CHAPTER 1

Theories of Failure

In practice, engineering materials have been observed to fail either by yielding or fracture. Yielding or permanent deformation is a pronounced sliding on planes through the crystalline grains of the material. It takes place without the actual rupture of the material. The functional utility for most machine parts is lost after a particular amount of yielding has taken place. Therefore, for all practical purposes yielding may be considered the criterion of failure for ductile materials. Fracture, on the other hand, is a failure in which separation occurs on a cross-section normal to the direction of tensile stress. The fracture criterion of failure is applicable to brittle materials. In practice, a limit of about 5 per cent elongation is usually taken as parting line between ductile and brittle materials.

For a machine part subjected to a uniaxial system of stress, the limiting allowable stress for design may be obtained from the mechanical testing of materials in simple tension. Usually the yield point stress is the deciding factor in such cases. But in majority of the cases, the parts are subjected to complex stress system and as such this simple approach is not applicable because the behaviour of material is greatly affected by the state of stress, type of loading, heat treatment process etc. Therefore, it is important to establish criterion for behaviour of materials under combined state of stress.

In a simple tensile test when specimen just starts yielding the following quantities are attained simultaneously :

(*a*) The principal stress $\sigma_{\max}$ reaches the yield point stress σ_{yp} or σ_{ult} of the material.

(*b*) The maximum shear stress $\left[\tau_{max} = \frac{\sigma_{max}}{2}\right]$ reaches the yield point stress in shear $\tau_{yp} = \frac{\sigma_{yp}}{2}$.

(*c*) The tensile strain ε reaches the yield point strain ε_{yp}.

(*d*) The total strain energy U absorbed by the unit volume of material reaches the value $U_{yp} = \frac{1}{2E} \cdot \sigma_{yp}^2$.

(*e*) The strain energy of distortion U_d absorbed per unit volume of material reaches a value $U_{dyp} = \frac{1+\mu}{3E}\sigma_{yp}^2$.

(*f*) The octahedral shearing stress reaches the value

$$\tau_{0yp} = (\sqrt{2/3})\sigma_{yp} = 0.47\ \sigma_{yp}.$$

In case of multi-axial state of stress the above values will not be attained simultaneously and as such it is of utmost importance in design to choose any one of the above quantities to calculate the limiting load which will not cause the inelastic action in the material with greatest possible economy.

We will now discuss, one by one, the theories of failure based on the above quantities.

1.1 MAXIMUM PRINCIPAL STRESS THEORY

This theory put forward by Rankine asserts that failure or fracture of a material occurs when the maximum principal stress at a point in a complex system attain a critical value regardless of the other stresses. The critical value of stress σ_{ult} is usually determined in a simple tensile test, where the failure of a specimen is defined to be due to either excessively large elongation or fracture, usually the latter is implied.

For complex stress system the major principal stress

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= \sigma_{ult} \text{ in simple tension} \qquad \text{...(1.1)}$$

The maximum principal stress theory can be represented graphically by a square *ABCD* (Fig. 1.1) the sides being defined by

$$\frac{\sigma_1}{\sigma_{ult}} = \pm 1 \text{ and } \frac{\sigma_2}{\sigma_{ult}} = \pm 1 \qquad \text{...(1.2)}$$

Failure occurs if point falls on the periphery of the square *ABCD*.

Experimental work indicates that this theory gives quite good results for brittle materials in all ranges of stresses, provided that both the principal stresses are of tensile nature. Failure is by fracture in such cases.

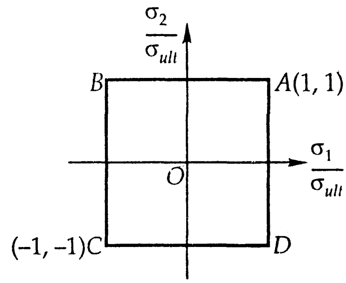

Fig. 1.1. Graphical representation of maximum principal stress theory.

1.2 MAXIMUM SHEARING STRESS THEORY

This theory by Guest and Tresca is based on the observation that in ductile material slipping occurs during yielding along critically oriented planes. It is assumed in this theory that the maximum stress alone produce inelastic action and that the equal tensile stresses ($\sigma_1 = \sigma_2$) have no influence in starting inelastic action. This implies that failure will occur when the maximum shearing stress τ_{max} in the complex system reaches the value of the maximum shearing stress in simple tension at the yield point.

Assuming biaxial stress system as shown in Fig. 1.2 (*a*), we have

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \tau_{yp}$$

$$= \frac{\sigma_{yp}}{2} \text{ in simple tension} \quad \ldots(1.3a)$$

or $\sigma_1 - \sigma_2 = \sigma_{yp}$ when σ_1 and σ_2 are tensile ...(1.3*b*)

or $-(\sigma_1 - \sigma_2) = \sigma_{yp}$ when σ_1 and σ_2 are compressive ...(1.3*c*)

Equations 1.3 (*b*) and (*c*) may be combined together as given below

$$\sigma_1 - \sigma_2 = \pm\, \sigma_{yp} \quad \ldots(1.3d)$$

or

$$\frac{\sigma_1}{\sigma_{yp}} - \frac{\sigma_2}{\sigma_{yp}} = \pm 1 \quad \ldots(1.4)$$

When $\sigma_2 = 0,\ \sigma_1 = \pm\, \sigma_{yp}$

and when $\sigma_1 = 0,\ \sigma_2 = \pm\, \sigma_{yp}$.

Equation (1.4) represents straight lines in II and IV quadrants and may be expressed graphically as shown in Fig. 1.3.

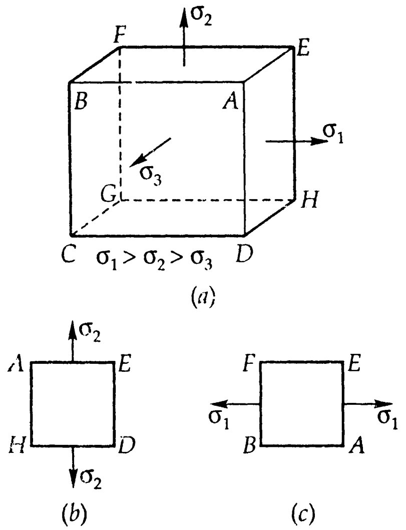

Fig. 1.2

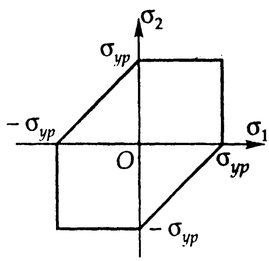

Fig. 1.3. Graphical representation of maximum shearing stress theory.

Now consider plane *AEHD* as shown in Fig. 1.2 (*b*). For yielding to occur in this plane

$$\pm \sigma_2 = \sigma_{yp} \qquad \text{...(1.5}a\text{)}$$

Similarly yielding to occur in plane *ABFE*

$$\pm \sigma_1 = \sigma_{yp} \qquad \text{...(1.5}b\text{)}$$

Equations 1.5 (*a*) and (*b*) represent straight lines as shown graphically in quadrants I and III of Fig. 1.3.

If a point having co-ordinates as σ_1 and σ_2 lies inside the hexagon of Fig. 1.3, it may be presumed that no yielding of material has occurred. When this point falls on the periphery of hexagon, one should take it for granted that the material has undergone inelastic deformation.

If the state of stress consists of triaxial tensile stress of nearly same magnitude, shearing stress in such a case will be of very small magnitude and failure would be by fracture rather than by yielding and hence maximum principal stress theory should be applied.

The maximum shearing stress theory gives fairly good results for ductile materials and for state of stress in which comparatively large shearing stresses are developed. However, for the pure shear as in torsion test, where maximum shear stress is developed, the shearing elastic limit of ductile metals is on an average found to be 0.57 of the tensile elastic limit. Hence in such cases, the maximum shearing stress theory gives results on the positive side.

1.3 MAXIMUM STRAIN THEORY

This theory suggested by St. Venant states that yielding at a point in a material begins when the maximum strain corresponding to a particular complex state of stress exceeds the strain corresponding to the yield point. If σ_1 and σ_2 are the two principal stresses ($\sigma_1 > \sigma_2$), then the strain in the direction of σ_1 is given by

$$\varepsilon_2 = \frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E} \qquad \text{...(1.6)}$$

The limiting value of ε_1 should not be more than $\frac{\sigma_{yp}}{E}$ in simple tension. Hence we may write

$$\frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E} = \frac{\sigma_{yp}}{E} \quad \text{or} \quad \sigma_1 - \mu\sigma_2 = \sigma_{yp}.$$

In Eq. (1.7) if σ_1 and σ_2 both are tensile then σ_1 can be higher than σ_{yp} but if σ_2 is compressive then σ_1 will have a value smaller than σ_{yp}. Therefore, in the former case, according to this theory, σ_1 can be increased beyond σ_{yp} without causing yielding in the material.

Maximum strain theory is an improvement over the maximum principal stress theory, even then it doesn't give satisfact-ory results for ductile materials. It is primarily used in cases where failure occurs by brittle fracture.

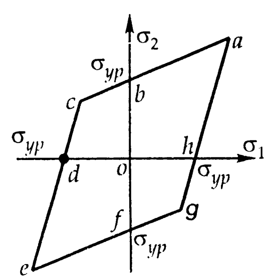

Fig. 1.4. Graphical representation of maximum strain theory.

This theory is represented graphically as shown in Fig. 1.4, where the different portions of the graphs are governed by the equations as given below :

$$\sigma_2 - \mu\sigma_1 = \sigma_{yp} \quad \text{for } ab$$
$$\sigma_1 - \mu\sigma_2 = \sigma_{yp} \quad \text{for } ah$$
$$\sigma_1 - \mu\sigma_2 = -\sigma_{yp} \quad \text{for } ed$$
$$\sigma_2 - \mu\sigma_1 = -\sigma_{yp} \quad \text{for } ef.$$

For unlike stresses, we have

$$\sigma_2 + \mu\sigma_1 = \sigma_{yp} \qquad \text{for } cb$$
$$\sigma_1 + \mu\sigma_2 = -\sigma_{yp} \qquad \text{for } cd$$
$$\sigma_1 + \mu\sigma_2 = \sigma_{kp} \qquad \text{for } gh$$
$$\sigma_2 + \mu\sigma_1 = -\sigma_{yp} \qquad \text{for } gf.$$

1.4 TOTAL STRAIN ENERGY THEORY

This theory proposed by Haigh states that inelastic action or yielding at a point in a material begins only when the energy per unit volume absorbed at a point is equal to the energy under uniaxial state of stress as in the case of simple tensile test. Thus in this case failure doesn't depend on the state of stress but governed by the energy stored in the material per unit volume.

Let us consider triaxial stress system ($\sigma_1 > \sigma_2 > \sigma_3$). For this state of stress, we have

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] \\ \varepsilon_2 &= \frac{1}{E}[\sigma_2 - \mu(\sigma_1 + \sigma_3)] \\ \varepsilon_3 &= \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)] \end{aligned}\right\} \qquad \text{...(1.8)}$$

Strain energy per unit volume can be expressed as

$$U = \tfrac{1}{2}\sigma_1\varepsilon_1 + \tfrac{1}{2}\sigma_2\varepsilon_2 + \tfrac{1}{2}\sigma_3\varepsilon_3 \qquad \text{...(1.9}a\text{)}$$

Substituting strain in terms of stresses from Eq. (1.8), we get

$$U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \qquad \text{...(1.9}b\text{)}$$

For biaxial stress system put $\sigma_3 = 0$ and the strain energy expression is modified as under.

$$U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2] \qquad \text{...(1.9}c\text{)}$$

For uniaxial stress system at yield point, we have

$$\sigma_2 = \sigma_3 = 0 \quad \text{and} \quad \sigma_1 = \sigma_{yp}$$

$$\therefore \qquad U_{yp} = \frac{\sigma_{yp}^2}{2E} \qquad \text{...(1.9}d\text{)}$$

Thus for failure by total strain energy theory, expressions (1.9*b*) or (1.9*c*), as the case may be, should be equated to (1.9*d*). For triaxial stress system, we have

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_{yp}^2 \qquad \text{...(1.10}a\text{)}$$

For biaxial stress system, we have

$$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \sigma_{yp}^2 \qquad \text{...(1.10}b\text{)}$$

Equation (1.10*b*) represents an ellipse with major and minor principal axes at 45° to σ_1 and σ_2 axes.

for $\sigma_2 = 0$, $\sigma_1 = \pm\,\sigma_{yp}$

for $\sigma_1 = 0$, $\sigma_2 = \pm\,\sigma_{yp}$

for $\sigma_1 = \sigma_2 = \sigma$, $\sigma = \pm\dfrac{\sigma_{yp}}{\sqrt{2(1-\mu)}}$

Assuming $\mu = \frac{1}{3}$, $\sigma = 0.866\sigma_{yp}$.

If $\sigma_1 = \sigma$ and $\sigma_2 = -\,\sigma$ as in the case of unlike stresses, we have

$$\sigma = \pm\frac{\sigma_{yp}}{\sqrt{2(1+\mu)}}$$

For $\mu = \frac{1}{3}$, $\sigma = \pm\,0.613\sigma_{yp}$.

The plot of this ellipse representing total strain energy theory of failure is shown in Fig. 1.5.

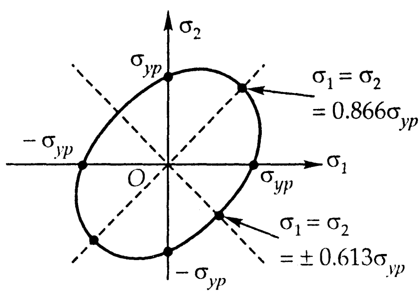

Fig. 1.5. Graphical representation of maximum strain energy theory.

1.5 MAXIMUM DISTORTION ENERGY THEORY

The total strain energy of a body consists of two parts, one associated with the volumetric changes in the body and the other due to change in shape or distortion. According to this theory by Von Mises and Hencky the inelastic action or yielding at any point in a body under any combination of stress begins only when the strain energy of distortion per unit volume absorbed at the point is equal to the strain energy of distortion per unit volume corresponding to the yield point stress in simple tension test.

For a triaxial state of stress at a point, we have total strain energy from Eq. 1.9 (*b*).

$$U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \qquad \text{...(1.9}b\text{)}$$

Neglecting small quantities of second and third order, the volumetric strain may be expressed as

$$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \qquad \text{...(1.11}a\text{)}$$

Substituting from Eq. (1.8), we get

$$\varepsilon_v = \frac{1-2\mu}{E}(\sigma_1 + \sigma_2 + \sigma_3) \qquad \text{...(1.11}b\text{)}$$

Equation 1.11 (*b*) states that volumetric strain is proportional to the summation of the three principal stresses. If this summation is zero, the volume change vanishes and the body is subjected to only distortion.

Therefore, the condition for zero volumetric change is

$$\sigma_1 + \sigma_2 + \sigma_3 = 0 \qquad \text{...(1.12)}$$

If $\sigma_1 = \sigma_2 = \sigma_3 = p$, then there will be no distortion in the body. We have from the above

$$p = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

The triaxial stress system of Fig. 1.6 (*a*) is equivalent to the superimposed stresses of Fig. 1.6 (*b*) and (*c*). Thus we have

$$\sigma_1 = p + \sigma_1' \,;\ \sigma_2 = p + \sigma_2' \quad \text{and} \quad \sigma_3 = p + \sigma_3'.$$

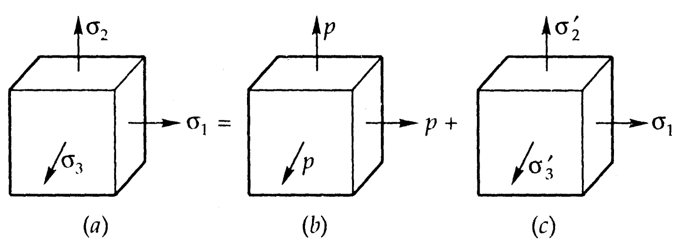

Fig. 1.6

Adding all the three principal stresses of Fig. 1.6, we have

$$\sigma_1 + \sigma_2 + \sigma_3 = 3p + \sigma_1' + \sigma_2' + \sigma_3'$$

or

$$\sigma_1' + \sigma_2' + \sigma_3' = 0$$

Which shows that the state of stress in Fig. 1.6 (*c*) will cause only distortion and no volumetric change.

Let us put $\sigma_1 = \sigma_2 = \sigma_3 = p$ in Eq. (1.9*b*) for the total strain energy, which will result in the expression for volumetric strain energy.

$$U_v = \frac{3(1-2\mu)}{2E} p^2 \qquad \text{...(1.13a)}$$

Since $$p = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

we have no substitution

$$U_v = \frac{1-2\mu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 \qquad \text{...(1.13b)}$$

The distortion energy may now be obtained by subtracting Eq. (1.13*b*) from Eq. (1.9*b*).

$$\begin{aligned} U_d &= U - U_v \\ &= \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \\ &\qquad - \frac{1-2\mu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 \\ &= \frac{1+\mu}{3E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1] \qquad \text{...(1.14a)} \\ &= \frac{1+\mu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \qquad \text{...(1.14b)} \end{aligned}$$

Distortion energy for uniaxial state of stress, as in simple tension test, at yield point may be obtained by putting

$$\sigma_2 = \sigma_3 = 0$$

and $$\sigma_1 = \sigma_{yp} \text{ in Eq. (1.14a)}$$

$$Ud_{yp} = \frac{1+\mu}{3E}\sigma_{yp}^2 \qquad \text{...(1.14c)}$$

Then the condition for yielding according to maximum distortion energy theory is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_{yp}^2 \qquad \text{...(1.15a)}$$

For plane stress $\sigma_3 = 0$ and Eq. (1.15*a*) be may written as under

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_{yp}^2 \qquad \text{...(1.15b)}$$

This is an equation for ellipse. It may be plotted as shown in Fig. 1.7.

A good agreement has been found between the maximum distortion energy theory and experimental results for ductile materials. The maximum principal stress theory appears to be the best for brittle materials but can be unsafe for ductile materials.

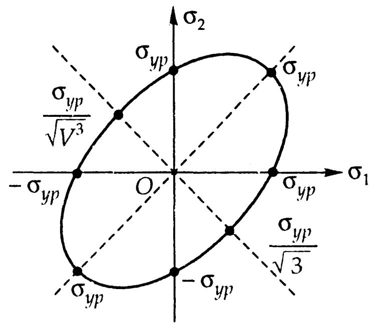

Fig. 1.7. Graphical representation of maximum distortion energy theory.

If one of the principal stresses, at a point is large in comparison with the other, all theories lead practically to the same results. The discrepancy between the theories is greatest in the second and fourth quadrants, when both principal stresses are numerically equal.

1.6 OCTAHEDRAL SHEARING STRESS THEORY

Eichinger has shown that the condition for yielding as given by the maximum distortion energy theory [Eq. (1.15*a*)] may also be obtained by considering the shearing stress acting on an octahedral plane such as *ABC* in Fig. 1.8.

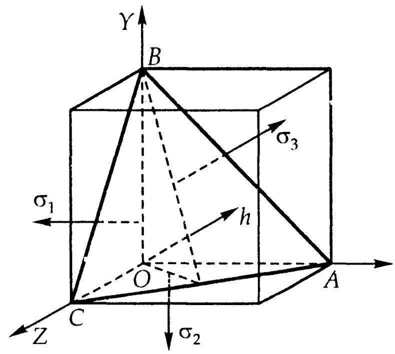

Fig. 1.8

σ_1, σ_2 and σ_3 are the principal stresses acting on an element. From geometry it can be seen that the cosine of the angle between the normal n to the octahedral plane and the co-ordinate axes x, y and z is equal to $1/\sqrt{3}$. If S is the unit resultant stress acting on the octahedral plane and S_n is the normal component of the stress S, shearing stress (τ_0) on the octahedral plane is given as

$$\tau_c = \sqrt{S^2 - S_n^2} \quad ...(1.16)$$

Resolving the resultant unit stress S into three components S_x, S_y, S_z and considering equilibrium, of the octahedral element we find

$$S_x = \frac{\sigma_1}{\sqrt{3}}, \quad S_y = \frac{\sigma_2}{\sqrt{3}}, \quad S_z = \frac{\sigma_3}{\sqrt{3}} \qquad ...(1.17)$$

and the resultant stress acting on the octahedral plane is given as

$$S = \sqrt{S_x^2 + S_y^2 + S_z^2}$$

$$= \frac{1}{\sqrt{3}}\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \qquad ...(1.18)$$

The normal component S_n of the stress S may be obtained by resolving S_x, S_y and S_z [Eq. (1.17)] in the direction of the normal n, which gives

$$S_n = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad ...(1.19)$$

Substituting S and S_n from Eqs. (1.18) and (1.19) in Eq. (1.16), the shearing stress on the octahedral plane is obtained as

$$\tau_0 = \tfrac{1}{3}\sqrt{3(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - (\sigma_1 + \sigma_2 + \sigma_3)^2}$$

$$\tau_0 = \tfrac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \qquad ...(1.20)$$

The octahedral shearing stress theory gives the same results as the distortion energy theory and hence may be called as equivalent stress theory.

On comparing Eq. (1.20) with Eq. (1.15*a*) it can be seen that the condition for yielding based on the distortion energy theory is equivalent to the statement that yielding begins when the octahedral shear stress reaches a critical value equal to

$$(\tau_0)_{cr} = \frac{\sqrt{2}}{3}\sigma_{yp} = 0.47\sigma_{y.p} \qquad ...(1.21)$$

Thus an octahedral stress theory may be stated as follows :

Inelastic action at any point in a body under any combination of stresses begins only when the octahedral shearing stress τ_0 becomes equal to $0.47\sigma_{yp}$ where σ_{yp} is the yield strength of material as determined from the standard tension test.

The octahedral shearing stress theory of failure enables us to apply the distortion energy theory of failure by dealing only with stresses instead of dealing with energy. This procedure seems to be more convenient because stress is a more familiar quantity in engineering design than the energy.

1.7 GRAPHICAL COMPARISON OF THEORIES OF FAILURE

Figure 1.9 shows the graphical comparison of different theories of failure discussed above, for a material having the same yield point stress in tension and compression and subjected to two dimensional stress system.

$$\sigma_3 = 0$$

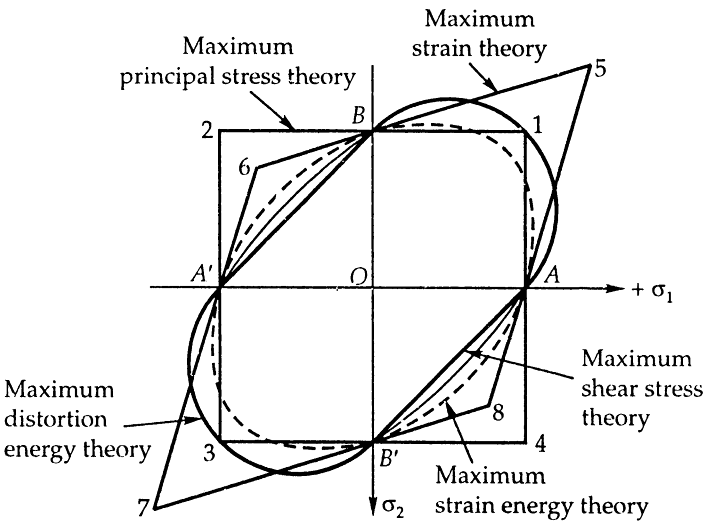

Fig. 1.9

The curves in Fig. 1.9 represent the limiting values of σ_1 and σ_2 according to various theories of failure at which yielding begins. The maximum stress theory is represented by the square 1234 and the maximum strain theory is represented by rhombus 5678. Since tension in one direction reduces the strain in the perpendicular direction, the strain theory indicates that two equal tensions will cause yielding at much higher value (point 5) then indicated by the maximum stress theory (point 1). The co-ordinates of the point 5 from Eq. (1.7) are

$$\sigma_1 - \mu\sigma_2 = \sigma_{yp}$$

or
$$\sigma_1 - \mu\sigma_1 = \sigma_{yp}$$

or
$$\sigma_1 = \sigma_2 = \sigma_{yp}/(1 - \mu)$$

for point 5 the two principal stresses are equal in magnitude and sign. In case of unlike stresses, the maximum strain theory indicates yielding begins at points 6 and 8 which have co-ordinates equal to $\sigma_{yp}/(1 + \mu)$. The values of stresses at these points are, therefore, lower than those indicated by the maximum stress theory (points 2 and 4).

The irregular hexagon *A*1*B*4′3*B*′*A*, which is constructed on the basis of Eq. (1.3*d*) represents the maximum shearing stress theory. This theory coincides with the maximum stress theory whenever both principal stresses have the same sign, but there is a considerable difference when the principal stresses have opposite sign.

By plotting Eq. (1.10*b*), we obtain the ellipse shown in Fig. 1.9. The ellipse deviates by only a comparatively small amount from the hexagon representing the maximum shearing stress theory.

ILLUSTRATIVE PROBLEMS

Example 1.1. *The major principal stress on an element of a steel member is 2000 kg/cm² and the minor principal stress is compressive. If tensile yield point of steel is 3000 kg/cm², find the minor principal stress at which failure will occur, according to following theories of failure :*

(*a*) *Maximum strain theory*

(*b*) *Maximum shearing stress theory*

(*c*) *Maximum strain energy theory*

(*d*) *Maximum distortion energy theory*

$$\mu = 0.25.$$

Solution. (*a*) For maximum strain theory of failure, the criteria of failure is given by Eq. (1.7)

$$\sigma_1 - \mu\sigma_2 = \sigma_{yp}$$

for σ_2 compressive this equation becomes

$$\sigma_1 + \mu\sigma_2 = \sigma_{yp}$$

or
$$\sigma_2 = (\sigma_{yp} - \sigma_1)/\mu$$
$$= (3000 - 2000)/0.25 = 4000 \text{ kg/cm}^2.$$

(*b*) For maximum shearing stress theory, governing equation is

$$\sigma_1 + \sigma_2 = \sigma_{yp}$$

or
$$\sigma_2 = \sigma_{yp} - \sigma_1 = 3000 - 2000 = 1000 \text{ kg/cm}^2.$$

(*c*) In case of maximum strain energy theory, using Eq. (1.10*b*), we get (putting $\sigma_2 = -$ ve)

$$\sigma_1^2 + \sigma_2^2 + 2\mu\sigma_1\sigma_2 = \sigma_{yp}^2$$

$$(2000)^2 + \sigma_2^2 + 2 \times 0.25(2000)\sigma_2 = (3000)^2$$

or $$\sigma_2^2 + 1000\sigma_2 - 5 \times 10^6 = 0$$

or $$\sigma_2 = 10^3\left[\frac{-1 \pm \sqrt{1+20}}{2}\right]$$

or $$\sigma_2 = 1790 \text{ kg/cm}^2.$$

(*d*) For maximum distortion energy theory using Eq. (1.15*b*), we get

$$\sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2 = \sigma_{yp}^2$$

$$(2000)^2 + \sigma_2^2 + 2000\sigma_2 = (3000)^2$$

or $$\sigma_2^2 + 2000\sigma_2 - 5 \times 10^6 = 0$$

or $$\sigma_2 = 1450 \text{ kg/cm}^2.$$

Example 1.2. *A cylindrical shaft made of steel for which σ_{yp} in tension is 7000 kg/cm², is subjected to static loads consisting of a bending moment M = 1200 kg/m and a torque T = 3600 kg/m. Determine the diameter d which the shaft must have for a factor of safety of 2. Apply distortion energy, maximum shear stress and octahedral shearing stress theories.*

Solution.

$$\sigma = \frac{Md}{2I} = \frac{1200 \times 100d}{\dfrac{2 \times \pi d^4}{64}} = \frac{384 \times 10^4}{\pi d^4} \text{ kg/cm}^2$$

$$\tau = \frac{Td}{2I_p} = \frac{3600 \times 100d}{\dfrac{2 \times \pi d^4}{32}} = \frac{576 \times 10^4}{\pi d^3} \text{ kg/cm}^2$$

Maximum shear stress

$$= \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{384 \times 10^4}{2\pi d^3}\right)^2 + \left(\frac{576 \times 10^4}{\pi d^3}\right)^2}$$

$$= \frac{192\sqrt{10} \times 10^4}{\pi d^3} = \frac{192 \times 3.16 \times 10^4}{\pi d^3}$$

Let us now determine the two principal stresses σ_1 and σ_2

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{384 \times 10^4}{2\pi d^3} + \frac{192 \times 3.16 \times 10^4}{\pi d^3}$$

$$= \frac{192 \times 4.16 \times 10^4}{\pi d^3}$$

$$\sigma_2 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{384 \times 10^4}{2\pi d^3} - \frac{192 \times 3.16 \times 10^4}{\pi d^3}$$

$$= -\frac{192 \times 2.16 \times 10^4}{\pi d^3}$$

(*a*) Now Eq. (1.15*b*) gives

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \left(\frac{\sigma_{yp}}{FS}\right)^2 \left(\frac{192 \times 4.16 \times 10^4}{\pi d^3}\right)^2$$

$$+ \left(\frac{192 + 2.16 \times 10^4}{\pi d^3}\right)^2 + \frac{192 \times 4.16 \times 10^4}{\pi d^3}$$

$$\times \frac{192 \times 2.16 \times 10^4}{\pi d^3} = \left(\frac{7000}{2}\right)^2$$

or $$\left(\frac{192 \times 10^4}{\pi d^3}\right)^2 [(4.16)^2 + (2.16)^2 + 4.16 \times 2.16] = (3500)^2$$

or $$d^6 = \left(\frac{192 \times 10^4}{\pi \times 3500}\right)^2 \times 30.95$$

d = 9.906 cm.

(*b*) Let us now determine the diameter by maximum shear stress theory which gives somewhat conservative figure.

Equating the maximum shear stress to shear stress corresponding to σ_{yp}, we have

$$\frac{192 \times 3.16 \times 10^4}{\pi d^3} = \left(\frac{7000}{2 \times FS}\right) = \frac{7000}{4}$$

or $$d^3 = \frac{192 \times 3.16 \times 4 \times 10^4}{\pi \times 7000}$$

or **d = 10.34 cm.**

(*c*) Let us now determine the diameter of the shaft by octahedral sharing stress theory.

For plane stress $\sigma_3 = 0$ and Eq. (1.20) may be written as

$$\tau_0 = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2}$$

$$= \frac{\sqrt{2}}{3}\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$$

Equating the maximum octahedral shear stress with the safe octahedral shear stress in simple tension (Eq. 1.21), we get

$$\frac{\sqrt{2}}{3}\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} = \frac{\sqrt{2}}{3}(\sigma_{yp} / FS)$$

or $$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = (\sigma_{yp} / FS)^2.$$

Substituting σ_1 and σ_2 and calculating the diameter d of the shaft, we will get the diameter same as determined by maximum distortion energy theory.

Example 1.3. *A load P of 5000 kg on the crank pin of the crank shaft as shown in Fig. 1.10 is required to turn the shaft at constant speed. The crank shaft is made of ductile steel having a yield strength of 2800 kg/cm² as determined in simple tensile test. Calculate the diameter of the shaft based on a factor of safety of 2.5.*

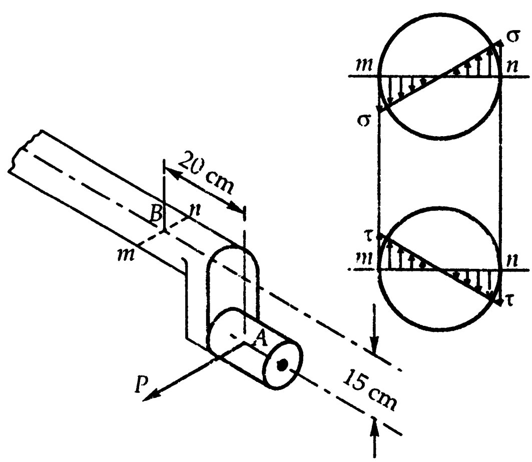

Fig. 1.10

Solution. The load P acting at A will subject the crank shaft to bending moment M and twisting moment T.

$$M = P \times 20$$

$$= 5000 \times 20 = 100{,}000 \text{ kg/cm}.$$

$$T = P \times 15$$
$$= 5000 \times 15 = 75{,}000 \text{ kg/cm.}$$

Bending stress at n

$$\sigma = \frac{32M}{\pi d^3} = \frac{32 \times 10^5}{\pi d^3} \text{ kg/cm}^2$$

Shearing stress at n

$$\tau = \frac{16T}{\pi d^3} = \frac{16 \times 7500}{\pi d^3} = \frac{12 \times 10^5}{\pi d^3} \text{ kg/cm}^2.$$

We will determine the diameter by applying maximum shear stress theory.

$$\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{32 \times 10^5}{2\pi d^3}\right)^2 + \left(\frac{12 \times 10^5}{\pi d^3}\right)^2}$$

$$= \frac{20 \times 10^5}{\pi d^3} \text{ kg/cm}^2$$

$$\tau_{yp} = \frac{2800}{2} = 1400 \text{ kg/cm}^2$$

We have $$\frac{20 \times 10^5}{\pi d^3} = \frac{\tau_{yp}}{FS} = \frac{1400}{2.5}$$

$\therefore$ $$d^3 = \frac{20 \times 10^5 \times 2.5}{1400\pi}$$

$$\mathbf{d = 10.44 \text{ cm.}}$$

Example 1.4. *Compare the permissible diameter of the steel circular shaft, subjected to torsion, according to following theories of failure,* $\mu = 0.3$.

(*a*) *Maximum stress theory,*

(*b*) *Maximum strain theory,*

(*c*) *Maximum shearing stress theory,*

(*d*) *Maximum strain energy theory.*

Solution. Assuming that the material has the same yield point in tension and compression, the conditions for yielding according to the stress theory, strain theory, shear stress theory and strain energy theory respectively are

$$\sigma_1 = \sigma_{yp}$$
$$\sigma_1 - \mu\sigma_2 = \sigma_{yp}$$

$$\sigma_1 - \sigma_2 = \sigma_{yp}$$

$$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \sigma_{yp}^2$$

This being a case of pure shear, we have

$$\sigma_1 = -\sigma_2 = \tau$$

and the above equations can be written as,

$$\tau_{yp} = C_{yp}$$

$$\tau_{yp} = \sigma_{yp}/(1 + \mu)$$

$$\tau_{yp} = \sigma_{yp}/2$$

$$\tau_{yp} = \sigma_{yp} / \sqrt{2(1+\mu)}$$

Putting $\mu = 0.3$ for steel we find the following results :

Maximum stress theory $\tau_{yp} = \sigma_{yp}$

Maximum strain theory $\tau_{yp} = 0.77\sigma_{yp}$

Maximum shear stress theory $\tau_{yp} = 0.5\sigma_{yp}$

Maximum strain energy theory $\tau_{yp} = 0.62\sigma_{yp}$.

For the design of a circular shaft in tension, the allowable value of working stress in shear can be assumed as $\tau_w = \tau_{yp}/FS$ and the diameter of shaft can be determined by the following equation :

$$\tau_w = \frac{16M_t}{\pi d^3}$$

Using maximum stress theory, we have

$$\tau_w = \frac{\tau_{yp}}{FS} = \frac{16M_t}{\pi d_1^3} = \frac{\sigma_{yp}}{FS} \qquad ...(A)$$

Using maximum strain theory, we have

$$\frac{16M_t}{\pi d_2^3} = \frac{0.77\,\sigma_{yp}}{FS}$$

or

$$\frac{\sigma_{yp}}{FS} = \frac{16M_t}{0.77\,\pi d_2^3} \qquad ...(B)$$

Comparing Eqs. (*A*) and (*B*), we have

$$d_1 : d_2 = 1 : 1.09$$

Using maximum shear stress theory, we get

$$\frac{16M_t}{\pi d_3^3} = \frac{0.5\,\sigma_{yp}}{FS}$$

or

$$\frac{\sigma_{yp}}{FS} = \frac{16M_t}{0.5\,\pi d_3^3} \qquad ...(C)$$

Comparing Eqs. (A) and (C), we get

$$d_1 : d_3 = 1 : 1.26$$

By strain energy theory, we have

$$\frac{16M_t}{\pi d_4^3} = \frac{0.62\sigma_{yp}}{FS}$$

or

$$\frac{\sigma_{yp}}{FS} = \frac{16M_t}{0.62\pi d_4^3} \qquad ...(D)$$

Comparing Eqs. (A) and (D), we have

$$d_1 : d_4 = 1 : 1.17$$

Hence considering the four theories of failure the following ratios of the diameters are obtained :

1 : 1.09 : 1.26 : 1.77. **Ans.**

SUPPLEMENTARY PROBLEMS

1.5. A piece of material is subjected to two perpendicular stress σ_1 tensile and σ_2 compressive. Find an expression for the strain energy per unit volume.

If a stress of 1250 kg/cm^2 acting alone gives the same value of strain energy as the expression already found, find the value of σ_2 when σ_1 is 1100 kg/cm^2. Poisson's ratio = 0.33.

[**Ans.** σ_2 = 332 kg/cm^2]

1.6. A cylindrical bar 25 mm in diameter is subjected to an end thrust of 2000 kg and is encased in a closely fitting sheath which reduces lateral strain by one-third of its value if free. Determine the strain energy per unit volume $E = 2 \times 10^6$ kg/cm^2; $\mu = 0.3$.

[**Ans.** 3.818 × 10^2 kg/cm^2]

1.7. The load on a bolt consists of an axial pull of 1000 kg together with a tranverse shear force of 500 kg. Estimate the diameter of bolt required according to :

1. Maximum shear stress theory and
2. Distortion energy theory.

Elastic limit in tension is 2800 kg/cm^2 and a factor of safety of 2.5 is to be applied. [**Ans.** (1) 1.377 and (2) 1.735 cm]

1.8. A thin-walled tube with an internal diameter of 35 cm and a wall thickness 6 mm subjected to an internal pressure of 70 kg/cm^2, an axial tensile load of 10,000 kg and a twisting moment. The yield point stress in tension is 2800 kg/cm^2. Determine the maximum twisting moment that can be applied, based on the maximum shear stress theory. [**Ans.** T = 14,400 kg/cm]