1 Fundamental Concepts and Scope

1.1. Introduction

The subject matter of Mechanics of Solids consists in the study of mechanical behaviour of solid deformable bodies under the action of external forces taking into account the internal forces and deformations. It is based on the principles of the familiar Newtonian mechanics and the mechanical properties of materials and has been a subject of human interest since long. Although the use of the name Mechanics of Solids may be only about half a century old, the subject has been studied under the names Strength of Materials, Mechanics of Materials, Structural Mechanics, Theory of Elasticity and Plasticity, Structural Stability Theory etc. for a fairly long time.

Theory of Elasticity and Structural Stability Analysis have been subjects of interest of mathematicians and physicists and have been treated with mathematical rigour and general concepts involved in each topic have been developed. Strength of Materials, however, has been studied by practising engineers and academicians on the basis of simplifying assumptions and experimental results of the tests related to mechanical properties. The name Mechanics of Solids is of relatively recent origin and is a compromise between the two streams and is an outcome of the recent engineering community handling the subject giving it a sound analytical footing while developing the relations and formulae with simplifying assumptions. This is in fact, the modern trend in all sciences.

A structure or a machine has various components that may be subjected to one or more of the structural actions like axial pull or thrust, shear, bending and twisting. Each component must be provided with a proper size so that under the action of loads the stresses must not exceed the required strength for the components. Secondly, the deformations in each part must not exceed a permissible value in order to avoid misfit of components, cracks etc. in any part or distortion in any zone. This aspect is described as providing adequate stiffness. Thirdly, if any part or zone of the machine or structure is in compression, buckling may take place at stresses below the crushing strength. Hence the designer must see that this kind of deformation is avoided, that is to say, stability of the structure is ensured.

Lastly, if the loads are of dynamic nature, the components must possess the capacity to absorb a certain amount of energy without failure, which is said to be a measure of its toughness. The study of solid mechanics helps in prescribing the size of components of a machine or structure ensuring adequate strength, stiffness, stability and toughness with minimum weight of material used.

Mechanics of Solids has a vast scope in the engineering world and has an interdisciplinary character. A mechanical engineer requires it in machine design, a civil engineer in the design of structures, an aeronautical engineer in the design of aircrafts and naval engineers in the design of the ship. Metallurgists use it in the study of mechanical properties of metals and electrical engineers in the design of transmission towers etc. It also encompasses the studies in material science and physics of metals. The subject, though one of the oldest, has a vast scope of growth and expansion.

1.2. Basic Assumptions

In the study of solid mechanics the following assumptions are common to all topics.

(*a*) **The material of the body under consideration is continuous.** This assumption implies that the space occupied by the body is continuously filled by the material and there is no void i.e. matter of the body is continuously distributed over the volume and that no cracks or holes develop on the application of loads. Such a medium is called a "Continuum" hence sometimes the term continuum mechanics is used. This continuum hypothesis enables us to isolate an infinitesimal element of the body and form differential equations whose solution will apply to the whole body subject to boundary conditions.

In reality, however, there is no perfectly continuous body ; intergrannular voids do exist. But analysis based on perfect continuity gives error free results.

(*b*) **The distribution of material in the body is homogeneous.** According to this assumption distribution of material in the body is even and smallest element possesses the same physical properties as the entire body. Engineering materials like metals are close to perfect homogeneity. But analysis based on homogeneity for materials with even less homogeneous character like concrete, timber etc. do not lead to any appreciable error.

(*c*) **Material of the body is isotropic.** By isotropy is meant the same mechanical properties in all directions. Again metals, though crystalline, possess isotropy near to perfection, materials like timber due to fibrous character or concrete due to imperfect homogeneity are not perfectly isotropic. Still analysis based on isotropy gives results without appreciable error and the formulae based on this assumption are applicable in engineering practice.

(*d*) **The deformations are small.** Engineering analysis, in general, is based on small deformations so that geometry of the structure remains unchanged and the basic laws like linear relation between load and deformation and superposition principle etc. are applicable. However, if there is any deviation *i.e.* if any large deformations or displacements are to be considered a mention may be made and analysis carried out accordingly.

(*e*) **The body under consideration is free from internal forces before loading.** This assumption enables the analyst to start with the deformations and internal forces caused by the loading. However, there may exist internal forces due to molecular disturbance and interaction on account of uneven temperature distribution in metals, setting in concrete etc; but these effects may be ignored if small. An experimental determination may be required to include these effects if appreciable.

1.3. Types of Forces

Forces that act on a body may be broadly classified into (*a*) External forces and (*b*) Internal forces.

1.3.1. External forces. External forces are generated due to interaction between two bodies. These may either be surface forces or body forces. Surface forces are spread over an area of the surface and may be contact pressure between two solid bodies or pressure of a fluid on a solid body, frictional force etc. Body forces are distributed over the entire volume of the body and act on each particle *e.g.* gravitational force, magnetic force, inertia force, electrostatic force etc. Body force is expressed as force per unit volume. Surface forces are generally expressed as force per unit area. Resultant of surface forces spread over an area may be given as concentrated force or a concentrated force may be applied by spreading it over a small area.

The external forces or loads may be either static or dynamic in nature. Most of the topics covered in this text deal with static loads.

1.3.2. Internal Forces. A body is composed of particles called molecules. When external forces act on the body, forces of interaction develop between the particles to resist deformation caused by the external forces. The interaction between particles is equal and opposite and keeps them in equilibrium. As long as two particles are in contact, the forces of interaction between them is internal but if they are separated the interaction force acting on each becomes external force for it.

1.4. Method of Sections

When a solid body is subjected to a set of external loads the body undergoes deformation and internal resistance develops to balance the effect of external loads. To study the nature of internal forces, first a diagram showing the body acted upon by all the externally applied loads, weight of the body, reactions of supports etc. is made isolating it from the environment. Thus let a body shown in Fig. 1.1 (*a*) be subjected to loads $P_1, P_2, P_3 \ldots P_6$ which includes all the forces described above.

This sketch of the body acted upon by all the loads is called its free body diagram. Since the body is in equilibrium under the action of these forces, they will satisfy the equations of equilibrium.

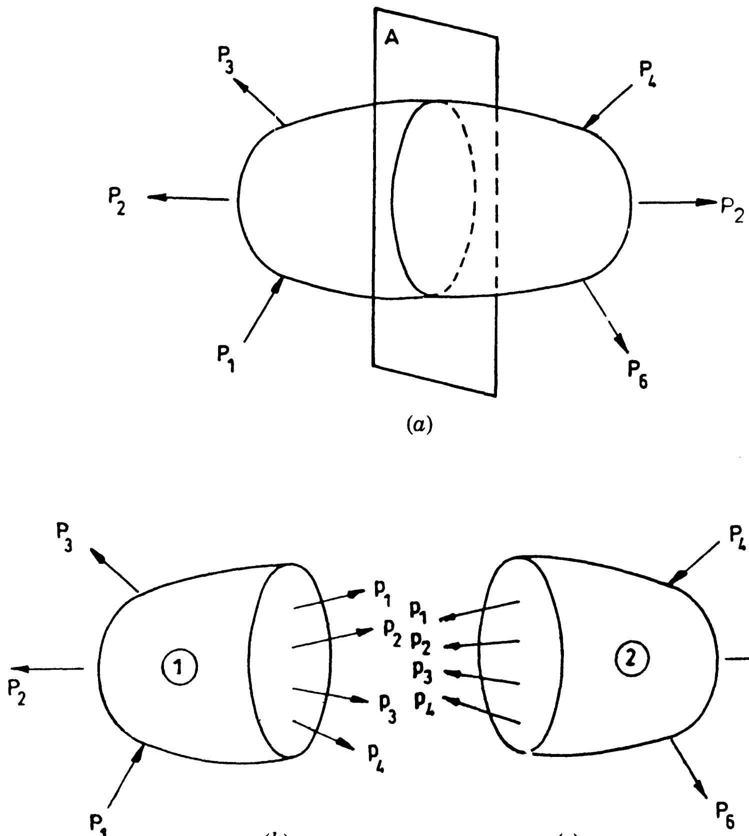
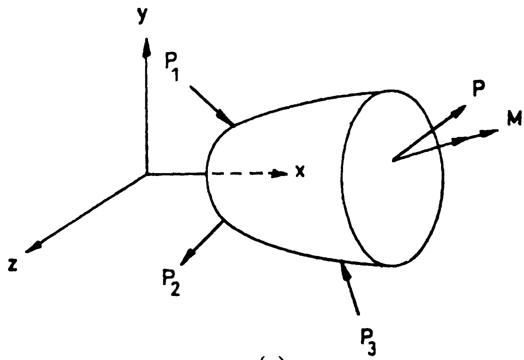


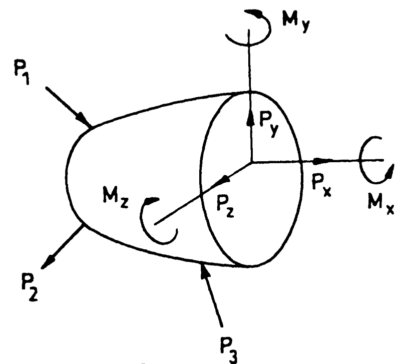
Fig. 1.1

To study the internal forces, we divide the body into two parts by passing an arbitrary plane through it. This separation exposes two identical faces as shown in Fig. 1.1 (b) and (c). Since the body is in equilibrium each part of it also must be in equilibrium. To keep any such part in equilibrium there must act some forces on the section Equal and opposite forces act on the exposed section of the other part. The external forces acting on each part are balanced by the internal forces which can be determined by the equations of equilibrium. This is known as the method of sections. The distribution of forces on an exposed section will result in general into a force and a moment which equilibrate the external loads.

It is convenient to take a section perpendicular to one of the co-ordinate axes x, y, or z. In Fig. 1.2, a section perpendicular to x-axis is shown with resulting force P and moment M. The components of P in x, y and z directions are P_x, P_y and P_z and the components of moment M about these axes are M_x, M_y and M_z respectively.



(a)



(b)

Fig. 1.2

These forces and moments can be determined by using the six equations of equilibrium, namely

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma F_z = 0$$
$$\Sigma M_x = 0, \quad \Sigma M_y = 0, \quad \Sigma M_z = 0 \qquad \text{...(1.1 } a\text{)}$$

In case of a planar problem these reduce to

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M_z = 0 \qquad \text{...(1.1 } b\text{)}$$

1.5. Concept of Stress

It has been shown in Art. 1.4 that a section of a body subjected to external loads has an internal force distribution over the area of the section. This force distribution is, in general, of varying magnitude and direction. It is desirable that the intensity of this force distribution on the various portion of the section be determined.

In Fig. 1.3 (*a*) is shown a part of a body to one side of a section *m–n* (say). The free body diagram of this part shows the external loads and the distribution of internal forces on section *m–n*. Let at a point *P* of this section the internal force over a small area ΔA be ΔP.

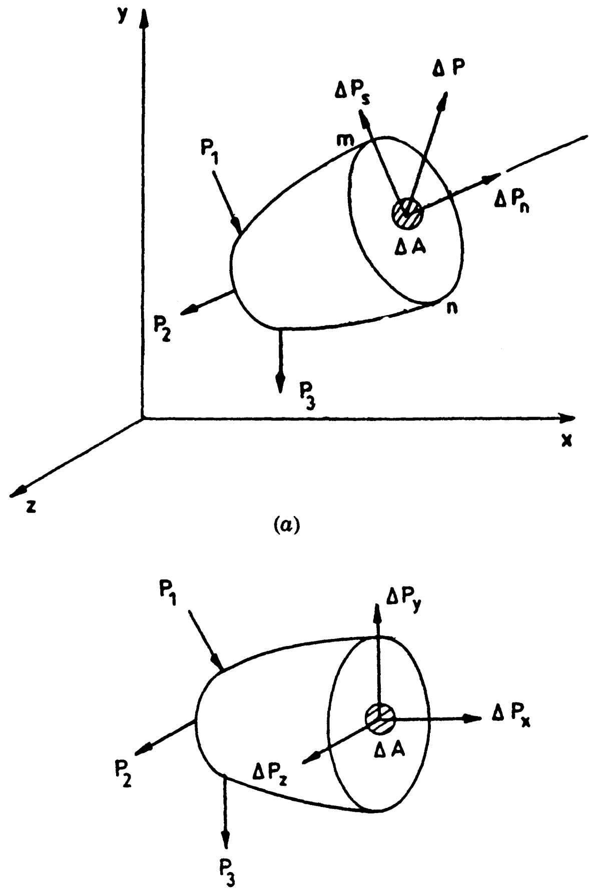


Fig. 1.3

Then stress at point P is defined as

$$\sigma = \underset{\Delta A \to 0}{\mathrm{Lt}} \frac{\Delta P}{\Delta A} \qquad \text{...(1.2)}$$

Here ΔP will in general be inclined to the normal to area ΔA. This stress will have two components-one along the normal, called the normal stress and the other tangential to the area, called the tangential or shear stress. If ΔP_n and ΔP_S be the normal and tangential components of ΔP, then

normal stress $$\sigma_{nn} = \underset{\Delta A \to 0}{\mathrm{Lt}} \frac{\Delta P_n}{\Delta A} \qquad \text{...(1.3 a)}$$

and shear stress $$\tau_{ns} = \underset{\Delta A \to 0}{\mathrm{Lt}} \frac{\Delta P_S}{\Delta A} \qquad \text{...(1.3 b)}$$

Here normal and shear stresses have been denoted by letters σ and τ respectively. The first suffix indicates the direction of the perpendicular to the plane on which the stress is acting and second suffix gives the direction of the stress.

In the perceding discussion the orientation of the plane was arbitrary. It is, however, convenient to cut sections perpendicular to coordinate axes x, y and z and determine the stresses on these. For example, in Fig. 1.3 (*b*) is shown a section perpendicular to x-axis. At point P the force on small area ΔA has components ΔP_x, ΔP_y and ΔP_z in x, y, z directions respectively and the stresses in these directions are given by

$$\sigma_{xx} = \operatorname*{Lt}_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A}, \quad \tau_{xy} = \operatorname*{Lt}_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A},$$
$$\tau_{xz} = \operatorname*{Lt}_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A} \qquad \text{...(1.4)}$$

The suffixes of the stresses follow the same pattern as explained. Each of the above stresses acts on the plane perpendicular to the x-axis, hence the first suffix x in each case indicates the plane on which the stress acts whereas the second suffixes are x, y, z respectively indicating the direction in which they act. Note that σ_{xx} is normal stress whereas τ_{xy} and τ_{xz} are shear stresses.

In a similar manner stresses on planes perpendicular to y and z axes can be given as (σ_{yy}, τ_{yx}, τ_{yz}) and (σ_{zz}, τ_{zx}, τ_{zy}) respectively. The nine components of stress at a point may thus be given by the matrix

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Some authors prefer to use a single letter for both normal and shear stresses ; such as

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

It is seen that repeated suffix appears in normal stresses and different suffixes in shear stresses. Sometimes a single suffix will be used for normal stress and double sufix for shear stress. Thus the stress components may be given as

$$\begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

This is also known as state of stress at a point and can be shown on the six faces of a rectangular element through the point as in Fig. 1.4.

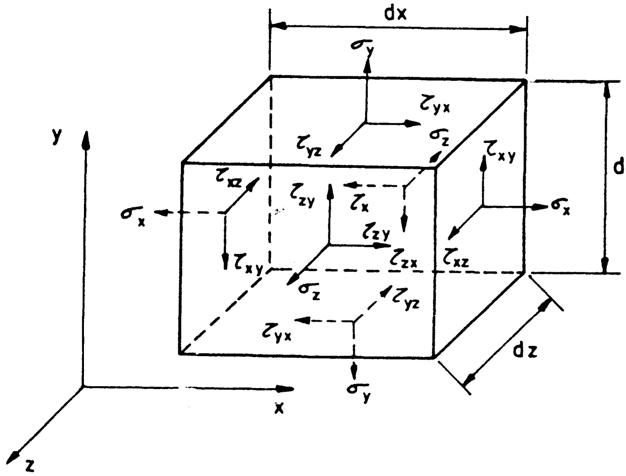


Fig. 1.4

Sometimes these stresses are given in the index form τ_{ij} $(i, j = x, y, z)$ which means that i and j can be designated by x, y and z in turn and permuted to give all the nine stress components at a point.

It will now be shown that the stress matrix is symmetric. For this we use the moment equilibrium equations about the axes passing through the centre of the element and parallel to the x, y, z axes. Taking sides of the element in Fig. 1.4 as dx, dy, dz along x, y, z axes and considering equilibrium of moments about a central axis parallel to z axis gives

$$\tau_{xy}\,(dy \cdot dz)\,dx - \tau_{yx}\,(dx \cdot dz)\,dy = 0$$

or $$\tau_{xy} = \tau_{yx} \qquad \text{...(1.5 } a)$$

In a similar manner it can be shown that

$$\tau_{yz} = \tau_{zy} \qquad \text{...(1.5 } b)$$

and $$\tau_{xz} = \tau_{zx} \qquad \text{...(1.5 } c)$$

Thus there remain only six independent stress component σ_x, σ_y, σ_z, $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$. This shows that the matrix representing the stress tensor* at a point is symmetric or suffixes of the shear stresses are interchangeable.

This result can be interpretted by means of a simplified two-dimensional case in which only shear stresses have been shown in Fig. 1.5. A

*A physical quantity obeying certain transformation law is called a tensor. A scalar quantity is a zero order tensor, a vector is a first order tensor and the stress matrix at a point represents a second order tensor.

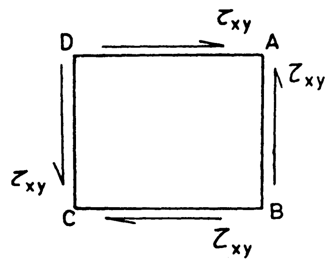


Fig. 1.5

shear stress τ_{xy} acting on one face, say AB, decides that for equilibrium the shear stresses on all the other faces will be the same and will act in the directions as shown. This is also known as the "principle of complementary shear". The arrowheads showing the direction of the shear stresses will always meet at two opposite corners and their tails on the other two opposite corners.

A three dimensional state of stress at a point can now be written as

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

and two dimensional state of stress in the x–y plane can be given by

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

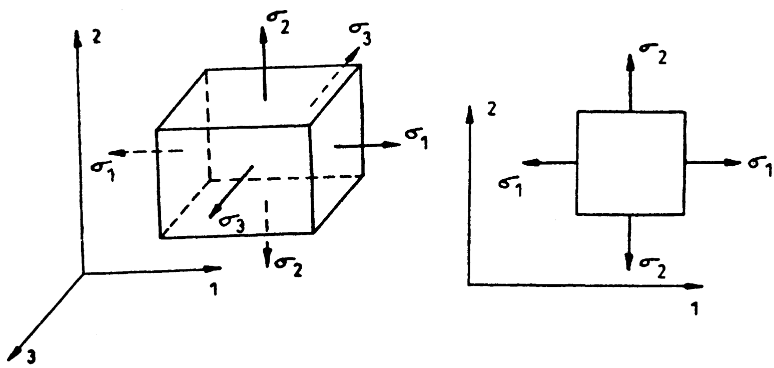
1.6. Principal planes and principal stresses

In describing the state of stress at a point we have seen that three stress components—one normal and two shear streses exist on each of the three planes perpendicular to the axes. A plane may be so oriented that only normal stress exists on it. Such a plane is called a principal plane and the normal stress acting on it is called principal stress. In fact three such planes exist at every point and normal stresses acting on them constitute the principal stresses at the point. It is sometimes convenient to orient the axes along the principal stress directions and deal with the stresses in terms of principal stresses only. The same may be done in case of two-dimensional stress system also. Principal stresses may be represented in the matrix form as

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

and are shown in Fig. 1.6 (*a*) and (*b*).

A detailed discussion on the two-dimensional case will be taken up in Chapter 8.



(*a*) Principal stresses in three-dimensional case.

(*b*) Principal stresses in two-dimensional case.

Fig. 1.6

1.7. Concept of strain

It has been discussed in the preceding articles that deformations take place when a set of external loads is applied on a body. Analogous to study of internal forces in terms of intensity of forces, we are interested in intensity of deformation or unit deformation. This sets out a concept known as strain. Extension of a line element per unit length is called longitudinal strain. Another type of strain that is encountered is unit rotation in a plane containing the line element and is defined by the change in right angle formed by the line element and a line perpendicular to it in the aforesaid plane. This change in right angle is called shear strain. A generalised mathematical description and analysis of strain will now follow.

1.7.1. Components of strain at a point. Let displacement components in the directions x, y, z at a point in a body be u, v, w* respectively. It is assumed that there is adequate restraint so that no rigid body displacement takes place and that the displacements are small. We consider an infinitesimal rectangular element *ABCD* of sides dx and dy in x–y plane as shown in Fig. 1.7. The element deforms into $A'B'C'D'$ and displacement of various points in the two coordinate directions are indicated.

Longitudinal strain in x-direction is

$$\varepsilon_{xx} = \frac{u + \frac{\partial u}{\partial x} dx - u}{dx} = \frac{\partial u}{\partial x} \qquad ...(1.7\ a)$$

Longitudinal strain in y-direction

$$\varepsilon_{yy} = \frac{v + \frac{\partial v}{\partial y} dy - v}{dy} = \frac{\partial v}{\partial y} \qquad ...(1.7\ b)$$

*Each of the displacements u, v and w is in general, function of x, y and z.

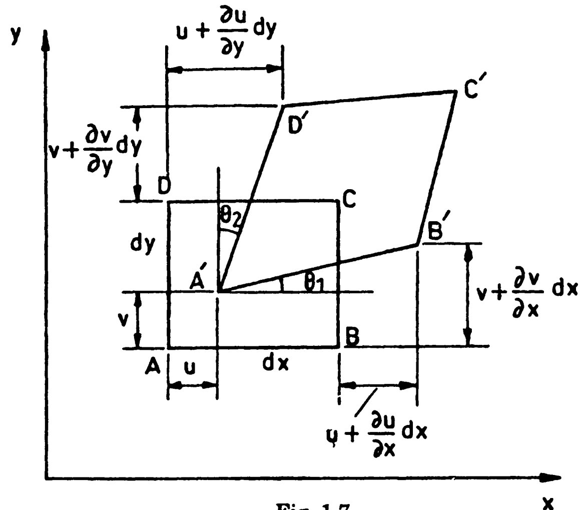


Fig. 1.7

The shear strain in the x–y plane is equal to the change in right angle at A and can be given by

Thus $$\gamma_{xy} = \gamma_{yx} = \theta_1 + \theta_2$$

$$= \frac{v + \frac{\partial v}{\partial x} dx - v}{dx} + \frac{u + \frac{\partial u}{\partial y} dy - u}{dy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad ...(1.7\ c)$$

The first suffix in the longitudinal strain gives the orientation of the line element and the second suffix is the direction in which strain is considered. Thus a longitudinal strain will always have repeated suffix. The two suffixes of the shear strain indicate the plane in which the shear strain takes place.

Analogously we can write

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \quad ...(1.7d)$$

$$\gamma_{xz} = \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad ...(1.7\ e)$$

and $$\gamma_{yz} = \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad ...(1.7\ f)$$

The suffixes of the shear strain are exchangeable, since two permutations give the same shear strain. The nine components of strain at a point (although only six independent ones) are given by the matrix

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_{zz} \end{bmatrix}$$

and are used by engineers as state of strain at a point. However, mathematicians use $\varepsilon_{xy} = \frac{\gamma_{xy}}{2}$, $\varepsilon_{xz} = \frac{\gamma_{xz}}{2}$ and $\varepsilon_{yz} = \frac{\gamma_{yz}}{2}$ as shear strains and thus

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_{yy} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_{zz} \end{bmatrix}$$

is the strain matrix used by them. The reason for this is that only in this form it can be used as a strain tensor. A rigorous explanation and discussion on this can be found in any of the textbooks on mathematical theory of elasticity.*

In further discussions only a single suffix for longitudinal or normal strains will be used such as ε_x, ε_y.

The planar strain at a point is given by $\begin{bmatrix} \varepsilon_x & \gamma_{xy} \\ \gamma_{xy} & \varepsilon_y \end{bmatrix}$

1.8. Principal Strains

Analogous to principal stresses, we can define the principal strain and their directions. The line segment along which there is only elongation or contraction and no rotation is called a principal direction and the longitudinal strain in this direction is called a principal strain. In a two-dimensional case there are only two principal directions and principal strains. The strain in the two cases may be given as

$$\begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$$

A detailed discussion on two-dimensional case will be taken up in Chapter 8.

1.9. Saint Venant's Principle

In 1855, the French mechanicist Barre de Saint Venant enunciated an important principle which states that distribution of stress or strain on sections of a body at a sufficient distance from the surface of application of a load system is independent of the manner of distribution of the load system. Thus any distribution of load can be replaced by a statically equivalent loading convenient for analysis of stresses or strains not in the near vicinity of loading. The term sufficient or appreciable distance, vaguely stated requires explanation. In practice it means a distance comparable to the size of the body. For example, in case of a bar it may be taken as three times its diameter or lateral dimension.

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*Mathematical Theory of Elasticity, SOKOLNIKOFF, I.S.

Two systems of loads or force are said to be statically equivalent when they result in the same force or couple or both.

When this principle is applied to the bending of a cantilever subjected to a bending moment applied at its end then the bending stress distribution on a section at a distance greater than its lateral dimensions will be independent of the distribution of the surface traction of which the bending moment is the resultant. Another simple example of

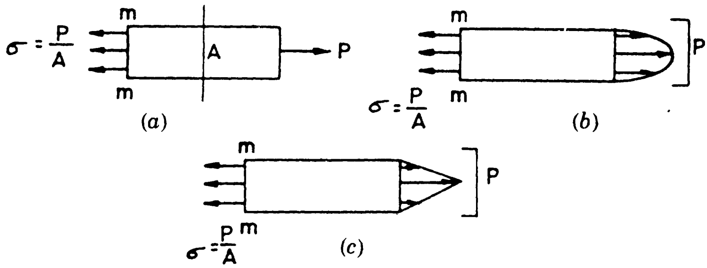


Fig. 1.8

application of St. Venant's Principle is the bar subjected to an axial load P. In Fig. 1.8 the end loads in (*a*), (*b*) and (*c*) are equivalent to an axial load P. If the cross-sectional area of the bar is at a distance greater than three times the dia. of the bar, say, then the resultant normal stress will be P/A uniformly distributed over the cross-section for any of the three manners of loading.

1.10. Constitutive Laws

Although stress and strain have been separately considered in defining and developing expressions for them, it may be logically concluded that there must exist some relationship between them. The relation between stress and strain depends upon the constitution of the material of the body and hence the laws governing the stress-strain relations are called constitutive equations or laws. The constitutive relations are closely related to the mechanical properties of the material which depends on the molecular structure and hence upon its microscopic behaviour and is subject matter of material science or solid state physics. In the mechanics of solids we are concerned with the average behaviour or the so called macroscopic manifestation of deformation under load. This is also referred as phenomonological approach to mechanical behaviour.

The various types of mechanical behaviour exhibited by engineering materials are briefly described below.

(*a*) Elastic Behaviour

If deformations disappear with the removal of the load and the body regains its original dimensions, the material is said to be elastic. This property is exhibited by most engineering materials upto a certain limit of load known as elastic limit. No material has been found to be perfectly elastic. Elastic behaviour may be of two types : linearly elastic and non-linearly elastic. In case of linear elasticity the load is proportional to displacement and is the basic statement of Hooke's Law, to be taken up separately. Non-linear elastic behaviour is described by a non-linear relationship between load and displacement or stress and strain.

(*b*) **Plastic Behaviour**

If the deformations set in a body are not recoverable after removal of the applied loads, the deformations are said to be plastic and the behaviour of the material is called plastic. Plasticity is exhibited by most of the materials beyond elastic limit. Different modes of plastic behaviour and related stress-strain diagrams will be discussed in the end of the chapter.

(*c*) **Viscous and Viscoelastic Behaviour**

If deformation in a material is such that load is a function of rate of deformation with respect to time rather than deformation and the deformation is permanent that is not recoverable on the removal of the load, the material is said to be viscous. At high temperature and under dynamic loading conditions, solid materials may exhibit such fluid type behaviour as in the case of Newtonian fluids.

A combination of viscous and elastic behaviour, that is, load being a function of both, deformation and time rate of deformation, is known as viscoelastic behaviour. Several metals and polymers exhibit this kind of behaviour in different manners and degree. Some of the special features of viscoelasticity are :

(*i*) **Anelasticity.** This behaviour also known as delayed elasticity is marked by deformations not being recovered instantaneously. In fact original undeformed shape is regained in infinite time.

(*ii*) **Stress Relaxation.** In this deformation remains constant while the load gradually decreases with time *i.e.* the stress relaxes hence the name of the phenomenon.

(*iii*) **Creep.** This is a phenomenon related to viscoelastic materials in which when the temperature is high compared to melting point of the material, the stretch continues or creeps at constant load or stress. The stretch or strain has two parts : elastic and viscous. The elastic strain disappears suddenly whereas the viscous strain is recovered gradually over an infinite time.

The stress-strain diagram under uniaxial loading for all the above cases will be discussed in Art. 1.13.

1.11. Principle of Superposition

This states that the resultant effect such as stress, strain, deflection or any other internal force factor in a linearly elastic body subjected to a number of forces is the algebraic sum of such effects when the forces are applied separately. The primary condition for this is that the deformations be small so that the geometry of the structure or the body is not altered and also the linear relationship between the stress and strain or load and deformation is maintained. The above postulate known as the principle of superposition also implies that the order in which the loads are applied is immaterial.

Fortunately deformations or displacements in most of the structures or bodies considered in engineering analysis are small and hence the principle of superposition is applicable. This serves as an important tool in analysing the structures subjected to a system of forces. If $P_1, P_2, P_3 ..., P_n$ be the forces and an effect such as deflection in a given direction at a point C separately due to these forces be δC_1, δC_2, $\delta C_3 ....$, δC_n respectively then the resultant deflection at point C in the same direction is given by $\delta C = \delta C_1 + \delta C_2 + \delta C_3 + ... + \delta C_n$.

1.12. Hooke's Law and Its General Form

Hooke's law is the constitutive law for a linear elastic material also called Hookean material. Robert Hooke, based on his experiments with springs under axial load pronounced that the force varies as the stretch (Ut tension sic vis) and also announced the notion of elasticity, a property due to which the body regains its original undeformed shape on removal of load. His above statement was later stated in the following form :

"In a linearly elastic body deformations are proportional to the force upto a certain limit". This is known as the Hooke's law and the load upto which the law is valid is known as the proportional limit.

The load displacement linear relationship can be given as

$$\delta = CP$$

where P is a force acting on a body constrained against any rigid body displacement, δ is displacement of a point and C is a constant depending on the material of the body, point at which the displacement is considered, direction of the displacement and direction of P. It, however, does not depend on the magnitude of P.

Hooke's law has also been used as stress-strain relationship for linear elastic materials according to which stress is proportional to strain upto a certain limit of stress known as proportional limit. Thus for uniaxial stress

$$\sigma = E\varepsilon \qquad ...(1.9)$$

where E is called Young's modulus or modulus of elasticity and is a constant for a given material. There are six components of stress and six components of strain existing at a point. The strains being small, the law of superposition can be extended to Hooke's law for a general state of stress in the form.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \dots & C_{16} \\ C_{21} & C_{22} & \dots & \dots & C_{26} \\ C_{31} & C_{32} & \dots & \dots & C_{36} \\ C_{41} & C_{42} & \dots & \dots & C_{46} \\ C_{51} & C_{52} & \dots & \dots & C_{56} \\ C_{61} & C_{62} & \dots & \dots & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \quad ...(1.10)$$

This is known as the generalised Hooke's law for an anistropic material. The simpler cases and the general case of three-dimensional stress in a linearly elastic isotropic material will be discussed in Chapter 2.

1.13. Stress-strain Diagrams and Constitutive Laws for Uniaxial Stress

As has been discussed in Art. 1.10 the constitutive relations depend on the mechanical behaviour of the material under loading. Although loading may be of many kinds, here we briefly discuss and classify the materials on the basis of stress-strain diagram under uniaxial tension and thus the relationship between the stress σ and the strain ε. The types of materials as classified in Art. 1.10 are taken up and both loading and unloading aspects are covered .

(*a*) **Elastic Material**

In an elastic material when the load ceases to act the deformations disappear and the body acquires its original shape and size. Although no material is perfectly elastic, most of the engineering materials exhibit elasticity upto a certain limit of stress called elastic limit. The stress-strain curves for linearly and non-linearly elastic materials are shown in Fig. 1.9 (*a*) and (*b*) respectively. The constitutive relation for linearly elastic material is

$$\sigma = E\varepsilon \quad ...(1.11\ a)$$

which is known as Hooke's law. For non-linearly elastic materials the relation is given as

$$\sigma = A\varepsilon^n \quad ...(1.11\ b)$$

where A and n are the constants depending on the material.

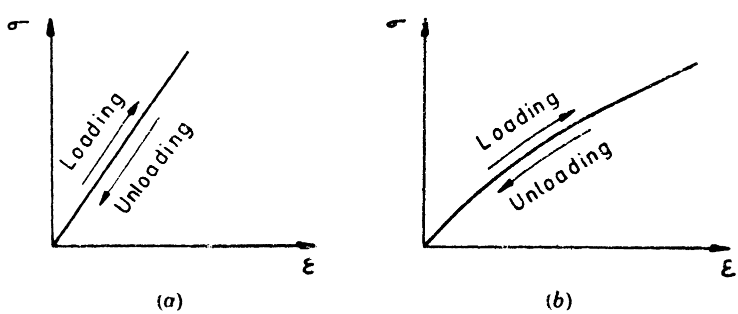


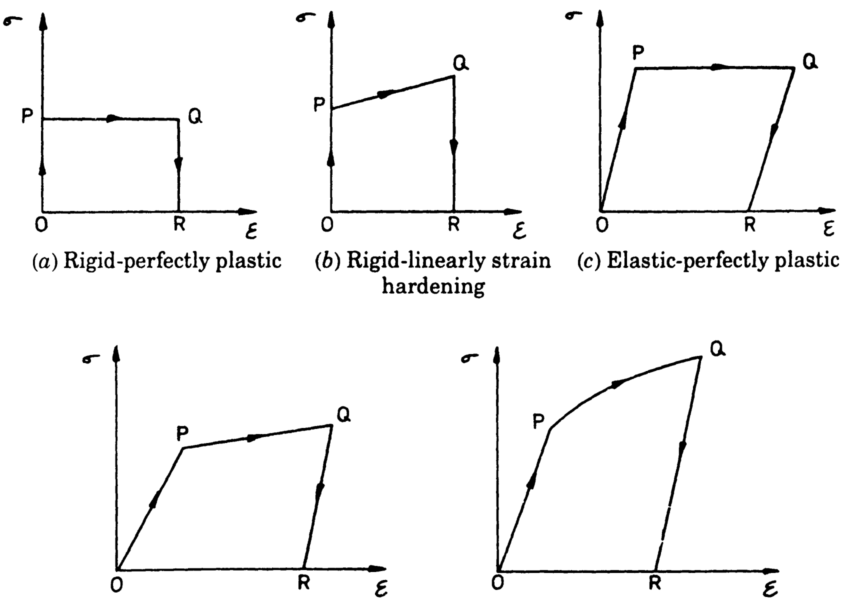
Fig. 1.9

In both the cases it is seen that the strain is completely recovered on unloading Fig. 1.9 (*a*) and (*b*)

(*b*) **Plastic Materials**

A material is said to be plastic when the deformations or strains caused remain as a permanent set and donot disappear on removal of the loads causing them.

Some materials have a small elastic deformation after which the plastic zone starts. If the small initial elastic deformation is ignored, the material may be treated as rigid and then plastic. Plastic deformation may be at constant stress or there may be stress hardening which is also sometimes called strain hardening. Rigid-perfectly plastic and Rigid-linearly strain hardening behaviour is shown by paths *OPQ* in Fig. 1.10 (*a*) and (*b*). If elastic zone is not ignorable, the behaviour is as



(*a*) Rigid-perfectly plastic (*b*) Rigid-linearly strain hardening (*c*) Elastic-perfectly plastic

(*d*) Elastic-linearly strain hardening (*d*) Elastic-nonlinearly strain hardening

Fig. 1.10

shown by *OPQ* in Fig. 1.10 (*c*) and (*d*), known as 'Elastic-perfectly plastic and Elastic-linearly strain hardening respectively. In Fig. 1.10 (*e*) the strain hardening is non-linear. Line *QR* is the recovery of strain on removal of stress which shows no recovery in (*a*) and (*b*) whereas slight elastic recovery in (*c*), (*d*) and (*e*). Analytical stress-strain equations for plastic behaviour are much involved and only idealised simple curves shown above are used by most practising engineers.

(*c*) **Viscoelastic Material**

In a viscous material the time rate of deformation is a function of the applied force. The deformation is not recoverable. Some of the solids exhibit viscous behaviour at high temperature or under dynamic loading condition. The stress-strain relationship for a viscous material may be given by

$$\sigma = \eta \frac{d\varepsilon}{dt} \qquad \text{...(1.12 } a\text{)}$$

as in the case of Newtonian fluids, the constant η being called the coefficient of viscosity. Some materials exhibit non-linear viscosity for which the constitutive law may be given as

$$\sigma = A\left[\frac{d\varepsilon}{dt}\right]^n \qquad \text{...(1.12 } b\text{)}$$

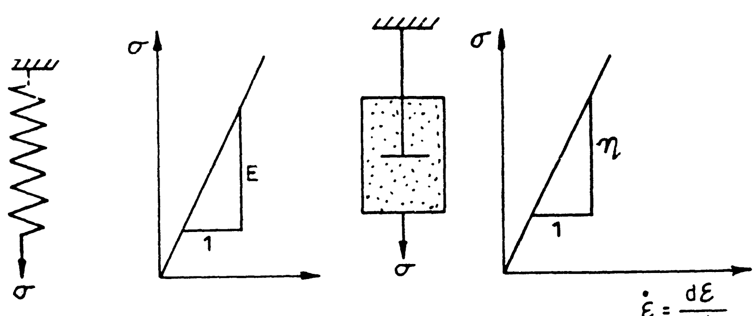
where A and n are constants.

Some engineering materials exhibit a combination of elastic and viscous behaviour which may be called a viscoelastic behaviour. The stress-strain relation or the so called constitutive relation for such material can be given in the simplest form as

$$\sigma = E\varepsilon + \eta \frac{d\varepsilon}{dt} \qquad \text{...(1.12 c)}$$

in which only linear viscosity has been taken into account. The viscoelastic behaviour is considered in detail in the subject known as Rheology* but here it will be treated only in brief. Mechanical models for elastic and viscous part are a Hookean spring and dash-pot respectively. The separate models and a number of their combinations are known as "rheological models" and will now be taken up for explaining the elastic, viscous and visoelastic behaviours.

In the absence of viscosity the elastic behaviour resembles that of a Hookean spring and the stress-strain curve is a straight line. Fig. 1.11 (a). For a purely viscous behaviour, the material follows the behaviour of the model represented by a dash-pot, Fig. 1.11 (b).



(a) (b)

Fig. 1.11

*For a detailed study on this topic the reader may like to refer to Rheology, Eirich, F.R. (Academic Press, New York, 1956).

For the study of viscoelastic behaviour the following models have been used :

(*i*) **Maxwell model**

In this a series combination of the Hookean spring and a linear dash-pot represents the material. If we use suffixes *s* and *d* for spring and dash-pot respectively, then for this model (Fig. 1.12 *a*)

$$\sigma_s = \sigma_d = \sigma$$

and $$\varepsilon = \varepsilon_s + \dot{\varepsilon}_d$$

Now for the spring, $$\dot{\varepsilon}_s = \frac{\dot{\sigma}_s}{E} = \frac{\dot{\sigma}}{E}$$

and for the dash-pot $$\dot{\varepsilon}_d = \frac{\sigma_d}{\eta} = \frac{\sigma}{\eta}$$

Hence $$\dot{\varepsilon} = \dot{\varepsilon}_s + \dot{\varepsilon}_d = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

or $$\dot{\sigma} + \frac{E}{\eta}\sigma = E\dot{\varepsilon} \qquad \text{...(1.13)}$$

When a constant stress σ_o is applied in the time range $t = 0$ to $t = t_1$, Fig. 1.12 (*b*), the strain response can be studied with the solution

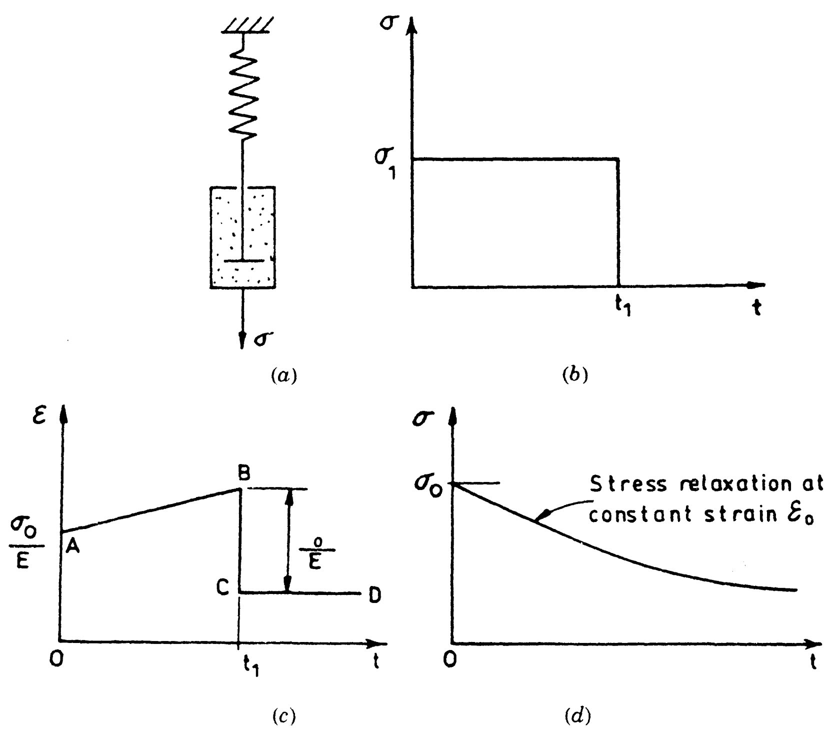


Fig. 1.12

of the governing Eqn. 1.13 which reduces to $\frac{d\varepsilon}{dt} = \frac{\sigma_o}{\eta}$ with initial condition $\varepsilon_o = \frac{\sigma_o}{E}$ and $\sigma = 0$. On integration, $\varepsilon = \frac{\sigma_o}{\eta} t + A$

where upon using the initial condition $A = \frac{\sigma_o}{E}$

Thus $$\varepsilon = \frac{\sigma_o}{E} + \frac{\sigma_o}{\eta} t \qquad ...(1.14)$$

This is shown in Fig. 1.12 (*c*).

The instantaneous strain is $\frac{\sigma_o}{E}$, it increases linearly with slope $\tan^{-1}\frac{\sigma_o}{\eta}$ and upon removal of stress at time t_1, the elastic strain $\frac{\sigma_o}{E}$ disappears and the viscous strain $\frac{\sigma_o}{\eta} t_1$ remains as permanent set. This is an illustration of an elementary creep problem.

In case of a constant strain ε_o, equation 1.13 modifies to $\frac{d\sigma}{dt} + \frac{E}{\eta}\sigma = 0$ with the initial condition that $\sigma = \sigma_o$ at $t = 0$.

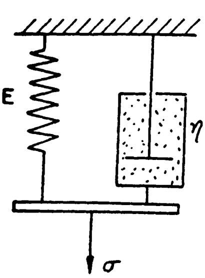
The solution is $\log_e \sigma = \frac{-E}{\eta} t + C_1$ or $\sigma = C_1 e^{-(E/\eta)t}$ and using the initial condition,

$$\sigma = \sigma_o e^{-(E/\eta)t} \qquad ...(1.15)$$

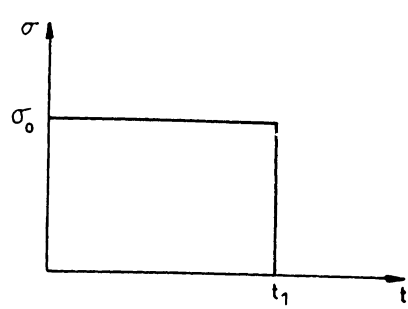
This is plotted in Fig. 1.12 (*d*) and describes a phenomenon known as stress-relaxation.

(*ii*) **Voigt-Kelvin Model**

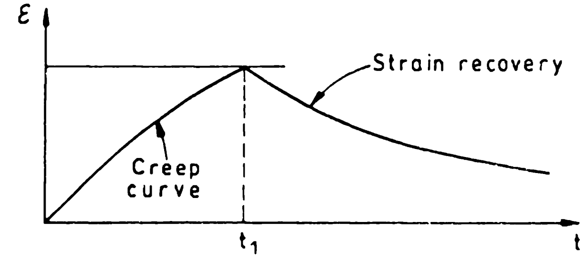
Another combination of the Hookean spring and the linear dashpot in parallel presents the Voigt-Kelvin model. Fig. 1.13(*a*). In this case the same strain $\varepsilon_s = \varepsilon_d = \varepsilon$ is induced in both elements and the total stress is given by



(*a*)



(*b*)



(c)
Fig. 1.13

$$\sigma = \sigma_s + \sigma_d$$

and the governing equation becomes

$$\frac{d\varepsilon}{dt} + \frac{E}{\eta}\varepsilon = \frac{\sigma}{\eta} \quad \text{...(1.16)}$$

Let a constant stress σ_o be applied in the time range $t = 0$ to t_1. Fig. 1.13 (*b*). Thus since $\sigma = \sigma_o$, a constant, the equation may be written as

$$\frac{d\varepsilon}{dt} + \frac{E}{\eta}\varepsilon = \frac{\sigma_o}{E}$$

The solution of which is $\varepsilon = Ce^{-(E/\eta)^t} + \dfrac{\sigma_o}{E}$

The initial strain being zero, the constant can be obtained from this condition.

$$0 = C + \frac{\sigma_o}{E}$$

or $$C = -\frac{\sigma_o}{E}$$

$\therefore$ $$\varepsilon = \left[1 - e^{-(E/\eta)\,t}\right]\frac{\sigma_o}{E} \quad \text{...(1.17)}$$

As is shown in Fig. 1.13 (*c*) the strain increases with time and becomes asymptotic to a horizontal line showing the maximum strain produced in the spring, a stage at which the spring takes all the applied load and the dashpot becomes inactive. When the stress is removed the strain decreases and the recovery curve becomes asymptotic to the time axis. Since the strain recovery is delayed, the property of the material is termed as delayed elasticity and the material is also called anelatic material.

(*iii*) **Standard Viscoelastic Model**

A combination of Maxwell model and Voigt-Kelvin model in series Fig. 1.14 (*a*) has been found to exhibit a more realistic behaviour of some metals at high temperature. Under a constant stress σ_o in the time range $t = 0$ to t_1 as shown in Fig. 1.14 (*b*), the strain response is described by Fig. 1.14 (*c*)

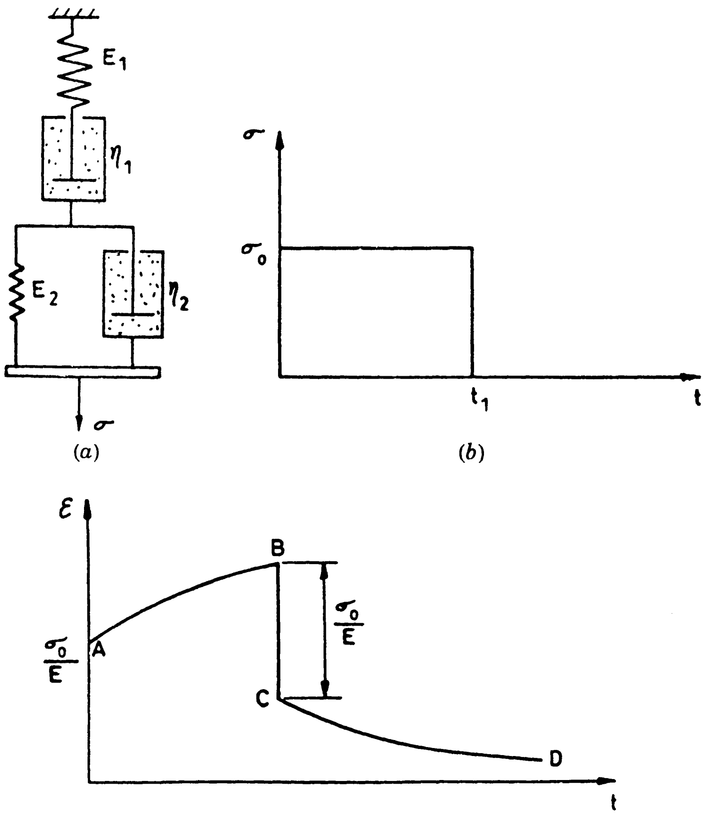


Fig. 1.14

OA is the initial strain, *AB* is the creep zone, *BC* is sudden elastic strain recovery whereas *CD* is the depiction of delayed elasticity in which strain recovery is time dependent and becomes asymptotic to the time axis.

1.14. The SI Units

An international organisation known as "The Conference Generale des Poids et Mesures" which governs all the matters related to metric system of units recommended in 1960 the 'Systeme Internationale de Unites' (The SI units) to be used the world over. This system of units which is an improved and refined version of metric system was approved by the 'International Organisation for Standardisation' in 1962. This is a logical and coherent system as all the derived units are obtained from the fundamental units by multiplication or division among the fundamental units without using any numerical factors.

The following physical quantities shown in Table 1.1 have been recognised as fundamental quantities and the units assigned to them as fundamental units.

Table 1.1

Physical Quantity	*The SI Units*	*Symbol*
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Luminous intensity	candela	Cd
Amount of substance	mole	mol
Plane angle	radian	rad
Solid angle	steradian	sr

Some of the derived units used in engineering practice are given in Table 1.2

Table 1.2

Physical Quantity	*Name of the Unit*	*Symbol*
Acceleration	metre per second squared	m/s^2
Area	metre squared	m^2
Angular acceleration	radian per second	rad/s
Density	kilogram per cubic metre	kg/m^3
Coefficient of linear expansion	per degree celcius	/°C
Current	ampere	A
Energy, work	joule, kilojoule, mega joule	J, kJ, MJ
	Watt hour, Kilowatt hour	Wh, kWh
Electrical charge	coulomb	C
Force	newton	N
Frequency	hertz	Hz
Moment	newton metre	Nm
Power	watt, kilowatt	W, kW
Pressure	Newton per metre squared	N/m^2
	bar	$10^5 N/m^2$
Potential difference	volt	V
Resistance	ohm	Ω
Resistivity	ohm metre	Ωm
Specific heat capacity	joule per kilogram degree celsius	J/kg °C
Surface tension	newton per metre	N/m
Temperature	kelvin	K
	degree celsius	°C
Torque	newton metre	Nm
Velocity	metre per second	m/s
Volume	cubic metre	m^3
Stress	newton per metre squared	N/m^2

Physical Quantity	*Name of the Unit*	*Symbol*
	Pascal	Pa
	mega pascal	MPa
Weight	newton	N
Weight density (Unit Weight)	newton per cubic metre	N/m^3
Elastic moduli E, G, K	newton per metre squared	N/m^2
Momentum	kilogram metre per second	kgm/s
Angular momentum (Moment of momentum)	kilogram metre squared per second	kgm^2/s
Impulse	newton second	Ns
Mass moment of inertia	kilogram metre squared	kgm^2
Area moment of inertia (Second moment of area)		m^4, mm^4
Section modulus	metre cubed, milimetre cubed	m^3, mm^3

Multiples and submultiples of S.I. units suggested above may sometimes be required to be used. For this, it is recommended that only the powers of 10 which are multiples of ± 3 be used. A list of such multiples and submultiples is given below in Table 1.3.

Table 1.3

Multiple / Submultiples	*Name*	*Symbol*
10^3	kilo	k
10^6	mega	M
10^9	giga	G
10^{12}	tera	T
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

Some examples of multiples or submultiples of SI units are :

$$
\begin{aligned}
kN/m^2 &= 10^3 N/m^2 \\
MN/m^2 &= 10^6 N/m^2 \\
MPa &= 10^6\ Pa \\
GN/m^2 &= 10^9\ N/m^2 \\
GPa &= 10^9\ Pa \\
kNm &= 10^3\ Nm \\
MW &= 10^6\ W\ (10^6 \text{watt}) \\
kJ &= 10^3\ J\ (\text{kilojoule}) \text{ etc.}
\end{aligned}
$$

1.15. Generalised Procedure for Solution

The solution of problems in mechanics of solids in general involves the following steps :

(*a*) The whole structure and the system of forces acting on it is idealised by making simplifying assumptions. The structure is approximated to that consisting of bars, plates, blocks and shells of simple form and the force systems to that consisting of concentrated or distributed loads of known pattern.

(*b*) A free body diagram of the whole structure is drawn and the reactions are determined with the use of equilibrium equations. In case of statically indeterminate problems additional equations arising out of compatibility considerations may be used.

(*c*) A part of the structure or an element of it may then be considered with the external and internal force factors (such as shear force, bending moment or direct force) acting on an exposed face. Since the whole body or structure is in equilibrium, such part or elements will also be in equilibrium.

In Fig. 1.15 (*a*), a structure is shown with the external loads and a particular kind of support system. A free body diagram of the structure is shown in Fig. 1.15 (*b*) and in Fig. 1.15 (*c*) and (*d*) f.b.d. of two parts of the structure are shown separately.

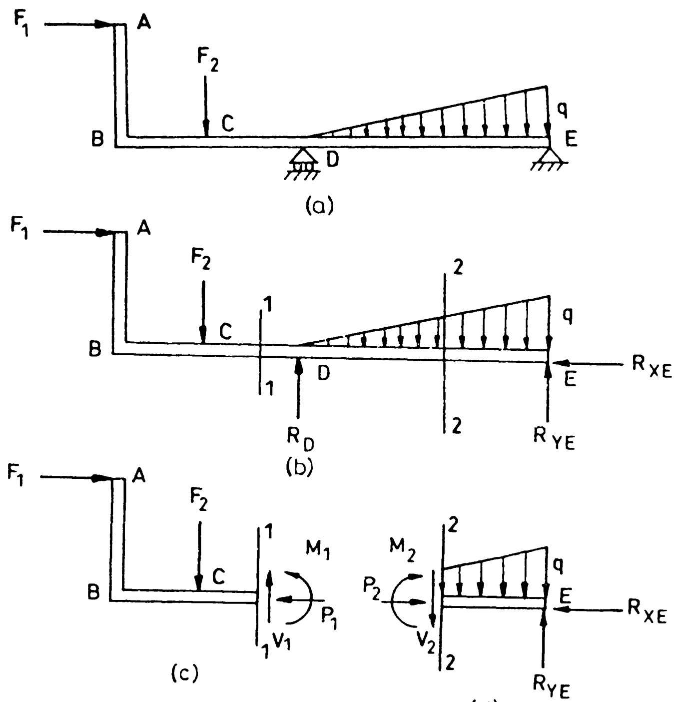


Fig. 1.15

(*d*) Having determined reactions and internal force factors (S.F., B.M., Torque etc.) at desired sections, stresses are determined using the formulae derived for a particular kind of problem.

(*e*) We may also be interested in the determination of displacement (*e.g.* slope and deflection in beams) for which differential equations of elements will be used, the constants of integration being determined from boundary conditions.

EXERCISE PROBLEMS

1.1. Discuss the subject matter of study in 'mechanics of solids' and briefly explain how it compares with the theory of elasticity and strength of materials in approach and rigour.

1.2. Elaborate on the concept of providing adequate strength, stiffness, stability and toughness while designing a structural member or a machine element.

1.3. What are the basic assumptions made in the analyses covered in mechanics of solids ?

1.4. What is continuum ? Is perfect continuity possible ?

1.5. Discuss the types of forces that act on a body.

1.6. Write an elaborate note on the method of sections.

1.7. Discuss the general state of stress at a point showing various components on a cubical element. What do the subscripts attached to the stresses stand for ?

1.8. Show that the stress matrix is symmetric or that the subscripts of shear stresses are interchangeable.

1.9. In a two-dimensional system explain the concept of complementary shear.

1.10. What do you mean by principal planes and principal stresses ?

1.11. Briefly explain the terms normal strain and shear strain.

1.12. Deduce expressions for components of strain at a point in terms of displacement components in the orthogonal coordinate directions.

1.13. Write the strain matrix usable in transformation equations. Write also the strain matrix containing only the principal strains in three-dimensional and two-dimensional cases.

1.14. State and explain St. Venant's principle. Show that the principle, although seemingly plausible, has proved to be an important tool for practical solutions.

1.15. Write short notes on (*i*) Generalised Hooke's Law, (*ii*) Principle of superposition, (*iii*) Constitutive Laws (*iv*) Method of sections, (*v*) Hooke's Law for elastic isotropic materials, (*vi*) Rheological models, (*vii*) Generalised procedure for solution of problems in solid mechanics and (*viii*) Purpose and scope of mechanics of solids.