frac{2 \cos \theta}{0} = \infty \quad \left[ \because \tan \frac{\pi}{2} = \infty \right] \\
 \tan 2A &= \tan \left( n\pi + \frac{\pi}{2} \right)
 \end{aligned}$$

$$\therefore A = \frac{n\pi}{2} + \frac{\pi}{4}$$

$$\text{Now } \tan 2iB = \tan \{(A+iB)-(A-iB)\}$$

$$\begin{aligned}
 &= \frac{\tan(A+iB) - \tan(A-iB)}{1 + \tan(A+iB) \tan(A-iB)} \\
 &= \frac{(\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta)}{1 + (\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)} \\
 &= \frac{2i \sin \theta}{1 + (\cos^2 \theta + \sin^2 \theta)} = i \sin \theta
 \end{aligned}$$

$$\therefore i \tanh 2B = i \sin \theta$$

$$\text{or } \tanh 2B = \sin \theta$$

$$\text{or } \frac{e^{2B} - e^{-2B}}{e^{2B} + e^{-2B}} = \frac{\sin \theta}{1}$$

Applying componendo-dividendo,

$$\begin{aligned}
 \frac{2e^{2B}}{2e^{-2B}} &= \frac{1 + \sin \theta}{1 - \sin \theta} \\
 \therefore e^{4B} &= \frac{1 + \cos \left( \frac{\pi}{2} - \theta \right)}{1 - \cos \left( \frac{\pi}{2} - \theta \right)} = \frac{2 \cos^2 \left( \frac{\pi}{4} - \frac{\theta}{2} \right)}{2 \sin^2 \left( \frac{\pi}{4} - \frac{\theta}{2} \right)} \\
 &= \cot^2 \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \\
 \therefore e^{4B} &= \cot \left( \frac{\pi}{4} - \frac{\theta}{2} \right)
 \end{aligned}$$

$$\therefore B = \frac{1}{2} \log \cot \left( \frac{\pi}{4} - \frac{\theta}{2} \right)$$

$$\therefore \tan^{-1}(\cos \theta + i \sin \theta) = \frac{n\pi}{2} + \frac{\pi}{4} + \frac{i}{2} \log \cot \left( \frac{\pi}{4} - \frac{\theta}{2} \right).$$

**Example 3.** If  $\cos^{-1}(u+iv)=\alpha+i\beta$ , prove that  $\cos^2 \alpha$  and  $\cosh^2 \beta$  are the roots of the equation  $x^2 - x(1+u^2+v^2) + u^2 = 0$ .

**Sol.** Now  $\cos^{-1}(u+iv)=\alpha+i\beta$

$$u+iv=\cos(\alpha+i\beta)$$

$$= \cos \alpha \cosh \beta - i \sin \alpha \sinh \beta$$

$$u=\cos \alpha \cosh \beta \quad \dots(i)$$

and

$$v=-\sin \alpha \sinh \beta$$

Now

$$\begin{aligned} u^2+v^2 &= \cos^2 \alpha \cosh^2 \beta + \sin^2 \alpha \sinh^2 \beta \\ &= \cos^2 \alpha \cosh^2 \beta + (1-\cos^2 \alpha)(\cosh^2 \beta - 1) \\ &= \cos^2 \alpha + \cosh^2 \beta - 1 \end{aligned}$$

or

$$1+u^2+v^2=\cos^2 \alpha + \cosh^2 \beta$$

also

$$u^2=\cos^2 \alpha \cosh^2 \beta. \quad \text{from (i)}$$

The equation which has roots  $\cos^2 \alpha$  and  $\cosh^2 \beta$  is,

$$x^2 - (\cos^2 \alpha + \cosh^2 \beta)x + \cos^2 \alpha \cosh^2 \beta = 0$$

or

$$x^2 - (1+u^2+v^2)x + u^2 = 0.$$

### 1.19. Logarithm of Complex Quantities

If  $z=x+iy$  and  $w=e^z$ , then  $z$  is defined as  $\log_w w$

$$\begin{aligned} \text{Now } e^{z+2n\pi i} &= e^z \cdot e^{2n\pi i} \\ &= e^z = w. \quad [ : e^{2n\pi i} = 1 ] \\ \therefore z+2n\pi i &= \log_w w \quad [\text{By definition}] \end{aligned}$$

Thus we see that  $\log_w w$  is a multivalued function. Hence logarithm of a complex quantity is a multivalued function.

The value of  $\log_w w$  corresponding to  $n=0$ , is called **principal value**. The general value of  $\log_w w$  denoted by  $\log_w w$ , and  $\log_w w = 2n\pi i + \log_w w$ , where  $\log_w w$  denotes the principal value.

**Example 1.** Separate  $\log_w(x+iy)$  into real and imaginary parts and hence obtain the real and imaginary parts of  $\log_w(x+iy)$ .

**Sol.** Let  $\log_w(x+iy)=x+iy$

$$\therefore x+iy=e^x+i\beta=e^x \cdot e^{i\beta}$$

$$\text{or } x+iy=e^x \cdot (\cos \beta + i \sin \beta)$$

Equating real and imaginary parts, we get

$$x = e^x \cdot \cos \beta \quad \dots(1)$$

$$\text{and } y = e^x \cdot \sin \beta \quad \dots(2)$$

Squaring and adding (1) and (2), we have

$$x^2+y^2=e^{2x}$$

$$\therefore \alpha = \log_e (x^2 + y^2)$$

or  $\alpha = \frac{1}{2} \log_e (x^2 + y^2)$

Also from (1) and (2), we have

$$\tan \beta = \frac{y}{x}$$

$$\text{or } \beta = \tan^{-1} \frac{y}{x}$$

$$\therefore \log_e (x+iy) = \frac{1}{2} \log_e (x^2 + y^2) + i \tan^{-1} (y/x).$$

$$\begin{aligned} \text{Now } \operatorname{Log}_e (x+iy) &= 2n\pi i + \log_e (x+iy) \\ &= \frac{1}{2} \log_e (x^2 + y^2) + i (2n\pi + \tan^{-1} y/x). \end{aligned}$$

**Example 2.** Find the principal values of  $\log(-1+i) - \log(-1-i)$ .

**Sol.** Here  $\log(-1+i) - \log(-1-i)$

$$\text{Let } -1 = r \cos \theta \text{ and } 1 = r \sin \theta$$

$$\therefore r = \sqrt{2}, \tan \theta = -1 \text{ or } \theta = \frac{3\pi}{4}$$

$$\begin{aligned} \therefore -1+i &= r(\cos \theta + i \sin \theta) = re^{i\theta} \\ &= \sqrt{2}e^{\frac{3\pi i}{4}} \end{aligned}$$

$$\begin{aligned} -1-i &= r(\cos \theta - i \sin \theta) = re^{-i\theta} \\ &= \sqrt{2}e^{-\frac{3\pi i}{4}} \end{aligned}$$

$$\therefore \log(-1+i) - \log(-1-i) = \log \frac{(-1+i)}{(-1-i)}$$

$$= \log \frac{\sqrt{2}e^{\frac{3\pi i}{4}}}{\sqrt{2}e^{-\frac{3\pi i}{4}}}$$

$$= \log \frac{1}{e^{-\frac{3\pi i}{2}}} = -\log e^{-\frac{3\pi i}{2}}$$

$$= -\log \left( e^{2\pi i - \frac{3\pi i}{2}} \right) = -\log e^{\frac{\pi i}{2}}$$

$$= -\frac{\pi i}{2}, \text{ is the principal value.}$$

**Example 3.** Show that (i)  $\operatorname{Log} i = \left( 2n + \frac{1}{2} \right) \pi i$   
(ii)  $\operatorname{Log}(-x) = (2n+1) \pi i + \log x$ .

**Sol.** (i)  $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = e^{\frac{\pi i}{2}}$

$$\log i = \log e^{\frac{\pi i}{2}} = \frac{\pi}{2} i$$

Now  $\begin{aligned} \text{Log } i &= 2n\pi i + \log i \\ &= 2n\pi i + \frac{\pi i}{2} \\ &= \left(2n + \frac{1}{2}\right)\pi i. \end{aligned}$

(ii)  $-x = x(\cos \pi + i \sin \pi) = xe^{i\pi}$

$$\begin{aligned} \log(-x) &= \log x + \log e^{i\pi} \\ &= \log x + \pi i \end{aligned}$$

$\therefore \begin{aligned} \text{Log } (-x) &= 2n\pi i + \log(-x) \\ &= 2n\pi i + \log x + \pi i \\ &= (2n+1)\pi i + \log x \end{aligned}$

**Example 4.** Prove that  $\tan\left(i \log \frac{a-ib}{a+ib}\right) = \frac{2ab}{a^2-b^2}$

**Sol.** Let  $a=r \cos \theta, b=r \sin \theta$

$$\therefore r = \sqrt{a^2+b^2}, \quad \tan \theta = \frac{b}{a}$$

$$\begin{aligned} \therefore a+ib &= r(\cos \theta + i \sin \theta) = re^{i\theta} \\ a-ib &= r(\cos \theta - i \sin \theta) = re^{-i\theta} \\ \frac{a-ib}{a+ib} &= \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta} \end{aligned}$$

$$\log \frac{a-ib}{a+ib} = \log e^{-2i\theta} = -2i\theta$$

$$i \log \frac{a-ib}{a+ib} = i(-2i\theta) = 2\theta$$

$$= 2 \tan^{-1} \frac{b}{a}$$

$$= \tan^{-1} \frac{b}{a} + \tan^{-1} \frac{b}{a}$$

$$\begin{aligned} \therefore \tan \left[ i \log \frac{a-ib}{a+ib} \right] &= \tan \left[ \tan^{-1} \frac{b}{a} + \tan^{-1} \frac{b}{a} \right] \\ &= \tan \left[ \tan^{-1} \frac{\frac{b}{a} + \frac{b}{a}}{1 - \frac{b^2}{a^2}} \right] \end{aligned}$$

$$= \tan \left[ \tan^{-1} \frac{2ab}{a^2 - b^2} \right]$$

$$\therefore \tan \left[ i \log \frac{a - ib}{a + ib} \right] = \frac{2ab}{a^2 - b^2}$$

**Example 5.** If  $i^{i^{i^{i^{\dots}}}} = A + iB$ , prove that

$$\tan \frac{\pi A}{2} = \frac{B}{A}$$

and

$$A^2 + B^2 = e^{-\pi B}$$

**Sol.** Here  $i^{i^{i^{\dots}}} = A + iB$

$$\therefore i^{A+iB} = A + iB$$

Takings logs on both sides, we have

$$(A+iB) \log i = \log (A+iB)$$

$$(A+iB) \left[ \frac{1}{2} \log 1 + i \tan^{-1} \frac{1}{0} \right] = \log (A+iB)$$

$$\left[ \because \log (A+iB) = \frac{1}{2} \log (A^2 + B^2) + i \tan^{-1} \frac{B}{A} \right]$$

$$(A+iB) \left( \frac{\pi}{2} i \right) = \frac{1}{2} \log (A^2 + B^2) + i \tan^{-1} \frac{B}{A}$$

Equating real and imaginary parts on both sides, we have

$$\therefore \frac{-B\pi}{2} = \frac{1}{2} \log (A^2 + B^2)$$

and  $\frac{\pi A}{2} = \tan^{-1} \frac{B}{A}$

$$\therefore A^2 + B^2 = e^{-\pi B}$$

and  $\tan \frac{\pi A}{2} = \frac{B}{A}$ .

### EXERCISE 1 (f)

1. Separate into real and imaginary parts.

- (i)  $\sin^{-1}(\cos \theta + i \sin \theta)$
- (ii)  $\cos^{-1}(\cos \theta + i \sin \theta)$
- (iii)  $\tanh^{-1}(x+iy)$ .

2. Prove that  $\tanh^{-1} x = \sinh^{-1} \left( \frac{x}{\sqrt{1-x^2}} \right)$ .

3. Prove that  $\coth^{-1} \frac{2}{x} = \sinh^{-1} \left( \frac{x}{\sqrt{4-x^2}} \right)$ .

4. Show that  $\sin^{-1}(ix) = 2n\pi + i \log [\sqrt{1+x^2} + x]$  and hence show that

$$\sin^{-1}(i) = 2n\pi + i \log (\sqrt{2} + 1).$$

5. Prove that  $\tan^{-1} i \left( \frac{x-a}{x+a} \right) = -\frac{i}{2} \log \left( \frac{a}{x} \right)$ .

6. Show that  $\sin^{-1} (\csc \theta)$

$$= [2n + (-1)^n] \frac{\pi}{2} + i(-1)^n \log \cot \frac{\theta}{2}$$

7. Prove that  $\log \sin(x+iy)$

$$= \frac{1}{2} \log \left( \frac{\cosh 2y - \cos 2x}{2} \right) + i \tan^{-1} (\cot x \tanh y).$$

8. Prove that  $\log \tan \left( \frac{\pi}{4} + \frac{ix}{2} \right) = i \tan^{-1} (\sinh x)$

9. Prove that  $\log \left( \frac{a+ib}{a-ib} \right) = 2i \tan^{-1} \left( \frac{b}{a} \right)$ .

10. If  $\cosh^{-1}(x+iy) + \cosh^{-1}(x-iy) = \cosh^{-1}(a)$ , show that  $2(a-1)x^2 + 2(a+1)y^2 = a^2 - 1$ .

11. Prove the  $\log(i^i) = -\frac{\pi}{2}$

12. Find the general value of  $(1+i \tan \alpha)^{-i}$

13. Find the real part of the principal value of  $(i)^{\log(1+i)}$

14. If  $\log \sin(\theta + i\phi) = \alpha + i\beta$ , prove that

$$2 \cos 2\theta = e^{i\phi} - e^{-i\phi} - 4e^{2\alpha}. \quad (D.U. 1983)$$

## 1.20. Summation of Trigonometric Series ( $C+iS$ method)

This method is very useful in finding the sum of a series (*finite or infinite*) containing sines or cosines of multiple angles. The method consists in selecting another series, called the auxiliary series. The auxiliary series when combined with the given series, yields a new series which can be summed up easily. For example to find the sum of the series

$$S = a_0 \sin \alpha + a_1 \sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + \dots \quad (i)$$

We select another series

$$C = a_0 \cos \alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta) + \dots, *$$

Then

$$C + iS = a_0 (\cos \alpha + i \sin \alpha) + a_1 [\cos(\alpha + \beta) + i \sin(\alpha + \beta)] + a_2 [\cos(\alpha + 2\beta) + i \sin(\alpha + 2\beta)] + \dots$$

or  $C + iS = a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha+2\beta)} + \dots \quad (ii)$

The series on R.H.S. of (ii) can be summed up easily by using the following standard series.

### Geometric Series

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

\*If the sum of the sine series is required, we choose auxiliary series as a corresponding cosine series and vice-versa. The sine series is denoted by S and the cosine series by C.

or  $a+ar+ar^2+\dots\infty = \frac{a}{1-r}$ , where  $|r| < 1$

(b) **Binomial Series**

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\infty=(1+x)^n$$

(c) **Sine and Cosine Series**

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\infty=\sin x$$

and  $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\infty=\cos x$

(d) **Sinh or Cosh Series**

$$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots\infty=\sinh x$$

$$1+\frac{x^1}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\infty=\cosh x$$

(e) **Exponential Series**

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\infty=e^x$$

(f) **Logarithmic Series**

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\infty=\log(1+x)$$

or  $-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\dots\infty=\log(1-x)$

(g) **Gregory's Series**

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\infty=\tan^{-1} x$$

The following examples illustrate the method.

**Example 1.** Find the sum of the series

$$\sin \alpha + \frac{1}{3} \sin 3\alpha + \frac{1}{3^2} \sin 5\alpha + \dots \infty$$

**Sol.** Let  $S = \sin \alpha + \frac{1}{3} \sin 3\alpha + \frac{1}{3^2} \sin 5\alpha + \dots \infty$

and  $C = \cos \alpha + \frac{1}{3} \cos 3\alpha + \frac{1}{3^2} \cos 5\alpha + \dots \infty$

$$\therefore C+iS = (\cos \alpha + i \sin \alpha) + \frac{1}{3} (\cos 3\alpha + i \sin 3\alpha) + \frac{1}{3^2} (\cos 5\alpha + i \sin 5\alpha) + \dots \infty$$

$$= e^{i\alpha} + \frac{1}{3} e^{3i\alpha} + \frac{1}{3^2} e^{5i\alpha} + \dots \infty \quad \dots (i)$$

The series on R.H.S. of (i) is a geometric series, with common ratio  $\frac{1}{3} e^{2i\alpha}$

$$\begin{aligned} \therefore C+iS &= \frac{e^{i\alpha}}{1 - \frac{1}{3} e^{2i\alpha}} = \frac{e^{i\alpha} \left( 1 - \frac{1}{3} e^{-2i\alpha} \right)}{\left( 1 - \frac{1}{3} e^{2i\alpha} \right) \left( 1 - \frac{1}{3} e^{-2i\alpha} \right)} \\ &\quad (\text{conjugate of } 1 - \frac{1}{3} e^{2i\alpha} \text{ is } 1 - \frac{1}{3} e^{-2i\alpha}) \\ &= \frac{e^{i\alpha} - \frac{1}{3} e^{-i\alpha}}{1 - \frac{1}{3} (e^{2i\alpha} + e^{-2i\alpha}) + \frac{1}{9}} \\ &= \frac{(\cos \alpha + i \sin \alpha) - \frac{1}{3} (\cos \alpha - i \sin \alpha)}{1 - \frac{2}{3} \cos 2\alpha + \frac{1}{9}} \\ &\quad (\because e^{2i\alpha} + e^{-2i\alpha} = 2 \cos 2\alpha) \\ &= \frac{\frac{2}{3} \cos \alpha + \frac{i}{3} \sin \alpha}{1 - \frac{2}{3} \cos 2\alpha + \frac{1}{9}} = \frac{6 \cos \alpha + i \cdot 12 \sin \alpha}{10 - 6 \cos 2\alpha} \end{aligned}$$

or  $C+iS = \frac{3 \cos \alpha + i 6 \sin \alpha}{5 - 3 \cos 2\alpha}$

Equating imaginary parts on both sides, we get

$$S = \frac{6 \sin \alpha}{5 - 3 \cos 2\alpha}, \text{ the required sum.}$$

**Example 2.** Sum the series

$$x \cos \theta - \frac{x^3}{2} \cos 2\theta + \frac{x^3}{3} \cos 3\theta - \dots \infty$$

**Sol.** Let  $C = x \cos \theta - \frac{x^3}{2} \cos 2\theta + \frac{x^3}{3} \cos 3\theta - \dots \infty$

and  $S = x \sin \theta - \frac{x^3}{2} \sin 2\theta + \frac{x^3}{3} \sin 3\theta - \dots \infty$

$$\therefore C+iS = x(\cos \theta + i \sin \theta) - \frac{x^3}{2} (\cos 2\theta + i \sin 2\theta)$$

$$+ \frac{x^3}{3} (\cos 3\theta + i \sin 3\theta) + \dots$$

$$= xe^{i\theta} - \frac{x^2 e^{2i\theta}}{2} + \frac{x^3 e^{3i\theta}}{3} \dots \infty$$

(The series on R.H.S. is a logarithmic series)  
 $= \log(1+xe^{i\theta}) = \log[1+x(\cos \theta + i \sin \theta)]$

$$\begin{aligned}
 &= \log [(1+x \cos \theta) + i \cdot x \sin \theta] \\
 &= \frac{1}{2} \log [(1+x \cos \theta)^2 + x^2 \sin^2 \theta] \\
 &\quad + i \tan^{-1} \left( \frac{x \sin \theta}{1+x \cos \theta} \right)
 \end{aligned}$$

or  $C+iS = \frac{1}{2} \log (1+2x \cos \theta + x^2) + i \tan^{-1} \left( \frac{x \sin \theta}{1+x \cos \theta} \right)$

Equating real parts on both sides, we get

$$C = \frac{1}{2} \log (1+2x \cos \theta + x^2)$$

**Example 3.** Find the sum of the series

$$\sin \alpha + x \sin (\alpha + \beta) + \frac{x^2}{2!} \sin (\alpha + 2\beta) + \dots \infty$$

**Sol.** Let  $S = \sin \alpha + x \sin (\alpha + \beta) + \frac{x^2}{2!} \sin (\alpha + 2\beta) + \dots \infty$

and  $C = \cos \alpha + x \cos (\alpha + \beta) + \frac{x^2}{2!} \cos (\alpha + 2\beta) + \dots \infty$

$$\begin{aligned}
 \therefore C+iS &= (\cos \alpha + i \sin \alpha) + x[\cos(\alpha+\beta) + i \sin(\alpha+\beta)] \\
 &\quad + \frac{x^2}{2!} [\cos(\alpha+2\beta) + i \sin(\alpha+2\beta)] + \dots \infty
 \end{aligned}$$

$$= e^{i\alpha} + x e^{i(\alpha+\beta)} + \frac{x^2}{2!} e^{i(\alpha+2\beta)} + \dots \infty$$

$$= e^{i\alpha} \left[ 1 + x e^{i\beta} + \frac{x^2}{2!} e^{2i\beta} + \dots \infty \right]$$

(The series on R.H.S. is an exponential series)

$$= e^{i\alpha} \cdot (e^{xe^{i\beta}})$$

$$= e^{i\alpha} \cdot [e^{x(\cos \beta + i \sin \beta)}].$$

$$= e^{\alpha} \cos \beta e^{i(\alpha+x \sin \beta)}$$

or  $C+iS = e^{\alpha} \cos \beta [\cos(\alpha+x \sin \beta) + i \sin(\alpha+x \sin \beta)]$

Equating imaginary parts on both sides, we get

$$S = e^{\alpha} \cos \beta \sin(\alpha+x \sin \beta)$$

**Example 4.** Sum to  $n$  terms the series

$$1+x \cos \theta + x^2 \cos 2\theta + x^3 \cos 3\theta + \dots \quad (x < 1).$$

**Sol.** Let  $C = 1+x \cos \theta + x^2 \cos 2\theta + x^3 \cos 3\theta + \dots + x^{n-1} \cos(n-1)\theta \quad (x < 1)$

and  $S = x \sin \theta + x^2 \sin 2\theta + x^3 \sin 3\theta + \dots + x^{n-1} \sin(n-1)\theta$

$$\begin{aligned}
 \therefore C+iS &= 1+x(\cos \theta + i \sin \theta) + x^2(\cos 2\theta + i \sin 2\theta) \\
 &\quad + x^3(\cos 3\theta + i \sin 3\theta) + \dots + x^{n-1}[\cos(n-1)\theta \\
 &\quad + i \sin(n-1)\theta]
 \end{aligned}$$

$$= 1+x e^{i\theta} + x^2 e^{2i\theta} + x^3 e^{3i\theta} + \dots + x^{n-1} e^{(n-1)i\theta}$$

(The series is a G.P.)

$$= \frac{1-x^n e^{ni\theta}}{1-x e^{i\theta}} = \frac{(1-x^n e^{ni\theta})(1-x e^{-i\theta})}{(1-x e^{i\theta})(1-x e^{-i\theta})}$$

$$\begin{aligned}
 &= \frac{1 - xe^{-i\theta} - x^n e^{ni\theta} + x^{n+1} e^{i(n-1)\theta}}{1 - x(e^{i\theta} + e^{-i\theta}) + x^2} \\
 &= \frac{1 - x(\cos \theta - i \sin \theta) - x^n (\cos n\theta + i \sin n\theta)}{1 - 2x \cos \theta + x^2} \\
 &\quad + x^{n+1} [\cos(n-1)\theta + i \sin(n-1)\theta] \\
 \text{or } C+iS &= \frac{1 - x \cos \theta - x^n \cos n\theta + x^{n+1} \cos(n-1)\theta}{1 - 2x \cos \theta + x^2} \\
 &\quad + i \cdot \frac{x \sin \theta - x^n \sin n\theta + x^{n+1} \sin(n-1)\theta}{1 - 2x \cos \theta + x^2}
 \end{aligned}$$

Equating real parts on both sides, we get

$$C = \frac{1 - x \cos \theta - x^n \cos n\theta + x^{n+1} \cos(n-1)\theta}{1 - 2x \cos \theta + x^2},$$

the required sum.

**Example 5.** Sum the series

$$\sin \alpha + \frac{1}{2} \sin 3\alpha + \frac{1 \cdot 3}{2 \cdot 4} \sin 5\alpha + \dots \infty$$

$$\text{Sol. Let } S = \sin \alpha + \frac{1}{2} \sin 3\alpha + \frac{1 \cdot 3}{2 \cdot 4} \sin 5\alpha + \dots \infty$$

and

$$C = \cos \alpha + \frac{1}{2} \cos 3\alpha + \frac{1 \cdot 3}{2 \cdot 4} \cos 5\alpha + \dots \infty$$

$$\begin{aligned}
 \therefore C+iS &= (\cos \alpha + i \sin \alpha) + \frac{1}{2} (\cos 3\alpha + i \sin 3\alpha) \\
 &\quad + \frac{1 \cdot 3}{2 \cdot 4} (\cos 5\alpha + i \sin 5\alpha) + \dots \infty \\
 &= e^{i\alpha} + \frac{1}{2} e^{3i\alpha} + \frac{1 \cdot 3}{2 \cdot 4} e^{5i\alpha} + \dots \\
 &= e^{i\alpha} \left[ 1 + \frac{1}{2} e^{2i\alpha} + \frac{1 \cdot 3}{2 \cdot 4} e^{4i\alpha} + \dots \right]
 \end{aligned}$$

(The series with in brackets is a Binomial series)  
 $= e^{i\alpha} (1 - e^{2i\alpha})^{-1/2}$

$$\begin{aligned}
 &= (\cos \alpha + i \sin \alpha) (1 - \cos 2\alpha - i \sin 2\alpha)^{-1/2} \\
 &= (\cos \alpha + i \sin \alpha) \cdot (2 \sin^2 \alpha - 2i \sin \alpha \cos \alpha)^{-1/2} \\
 &= (2 \sin \alpha)^{-1/2} (\cos \alpha + i \sin \alpha) \cdot
 \end{aligned}$$

$$(\sin \alpha - i \cos \alpha)^{-1/2}$$

$$\begin{aligned}
 &= (2 \sin \alpha)^{-1/2} \cdot (\cos \alpha + i \sin \alpha) \cdot \left[ \cos \left( \frac{\pi}{2} - \alpha \right) \right. \\
 &\quad \left. - i \sin \left( \frac{\pi}{2} - \alpha \right) \right]^{-1/2}
 \end{aligned}$$

$$= (2 \sin \alpha)^{-1/2} \cdot (\cos \alpha + i \sin \alpha) \cdot \left[ \cos \left( \frac{\pi}{4} - \frac{\alpha}{2} \right) + i \sin \left( \frac{\pi}{4} - \frac{\alpha}{2} \right) \right]$$

(By Demoivre's theorem)

or  $C+iS = (2 \sin \alpha)^{-1/2} \cdot \left[ \cos \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) + i \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right) \right]$

(Amplitudes are added on multiplication.)

Equating imaginary parts on both sides, we get

$$S = (2 \sin \alpha)^{-1/2} \cdot \sin \left( \frac{\pi}{4} + \frac{\alpha}{2} \right)$$

### EXERCISE 1 (g)

Sum the following series.

1.  $\sin \alpha + \frac{1}{2} \sin 2\alpha + \frac{1}{2^2} \sin 3\alpha + \dots \infty$

2.  $1 + x \cos y + x^2 \cos 2y + \dots \infty$

3.  $\cos \alpha \sin \alpha + \cos^2 \alpha \sin 2\alpha + \cos^3 \alpha \sin 3\alpha + \dots \infty$

[Hint.  $C = \cos \alpha \cos \alpha + \cos^3 \alpha \cos 2\alpha + \cos^3 \alpha \cos 3\alpha + \dots$ ]

4.  $\cos \alpha + x \cos(\alpha + \beta) + \frac{x^3}{2!} \cos(\alpha + 2\beta) + \dots \infty$

5.  $\sin \theta - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \dots$

6.  $c \sin \alpha + \frac{c^3}{3} \sin 3\alpha + \frac{c^5}{5} \sin 5\alpha + \dots \infty$

7.  $\cos \alpha - \frac{1}{3} \cos 3\alpha + \frac{1}{5} \cos 5\alpha - \dots \infty$

8.  $1 + "C_1 \cos \alpha + "C_2 \cos 2\alpha + "C_3 \cos 3\alpha + \dots \infty$

9.  $x \sin \theta - x^2 \sin 2\theta + x^3 \sin 3\theta - \dots$  to  $n$  terms

10.  $1 + n \sin \theta + \frac{n(n-1)}{2!} \sin 2\theta + \frac{n(n-1)(n-2)}{3!} \sin 3\theta + \dots + \sin n\theta.$