PART I

ELASTIC ANALYSIS

1

Stress Analysis

1.1 Introduction

This chapter presents the three-dimensional theory of stress of a continuous medium. A continuous medium is a material in which each volume of substance is sufficiently dense so that the concepts such as mass density, stress etc. have meaning at every point in the region occupied by the material. The theory of stress depends upon Newton's laws of motion, which are independent of the nature of continuous materials. Therefore, the relationships derived here are applicable to all continuous materials, whether they behave elastically, plastically, viscoelastically, or in any other manner.

1.2 Force Distribution

The forces acting on a body can be distinguished as follows :

(*a*) Internal forces.

(*b*) External forces.

(*a*) **Internal forces.** The internal mechanical state of a body is described by the stresses and strains at all points in it. The internal forces are the reactive forces which are set up due to external applied forces. And within elastic limits, the internal forces are numerically equal to the external forces.

(*b*) **External forces.** The state of stress and strain in a body arises due to external influences. The external forces acting on a body can be divided into two types.

(*i*) *Surface forces*—which are the forces distributed over the surface or boundary of the body and acting from the surrounding medium, like atmospheric pressure, hydraulic pressure or contact pressure exerted by one body on another. An exterior surface is any surface comprising a part of the bounding surface of the body and includes for example, the surface surrounding an internal cavity. The surface forces acting on the external surface of the body are described in terms of forces per unit area and, as such, are called applied stresses. Their components along the x, y and z axes of reference are represented by S_x, S_y and S_z, such that

$$\left.\begin{aligned} S_x &= \operatorname*{Limit}_{\Delta A \to 0} \frac{\Delta Fx}{\Delta A} \\ S_y &= \operatorname*{Limit}_{\Delta A \to 0} \frac{\Delta Fy}{\Delta A} \\ S_z &= \operatorname*{Limit}_{\Delta A \to 0} \frac{\Delta Fz}{\Delta A} \end{aligned}\right\} \qquad \text{...(1.1)}$$

where ΔFx, ΔFy and ΔFz are the components of the surface force ΔF along the x, y and z co-ordinate axes respectively and acting on elementary area ΔA. Surface forces are also called the contact forces.

An internal surface is created only by an imaginary cut through the body and lies, prior to such a cut, entirely inside the boundaries of the body. In order to distinguish between surface force distribution at actual and imaginary boundaries, we call the surface force distribution on the actual boundary, the traction force and on the imaginary boundary, the surface force.

(*ii*) *Body forces*—which act throughout the body and are not produced by physical contact with other bodies. Body force intensities are designated as forces per unit volume or per unit mass. For example, gravitational forces electromagnetic forces, centrifugal forces and inertia forces are body forces. They are represented by B_x, B_y and B_z, such that

$$\left.\begin{aligned} B_x &= \operatorname*{Limit}_{\Delta V \to 0} \frac{\Delta Fx}{\Delta V} \text{ or } \operatorname*{Limit}_{\Delta m \to 0} \frac{\Delta Fx}{\Delta m} \\ B_y &= \operatorname*{Limit}_{\Delta V \to 0} \frac{\Delta Fy}{\Delta V} \text{ or } \operatorname*{Limit}_{\Delta m \to 0} \frac{\Delta Fy}{\Delta m} \\ B_z &= \operatorname*{Limit}_{\Delta V \to 0} \frac{\Delta Fz}{\Delta V} \text{ or } \operatorname*{Limit}_{\Delta m \to 0} \frac{\Delta Fz}{\Delta m} \end{aligned}\right\} \qquad \text{...(1.2)}$$

where ΔV and Δm are the elementary volume or elementary mass respectively.

1.3 Definition of Stress

Consider a body as shown in Fig. 1.1, which in static equilibrium under the action of the system of external forces F_1, F_2, ..., F_n. Suppose now that the body is cut into two parts 1 and 2 by an imaginary plane AB. Then along the plane AB there are forces between the particles in part 1 and those in part 2, that tend to hold the body together. F_{21} is the sum of all the reactions of the particles in part 1 on the particles in part 2. For the body to remain in one piece and for its equilibrium to be maintained, F_{21} must be opposed by an equal and opposite force F_{12}, which is equal to the sum of all the particle forces in part 2 on the particles in part 1.

Thus $$F_{21} = -F_{12}$$

or $$F_{21} + F_{12} = 0$$

Therefore, when a body is in equilibrium under the action of external forces, any arbitrary portion of that body must be in equilibrium under the action of the external and internal forces on that portion.

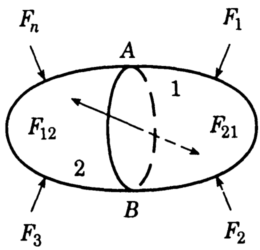


Fig. 1.1 Internal forces in a continuous body.

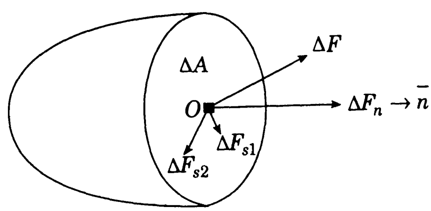


Fig. 1.2 Normal and shear force components of the resultant force Δ F acting on an elementary area Δ A at a point O.

Consider now an element of area ΔA on this section, as shown in Fig. 1.2. Let the resultant internal forces acting on this area be ΔF, such that $\Sigma \Delta A.\Delta F = F_{21}$. The component of this force in the direction of unit normal $\bar{n}$ is ΔF_n. And the other two components at right angles to each other in the plane of the section are ΔF_{s_1} and ΔF_{s_2}. Then the normal component of stress at point O is defined as :

$$\sigma = \underset{\Delta A \to 0}{\text{Limit}} \frac{\Delta F_n}{\Delta A} \qquad \text{...(1.3)}$$

and the shear components of stress are

$$\left.\begin{aligned} \tau_{s_1} &= \underset{\Delta A \to 0}{\text{Limit}} \frac{\Delta F_{s_1}}{\Delta A} \\ \tau_{s_2} &= \underset{\Delta A \to 0}{\text{Limit}} \frac{\Delta F_{s_2}}{\Delta A} \end{aligned}\right\} \qquad \text{....(1.4)}$$

Therefore, on a plane at a point there are three stress components, one normal to the plane and the other two tangential to the plane. It should be remembered that if a different plane is passed through the same point in the body, then the stresses will be different. Further, we have taken only two directions mutually perpendicular to each other in the section to determine only two components of shear stress. However, there are infinite number of directions along which shear stresses may be determined.

Since in addition to magnitude and direction, a plane of reference must also be specified for the stress, therefore, we can say that stress is not a vector but a tensor quantity. A tensor quantity is a more general quantity than a vector quantity since it needs more than three components for its complete definition (a vector needs only three quantities for its complete definition). For the stress tensor, six components must be known, three of these are the vector quantities, direction and magnitude, and the other three components are those necessary to define a plane of reference to which the stress is referred.

1.4 Stress Notations

In order to understand clearly the state of stress at a point, double subscript system representing stress components is utilised. The first subscript denotes the direction of the outward drawn normal on the plane on which the stress acts, and the second subscript denotes the direction towards which the stress acts. Therefore, on the positive x-face of an elementary parallelopiped, the stresses are τ_{xx}, τ_{xy} and τ_{xz} in the Cartesian co-ordinate system of axes. Out of which τ_{xx} is the normal stress, because the first subscripted x, denotes the direction (*i.e.*, x-direction) of the outward drawn normal on the plane-x and the second subscripted x, denotes the direction (*i.e.*, x-direction) along which the stress acts. Whereas τ_{xy} and τ_{xz} are the shear stresses, because the subscripted x gives the direction of the outward drawn normal, *i.e.*, the x-direction and the subscripts y and z give the directions (*i.e.*, y and z-directions) along which the stresses are measured. The stress components on a plane will be all positive if the outward drawn normal on that plane is in the positive direction of the co-ordinate axes.

1.5 Stress Tensor at a Point

The state of stress at a point in the Cartesian co-ordinates can be represented by the following nine components of stress known as the stress tensor :

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

The first subscript in τ_{ij} can be considered to identify a row of the array of terms whereas the second subscript can be considered to identify a column of the array. The principle or leading diagonal terms are all normal stresses and all off-diagonal terms are shear stresses. The normal stresses can also be written as σ_{xx}, σ_{yy}, σ_{zz} or σ_x, σ_y, σ_z instead of τ_{xx}, τ_{yy}, τ_{zz} or τ_x, τ_y, τ_z. Throughout this book we shall write the normal stresses as σ_x, σ_y and σ_z. Therefore, the stress tensor may be written as

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

All the stress components have been shown on a parallelopiped in Fig. 1.3. It can be noted that there are nine components of stress in total for a general state of stress.

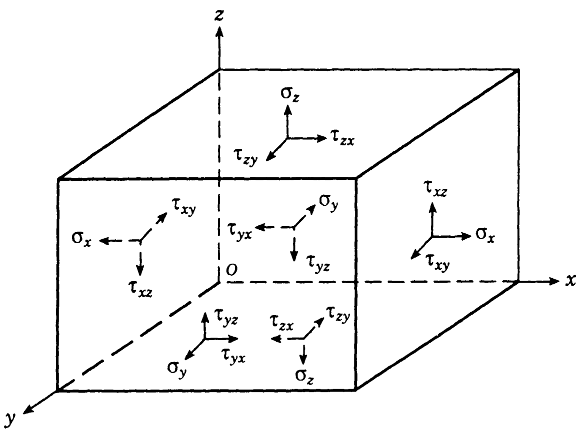


Fig. 1.3 Components of stress tensor acting on the faces of a rectangular parallelopiped.

The other more frequently used notations for components of the stress tensor are given in Table 1.1.

Table 1.1 Summary of Stress Notations

Engineering	σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
Some American writers	σ_{xx}	σ_{yy}	σ_{zz}	σ_{xy}	σ_{yz}	σ_{zx}
Some Russian writers	X_x	Y_y	Z_z	X_y	Y_z	Z_x
Some English writers	P	Q	R	S	T	U
In Tensor notations	σ_{11}	σ_{22}	σ_{33}	σ_{12}	σ_{23}	σ_{31}

In the same manner the stress tensors in cylindrical (r, θ, z) and spherical (r, θ, ϕ) co-ordinates may be written as follows :

$$\begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_z \end{bmatrix} \text{ and } \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \sigma_\phi \end{bmatrix}$$

1.6 Stress Gradient

The variation of stress with distance is called stress gradient. If the stress tensor at a point is known then the stress components in the neighbourhood of the point can be known by expanding the stress components by Taylor series expansion. Thus if σ_x is the stress at a point then the stress at a point at an infinitely small distance δx will be given by

$$\sigma_x + \frac{\partial}{\partial x}\sigma_x \,.\, \delta_x + \text{higher order terms.}$$

Neglecting higher order terms, we get $\sigma_x + \dfrac{\partial \sigma_x}{\partial x} \,.\, \delta x$

In general it can be written as $\tau_{ij} + \dfrac{\partial \tau_{ij}}{\partial x_j} \,.\, \delta x_j$.

Taylor series expansion of a function $f(x)$ at $(x + h)$, where h is an infinitely small distance, is given by

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2\,!} f''(x) + \frac{h^3}{3\,!} f'''(x) + \ldots\ldots$$

where dashes represent differentiation with respect to x.

1.7 Nature of Stress Tensor

Consider an infinitesimal parallelopiped of sides dx, dy and dz, as shown in Fig. 1.4. The shear stress τ_{xy} and τ_{yx} are shown acting on its faces perpendicular to the x and y-axes. By expressing the condition of equilibrium of moments $\Sigma M_z = 0$ about point O, we get

$$(\tau_{xy}\, dydz),\, dx - (\tau_{yx} \,.\, dxdy)\, dz = 0$$

$$\therefore \quad \tau_{xy} = \tau_{yx}$$

Similarly by considering the equilibrium of moments

$$\Sigma M_x = 0,\ \Sigma M_y = 0, \text{ we get}$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{xz} = \tau_{zx}$$

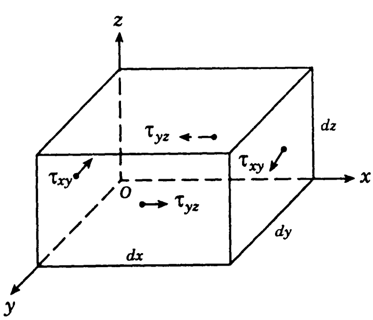


Fig. 1.4 Moment equilibrium of the stress component acting on the faces of a rectangular parallelopiped.

Therefore, the shear stresses are complimentary in nature and hence the stress tensor is symmetric in nature.

$$\therefore \qquad \tau_{ij} = \tau_{ji}\ ;\ i \neq j \ \text{ and } \ i, j = x, y, z \qquad ...(1.5)$$

Hence, the stress tensor in Cartesian co-ordinates may be written as

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}.$$

Therefore, the components of stress tensor reduces from 9 to 6 independent components.

If we consider stress gradients in the parallelopiped, then the higher order terms like $\frac{\partial \tau_{xy}}{\partial x} dx$ etc., contribute only higher order terms in dx, dy, dz to the moments. Also, the acceleration effects, being proportional to the mass moment of inertia of the element, are of higher degree in dx, dy, dz, and hence do not contribute to the moments.

1.8 Equilibrium Equations in Cartesian Co-ordinates

Consider an infinitesimal rectangular parallelopiped, as shown in Fig. 1.5, having sides dx, dy and dz under the action of surface inertia forces and body forces.

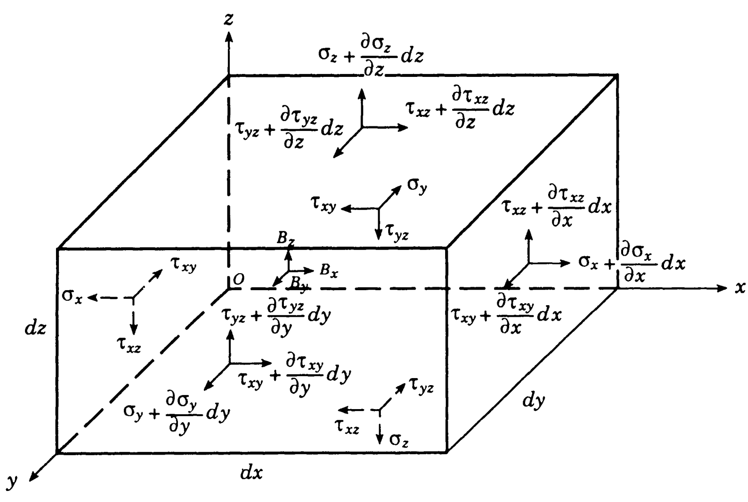


Fig. 1.5 Stress components acting on the faces of an elementary rectangular parallelopiped.

Consider the equilibrium of forces in the x-direction. By using Newton's second law of motion, we get

$$\left(\sigma_x + \frac{\partial\sigma_x}{\partial x} . dx\right) dydz - \sigma_x . dydz$$
$$+ \left(\tau_{xy} + \frac{\partial\tau_{xy}}{\partial y} dy\right) dxdz - \tau_{xy} dxdz$$
$$+ \left(\tau_{xz} + \frac{\partial\tau_{xz}}{\partial z} dz\right) dxdy - \tau_{xz} dxdy$$
$$+ B_x . dxdydz = \rho dxdydz . \bar{a}_x$$

where B_z = component of the body force in the x-direction, expressed per unit volume.

$\bar{a}_x$ = acceleration in the x-direction.

ρ = density of the material of the parallelopiped.

Simplifying, we get $\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + B_x = \rho\bar{a}_x$

Similarly considering the equilibrium of forces in the y and z directions, we get respectively

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + B_y = \rho\bar{a}_y$$
$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + B_z = \rho\bar{a}_z$$

In the absence of inertia and body forces, we get the equilibrium equations as :

$$\left.\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0 \\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} = 0 \\ \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} = 0 \end{aligned}\right\} \quad ...(1.6)$$

In the form of tensor notations, the equilibrium equations may be written as :

$$\frac{\partial\tau_{ij}}{\partial x_j} + B_i = 0. \quad ...[1.6\ (a)]$$

Example 1.1 *The state of stress at a point is given by :*

$$\sigma_x = x^2y + 20\ ; \quad \tau_{xy} = 3x^2y$$
$$\sigma_y = x^3z + y^2\ ; \quad \tau_{yz} = yz$$
$$\sigma_z = yz^2 + 10\ ; \quad \tau_{xz} = xz.$$

Determine the body force distribution at the point (1, 2, 3) so that the stresses are in equilibrium.

Solution. Using the equilibrium equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0$$

we get $$2xy + 3x^2 + x + B_x = 0$$

$\therefore$ $$B_x = -(2xy + 3x^2 + x)$$

At the point (1, 2, 3), B_x becomes

$$B_x = -(2 \times 1 \times 2 + 3 \times 1 + 1)$$
$$= -(4 + 3 + 1) = -8$$

Similarly, using the other two equilibrium equations, we get

$$B_y = -(6xy + 2y + y) = -18$$
$$B_z = -(2z + 2yz) = -18$$

$\therefore$ The body force distribution required for equilibrium becomes :

$$B = -8\hat{i} - 18\hat{j} - 18\hat{k}$$

1.9 Direction Cosines Relationships

Direction cosine is the property of the angle between two lines and is equal to the cosine of the angle between them.

Consider two sets of rectilinear orthogonal co-ordinate axes x_1, x_2, x_3 and x_1', x_2', x_3', as shown in Fig. 1.6. The relationship between the angles of the new geometry and old geometry may be defined by nine components $a_{ij} = \cos (x_i', x_j)$, where a_{ij} are the direction cosines and $i, j = 1, 2, 3$.

Choose an arbitrary vector $\overline{A}$ having components A_1, A_2, A_3 referred to the original co-ordinate system of axes x_1, x_2, x_3 respectively. The components of this vector in the new co-ordinate system x_1', x_2', x_3' denoted as A_1', A_2', A_3' can be determined from the old unprimed components through the following transformation equations :

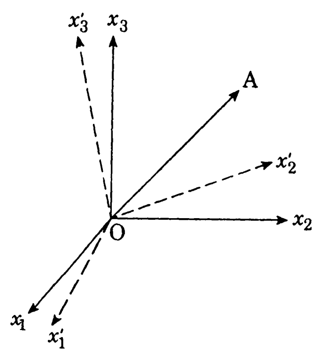


Fig. 1.6 Direction cosine relationship.

$$A_1' = a_{11}A_1 + a_{12}A_2 + a_{13}A_3$$
$$A_2' = a_{21}A_1 + a_{22}A_2 + a_{23}A_3 \quad \ldots(1.7)$$
$$A_3' = a_{31}A_1 + a_{32}A_2 + a_{33}A_3$$

These equations in the tensor notation form can be written as :

$$A_i' = a_{ij}A_j \; ; i = 1, 2, 3, j = 1, 2, 3 \quad \ldots(1.8)$$

The repeated index j indicates summation over the integers 1, 2, 3, index i appearing singly on each side of the equation, is a free index that ranges over these same integers to yield the three separate equations.

The reverse transformation can be written as :

$$A_i = a_{ji}A_j' = [a_{ij}]^T A_j' \quad ...(1.9)$$

where T stands for transpose.

The set of nine direction cosine coefficients constitute a 3 × 3 matrix, called the transformation matrix, which may be written as

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) & \cos(x_1', x_3) \\ \cos(x_2', x_1) & \cos(x_2', x_2) & \cos(x_2', x_3) \\ \cos(x_3', x_1) & \cos(x_3', x_2) & \cos(x_3', x_3) \end{bmatrix}$$

These nine direction cosine coefficients satisfy the following relation :

$$a_{ik}a_{jk} = \delta_{ij} \quad ...(1.10)$$

where δ_{ij} = Kronecker delta or unit tensor defined as

$$\delta_{ij} = \begin{cases} 1 ; i = j \\ 0 ; i \neq j \end{cases} \quad ...(1.11)$$

Expanding equation (1.10), six independent conditions are obtained on the elements of the transformation matrix. The normality conditions are :

$$\begin{aligned} a^2{}_{11} + a^2{}_{12} + a^2{}_{13} &= 1 \\ a^2{}_{21} + a^2{}_{22} + a^2{}_{23} &= 1 \\ a^2{}_{31} + a^2{}_{32} + a^2{}_{33} &= 1 \end{aligned} \quad ...(1.12)$$

and the orthogonality conditions give,

$$\begin{aligned} a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0 \\ a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} &= 0 \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0 \end{aligned} \quad ...(1.13)$$

The normality and orthogonality conditions must be satisfied for the transformation to be rectilinear and orthogonal.

1.10 Normal and Shearing Stresses

Consider a rectangular parallelopiped, as shown in Fig. 1.7. Let σ_r be the resultant stress at a point O and σ_{rx}, σ_{ry}, σ_{rz} are its

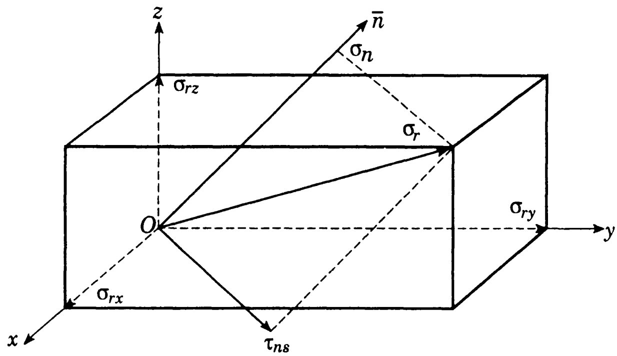


Fig. 1.7 Stresses on an oblique plane.

components along the three axes of reference. Let a_{rx}, a_{ry}, a_{rz} be the direction cosines of the resultant stress σ_r, so that,

$$a_{rx} = \frac{\sigma_{rx}}{\sigma_r}$$

$$a_{ry} = \frac{\sigma_{ry}}{\sigma_r} \qquad \text{...(1.14)}$$

$$a_{rz} = \frac{\sigma_{rz}}{\sigma_r}$$

The resultant stress σ_r can be resolved into normal stress σ_n along the unit normal $\bar{n}$ having direction cosines a_{nx}, a_{ny}, a_{nz} and shear stress τ_{ns}.

Now $\quad \sigma_n = \sigma_r \cdot a_{nr}$

where $\quad a_{nr} = a_{nx} \cdot a_{rx} + a_{ny} \cdot a_{ry} + a_{nz} \cdot a_{rz}$

$\therefore \quad \sigma_n = \sigma_r (a_{nx} \cdot a_{rx} + a_{ny} \cdot a_{ry} + a_{nz} \cdot a_{rz})$

$\quad \sigma_n = \sigma_{rx} \cdot a_{nx} + \sigma_{ry} \cdot a_{ny} + \sigma_{rz} \cdot a_{nz}$

Also $\quad \sigma_{rx} = \sigma_x \cdot a_{nx} + \tau_{xy} \cdot a_{ny} + \tau_{xz} \cdot a_{nz}$

$\quad \sigma_{ry} = \tau_{xy} \cdot a_{nx} + \sigma_y \cdot a_{ny} + \tau_{yz} \cdot a_{nz}$

$\quad \sigma_{rz} = \tau_{xz} \cdot a_{nx} + \tau_{yz} \cdot a_{ny} + \sigma_z \cdot a_{nz}$

These are called the *stress vectors*.

Hence the normal stress σ_n becomes

$$\sigma_n = \sigma_x a_{nx}^2 + \sigma_y \cdot a_{ny}^2 + \sigma_z \cdot a_{nz}^2 + 2(\tau_{xy} . a_{nx} a_{ny} + \tau_{yz} . a_{ny} a_{nz} + \tau_{xz} . a_{nx} a_{nz}) \qquad \text{...(1.15)}$$

Also $\sigma_n^2 + \tau_{ns}^2 = \sigma_r^2$

$$\therefore \quad \tau_{ns} = \sqrt{\sigma_r^2 - \sigma_n^2}$$

where $$\sigma_r = \sqrt{\sigma_{rx}^2 + \sigma_{ry}^2 + \sigma_{rz}^2}$$

Let a_{sx}, a_{sy}, a_{sz} be the direction cosines of τ_{ns}. For equilibrium of forces in the x-direction, we have

$$\sigma_n.a_{nx} + \tau_{ns}.a_{sx} = \sigma_{rx}$$

$$\therefore \quad a_{sx} = \frac{1}{\tau_{ns}}[\sigma_{rx} - \sigma_n.a_{nx}]$$

$$= \frac{1}{\tau_{ns}}[(\sigma_x - \sigma_n)\,a_{nx} + \tau_{xy}a_{ny} + \tau_{xz}a_{nz}]$$

Similarly $$a_{sy} = \frac{1}{\tau_{ns}}[\tau_{xy}a_{nx} + (\sigma_y - \sigma_n)\,a_{ny} + \tau_{yz}a_{nz}]$$

$$a_{sz} = \frac{1}{\tau_{ns}}[\tau_{xz}a_{nx} + \tau_{yz}a_{ny} + (\sigma_z - \sigma_n)\,a_{nz}]$$

Example 1.2 *The stress components at a point are : $\sigma_x = 20$, $\sigma_y = 10$, $\sigma_z = 5$, $\tau_{xy} = 10$, $\tau_{yz} = 4$ and $\tau_{xz} = 5$ MPa. Determine the normal and shearing stresses on the plane whose direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. Also calculate the direction of the shear stress.*

Solution. Resolving the stresses along the three co-ordinate axes, we have

$$\sigma_{rx} = \sigma_x.a_{nx} + \tau_{xy}.a_{ny} + \tau_{xz}.a_{nz}$$

$$= \frac{1}{\sqrt{3}}(20 + 10 + 5) = \frac{35}{\sqrt{3}}\text{ MPa}$$

$$\sigma_{ry} = \tau_{xy}.a_{nx} + \sigma_y.a_{ny} + \tau_{yz}.a_{nz}$$

$$= \frac{1}{\sqrt{3}}(10 + 10 + 4) = \frac{24}{\sqrt{3}}\text{ MPa}$$

$$\sigma_{rz} = \tau_{xz}.a_{nx} + \tau_{yz}.a_{ny} + \sigma_z.a_{nz}$$

$$= \frac{1}{\sqrt{3}}(5 + 4 + 5) = \frac{14}{\sqrt{3}}\text{ MPa}$$

Resultant stress, $\sigma_r = \sqrt{\sigma_{rx}^2 + \sigma_{ry}^2 + \sigma_{rz}^2}$

$$= \sqrt{\tfrac{1}{3}[(35)^2 + (24)^2 + (14)^2]}$$

$$= \sqrt{\frac{1}{3}(1225 + 576 + 196)}$$

$$= \sqrt{\frac{1997}{3}} = 25.8 \text{ MPa}$$

Normal stress, $\sigma_n = \sigma_{rx}.a_{nx} + \sigma_{ry}.a_{ny} + \sigma_{rz}.a_{nz}$

$$= \frac{1}{3}(35 + 24 + 14)$$

$$= \frac{73}{3} = 24.3 \text{ MPa}$$

$\therefore$ Shear stress, $\tau_{ns} = \sqrt{\sigma_r^2 - \sigma_n^2}$

$$= \sqrt{665.67 - 592.11}$$

$$= \sqrt{73.56} = 8.576 \text{ MPa}$$

The direction of the shear stress is obtained as follows :

$$a_{sx} = \frac{1}{\tau_{ns}}[\sigma_{rx} - \sigma_n.a_{nx}]$$

$$= \frac{1}{8.576}\left[\frac{35}{\sqrt{3}} - \frac{24.3}{\sqrt{3}}\right]$$

$$= \frac{10.7}{\sqrt{3} \times 8.576} = 0.7203$$

$$a_{sy} = \frac{1}{\tau_{ns}}[\sigma_{ry} - \sigma_n.a_{ny}]$$

$$= \frac{1}{8.576}\left[\frac{24}{\sqrt{3}} - \frac{24.3}{\sqrt{3}}\right]$$

$$= \frac{-0.3}{\sqrt{3} \times 8.576} = -0.0202$$

$$a_{sz} = \frac{1}{\tau_{ns}}[\sigma_{rz} - \sigma_n.a_{nz}]$$

$$= \frac{1}{8.576}\left[\frac{14}{\sqrt{3}} - \frac{24.3}{\sqrt{3}}\right]$$

$$= \frac{-10.3}{\sqrt{3} \times 8.576} = -0.6934$$

1.11 Transformation Equations for Stresses

Consider an infinitesimal tetrahedron *OABC* cut out of a continuous medium, as shown in Fig. 1.8. Let the orthogonal edges of the tetrahedron be of lengths Δx, Δy and Δz. Positive shear and

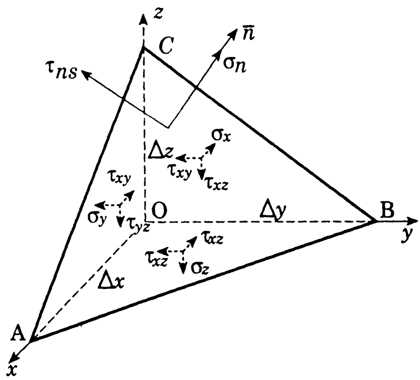


Fig. 1.8 Stress components acting on the faces of an elementary tetrahedron.

normal stresses have been shown on the faces of the tetrahedron parallel to the reference planes. On the inclined surface ABC whose unit outward drawn normal is $\bar{n}$, having direction cosines a_{nx}, a_{ny} and a_{nz}, the normal and total shear stresses are σ_n and τ_{ns} respectively. Then

$$a_{nx} = \frac{\text{area } BOC}{\text{area } ABC}$$

$$a_{ny} = \frac{\text{area } AOC}{\text{area } ABC}$$

$$a_{nz} = \frac{\text{area } AOB}{\text{area } ABC} .$$

By using Newton's second law of motion in the direction of $\bar{n}$, we get

$$\sigma_n . ABC - \sigma_x . BOC . a_{nx} - \tau_{xy} . BOC.a_{ny}$$

$$- \tau_{xz} . BOC . a_{nz} - \sigma_y . AOC . a_{ny} - \tau_{yz} . AOC . a_{nz}$$

$$- \tau_{xy} . AOC . a_{nx} - \sigma_z . AOB . a_{nz} - \tau_{xz} . AOB . a_{ax}$$

$$\tau_{yz} AOB . a_{ny} - \frac{\gamma}{6} . \Delta x \Delta y \Delta z . a_{nz} = \rho . \bar{a}_n . \frac{\Delta x \, \Delta y \, \Delta z}{6}$$

where $\bar{a}_n$ = Acceleration in $\bar{n}$ direction.

ρ = Density of the tetrahedron material.

γ = Specific weight of tetrahedron material.

and $\frac{\Delta x \, \Delta y \, \Delta z}{6}$ is the volume of the tetrahedron.

Dividing throughout by ABC, we get

$$\sigma_n - \sigma_x a^2{}_{nx} - \tau_{xy} a_{nx} . a_{ny} - \tau_{xz} . a_{nx} . a_{nz}$$

$$- \sigma_y a^2{}_{ny} - \tau_{yz} a_{ny} a_{nz} - \tau_{xy} a_{nx} a_{ny} - \sigma_z . a^2{}_{nz}$$

$$- \tau_{xz} a_{nx} a_{nz} - \tau_{yz} a_{ny} a_{nz} - \frac{\gamma}{6} . \frac{\Delta x \, \Delta y \, \Delta z \, . \, a_{nz}}{ABC}$$

$$= \rho . \frac{\bar{a}_n}{6} . \frac{\Delta x \, \Delta y \, \Delta z}{ABC}$$

In the absence of inertia and body forces, we get

$$\sigma_n = \sigma_x a^2{}_{nx} + \sigma_y a^2{}_{ny} + \sigma_z a^2{}_{nz} + 2(\tau_{xy}.a_{nx}a_{ny} + \tau_{yz}a_{ny}a_{nz} + \tau_{xz}.a_{nx}a_{nz}] \quad ...(1.17)$$

Let a_{sx}, a_{sy} and a_{sz} be the direction cosines of the shear stress τ_{ns}. Then since the normal and shearing stresses are at right angles, therefore,

$$a_{nx}.a_{sx} + a_{ny}.a_{sy} + a_{nz}.a_{sz} = 0$$

Also $$a^2{}_{sx} + a^2{}_{sy} + a^2{}_{sz} = 1$$

Similarly, applying Newton's second law in the direction of the shear stress, we get

$$\begin{aligned}\tau_{ns} = \sigma_x.a_{nx}a_{sx} &+ \sigma_y a_{ny}a_{sy} + \sigma_z a_{nz}a_{sz} \\ &+ \tau_{xy}(a_{nx}a_{sy} + a_{ny}a_{sx}) \\ &+ \tau_{yz}(a_{ny}a_{sz} + a_{nz}a_{sy}) \\ &+ \tau_{xz}(a_{nx}a_{sz} + a_{nz}a_{sx})\end{aligned} \quad ...(1.18)$$

In tensor notation, the normal and shearing stresses may be written as

$$\left.\begin{aligned}\sigma_n &= a_{(n)i}\, a_{(n)j}\, \tau_{ij} \quad ...(1.19)\\ \tau_{ns} &= a_{ni}a_{sj}\tau_{ij} \quad ...(1.20)\end{aligned}\right\} \quad i, j = x, y, z$$

where the subscript n within the bracket implies that submission is not to be taken over n, Or in general, if we are given a stress tensor τ_{lm} in the old co-ordinate system x, y, z, then the transformed stress tensor $\tau_{i'j'}$ in the new co-ordinate system x', y', z' may be written in the tensor notation form as

$$\begin{aligned}\tau_{i'j'} &= a_{il}a_{jm}\tau_{lm} \quad ...(1.21)\\ &= (a_{il}\tau_{lm})\, a_{jm} = \tau_{im}.a_{jm} = \tau_{im}.(a_{mj})^T\end{aligned}$$

In terms of Cartesian co-ordinates, the components of the transformed stress tensor may be written as

$$\begin{aligned}\sigma_{x'} &= a_{(x')i}\, a_{(x')j}\, \tau_{ij}\\ &= \sigma_x.a^2{}_{x'x} + \sigma_y a^2{}_{x'y} + \sigma_z a^2{}_{x'z} + 2(\tau_{xy}\, a_{x'x}\, a_{x'y} + \tau_{yz}a_{x'y}a_{x'z} + \tau_{xz}a_{x'x}a_{x'z})\end{aligned} \quad ...(1.22)$$

$$\begin{aligned}\sigma_{y'} &= a_{(y')i}\, a_{(y')j}\, \tau_{ij}\\ &= \sigma_x a^2{}_{y'x} + \sigma_y a^2{}_{y'y} + \sigma_z a^2{}_{y'z} + 2(\tau_{xy}.a_{y'x}a_{y'y} + \tau_{yz}.a_{y'y}a_{y'z} + \tau_{xz}a_{y'x}.a_{y'z})\end{aligned} \quad ...(1.23)$$

$$\begin{aligned}\sigma_{z'} &= a_{(z')i}\, a_{(z')j}\, .\, \tau_{ij}\\ &= \sigma_x.a^2{}_{z'x} + \sigma_y a_{z'y} + \sigma_z a^2{}_{z'z} + 2(\tau_{xy}.a_{z'x}\, a_{z'y} + \tau_{yz}\, a_{z'y}\, a_{z'z} + \tau_{xz}\, a_{z'x}\, a_{z'z}\end{aligned} \quad ...(1.24)$$

$$\begin{aligned}
\tau_{x'y'} &= a_{x'i}a_{y'j}\tau_{ij} \\
&= \sigma_x a_{x'x}\, a_{y'x} + \sigma_y a_{x'y}a_{y'y} + \sigma_z a_{x'z}a_{y'z} \\
&+ \tau_{xy}\,(a_{x'x}a_{y'y} + a_{x'y}a_{y'x}) \\
&+ \tau_{yz}\,(a_{x'y}a_{y'z} + a_{x'z}a_{y'y}) \\
&+ \tau_{xz}\,(a_{x'x}a_{y'z} + a_{x'z}a_{y'x}) \qquad \ldots(1.25)
\end{aligned}$$

$$\begin{aligned}
\tau_{y'z'} &= a_{y'i}a_{z'j}\tau_{ij} \\
&= \sigma_x a_{y'x}a_{z'x} + \sigma_y a_{y'y}a_{z'y} + \sigma_z . a_{y'z}a_{z'z} \\
&+ \tau_{xy}\,(a_{y'x}a_{z'y} + a_{y'y}a_{z'x}) \\
&+ \tau_{yz}\,(a_{y'y}a_{z'z} + a_{y'z}a_{z'y}) \\
&+ \tau_{xz}\,(a_{y'x}a_{z'z} + a_{y'z}a_{z'x}) \qquad \ldots(1.26)
\end{aligned}$$

$$\begin{aligned}
\tau_{x'z'} &= a_{x'i}a_{z'j}\tau_{ij} \\
&= \sigma_x a_{x'x}a_{z'x} + \sigma_y a_{x'y}a_{z'y} + \sigma_z a_{x'z}a_{z'z} \\
&+ \tau_{xy}\,(a_{x'x}a_{z'y} + a_{x'y}a_{z'x}) \\
&+ \tau_{yz}\,(a_{x'y}a_{z'z} + a_{x'z}a_{z'x}) \\
&+ \tau_{xz}\,(a_{x'x}a_{z'z} + a_{x'z}a_{z'x}) \qquad \ldots(1.27)
\end{aligned}$$

Example 1.3 *The stress components at a point are :* $\sigma_x = 50$, $\sigma_y = 30$, $\sigma_z = 15$, $\tau_{xz} = z0$, $\tau_{yz} = 5$, $\tau_{xz} = 10$ *MPa with respect to xyz co-ordinate system. If the co-ordinate system is rotated about z-axis in the anti-clockwise direction through 30°, determine the new stress components.*

Solution. Let the new co-ordinate system be $x'y'z'$, as shown in Fig. 1.9. Then the direction cosines are

	x	y	z
x'	$\sqrt{3}/2$	$1/2$	0
y'	$-1/2$	$\sqrt{3}/2$	0
z'	0	0	1

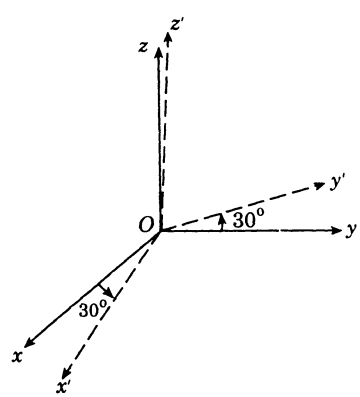


Fig. 1.9 Rotation about z-axis.

Using transformation equations for stresses, we get

$$\begin{aligned}
\sigma_{x'} &= \sigma_x . a^2_{x'x} \\
&+ \sigma_y . a^2_{x'y} + \sigma_z . a^2_{x'z} \\
&+ 2(\tau_{xy} . a_{x'x}a_{x'y} + \tau_{yz}a_{x'y}a_{x'z} + \tau_{xz}a_{x'x}a_{x'z})
\end{aligned}$$

$$= 50 \times \frac{3}{4} + 30 \times \frac{1}{4} + 15 \times 0 + 2\left(20 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} + 0 + 0\right)$$

$$= \frac{150}{4} + \frac{30}{4} + 10\sqrt{3}$$

$$= 37.5 + 7.5 + 17.32$$

$$= 62.32 \text{ MPa}$$

$$\sigma_{y'} = \sigma_x.a^2_{y'x} + \sigma_y.a^2_{y'y} + \sigma_z.a^2_{y'z} + 2(\tau_{xy}.a_{y'x}.a_{y'y} + \tau_{yz}.a_{y'y}.a_{y'z} + \tau_{xz}.a_{y'x}.a_{y'z})$$

$$= 50 \times \frac{1}{4} + 30 \times \frac{3}{4} + 0 + 2\left[20 \times \left(-\frac{1}{2}\right) \times \frac{\sqrt{3}}{2} + 0 + 0\right]$$

$$= \frac{50}{4} + \frac{45}{2} - 10\sqrt{3}$$

$$= 12.5 + 22.5 - 17.32$$

$$= 17.68 \text{ MPa}$$

$$\sigma_{z'} = \sigma_x.a^2_{z'x} + \sigma_y.a^2_{z'y} + \sigma_z.a^2_{z'z} + 2(\tau_{xy}.a_{z'x}.a_{z'y} + \tau_{yz}.a_{z'y}.a_{z'z} + \tau_{xz}.a_{z'x}.a_{z'z})$$

$$= 0 + 0 + 15 \times 1 + 2(0 + 0 + 0)$$

$$= 15 \text{ MPa}$$

$$\tau_{x'y'} = \sigma_x.a_{x'x}.a_{y'x} + \sigma_y.a_{x'y}.a_{y'y} + \sigma_z.a_{x'z}.a_{y'z} + \tau_{xy}(a_{x'x}a_{y'y} + a_{x'y}a_{y'z}) + \tau_{yz}(a_{x'y}.a_{y'z} + a_{x'z}.a_{y'y}) + \tau_{xz}(a_{x'x}.a_{y'z} + a_{x'z}.a_{y'x})$$

$$= 50 \times \frac{\sqrt{3}}{2} \times \left(-\frac{1}{2}\right) + 30 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} + 15 \times 0 \times 0 + 20\left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}\right) + 5\left(\frac{1}{2} \times 0 + 0 \times \frac{\sqrt{3}}{2}\right) + 10\left(\frac{\sqrt{3}}{2} \times 0 - 0 \times \frac{1}{2}\right)$$

$$= \frac{-25\sqrt{3}}{2} + \frac{15\sqrt{3}}{2} + 20\left(\frac{3}{4} - \frac{1}{4}\right)$$

$$= \frac{-10\sqrt{3}}{2} + 10 = -5\sqrt{3} + 10$$

$$= 10 - 8.660 = 1.34 \text{ MPa}$$

$$\tau_{y'z'} = \sigma_x.a_{y'x}.a_{x'x} + \sigma_y.a_{y'y}.a_{z'y} + \sigma_z.a_{y'z}.a_{z'z}$$
$$+ \tau_{xy}(a_{y'x}.a_{z'y} + a_{y'y}.a_{z'x})$$
$$+ \tau_{yz}(a_{y'y}.a_{z'z} + a_{y'z}.a_{z'y})$$
$$+ \tau_{xz}(a_{y'x}.a_{z'z} + a_{y'z}.a_{z'x})$$
$$= 0 + 0 + 0 + 0 + 5 \times \frac{\sqrt{3}}{2} \times 1 + 10\left(-\frac{1}{2}\right) \times 1$$
$$= \frac{5\sqrt{3}}{2} - 5 = 4.33 - 5 = -0.67 \text{ MPa}$$

$$\tau_{x'z'} = \sigma_x.a_{x'x}.a_{z'x} + \sigma_y.a_{x'y}.a_{z'y} + \sigma_z.a_{x'z}.a_{z'z}$$
$$+ \tau_{xy}(a_{x'x}.a_{z'y} + a_{x'y}.a_{z'x})$$
$$+ \tau_{yz}(a_{x'y}.a_{z'z} + a_{x'z}.a_{z'y})$$
$$+ \tau_{xz}(a_{x'x}.a_{z'z} + a_{x'z}.a_{z'x})$$
$$= 0 + 0 + 0 + 0 + 5 \times \frac{1}{2} \times 1 + 10 \times \frac{\sqrt{3}}{2} \times 1$$
$$= 2.5 + 5\sqrt{3} = 2.5 + 8.66 = 11.16 \text{ MPa}$$

Otherwise by matrix multiplication method. We know that

$$\tau_{i'j'} = a_{il}a_{jm}\,\tau_{lm}$$
$$= (a_{il}.\tau_{lm})\,a_{jm} = \tau_{im}.a_{jm} = \tau_{im}.\,(a_{mj})^T$$
$$\tau_{im} = a_{il}.\tau_{lm}$$
$$= \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 50 & 20 & 10 \\ 20 & 30 & 5 \\ 10 & 5 & 15 \end{bmatrix}$$
$$= \begin{bmatrix} 53.3 & 32.32 & 11.16 \\ -7.8 & 15.98 & -0.67 \\ 10 & 5 & 15 \end{bmatrix}$$
$$\tau_{i'j'} = \tau_{im}\,(a_{mj})^T$$
$$= \begin{bmatrix} 53.3 & 32.32 & 11.16 \\ -7.8 & 15.98 & -0.67 \\ 10 & 5 & 15 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ \frac{1}{2} & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 62.32 & 1.34 & 11.16 \\ 1.34 & 17.68 & -0.67 \\ 11.16 & -0.67 & 15 \end{bmatrix} \text{MPa}$$

Method of multiplication of two matrices.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{22}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

To determine the transpose of the matrix, change rows into columns.

1.12 Principal Stresses and Principal Planes

Consider an infinitesimal tetrahedron having three faces with known stresses on the reference planes, as shown in Fig. 1.10.

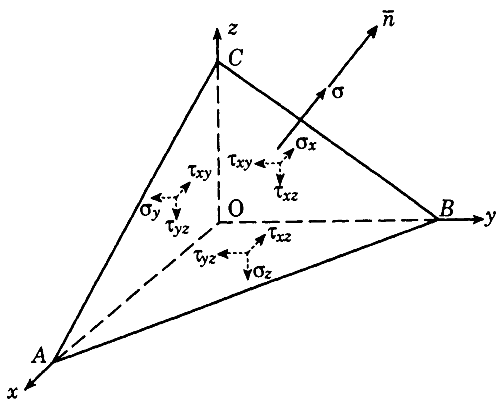


Fig. 1.10 Stress components acting on the faces of an elementary tetrahedron.

Assume that the inclined face *ABC* is a principal plane. The principal stress σ on this plane is along the normal $\bar{n}$ to this plane having direction cosines a_{nx}, a_{ny} and a_{ns}.

Applying Newton's second law of motion in the z-direction and neglecting body and inertia forces, we get

$$\sigma.ABC.a_{nz} - \sigma_z.AOB - \tau_{yz}.AOC - \tau_{xz}.BOC = 0$$

Dividing throughout by area *ABC*, we get

$$\sigma.a_{nz} - \sigma_z.a_{nz} - \tau_{yz}.a_{ny} - \tau_{xz}.a_{nz} = 0$$

$$\tau_{xz}.a_{nx} + \tau_{yz}.a_{ny} + (\sigma_z - \sigma)\, a_{nz} = 0$$

Similarly, by considering the equilibrium of forces in the x and y directions, we get

$$(\sigma_x - \sigma)\, a_{nx} + \tau_{xy}\, a_{ny} + \tau_{xz}\, a_{nz} = 0$$

$$\tau_{xy}.a_{nx} + (\sigma_y - \sigma)\, a_{ny} + \tau_{yz} a_{nz} = 0$$

Therefore, the three simultaneous homogeneons equations can be written as :

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By using Cramer's rule, we get

$$a_{nx} = \frac{\begin{vmatrix} 0 & \tau_{xy} & \tau_{xz} \\ 0 & \sigma_y - \sigma & \tau_{yz} \\ 0 & \tau_{yz} & \sigma_z - \sigma \end{vmatrix}}{\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix}}$$

a_{nx} will be zero, as will the other direction cosines, unless the denominator in the preceeding equation is zero so as to permit an indeterminate result. But all the direction cosines cannot be zero, because

$$a^2_{nx} + a^2_{ny} + a^2_{nz} = 1$$

Thus a necessary condition required for the solution of this problem is :

$$D = \begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

Expanding the determinant, we get

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\,\sigma^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau^2_{xy} - \tau^2_{yz} - \tau^2_{xz})\,\sigma - (\sigma_x\sigma_y\sigma_z - \sigma_x\tau^2_{yz} - \sigma_y\tau^2_{xz} - \sigma_z\tau^2_{xy} + 2\tau_{xy}\tau_{yz}\tau_{zx}) = 0 \qquad ...(1.28)$$

This is a cubic equation and must have three distinct roots, giving three principal stresses. Further it may be borne in mind that principal stresses depend only on the state of stress at a point in a body and do not depend on the orientation of the co-ordinate axes.

1.12.1 Stress Invariants

The combination of stresses at a point which do not change with the orientation of the co-ordinate axes are called stress invariants. Therefore, we define,

$$\sigma_x + \sigma_y + \sigma_z = I_1 = \text{First invariant of stress} \quad ...(1.29)$$

$$\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau^2_{xy} - \tau^2_{yz} - \tau^2_{xz} = I_2$$
$$= \text{Second invariant of stress} \quad ...(1.30)$$

$$\sigma_x\sigma_y\sigma_z - \sigma_x\tau^2_{yz} - \sigma_y\tau^2_{xz} - \sigma_z\tau^2_{xy} + 2\tau_{xy}\tau_{yz}\tau_{xz} = I_3$$
$$= \text{Third invariant of stress} \quad ...(1.31)$$

It can be noticed that

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \text{trace of the stress tensor}$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$$

= value of the determinant of the stress tensor.

Therefore, the cubic equation can be written as

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \quad ...(1.34)$$

The roots of this equation can be determined by hit and trial method. This is more time consuming and cumbersome. Therefore, it should always be preferred to determine the roots by a numerical technique. For the above equation, Newton-Raphson method may be used to determine the roots. This method can give the roots with a sufficiently high accuracy in three or four iterations and the convergance is also very rapid.

Solution of Cubic Equation

1. First Method

The cubic equation is :

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

Put $\sigma = r\cos\theta + \frac{I_1}{3}$

$$r^3\cos^3\theta + \frac{I_1^3}{27} + I_1 r^2\cos^2\theta + \frac{I_1^2}{3} r\cos\theta - I_1 r^2\cos^2\theta$$

$$-\frac{I_1^3}{9} - \frac{2}{3} I_1^2 r\cos\theta + I_2 r\cos\theta + \frac{I_1 I_2}{3} - I_3 = 0$$

$$r^3\cos^3\theta - \left(\frac{I_1^2}{3} - I_2\right) r\cos\theta - \left(\frac{2}{27} I_1^3 - \frac{I_1 I_2}{3} + I_3\right) = 0$$

$$\cos^3\theta - \frac{1}{r^2}\left(\frac{I_1^2}{3} - I_2\right)\cos\theta - \frac{1}{r^3}\left(\frac{2}{27}I_1^3 - \frac{I_1I_2}{3} + I_3\right) = 0 \quad ...(a)$$

Now $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

or $$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0 \quad ...(b)$$

Comparing Eqs. (*a*) and (*b*), we have

$$\frac{1}{r^2}\left(\frac{I_1^2}{3} - I_2\right) = \frac{3}{4}$$

$$r = \left[\frac{4}{3}\left(\frac{I_1^2}{3} - I_2\right)\right]^{1/2} \quad ...(c)$$

and $$\frac{1}{r^3}\left(\frac{2}{27}I_1^3 - \frac{I_1I_2}{3} + I_3\right) = \frac{1}{4}\cos 3\theta$$

or $$\cos 3\theta = \frac{4}{r^3}\left(\frac{2}{27}I_1^3 - \frac{I_1I_2}{3} + I_3\right)$$

$$\theta_1 = \frac{1}{3}\cos^{-1}\left[\frac{4}{r^3}\left(\frac{2}{27}I_1^3 - \frac{I_1I_2}{3} + I_3\right)\right] \quad ...(d)$$

$$\theta_2 = 120° - \theta_1$$

$$\theta_3 = 120° + \theta_1$$

Then $$\sigma_1 = r\cos\theta_1 + \frac{I_1}{3}$$

$$\sigma_2 = r\cos\theta_2 + \frac{I_1}{3}$$

$$\sigma_3 = r\cos\theta_3 + \frac{I_1}{3}$$

2. Second Method

Consider the cubic equation :

$$y^3 + py^2 + qy + r = 0$$

Put $y = x - \frac{p}{3}$, we get

$$x^3 + \left(q - \frac{p^2}{3}\right)x + \left(\frac{2}{27}p^3 - \frac{pq}{3} + r\right) = 0$$

or $$x^3 + ax + b = 0$$

where $$a = q - \frac{p^2}{3}$$

$$b = \frac{2}{27}p^3 - \frac{pq}{3} + r$$

Put $$\cos\theta = \frac{-b}{2\left(-\frac{a^3}{27}\right)^{1/2}}$$

$$g = 2\left(-\frac{a}{3}\right)^{1/2}$$

Then $$y_1 = g\cos\frac{\theta}{3} - \frac{p}{3}$$

$$y_2 = g\cos\left(120° + \frac{\theta}{3}\right) - \frac{p}{3}$$

$$y_3 = g\cos\left(240° + \frac{\theta}{3}\right) - \frac{p}{3}$$

Here $p = -I_1, \quad q = I_2, \quad r = -I_3$

$y_1 = \sigma_1, \quad y_2 = \sigma_2, \quad$ and $\quad y_3 = \sigma_3$

3. Newton-Raphson Method

Let $$f(\sigma) = \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3$$

If σ_i is the approximate root chosen for the above equation, then the better approximated root will be given by,

$$\sigma_{i+1} = \sigma_i - \frac{f(\sigma_i)}{f'(\sigma_i)}$$

where $i = 0, 1, 2, \ldots\ldots$

The iteration process may be continued till the difference between the two successive roots is not appreciable as demanded by the accuracy.

Thus, the three principal stresses σ_1, σ_2, σ_3 can be determined. Each principal stress when substituted back will give three direction cosines. For σ_1 to be substituted, let

$$A_1 = \begin{bmatrix} \sigma_y - \sigma_1 & \tau_{yz} \\ \tau_{yz} & \sigma_z - \sigma_1 \end{bmatrix}$$

= cofactor of first term of determinant D.

$$B_1 = -\begin{bmatrix} \tau_{xy} & \tau_{yz} \\ \tau_{xz} & \sigma_z - \sigma_1 \end{bmatrix}$$

= cofactor of second term of determinant D.

$$C_1 = \begin{bmatrix} \tau_{xy} & \sigma_y - \sigma_1 \\ \tau_{xz} & \tau_{yz} \end{bmatrix}$$

= cofactor of third term of determinant D.

Then the non-trivial (non-zero) solution is :

$$\frac{a_{nx_1}}{A_1} = \frac{a_{ny_1}}{B_1} = \frac{a_{nz_1}}{C_1} = K_1$$

where K_1 = a constant $\neq 0$

$$K_1 = \frac{\sqrt{a^2_{nx_1} + a^2_{ny_1} + a^2_{nz_1}}}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm \frac{1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$$

Therefore, the direction cosines for principal stress σ_1 become :

$$a_{nx_1} = \frac{A_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$$

$$a_{ny_1} = \frac{B_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} \qquad \text{...(1.35)}$$

$$a_{nz_1} = \frac{C_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$$

Similarly, direction cosines for other principal stresses can be determined.

1.12.2 Discussion

(*i*) *Case 1.* If the three roots have distinct values, *i.e.*, $\sigma_1 \neq \sigma_2 \neq \sigma_3$, then the three corresponding principal stresses occur on a single set of mutually perpendicular principal planes.

(*ii*) *Case 2.* If two of the three roots are equal (say $\sigma_1 \neq \sigma_2 = \sigma_3$) then all planes perpendicular to the principal planes on which σ_1 acts are principal planes acted on by the principal stresses $\sigma_2 = \sigma_3$.

(*iii*) *Case 3.* If the three roots are equal, *i.e*, $\sigma_1 = \sigma_2 = \sigma_3$ then every plane is a principal plane. This type of state of stress is called Hydrostatic.

If the three principal stresses are distinct then all the nine direction cosines for the three principal stresses can be determined. However, when two of the principal stresses are equal to each other or there is a hydrostatic state of stress, then all the nine direction cosines cannot be determined directly. For such a case, the following procedure may be followed.

(*iv*) Let $\sigma_1 \neq \sigma_2 = \sigma_3$.

Let the direction cosines for the three principal stresses, σ_1, σ_2 and σ_3 be given by a_{nx_1}, a_{ny_1} , a_{nz_1} ; a_{nx_2}, a_{ny_2} , a_{nz_2} ; a_{nx_3}, a_{ny_3}, a_{nz_3} respectively. Then direction cosines for σ_1 and σ_2 can be determined directly by the procedure as outlined above. In order to determine the direction cosines for σ_3, the following equations may be used.

$$a^2_{nx3} + a^2_{ny3} + a^2_{nz3} = 1$$

$$a_{nx1}.a_{nx3} + a_{ny1}.a_{ny3} + a_{nz1}.a_{nz3} = 0$$

$$a_{nx2}.a_{nx3} + a_{ny2}.a_{ny3} + a_{nz2}.a_{nz3} = 0$$

From these three equations, we can determine a_{nx3}, a_{ny3}, a_{nz3}.

(*v*) Let $\sigma_1 = \sigma_2 = \sigma_3$.

For such a state of stress, every direction is a principal direction. Since we are limiting our study to the three-coordinate system, therefore, we are interested only to determine three directions which are mutually perpendicular to each other. To determine these three directions the following procedure may be adopted.

Let the direction cosines for σ_1 be a_{nx1}, a_{ny1}, a_{nz1}. Then any two other directions (say X_2 and X_3') can be assumed, which are at right angles to the direction of σ_1 by using the orthogonality condition. Then in order to ensure that X_2 and X_3' are also perpendicular to each other, the following linear relation must hold good.

$$(X_2 + kX_3') \perp X_2$$

where k is a constant and can be determined by using orthogonality condition. Then the X_3 direction which is perpendicular to the direction of σ_1 and direction X_2 is

$$X_3 = X_2 + kX_3'$$

Hence the three mutually perpendicular directions can be determined.

1.13 Computer Program for Principal Stresses and Principal Planes

In order to demonstrate the use of the computer to determine the principal stresses and principal planes, a computer program has been developed and written for an IBM PC/XT machine using FORTRAN-77 language. The program has been illustrated by feeding numerical data. To assist the reader in understanding the program, a legend of symbols and description is given below :

$$SX = \sigma_x, \quad SY = \sigma_y, \quad SZ = \sigma_z$$
$$SXY = \tau_{xy}, \quad SYZ = \tau_{yz}, \quad SXZ = \tau_{xz}$$
$$a_1 = I_1, \quad a_2 = I_2, \quad a_3 = I_3$$
$$S(1) = \sigma_1, \quad S(2) = \sigma_2, \quad S(3) = \sigma_3$$
$$l(j) = a_{nxj}, \quad m(j) = a_{nyj}, \quad n(j) = a_{nzj}.$$

```
C  Program for principal stresses
C
C  Program developed by Dr. sadhu singh
C  Pantnager University
C
   dimension a(3), b(3), c(3), s(3)
   integer x, y
   real l(3), m(3), n(3), k(3)
   open (unit = 7, file = 'ps.in', status = 'old')
   open (unit = 8, file = 'ps.out', status = 'new')
   write (8, 5)
 5 format  (1x,  'input',  //,  'given stress components
   are :', //)
   read (7, *) sx, sy, sz, sxy, syz, sxz
   write (8, 10) sx, sy, sz
10 format  (1x, 'sx = ', f7.3, 5x, 'sy = ', f7.3, 5x, 'sz
   1 = ', 1f7.3, /)
   write (8, 20) sxy, syz, sxz
20 format (1x, 'sxy = ', f7.3, 5x, 'syz = ', f7.3, 5x, 'sxz
   1 = ', 1f7.3, //)
   a1 = sx + sy + sz
   a2 = sx*sy + sy*sz + sx*sz - sxy**2 - syz**2 - sxz**2
   a3 = sx*sy*sz + 2.*sxy*syz*sxz - sx*syz**2
   1 - sy*sxz**2 - sz*sxy**2
   Write (8, 15)
15 format (1x, 'output', //, 'principal stresses are :')
   r = (1./3.)*a1**2 - a2
   t = sqrt [(1./27.)*r**3)]
   q = (1./3.)*a1*a2 - a3 - (2./27.)*a1**3
   st = sqrt ((1./3.)*r)
   alpha = acos [- q/(2.*t)]
   s(1) = 2.*st*cos (alpha/3.) + (1./3.)*a1
   s(2) = 2. *st*cos [(alpha/3.) + 2.0944] + (1.3)*a1
   s(3) = 2.*st*cos [(alpha/3.) + 4.1888] + (1./3.)*a1
```

```
      do 50x = 1, 2
      do 40y = x, 3
      if [s(x).lt.s(y)] go to 30
      go to 40
30    temp = s(x)
      s(x) = s(y)
      s(y) = temp
40    continue
50    continue
      write (8, 60) s(1), s(2), s(3)
60    format (/, 1x, 's1 = ', f 7.3, 5x, 's2 = ', f 7.3, 5x,
      1 's3 = ', 1f 7.3, /)
      write (8, 65)
65    format (/, 'direction cosines are :' , //)
      do 80j = 1, 3
      a(j) = [(sy - s(j)]*[sz - s(j)] - syz**2
      b(j) = - (sxy*(sz - s(j)] - sxz*syz)
      c(j) = sxy*syz - sxz* [sy - s(j)]
      k(j) = 1./(sqrt [a (j)**2 + b(j)**2 + c(j)**2)]
      l(j) = a(j)*k(j)
      m(j) = b(j)*k(j)
      n(j) = c(j)*k(j)
      write (8, 70)j, l(j), j, m(j), j, n(j)
70    format (1x, 'l (', i1',) = ' , f 7.4, 10x, 'm (' , i1,')
      1 = ' , f 7.4, 10x, 'n (' , i1, ') = ' , f 7.4)
80    continue
      stop
      end
```

---

input

given stress components are :

$sx = 21.450$ $sy = 5.740$ $sz = -6.850$

$sxy = -5.720$ $syz = 11.750$ $sxz = 5.500$

output

principal stresses are :

$s1 = 23.428$ $s2 = 12.391$ $s3 = -15.479$

direction cosines are :

$l(1) = .9621$ $m(1) = -.2628$ $n(1) = .0728$

$l(2) = -.1810$ $m(2) = -.8155$ $n(2) = -.5497$

$l(3) = .2038$ $m(3) = .5157$ $n(3) = -.8321$

Example 1.4. *The state of stress at a point is given by the following array of terms :*

$$\begin{bmatrix} 9 & 6 & 3 \\ 6 & 5 & 2 \\ 3 & 2 & 4 \end{bmatrix} MPa$$

Determine the principal stresses and principal directions.

Solution. Principal stresses are the roots of the cubic equation :

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where $I_1 = \sigma_x + \sigma_y + \sigma_z = 9 + 5 + 4 = 18$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z + \tau^2_{xy} - \tau^2_{yz} - \tau^2_{xz}$$

$$= 9 \times 5 + 5 \times 4 + 9 \times 4 - 6^2 - 2^2 - 3^2$$

$$= 45 + 20 + 36 - 36 - 4 - 9 = 52$$

$$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau^2_{yz} - \sigma_y\tau^2_{xz} - \sigma_z\tau^2_{xy} + 2\tau_{xy}\tau_{yz}\tau_{xz}$$

$$= 9 \times 5 \times 4 - 9 \times 4 - 5 \times 9 - 4 \times 36 + 2 \times 6 \times 2 \times 3$$

$$= 180 - 36 - 45 - 144 + 72$$

$$= 27$$

∴ The cubic equation becomes

$$\sigma^3 - 18\sigma^2 + 52\sigma - 27 = 0$$

We shall solve this equation by Newton Raphson method.
Let the approximate root be assumed = 3. Then we have from,

First approximation $= 3 - \dfrac{f(3)}{f'(3)}$

$$= 3 - \frac{27 - 162 + 156 - 27}{27 - 108 + 52} = 3 - \frac{6}{29}$$

$$= 3 - 0.207 = 2.793$$

Second approximation

$$= 2.793 - \frac{f(2.793)}{f'(2.793)}$$

$$= 2.793 - \frac{0.387}{25.148}$$

$$= 2.793 - 0.01538 = 2.777$$

Third approximation

$$= 2.777 - \frac{f(2.777)}{f'(2.777)}$$

$$= 2.777 - \frac{0.004}{-24.837}$$

$$= 2.777 + 0.00016 = 2.776$$

Therefore, we take one root as 2.776. Dividing the cubic equation by $(\sigma - 2.776)$, we get

$$\sigma^2 - 15.224\sigma + 9.74$$

The roots of this quadratic equation are 14.554 and 0.669. Hence the three principal stresses are

$$\sigma_1 = 14.554 \text{ MPa} ; \sigma_2 = 2.776 \text{ MPa and } \sigma_3 = 0.669 \text{ MPa}$$

Now we determine the principal directions for σ_1 stress. For σ_1 to be substituted, the determinant D becomes

$$\begin{vmatrix} 9-14.554 & 6 & 3 \\ 6 & 5-14.554 & 2 \\ 3 & 2 & 4-14.554 \end{vmatrix} = \begin{vmatrix} -5.554 & 6 & 3 \\ 6 & -9.554 & 2 \\ 3 & 2 & -10.554 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} -9.554 & 2 \\ 2 & -10.554 \end{vmatrix} = 100.83 - 4 = 96.83$$

$$B_1 = -\begin{vmatrix} 6 & 2 \\ 3 & -10.554 \end{vmatrix} = -(-63.324 - 6) = 69.324$$

$$C_1 = \begin{vmatrix} 6 & -9.554 \\ 3 & 2 \end{vmatrix} = 12 + 28.662 = 40.662$$

$$\sqrt{A_1^2 + B_1^2 + C_1^2} = \sqrt{(96.83)^2 + (69.324)^2 + (40.662)^2}$$

$$= \sqrt{9376 + 4805.8 + 1653.4}$$

$$= \sqrt{15835.2} = 125.83$$

$$\therefore \quad a_{nx_1} = \frac{A_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{96.83}{125.83} = 0.769$$

$$a_{ny_1} = \frac{B_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{69.324}{125.83} = 0.550$$

$$a_{nz_1} = \frac{C_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{40.662}{125.84} = 0.325$$

Similarly, $a_{nx_2} = -0.226$; $\quad a_{nx_3} = 0.596$

$a_{ny_2} = -0.177$; $\quad a_{ny_3} = -0.800$

$a_{nz_2} = 0.944$; $\quad a_{nz_3} = 0.057$

Example 1.5 *The stress tensor at a point with reference to axes (x, y, z) is given by the array*

$$\begin{bmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix} MPa$$

Show that by transformation of the axes by 45° about the z-axis, the stress invariants remain unchanged.

Solution. The stress invariants are :

$$I_1 = 4 + 6 + 8 = 18$$

$$I_2 = 4 \times 6 + 6 \times 8 + 4 \times 8 - 1 \times 1 - 2 \times 2 - 0$$

$$= 24 + 48 + 32 - 1 - 4 = 99$$

$$I_3 = 4 \times 48 - 1 \times 8 + 2 \times (-12)$$

$$= 192 - 8 - 24 = 160$$

The direction cosines for the transformation are given by

	x	y	z
x'	$1/\sqrt{2}$	$1/\sqrt{2}$	0
y'	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
z'	0	0	1

$$\sigma_x' = 4 \times \tfrac{1}{2} + 6 \times \tfrac{1}{2} + 0 + 2 \times 1 \times \tfrac{1}{2} + 0 + 0$$

$$= 2 + 3 + 1 = 6 \text{ MPa}$$

$$\sigma_y' = 4 \times \tfrac{1}{2} + 6 \times \tfrac{1}{2} + 0 - 2 \times 1 \times \tfrac{1}{2} + 0 + 0$$

$$= 2 + 3 - 1 = 4 \text{ MPa}$$

$$\sigma_z' = 0 + 0 + 8 \times 1 + 0 + 0 + 0$$

$$= 8 \text{ MPa}$$

$$\tau_{x'y'} = -4 \times \tfrac{1}{2} + 6 \times \tfrac{1}{2} + 0 + 1\left(\tfrac{1}{2} - \tfrac{1}{2}\right) + 0 + 0$$

$$= -2 + 3 = 1 \text{ MPa}$$

$$\tau_{y'z'} = 0 + 0 + 0 + 0 + 0 + 2\,(-1/\sqrt{2})$$

$$= -\sqrt{2} \text{ MPa}$$

$$\tau_{x'z'} = 0 + 0 + 0 + 0 + 0 + 2 \times 1/\sqrt{2}$$

$$= \sqrt{2} \text{ MPa}$$

Hence the new stress tensor becomes

$$\begin{bmatrix} 6 & 1 & \sqrt{2} \\ 1 & 4 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 8 \end{bmatrix} \text{MPa}$$

The new stress invariants are :

$$I_1' = 6 + 4 + 8 = 18$$

$$I_2' = 6 \times 4 + 4 \times 8 + 6 \times 8 - 1 - 2 - 2$$
$$= 24 + 32 + 48 - 5 = 99$$
$$I_3' = 6 \times 30 - 1 \times 10 + \sqrt{2}\,(-5\sqrt{2}) = 180 - 10 - 10 = 160$$

which remains unchanged. Hence proved.

1.14 Maximum Shearing Stresses

Let us choose the axes of coordinates *oxyz* in the direction of the normals to the principal areas, *i.e.*, along the principal stresses. Let us also choose an arbitrary area whose outward normal is $\bar{n}$ having direction cosines a_{nx}, a_{ny} and a_{nz}. The normal stress on this area is :

$$\sigma_n = \sigma_1 a^2_{nx} + \sigma_2 a^2_{ny} + \sigma_3 a^2_{nz}$$

The resultant stress is

$$\sigma^2_r = \sigma^2_n + \tau^2_{ns}$$

$$\therefore \quad \tau^2_{ns} = \sigma^2_r - \sigma^2_n$$

Also $\quad \sigma^2_r = \sigma_1^2 a^2_{nx} + \sigma_2^2 a^2_{ny} + \sigma_3^2 a^2_{nz}$

$$\therefore \quad \tau^2_{ns} = (\sigma_1^2 a^2_{nx} + \sigma_2^2 a^2_{ny} + \sigma_3^2 a^2_{nz}) - (\sigma_1 a^2_{nx} + \sigma_2 a^2_{ny} + \sigma_3 a^2_{nz})^2$$

Now $\quad a^2_{nz} = 1 - a^2_{nx} - a^2_{ny}$

$$\therefore \quad \tau^2_{ns} = \sigma_1^2 a^2_{nx} + \sigma_2^2 a^2_{ny} + \sigma_3^2 (1 - a^2_{nx} - a^2_{ny}) - [\sigma_1 a^2_{nx} + \sigma_2 a^2_{ny} + \sigma_3 (1 - a^2_{nx} - a^2_{ny})]^2$$

$$= (\sigma_1^2 - \sigma_3^2)\, a^2_{nx} + (\sigma_2^2 - \sigma_3^2)\, a^2_{ny} + \sigma_3^2 - [(\sigma_1 - \sigma_3) a^2_{nx} + (\sigma_2 - \sigma_3) a^2_{ny} + \sigma_3]^2$$

For τ_{ns} to be maximum or minimum,

$$\frac{\partial \tau^2_{ns}}{\partial a_{nx}} = \frac{2\tau_{ns}\, \partial \tau_{ns}}{\partial a_{nx}} = 0$$

$$\frac{\partial \tau^2_{ns}}{\partial a_{ny}} = \frac{2\tau_{ns}\, \partial \tau_{ns}}{\partial a_{ny}} = 0$$

$\therefore$ Either $\quad \tau_{ns} = 0$

or
$$\frac{\partial \tau_{ns}}{\partial a_{nx}} = \frac{\partial \tau_{ns}}{\partial a_{ny}} = 0$$

If $\tau_{ns} = 0$, then this obviously gives a principal area.

$$\frac{\partial \tau_{ns}}{\partial a_{nx}} = 0 \text{ gives}$$

$$2(\sigma_1^2 - \sigma_3^2)\, a_{nx} - 4[(\sigma_1 - \sigma_3)\, a^2_{nx} + (\sigma_2 - \sigma_3)\, a^2_{ny} + \sigma_3]\,(\sigma_1 - \sigma_3)\, a_{nx} = 0$$

Since $\sigma_1 \neq \sigma_3$

$$\therefore \quad (\sigma_1 + \sigma_3)\, a_{nx} - 2\,[(\sigma_1 - \sigma_3)^2 a^2_{nx} + (\sigma_2 - \sigma_3) a^2_{ny} + \sigma_3]\, a_{nx} = 0$$

$$\{\sigma_1 - \sigma_3 - 2\,[(\sigma_1 - \sigma_3)\, a^2_{nx} + (\sigma_2 - \sigma_3)\, a^2_{ny}]\}\, a_{nx} = 0$$

Similarly $\dfrac{\partial \tau_{ns}}{\partial a_{ny}} = 0$ gives

$$\{\sigma_2 - \sigma_3 - 2\,[(\sigma_1 - \sigma_3)\, a_{nx} + (\sigma_2 - \sigma_3)\, a^2_{ny}]\}\, a_{ny} = 0$$

Therefore, $a_{nx} = a_{ny} = 0$, $a_{nz} = 1$ must be dropped since it gives a principal area lying in the plane *oxy*. It will therefore be necessary to consider three cases.

(*i*) $a_{nx} \neq 0, \quad a_{ny} = 0$

(*ii*) $a_{nx} = 0, \quad a_{ny} \neq 0$

(*iii*) $a_{nx} \neq 0, \quad a_{ny} \neq 0$

For the first case, we get

$$\sigma_1 - \sigma_3 - 2\,(\sigma_1 - \sigma_3)\, a^2_{nx} = 0$$

$$\therefore \quad (\sigma_1 - \sigma_3)\,(1 - 2a^2_{nx}) = 0$$

$$\therefore \quad a_{nx} = \pm \frac{1}{\sqrt{2}}, \quad a_{ny} = 0, \quad a_{nz} = \pm \frac{1}{\sqrt{2}}$$

For the second case, $\sigma_2 - \sigma_3 - 2\,(\sigma_2 - \sigma_3)\, a^2_{ny} = 0$

$$\therefore \quad (\sigma_2 - \sigma_3)\,(1 - 2a^2_{ny}) = 0$$

$$\therefore \quad a_{ny} = \pm \frac{1}{\sqrt{2}}, \quad a_{nx} = 0, \quad a_{nz} = \pm \frac{1}{\sqrt{2}}$$

The third condition is impossible, since cancelling a_{nx} and a_{ny}, subtracting the resulting equations from each other, we get $\sigma_1 = \sigma_2$, which is contrary to the assumption of $\sigma_1 \neq \sigma_2 \neq \sigma_3$.

If we had eliminated a_{ny} instead of a_{nz} and repeated the analysis, then we would have obtained

$$a_{nx} = \pm \frac{1}{\sqrt{2}}, \quad a_{ny} = \pm \frac{1}{\sqrt{2}}, \quad a_{nz} = 0$$

Each of these two solutions determine two areas passing through one of the coordinate axes and inclined to the other at angles of 45° and 135°. Substituting back, we get

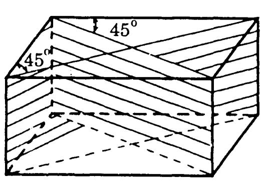


Fig. 1.11 Planes of maximum shear stress.

$$\tau^2_{ns} = \frac{\sigma_1^2 - \sigma_3^2}{2} - \left(\frac{\sigma_1 + \sigma_3}{2}\right)^2 + \sigma_3^2$$

$$= \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2$$

$$\therefore \qquad (\tau_{ns})_{max} = \pm\left(\frac{\sigma_1 - \sigma_3}{2}\right) \qquad \ldots(1.36)$$

The remaining two solutions give,

$$(\tau_{ns})_{max} = \pm\left(\frac{\sigma_2 - \sigma_3}{2}\right) \qquad \ldots(1.37)$$

and

$$(\tau_{ns})_{max} = \pm\left(\frac{\sigma_1 - \sigma_2}{2}\right) \qquad \ldots(1.38)$$

For $\sigma_1 > \sigma_2 > \sigma_3$

$$(\tau_{ns})_{max} = \pm\left(\frac{\sigma_1 - \sigma_3}{2}\right) \qquad \ldots(1.39)$$

The direction cosines for planes of $(\tau_{ns})_{max}$ and $(\tau_{ns})_{min}$ are given below and planes of maximum shear stress are shown in Fig. 1.11.

Direction cosine	$(\tau_{ns})_{max}$			$(\tau_{ns})_{min}$		
a_{nx}	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$	0	0	± 1
a_{ny}	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0	0	± 1	0
a_{nz}	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	± 1	0	0

Example 1.6 *The components of stress at a point are given by : $\sigma_x = 4$, $\sigma_y = 6$, $\sigma_z = 5$, $\tau_{xy} = 2$, $\tau_{yz} = 1$, $\tau_{zx} = 3$ MPa. Determine the principal stresses and principal planes.*

Solution. Now

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$
$$= 4 + 6 + 5 = 15$$
$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau^2_{xy} - \tau^2_{yz} - \tau^2_{zx}$$
$$= 4 \times 6 + 6 \times 5 + 4 \times 5 - 4 - 1 - 9$$
$$= 24 + 30 + 20 - 4 - 1 - 9$$
$$= 60$$
$$I_3 = -(\sigma_x\tau^2_{yz} + \sigma_y\tau^2_{zx} + \sigma_z\tau^2_{xy} - \sigma_x\sigma_y\sigma_z - 2\tau_{xy}\tau_{yz}\tau_{zx})$$
$$= -(4 \times 1 + 6 \times 9 + 5 \times 4 - 120 - 12)$$
$$= -(4 + 54 + 20 - 120 - 12)$$
$$= +54$$

$\therefore$ Cubic equation becomes

$$\sigma^3 - 15\sigma^2 + 60\sigma - 54 = 0$$

Now $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\therefore \quad \cos^3\theta - \tfrac{3}{4}\cos\theta - \tfrac{1}{4}\cos 3\theta = 0 \qquad ...(a)$$

Put $\sigma = r\cos\theta + \dfrac{I_1}{3} = r\cos\theta + 5$

∴ The cubic equation becomes :

$$r^3\cos^3\theta - 15r^3\cos^2\theta + 60r\cos\theta - 54 + 125$$
$$+ 15r^2\cos^2\theta + 75r\cos\theta - 375 - 150r\cos\theta + 300 = 0$$
$$r^3\cos^3\theta - 15r\cos\theta - 4 = 0$$
$$\cos^3\theta - \frac{15}{r^2}\cos\theta - \frac{4}{r^3} = 0 \qquad ...(b)$$

Hence, Eqs. (*a*) and (*b*) are identical if

$$\frac{15}{r^2} = \frac{3}{4}, \quad i.e., \quad r = \sqrt{20} = 4.47$$

and
$$\frac{4}{r^3} = \frac{\cos 3\theta}{4}$$

or
$$\cos 3\theta = \frac{16}{(4.47)^3} = \frac{16}{89.4} = 0.179$$

$$\therefore \quad \theta_1 = 26.6°, \theta_2 = 93.4°, \theta_3 = 146.6°$$
$$\therefore \quad r_1\cos\theta_1 = 4.47\cos 26.6° = 4.00$$
$$r_2\cos\theta_2 = 4.47\cos 93.4° = -0.265$$
$$r_3\cos\theta_3 = 4.47\cos 146.6° = -3.731$$
$$\therefore \quad \sigma_1 = r_1\cos\theta_1 + 5 = 4 + 5 = 9.0 \text{ MPa}$$
$$\sigma_2 = r_2\cos\theta_2 + 5 = -0.265 + 5 = 4.735 \text{ MPa}$$
$$\sigma_3 = r_3\cos\theta_3 + 5 = -3.731 + 5 = 1.268 \text{ MPa}$$

1.15 Octahedral Stresses

The normal and shearing stresses on a plane which is equally inclined to the three coordinate axes are called octahedral stresses.

The direction cosines of the octahedral plane are :

$$a_{nx} = \pm\frac{1}{\sqrt{3}}, \quad a_{ny} = \pm\frac{1}{\sqrt{3}}$$
$$a_{nz} = \pm\frac{1}{\sqrt{3}}$$

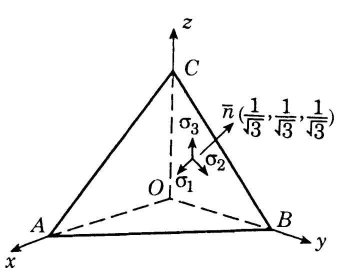


Fig. 1.12 Stresses on an octahedral plane.

The acute angles between the normal to the octahedral plane and the coordinate axes are :

$$\cos^{-1}\frac{1}{\sqrt{3}} = 54° \ 45'$$

Consider an elementary tetrahedron *OABC* as shown in Fig. 1.12. If σ_1, σ_2, σ_3 are the principal stresses on the octahedral plane, and the coordinate system is principal, then the components of stress along the three axes are :

$$T_x = \sigma_1 . a_{nx} = \frac{1}{\sqrt{3}} \sigma_1$$

$$T_y = \sigma_2 . a_{ny} = \frac{1}{\sqrt{3}} \sigma_2$$

$$T_z = \sigma_3 . a_{nz} = \frac{1}{\sqrt{3}} \sigma_3$$

Resultant stress on the octahedral plane becomes,

$$(\sigma_r)_{oct} = \sqrt{T^2_x + T^2_y + T^2_z}$$

$$= \sqrt{\frac{1}{3} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \qquad ...(1.40)$$

Octahedral normal stress,

$$(\sigma_n)_{oct} = \sigma_1 a^2_{nx} + \sigma_2 a^2_{ny} + \sigma_3 a^3_{nz}$$

$$= \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) \qquad ...(1.41)$$

∴ Octahedral shear stress,

$$\tau_{oct} = \sqrt{(\sigma_r)^2_{oct} - (\sigma_n)^2_{oct}}$$

$$= \frac{1}{3} \sqrt{(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2}$$

$$= \frac{1}{3} \sqrt{3(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_1\sigma_3)}$$

$$= \frac{1}{3} \sqrt{2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3)}$$

$$= \frac{1}{3} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$= \frac{1}{3} \sqrt{2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)}$$

$$= \frac{1}{3} \sqrt{2I_1^2 - 6I_2} \qquad ...(1.42)$$

1.16 Equilibrium Equations at Boundary

Consider an elementary tetrahedron *OABC*, as shown in Fig. 1.31. Let T_x, T_y, T_z be the components of the resultant stress acting on the face *ABC* along the three coordinate axes x, y, z respectively. In the absence of body and inertia forces, the equilibrium of forces in the x-direction give

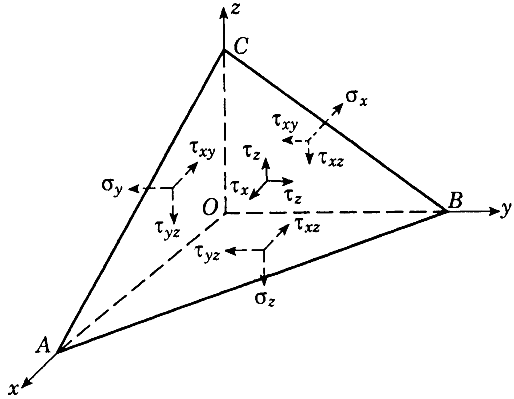


Fig. 1.13 Stresses acting on the faces of an elementary tetrahedron.

$$T_x.ABC = \sigma_x.BOC + \tau_{xy}, AOC + \tau_{xz}.AOB$$

If the direction cosines of the normal $\overline{n}$ on the face ABC are a_{nx}, a_{ny} and a_{nz}, then

$$BOC = ABC.a_{nx}$$
$$AOC = ABC.a_{ny}$$
$$AOB = ABC.a_{nz}$$

$$\therefore \quad T_x = \sigma_x.a_{nx} + \tau_{xy}.a_{ny} + \tau_{xz}.a_{nz}$$

Similarly considering equilibrium in the y and z directions, we get

$$T_y = \tau_{xy}.a_{nx} + \sigma_y.a_{ny} + \tau_{yz}.a_{nz} \qquad ...(1.43)$$
$$T_z = \tau_{xz}.a_{nx} + \tau_{yz}.a_{ny} + \sigma_z.a_{nz}$$

In the tensor notation these equation may be written as

$$T_i = \tau_{ij}.a_{nj} \qquad ...(1.44)$$

1.17 Hydrostatic and Pure Shear Components of Stress Tensor

The stress tensor at a point is given by

$$\tau_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

The stress tensor τ_{ij} can be decomposed into two stress tensors so that one of them contains pure shear components only, which is known as the *Deviator* stress tensor and produces distortion only in the body and the other contains hydrostatic components only, which is known as the *spherical* or *hydrostatic* stress tensor and produces volume change only.

$$\therefore \quad \tau_{ij} = \tau'_{ij} + \tau''_{ij}$$

= Deviator stress tensor + Spherical stress tensor.

$$= \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} \quad ...(1.45)$$

where $\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$ is the mean stress.

$$\therefore \quad \tau'_{ij} = \begin{bmatrix} \frac{2\sigma_x - \sigma_y - \sigma_z}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \frac{2\sigma_y - \sigma_z - \sigma_x}{3} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \frac{2\sigma_z - \sigma_x - \sigma_y}{3} \end{bmatrix}$$

$$\tau''_{ij} = \begin{bmatrix} \frac{\sigma_x + \sigma_y + \sigma_z}{3} & 0 & 0 \\ 0 & \frac{\sigma_x + \sigma_y + \sigma_z}{3} & 0 \\ 0 & 0 & \frac{\sigma_x + \sigma_y + \sigma_z}{3} \end{bmatrix}$$

If the coordinate system is the principal coordinate system, then,

$$\tau_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1 - \sigma_m & 0 & 0 \\ 0 & \sigma_2 - \sigma_m & 0 \\ 0 & 0 & \sigma_3 - \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

where $\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

$$\therefore \quad \tau'_{ij} = \begin{bmatrix} \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3} & 0 & 0 \\ 0 & \frac{2\sigma_2 - \sigma_3 - \sigma_1}{3} & 0 \\ 0 & 0 & \frac{2\sigma_3 - \sigma_1 - \sigma_2}{3} \end{bmatrix}$$

$$\tau''_{ij} = \begin{bmatrix} \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} & 0 & 0 \\ 0 & \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} & 0 \\ 0 & 0 & \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \end{bmatrix} \quad ...(1.46)$$

The stress invariants in the principal coordinate system for the deviator stress tensor become,

$$I_1' = 0$$

$$I_2' = -\frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad ...(1.47)$$

$$I_3' = \frac{1}{27}[(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2)]$$

and for the spherical stress tensor

$$I_1'' = 3\sigma_m = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2'' = 3\sigma^2_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)^2 \quad ...(1.48)$$

$$I_3'' = \sigma^3_m = \frac{1}{27}(\sigma_1 + \sigma_2 + \sigma_3)^3$$

Example 1.7 *Prove the following relationships :*

(*i*) $(\sigma_n)_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

(*ii*) $\tau^2_{oct} = \frac{1}{9}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$

(*iii*) $9\tau^2_{oct} = 2I_1^2 - 6I_2$.

Solution. The direction cosines of the octahedral plane are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. If $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses on the octahedral plane, then the components of the principal stresses along the principal coordinate axes are $\frac{1}{\sqrt{3}}\sigma_1, \frac{1}{\sqrt{3}}\sigma_2, \frac{1}{\sqrt{3}}\sigma_3$ respectively.

$\therefore$ Resultant stress $(\sigma_r)_{oct} = \sqrt{\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)}$

(*i*) $\sigma_n = \sigma_1 a^2_{nx} + \sigma_2 a^2_{ny} + \sigma_3 a^2_{nz}$

$\therefore$ $(\sigma_n)_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

(*ii*) $\tau_{oct} = \sqrt{(\sigma_r)^2_{oct} - (\sigma_n)^2_{oct}}$

$$= \sqrt{\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_2^2) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2}$$

$$\tau^2_{oct} = \frac{1}{9}[3(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_1\sigma_3)]$$

$$= \frac{1}{9}[(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2) + (\sigma_1^2 + \sigma_3^2 - 2\sigma_2\sigma_3) + (\sigma_3^2 + \sigma_1^2 - 2\sigma_1\sigma_3)]$$

$$= \frac{1}{9}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

(iii)

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$2I_1^2 - 6I_2 = 2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$= 2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1) - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$= 2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)$$

$$= (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2) + (\sigma_2^2 + \sigma_3^2 - 2\sigma_2\sigma_3) + (\sigma_3^2 + \sigma_1^2 - 2\sigma_3\sigma_1)$$

$$= (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 9\tau^2_{oct}$$

Hence proved.

Example 1.8 *The principal stresses at a point in a body are 50, 40 and – 30 MPa. Calculate the normal and shear stresses on the octahedral plane.*

Solution. The direction cosines of the octahedral plane are

$$a_{nx} = a_{ny} = a_{nz} = \frac{1}{\sqrt{3}}$$

Normal stress on the octahedral plane is,

$$(\sigma_n)_{oct} = \sigma^2{}_1 a_{nx} + \sigma_2 a^2{}_{ny} + \sigma_3 a^2{}_{nz}$$

$$= \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$= \frac{1}{3}(50 + 40 - 30) = 20 \text{ MPa}$$

Resultant stress on the octahedral plane is,

$$(\sigma_r)_{oct} = \sqrt{\sigma_1^2 a^2{}_{nx} + \sigma_2^2 a^2{}_{ny} + \sigma_3^2 a^2{}_{nz}}$$

$$= \sqrt{\frac{1}{3}[(50)^2 + (40)^2 + (-30)^2]}$$

$$= \sqrt{\frac{1}{3}(2500 + 1600 + 900)}$$

$$= \sqrt{\frac{5000}{3}} = \sqrt{1666.67} = 40.825 \text{ MPa}$$

∴ Shear stress on the octahedral plane becomes,

$$\tau_{oct} = \sqrt{(\sigma_r)^2_{oct} - (\sigma_n)^2_{oct}}$$

$$= \sqrt{\frac{5000}{3} - 400} = \sqrt{\frac{3800}{3}}$$

$$= \sqrt{1266.67} = 35.59 \text{ MPa}$$

Example 1.9 *The stress tensor at a point is given by the following array :*

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix} MPa$$

Calculate the deviator and spherical stress tensors.

Solution. Mean stress

$$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$$= \frac{1}{3}(4 + 3 + 2) = 3 \text{ MPa}$$

Deviator stress tensor

$$\tau'_{ij} = \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 4 \\ 3 & 4 & -1 \end{bmatrix} \text{MPa}$$

Spherical stress tensor

$$\tau''_{ij} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{MPa}$$

Example 1.10 *The stress tensor at a point is given by the following array :*

$$\begin{bmatrix} 5 & -2 & 4 \\ -2 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix} MPa$$

Determine the stress-vectors on the plane whose unit normal has direction cosines $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$.

Solution. The stress vectors are given by

$$\sigma_{rx} = \sigma_x . a_{nx} + \tau_{xy} . a_{ny} + \tau_{xz} . a_{nz}$$
$$= 5 \times \frac{1}{\sqrt{2}} - 2 \times \frac{1}{2} + 4 \times \frac{1}{2}$$
$$= 3.535 - 1 + 2 = 4.535 \text{ MPa}$$
$$\sigma_{ry} = \tau_{xy} . a_{nx} + \sigma_y . a_{ny} + \tau_{yz} . a_{nz}$$
$$= -2 \times \frac{1}{\sqrt{2}} + 2 \times \frac{1}{2} + 1 \times \frac{1}{2}$$
$$= -1.414 + 1 + 0.5 = 0.086 \text{ MPa}$$
$$\sigma_{rz} = \tau_{xz} . a_{nx} + \tau_{yz} . a_{ny} + \sigma_z . a_{nz}$$
$$= 4 \times \frac{1}{\sqrt{2}} + 1 \times \frac{1}{2} + 3 \times \frac{1}{2}$$
$$= 2.828 + 0.5 + 1.5 = 4.828 \text{ MPa}$$
$$\therefore \quad \sigma_r = (4.535\,\hat{i} + 0.086\,\hat{j} + 4.828\,\hat{k}) \text{ MPa}$$

1.18 Stress Distribution at a Point

In order to know as to how the stresses are distributed over all the possible areas passing through a given point in a body, the following procedure may be adopted.

1.18.1 Cauchy's Method

Place the origin of coordinates at some chosen point O in a body as shown in Fig. 1.14. Pass an elementary area with outward normal $\bar{n}$ having direction cosines a_{nx}, a_{ny}, a_{nz} through this point.

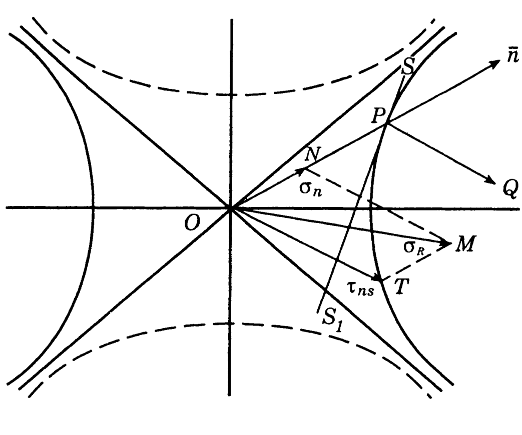


Fig. 1.14 Stress distribution at a point.

Along this normal construct a vector of length $OP = p$. The coordinates of the end of this vector are :

$$\xi = p.a_{nx}, \eta = p.a_{ny}, \zeta = p.a_{nz} \quad ...(1.49)$$

The normal stress on this area is :

$$\sigma_n = \sigma_x \cdot a^2_{nx} + \sigma_y.a^2_{ny} + \sigma_z.a^2_{nz} + 2\tau_{xy}a_{nx}a_{ny} + 2\tau_{yz}a_{ny}a_{nz} + 2\tau_{xz}a_{nx}a_{nz} \quad ...(1.50)$$

Substituting Eq. (1.49) in (1.50), we get

$$p^2\sigma_n = \sigma_x.\xi^2 + \sigma_y.\eta^2 + \sigma_z.\zeta^2 + 2\tau_{xy}\xi\eta + 2\tau_{yz}\eta\zeta + 2\tau_{xz}\xi\zeta \quad ...(1.51)$$

Now adjust the length of the vector so that

$$p^2\sigma_n = \pm c^2$$

where c is a constant. If σ_n is tensile, then take positive sign and if compressive then take negative sign.

$$\therefore \qquad p = \sqrt{\frac{\pm c^2}{\sigma_n}}$$

or

$$\sigma_n = \frac{\pm c^2}{p^2} \quad ...(1.52)$$

Introducing the relation :

$$\oint (\xi, \eta, \zeta) = \sigma_x\xi^2 + \sigma_y\eta^2 + \sigma_z\zeta^2 + 2\tau_{xy}\xi\eta + 2\tau_{yz}\eta\zeta + 2\tau_{xz}\xi\eta \quad ...(1.53)$$

we get

$$\oint (\xi, \eta, \zeta) = \pm c^2 \quad ...(1.54)$$

Eq. (1.54) represents a surface of the second order which is called the stress surface or Cauchy's quadric. This surface has the centre at the origin and it may therefore be (*i*) an ellipsoid (*ii*) a hyperboloid of one sheet or (*iii*) a hyperboloid of two sheets ; the ends of the vector $OP = \overline{p}$ lie on this surface.

Let the quadric be a hyperboloid of one sheet and that it has been constructed to simplify the drawing on the assumption that the outward drawn normal $\overline{n}$ to the area lies in this plane. Then

$$OP = \overline{p}$$

$$\therefore \qquad \sigma_n = ON = \pm \frac{c^2}{p^2}$$

In order to find the total stress σ_r on the given area, we take partial derivatives of function given by Eq. (1.53).

$$\frac{\partial \Phi}{\partial \xi} = 2\,(\sigma_x \xi + \tau_{xy}\eta + \tau_{xz}\zeta)$$

$$= 2p\,(\sigma_x a_{nx} + \tau_{xy} a_{ny} + \tau_{xz} a_{nz}) = 2p\sigma_{rx}$$

$$\frac{\partial \Phi}{\partial \eta} = 2\,(\sigma_y \eta + \tau_{xy}\xi + \tau_{yz}\zeta)$$

$$= 2p\,(\tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{yz} a_{nz}) = 2p\sigma_{ry}$$

$$\frac{\partial \Phi}{\partial \zeta} = 2\,(\sigma_z \zeta + \tau_{xz}\xi + \tau_{yz}\eta)$$

$$= 2p\,(\sigma_z a_{nz} + \tau_{xz} a_{nx} + \tau_{yz}.a_{ny}) = 2p\sigma_{rz}$$

These partial derivatives are proportional to the cosines of the angles that the normal PQ to the surface makes with the coordinate axes. On this basis the last equalities show that the cosines of the direction angles of the normal to the surface are proportional to the projections σ_{rx}, σ_{ry}, σ_{rz} of the total stress σ_r over the area concerned on the coordinate axes. Hence the total stress $OM = \sigma_r$ is perpendicular to the tangent plane SS_1 to the surface ; knowing its direction, we obtain its magnitude $\sigma_r = OM$, by drawing $NM \perp ON$. Shearing stress $\tau_{ns} = OT$ can be readily determined. Thus Cauchy's quadric surface enables the stress distribution at the given point O of the body to be fully investigated.

It is known from analytic geometry that by rotating the co-ordinate axes the equation of the surface of the second order can be transformed so as to eliminate the terms containing products of coordinates in a new coordinate system (α, β, γ). Then we shall obviously have

$$\tau_{\alpha\beta} = \tau_{\beta\gamma} = \tau_{\alpha\gamma} = 0$$

Hence the shearing stresses will vanish on three mutually perpendicular areas normal to the new axes (α, β, γ). These three ares are called principal, the normal stresses on them σ_α, σ_β, σ_γ are called principal stresses at the given point O in the body. We denote them more compactly as :

$$\sigma_\alpha = \sigma_1,\ \sigma_\beta = \sigma_2,\ \sigma_\gamma = \sigma_3$$

Then the equation of Cauchy's quadric referred to the new axes is,

$$\sigma_1\xi^2 + \sigma_2\eta^2 + \sigma_3\zeta^2 = \pm c^2 \qquad ...(1.55)$$

The final form of this surface depends upon the sign of the principal stresses σ_1, σ_2, σ_3 and on the associated sign of the right hand term of the equation. Consider two fundamental cases.

(*i*) all three principal stresses are of the same sign.

(*ii*) the signs of the principal stresses are different.

Case I. Let us assume that $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_3 > 0$, *i.e.*, all the principal stresses are tensile. Then the plus sign should apparently be chosen on the right hand side. Therefore, we get

$$\frac{\xi^2}{\left(\frac{c^2}{\sigma_1}\right)} + \frac{\eta^2}{\left(\frac{c^2}{\sigma_2}\right)} + \frac{\zeta^2}{\left(\frac{c^2}{\sigma_3}\right)} = 1 \qquad ...(1.56)$$

This is the equation of an ellipsoid with the semi-axes

$$\bar{a} = \frac{c}{\sqrt{\sigma_1}}, \bar{b} = \frac{c}{\sqrt{\sigma_2}}, \bar{c} = \frac{c}{\sqrt{\sigma_3}}$$

In this case, $\sigma_n = +\frac{c^2}{p^2} > 0$, *i.e.*, the normal stresses on all areas passing through the point in question are positive and consequently tensile. If all principal stresses are compressive, *i.e.*, $\sigma_1 < \sigma_2 < \sigma_3 < 0$, then minus sign should be taken before c^2, this again gives an ellipsoid, but now $\sigma_n = -\frac{c^2}{p^2} < 0$, *i.e.*, the normal stresses on all areas passing through the point in question are compressive.

Case II. Let us assume that $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 < 0$, *i.e.*, two of principal stresses σ_1 and σ_2 are tensile and the third stress σ_3 is compressive. Thus we can write now,

$$\sigma_1\xi^2 + \sigma_2\eta^2 - |\sigma_3|\zeta^2 = +c^2$$

$$\sigma_1\xi^2 + \sigma_2\eta^2 - |\sigma_3|\zeta^2 = -c^2$$

where $|\sigma_3|$ denotes the absolute value of the compressive principal stress. Above equations can be written as

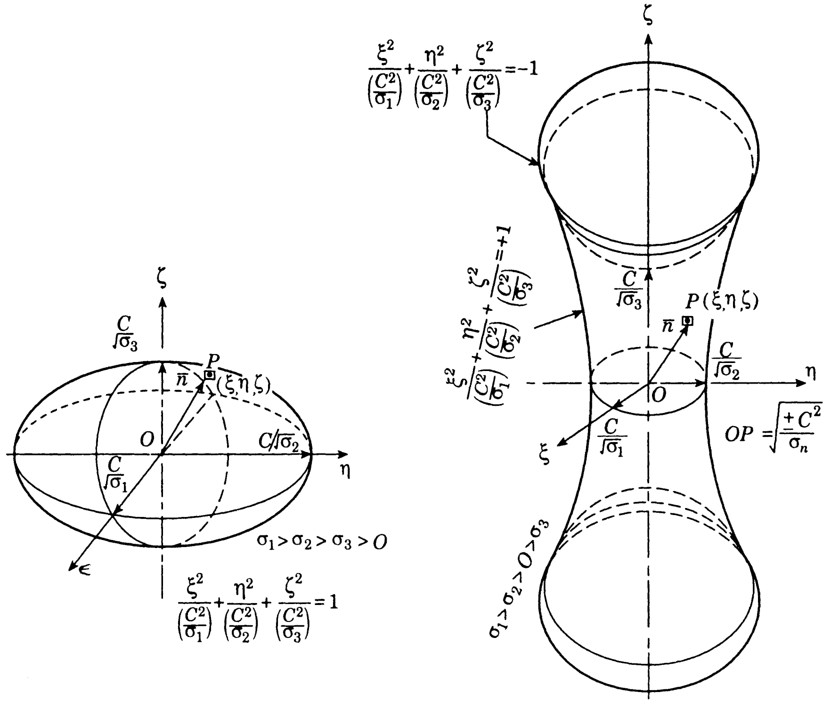
$$\frac{\xi^2}{\left(\frac{c^2}{\sigma_1}\right)} + \frac{\eta^2}{\left(\frac{c^2}{\sigma_2}\right)} - \frac{\zeta^2}{\left(\frac{c^2}{|\sigma_3|}\right)} = 1 \qquad ...(1.57)$$

$$-\frac{\xi^2}{\left(\frac{c^2}{\sigma_1}\right)} - \frac{\eta^2}{\left(\frac{c^2}{\sigma_2}\right)} + \frac{\zeta^2}{\left(\frac{c^2}{|\sigma_3|}\right)} = 1 \qquad ...(1.58)$$

Equations (1.57) and (1.58) give hyperboloids of one sheet and two sheets respectively. For $c^2 = 0$, we get,

$$\sigma_1\xi^2 + \sigma_2\eta^2 - |\sigma_3|\zeta^2 = 0 \qquad ...(1.59)$$

Equation (1.59) gives the asymptotic cone separating the above two hyperboloids and tending to approach them it infinity. These have been shown in Fig. 1.15.



(*a*) Stress ellipsoid. (*b*) one sheeted and two sheeted hyperboloids

Fig. 1.15 Cauchy's method.

If the end of the vector representing the normal stress on the area appears to be on the hyperboloid of one sheet, then this stress is positive, *i.e.*, tensile ; if on the other hand, it appears to be on the hyperboloid of two sheets, then it is negative, *i.e.*, compressive. In the intermediate case (for $c^2 = 0$), it may be directed along the generator of the asymptotic cone. In this case the length of the vector becomes infinite and according to Eq. (1.52), $\sigma_n = 0$. Hence, on areas normal to the generators of the asymptotic cone there act only shearing stresses.

Case III. Two principal stresses negative, *i.e.*, $\sigma_1 > 0$, $\sigma_2 < 0$, $\sigma_3 < 0$. In this case, the conic surface is split into two parts again.

But here the x-axis is the central axis, and all rays OP extending out to the hyperboloid of one sheet correspond to outer normals of planes having a compressive normal stress, and those extending out to the hyperboloid of two sheets correspond to planes of tensile stresses.

Case IV. All principal stresses negative, *i.e.* $\sigma_1 \leq \sigma_2 \leq \sigma_3 < 0$. In this case, negative sign must be taken before c^2. This case is similar to case *I* except that all stresses are negative.

Case V. One principal stress zero.

(*a*) $\sigma_1 \geq \sigma_2 > \sigma_3 = 0$

In this case, Eq. (1.56) becomes independent of ζ, and the surface degenerates to a cylinder with sides parallel to ζ axis. The cross-section of the cylinder has the shape,

$$\xi^2\sigma_1 + \eta^2\sigma_2 = c^2$$

or

$$\frac{\xi^2}{\left(\dfrac{c^2}{\sigma_1}\right)} + \frac{\eta^2}{\left(\dfrac{c^2}{\sigma_2}\right)} = 1$$

which is an ellipse or circle depending upon whether

$$\sigma_1 \neq \sigma_2 \quad \text{or} \quad \sigma_1 = \sigma_2$$

(*b*) $\sigma_1 > \sigma_2 = 0 > \sigma_3$

Both signs of c^2 are applicable and the equation becomes independent of η.

$$\frac{\xi^2}{\left(\dfrac{c^2}{\sigma_1}\right)} + \frac{\zeta^2}{\left(\dfrac{c^2}{\sigma_3}\right)} = \pm 1$$

which (since $\sigma_3 < 0$) represents a double set of hyperbolic cylinders with sides parallel to η axis and asymptotic to the planes.

$$\xi = \pm \sqrt{\frac{\sigma_3}{\sigma_1}} \cdot \zeta$$

Zero normal stresses are again evident when the ray OP lies in one of these asymptotic planes. If $|\sigma_3| = \sigma_1$, the asymptotic planes are inclined at 45° to the $\eta\zeta$ and $\xi\eta$ planes.

(*c*) $\sigma_1 = 0 > \sigma_2 \geq \sigma_3$

This is similar to the case V(*a*) except that the elliptic cylinder is parallel to ξ axis and normal stresses are negative on all planes.

Case VI. Two principal stresses are zero.

(*a*) $\sigma_1 > \sigma_2 = \sigma_3 = 0$

This case is that of a double set of infinite planes.

$$\sigma_1\xi^2 = c^2 \quad \text{or} \quad \xi = \pm\sqrt{\frac{c^2}{\sigma_1}}$$

which are parallel to the $\eta\zeta$ planes.

(*b*) $$\sigma_1 = \sigma_2 = 0 > \sigma_3$$

Here the two planes are given by

$$\sigma_3\zeta^2 = -c^2$$

or $$\zeta = \pm\sqrt{\frac{c^2}{|\sigma_3|}}$$

which are parallel to the $\xi\eta$ plane.

1.18.2 Lame's Method

If the principal areas at a given point have been found, then, along with Cauchy's quadric, it is possible to indicate another geometrical representation of stress distribution proposed by Lame's—the stress ellipsoid.

Let us assume that the coordinate planes *oxy, oyz* and *ozx* coincide with the principal areas at a given point and consequently on these areas

$$\tau_{xy} = \tau_{yz} = \tau_{xz} = 0 \quad \text{and} \quad \sigma_x = \sigma_1, \sigma_y = \sigma_2, \sigma_z = \sigma_3$$

$$\therefore \quad \left.\begin{aligned} \sigma_{rx} &= \pm\,\sigma_1\, a_{nx} \\ \sigma_{ry} &= \sigma_2\, a_{ny} \\ \sigma_{rz} &= \sigma_3\, a_{nz} \end{aligned}\right\} \qquad ...(1.60)$$

Relationships (1.60) may be represented geometrically. To this end, we lay off from the given point O, a vector $\overline{OP}$ equal to the total stress σ_r on the chosen area with outward normal $\overline{n}$ as shown in Fig. 1.16. The co-ordinates of the end of this vector are

$$x = \sigma_{rx}, \quad y = \sigma_{ry}$$
$$z = \sigma_{rz} \qquad ...(1.61)$$

As the inclination of the area is changed, the point P describes a certain surface that appears to be an ellipsoid. From Eqs. (1.60) and (1.61) we have

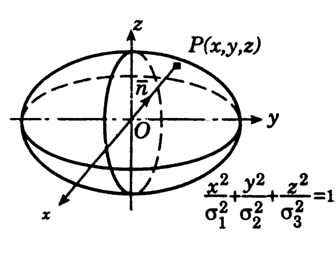


Fig. 1.16 Lames stress ellipsoid.

$$x = \sigma_1 . a_{nx}, y = \sigma_2 . a_{ny}, z = \sigma_3 . a_{nz}$$

$$\therefore \quad \frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} + \frac{z^2}{\sigma_3^2} = a^2_{nx} + a^2_{ny} + a^2_{nz} = 1 \qquad ...(1.62)$$

This is the equation of an ellipsoid referred to the principal axes. This ellipsoid is called the stress ellipsoid or Lame's ellipsoid. One of its three semi-axes is the longest, another the shortest, the third is half way between them : consequently, the three principal stresses are maximum, minimum and minimax respectively.

If two of the principal stresses are equal (*e.g.*, $\sigma_1 = \sigma_2$), Lame's ellipsoid is an ellipsoid of revolution and the state of stress at a given point is symmetric with respect to the third principal axis OZ. If all the principal stresses are equal, *i.e.* $\sigma_1 = \sigma_2 = \sigma_3$, then Lame's ellipsoid becomes a sphere and all the areas at the given point are principal and the stresses on them are equal. This is a case of hydrostatic state of stress or all round tension.

1.19 Graphical method for the Determination of Normal and Shear Stresses. (Three dimensional Mohr Circle).

Consider a point O in a stressed body where the planes of the xyz axes coincide with the principal axes of stress at the point. Consider an infinitesimal tetrahedron $OABC$, as shown in Fig. 1.17 at the point O whose face ABC has a unit normal $\bar{n}$ having direction cosines a_{nx}, a_{ny}, a_{nz}. Let σ_n and τ_{ns} be the normal and shearing stresses on the inclined face ABC, then the resultant stress σ_r becomes.

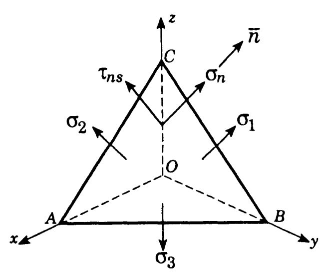


Fig. 1.17

$$\sigma^2_r = \sigma_n^2 + \tau^2_{ns}$$

Also $\sigma^2_r = \sigma_1^2 a^2_{nx} + \sigma_2^2 a^2_{ny} + \sigma_3^2 a^2_{nz}$

$$\therefore \quad \sigma_1^2 a^2_{nx} + \sigma_2^2 a^2_{ny} + \sigma_3^2 a^2_{nz} = \sigma_n^2 + \tau^2_{ns}$$

Further $\quad \sigma_n = \sigma_1 a^2_{nx} + \sigma_2 a^2_{ny} + \sigma_3 a^2_{nz}$

$$1 = a^2_{nx} + a^2_{ny} + a^2_{nz}$$

Thus we get three non-homogenous simultaneous equations, which may be written as :

$$\begin{bmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a^2_{nx} \\ a^2_{ny} \\ a^2_{nz} \end{bmatrix} = \begin{bmatrix} \sigma^2_n + \tau^3_{ns} \\ \sigma_n \\ 1 \end{bmatrix} \qquad ...(1.63)$$

These equations can be solved for $a^2_{nx}, a^2_{ny}, a^2_{nz}$ by Cramer's rule. Hence

$$a^2_{nx} = \frac{\begin{vmatrix} \sigma^2_n + \tau^2_{ns} & \sigma_2^2 & \sigma_3^2 \\ \sigma_n & \sigma_2 & \sigma_3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & 1 & 1 \end{vmatrix}}$$

$$a^2_{ny} = \frac{\begin{vmatrix} \sigma_1^2 & \sigma^2_n + \tau^2_{ns} & \sigma_3^2 \\ \sigma_1 & \sigma_n & \sigma_3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & 1 & 1 \end{vmatrix}}$$

$$a^2_{nz} = \frac{\begin{vmatrix} \sigma_1^2 & \sigma_2^2 & \sigma^2_n + \tau^2_{ns} \\ \sigma_1 & \sigma_2 & \sigma_n \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & 1 & 1 \end{vmatrix}}$$

Case I. All principal stresses are different, *i.e.*, $\sigma_1 \neq \sigma_2 \neq \sigma_3$ and $\sigma_1 > \sigma_2 > \sigma_3$, then

$$\begin{vmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$$

Solving for $a^2_{nx}, a^2_{ny}, a^2_{nz}$, we get

$$a^2_{nx} = \frac{(\sigma^2_n + \tau^2_{ns})(\sigma_2 - \sigma_3) - \sigma_2^2(\sigma_n - \sigma_3) + \sigma_3^2(\sigma_n - \sigma_2)}{\sigma_1^2(\sigma_2 - \sigma_3) - \sigma_2^2(\sigma_1 - \sigma_3) + \sigma_3^2(\sigma_1 - \sigma_2)}$$

or $$a^2_{nx} = \frac{\tau^2_{ns} + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \qquad ...(1.64)$$

$$\text{Similarly, } a^2_{ny} = \frac{\tau^2_{ns} + (\sigma_n - \sigma_3)(\sigma_n - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \quad \text{...(1.65)}$$

$$a^2_{nz} = \frac{\tau^2_{ns} + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \quad \text{...(1.66)}$$

From equation (1.64), we get

$$\tau^2_{ns} + \sigma^2_n - \sigma_n(\sigma_2 + \sigma_3) + \sigma_2\sigma_3 - a^2_{nx}(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) = 0$$

or
$$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 + a^2_{nx}(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)$$

This equation can be written as :

$$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = \left(\sigma_1 - \frac{\sigma_2 + \sigma_3}{2}\right)a^2_{nx} + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2(1 - a^2_{nx})$$

For given values of $\sigma_1, \sigma_2, \sigma_3$ this equation represents a family of circles in the σ_n and τ_{ns} plane with centre at $\left(\frac{\sigma_2 + \sigma_3}{2}, 0\right)$ and direction cosine a_{nx} as a parameter. Since $0 \le a^2_{nx} \le 1$, the family is bounded by two limiting circles,

$$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 \text{ for } a_{nx} = 0$$

and
$$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = \left(\sigma_1 - \frac{\sigma_2 + \sigma_3}{2}\right)^2 \text{ for } a_{nx} = 1$$

Similarly from Eqs. (1.65) and (1.66), we get respectively

$$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_3 + \sigma_1}{2}\right)^2 = \left(\frac{\sigma_3 - \sigma_1}{2}\right)^2 \text{ for } a_{ny} = 0$$

$$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_2 + \sigma_1}{2}\right)^2 = \left(\sigma_2 - \frac{\sigma_3 + \sigma_1}{2}\right)^2 \text{ for } a_{ny} = 1$$

with centre at
$$\left(\frac{\sigma_3 + \sigma_1}{2}, 0\right)$$

and $$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_1 + \sigma_2}{2}\right)^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 \text{ for } a_{nz} = 0$$

$$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_1 + \sigma_2}{2}\right)^2 = \left(\sigma_3 - \frac{\sigma_1 + \sigma_2}{2}\right)^2 \text{ for } a_{nz} = 1$$

with centre at $[(\sigma_1 + \sigma_2)/2, 0]$

In the $\sigma_n - \tau_{ns}$ plane, these limiting circles have been shown in Fig. 1.18. The point representing the normal and shear stresses σ_n and τ_{ns} must lie within the shaded area.

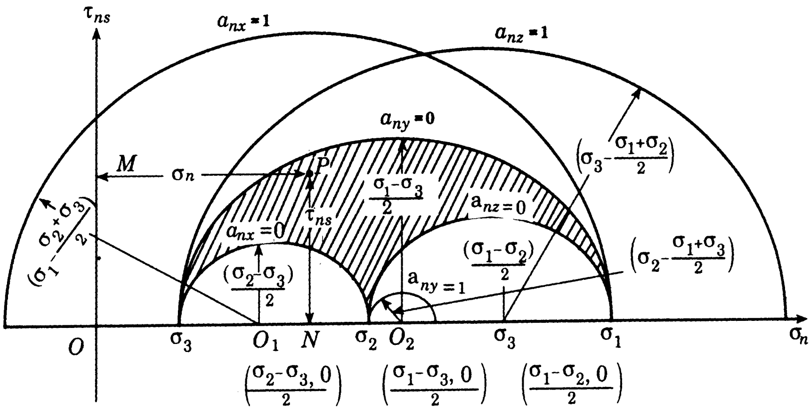


Fig. 1.18 Three-diamensional Mohr's circles.

The maximum shear stress,

$$(\tau_{ns})_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

and the normal stress on the same plane is $\frac{\sigma_1 + \sigma_3}{2}$.

Therefore, by substituting

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} \text{ and } \tau_{ns} = \frac{\sigma_1 - \sigma_3}{2}$$

in Eqs. (1.64) to (1.66), we get

$$a_{nx} = \pm \frac{1}{\sqrt{2}}, \; a_{ny} = 0 \text{ and } \sigma_{nz} = \pm \frac{1}{\sqrt{2}}$$

Therefore, the plane of the maximum shear stress bisects the angle formed by the planes of maximum and minimum normal stresses.

In order to determine the normal and shearing stresses on a plane whose direction cosines are given, the following radii may be laid off at points O_1, O_2 and O_3. The point of intersection of these radii will locate the point of interest giving normal and shearing stresses σ_n and τ_{ns} respectively.

$$r_1 = \sqrt{\left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 + a^2_{nx}(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \text{ at } O_1\left(\frac{\sigma_2 + \sigma_3}{2}, 0\right)$$

$$r_2 = \sqrt{\left(\frac{\sigma_3 - \sigma_1}{2}\right)^2 + a^2_{ny}(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \text{ at } O_2\left(\frac{\sigma_1 + \sigma_3}{2}, 0\right)$$

$$r_3 = \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + a^2_{nz}(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \text{ at } O_3\left(\frac{\sigma_1 + \sigma_2}{2}, 0\right)$$

Case II. Two principal stresses are equal $\sigma_1 = \sigma_2 > \sigma_3$.

In this case, we have

$$\sigma_1^2 (a^2_{nx} + a^2_{ny}) + \sigma_3^2 a^2_{nz} = \sigma^2_n + \tau^2_{ns}$$

$$\sigma_1 (a^2_{nx} + a^2_{ny}) + \sigma_3 a^2_{nz} = \sigma_n$$

$$a^2_{nx} + a^2_{ny} + a^2_{nz} = 1$$

These can be written as

$$\begin{bmatrix} \sigma_1^2 & \sigma_1^2 & \sigma_3^2 \\ \sigma_1 & \sigma_1 & \sigma_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a^2_{nx} \\ a^2_{ny} \\ a^2_{nz} \end{bmatrix} = \begin{bmatrix} \sigma^2_n + \tau^2_{ns} \\ \sigma_n \\ 1 \end{bmatrix}$$

The solution exists only if the determinants

$$\begin{vmatrix} \sigma^2_n + \tau^2_{ns} & \sigma_1^2 & \sigma_3^2 \\ \sigma_n & \sigma_1 & \sigma_3 \\ 1 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} \sigma_1^2 & \sigma^2_n + \tau^2_{ns} & \sigma_3^2 \\ \sigma_1 & \sigma_n & \sigma_3 \\ 1 & 1 & 1 \end{vmatrix}$$

and $\begin{vmatrix} \sigma_1^2 & \sigma_1^2 & \sigma^2_n + \tau^2_{ns} \\ \sigma_1 & \sigma_1 & \sigma_n \\ 1 & 1 & 1 \end{vmatrix}$ are all zero.

The third determinant does not give any solution as it is itself zero. The first two determinants give identical results as

$$\tau^2_{ns} + \left(\sigma_n - \frac{\sigma_1 + \sigma_3}{2}\right)^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2$$

This result is similar to Mohr's circle in two dimensions.

Case III. All the three principal stresses are equal.

$$\sigma_1 = \sigma_2 = \sigma_3$$

This is a hydrostatic state of stress. In this case,

$$\sigma_n = \sigma_1 = \sigma_2 = \sigma_3$$
$$\tau_{ns} = 0$$

1.20 Graphical Construction for the Determination of Normal and Shearing Stresses

The following steps may be followed (Fig. 1.19) :

1. Along the σ_n axis, locate σ_1, σ_2, σ_3 after choosing a convenient scale.

2. Locate the centres O_1, O_2 and O_3 at distances $\left(\frac{\sigma_2 + \sigma_3}{2}\right)$, $\left(\frac{\sigma_3 + \sigma_1}{2}\right)$ and $\left(\frac{\sigma_1 + \sigma_2}{2}\right)$ respectively from the origin.

3. Draw three circles with centres O_1, O_2 and O_3 and radii $\left(\frac{\sigma_2 + \sigma_3}{2}\right)$, $\left(\frac{\sigma_3 + \sigma_1}{2}\right)$ and $\left(\frac{\sigma_1 + \sigma_2}{2}\right)$ respectively.

4. Draw vertical lines at σ_1, σ_2 and σ_3. Set off the angle $\alpha = \cos^{-1} a_{nx}$ from the vertical at σ_1 in the anticlockwise direction and draw the line $\sigma_1 Q_3 Q_2$ cutting the circles with centres at O_2 and O_3 at Q_2 and Q_3 respectively.

5. With centre O_1 draw arc Q_2Q_3.

6. Set off the angle $\gamma = \cos^{-1} a_{nz}$ from the vertical at point σ_3 in the clockwise direction and draw line $\sigma_3 S_1 S_2$ cutting the circles with centres at O_1 and O_2 at S_1 and S_2 respectively.

7. With centre O_3 draw arc S_1S_2.

8. The intersection of the arcs Q_2Q_3 and S_1S_2 at point P is the required point to give the normal and shearing stresses.

9. To check the construction, set off $\beta = \cos^{-1} a_{ny}$ on each side of the vertical at point σ_2 to cut the circles with centres at O_1 and O_3 at T_1 and T_3 respectively. With centre O_2 draw arc T_1T_3. If the diagram is correct then the arc T_1T_3 will pass through point P.

10. From point P drop perpendiculars on σ_n and τ_{ns} axes. Then $PM = \sigma_n$ and $PN = \tau_{ns}$.

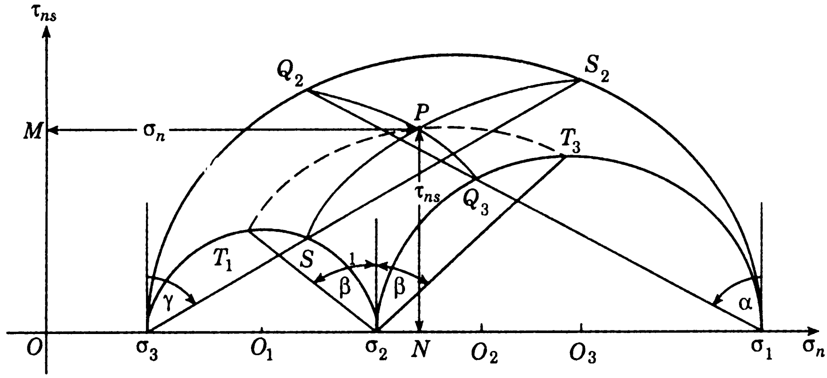


Fig. 1.19 Graphical construction for three dimensional Mohr's circles.

Proof. Draw $\sigma_1 Q_3 Q_2$ at angle $\alpha = \cos^{-1} a_{nx}$ with the vertical at point σ_1 in the counterclockwise direction (Fig. 1.20). Join $\sigma_3 Q_2$. Then $\angle\, Q_2\sigma_3 O_1 = \alpha$ [$\because$ $\angle\, \sigma_3 Q_2 \sigma_1 = 90°$ being the angle on a semi-circle].

Draw $O_1N \perp \sigma_3 Q_2$ and $O_1Q' \perp Q_2Q_3$. Also join $\sigma_2 Q_3$ and O_1Q_2 and O_1Q_3.

In $\Delta\, Q_2O_1N$, we have

$$(Q_2O_1)^2 = (Q_2N)^2 + (O_1N)^2$$

$$Q_2N = Q_2\sigma_3 - \sigma_3 N$$

$$= (\sigma_1 - \sigma_3)\cos\alpha - \sigma_3 O_1 \cos\alpha$$

$$= (\sigma_1 - \sigma_3)\, a_{nx} - \left(\frac{\sigma_2 - \sigma_3}{2}\right) a_{nx}$$

$$\therefore \quad (Q_2O_1)^2 = \left[(\sigma_1 - \sigma_3) - \left(\frac{\sigma_2 - \sigma_3}{2}\right)\right]^2 a^2_{nx} + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 \sin^2\alpha$$

$$= \left[(\sigma_1 - \sigma_3)^2 + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 - (\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)\right] a^2_{nx}$$

$$+ \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 (1 - a^2_{xn})$$

$$= [(\sigma_1 - \sigma_3)^2 - (\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)]\, a^2_{nx} + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

$$= (\sigma_1 - \sigma_3)(\sigma_1 - \sigma_2)\, a^2_{nx} + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

Since $O_1Q' \perp Q_2Q_3$. Therefore, $O_1Q_2Q_3$ is an isosceles triangle. Hence $Q_2Q' = Q_3Q'$ and $O_1Q_2 = O_1Q_3$.

$\therefore$ O_1Q_2 is the radius of the arc with centre at O_1 and the arc passes through the point Q_3. Similarly other arcs bear the same correspondence to the stress point.

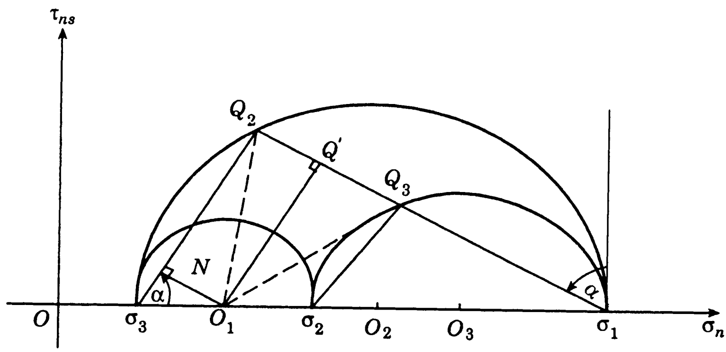


Fig. 1.20

Example 1.11 *The principal stresses on a plane are : $\sigma_1 = 9$, $\sigma_2 = 6$, $\sigma_3 = 3$ kPa. Determine the normal and shearing stresses on a plane whose direction cosines are $\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}$ by using three-dimensional Mohr's circle.*

Solution. First method. (Fig. 1.21).

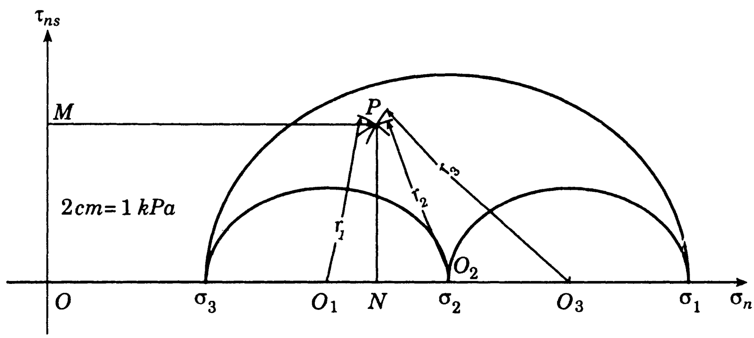


Fig. 1.21

Centres of Mohr's circles are :

$$O_1 : \left(\frac{\sigma_2 + \sigma_3}{2}, 0 \right) = (4.5, 0)$$

$$O_2 : \left(\frac{\sigma_3 + \sigma_1}{2}, 0 \right) = (6, 0)$$

$$O_3 : \left(\frac{\sigma_1 + \sigma_2}{2}, 0 \right) = (7.5, 0)$$

Now, $\quad a_{nx} = \frac{1}{2}, \quad a_{ny} = \frac{1}{2}, \quad a_{nz} = \frac{1}{\sqrt{2}}$

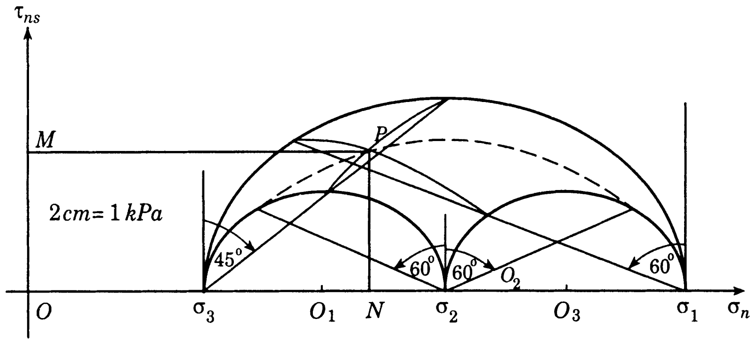


Fig. 1.22

The radii of circles are :

$$r_1 = \sqrt{\left(\frac{\sigma_2 - \sigma_3}{2} \right)^2 + a^2_{nx} (\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \text{ at } O_1$$

$$= \sqrt{\frac{9}{4} + \frac{1}{4} \times 3 \times 6} = \sqrt{2.25 + 4.5}$$

$$= \sqrt{6.75} = 2.6 \text{ kPa}$$

$$r_2 = \sqrt{\left(\frac{\sigma_3 - \sigma_1}{2} \right)^3 + a^2_{ny} (\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \text{ at } O_2$$

$$= \sqrt{\frac{36}{4} + \frac{1}{4} \times 3 \times (-3)}$$

$$= \sqrt{9 - \frac{9}{4}} = \sqrt{6.75} = 2.6 \text{ kPa}$$

$$r_3 = \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2} \right)^2 + a^2_{nz} (\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \text{ at } O_3$$

$$= \sqrt{\frac{9}{4} + \frac{1}{2}(-6)(-3)}$$

$$= \sqrt{2.25 + 9} = \sqrt{11.25} = 3.35 \text{ kPa}$$

Let 2 cm = 1.0 kPa

Take $O\sigma_1$ = 18 cm, $O\sigma_2$ = 12 cm and $O\sigma_3$ = 6 cm. Locate the centres O_1, O_2 and O_3. With centre O_1 draw a circle passing through σ_1 and σ_2, with centre O_2 draw the circle passing through σ_1 and σ_3 and with centre O_3 draw the circle passing through σ_2 and σ_3. Now with centres O_1, O_2 and O_3 draw arcs with radii r_1, r_2 and r_3 respectively to intersect at point P. Draw PN and PM perpendiculars on σ_n and τ_{ns} axes respectively. Then $\sigma_n = PM$ = 5.25 kPa and $\tau_{ns} = PN$ = 2.5 kPa.

Second Method. (Fig. 1.22).

$$\alpha = \cos^{-1} a_{nx} = \cos^{-1} \frac{1}{2} = 60°$$

$$\beta = \cos^{-1} a_{ny} = \cos^{-1} \frac{1}{2} = 60°$$

$$\gamma = \cos^{-1} a_{nz} = \cos^{-1} \frac{1}{\sqrt{2}} = 45°$$

Now follow the steps as outlined in article 1.20.

From the graphical construction, we get

$$\sigma_n = 5.25 \text{ kPa}, \quad \tau_{ns} = 2.5 \text{ kPa}.$$

1.21 Equilibrium Equations in Polar Coordinates

Consider an elementary area $ABCD$ between radii r and $r + dr$ and angular positions θ and $\theta + d\theta$, as shown in Fig. 1.23. Let B_r and B_θ be the body forces acting on the area $ABCD$ per unit area. Considering the equilibrium of forces in the r-direction by using Newton's second law of motion, we get

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right)(r + dr)\, d\theta - \sigma_r . r d\theta$$

$$+ \left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta\right) dr - \tau_{\theta r} dr - \sigma_\theta . dr \,.\, \frac{d\theta}{2}$$

$$- \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right) dr.\, \frac{d\theta}{2}$$

$$+ B_r . r d\theta . dr = \rho . \bar{a}_r . r d\theta . dr$$

where $\bar{a}_r$ = acceleration in the r-direction.

ρ = density.

Simplifying and neglecting small quantities and noting that $\sin \frac{d\theta}{2} \simeq \frac{d\theta}{2}$, $\cos \frac{d\theta}{2} \simeq 1$, we get

$$\frac{\partial \sigma_r}{d_r}.dr.rd\theta + \sigma_r.dr.d\theta + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta dr - \sigma_\theta.dr.\frac{d\theta}{2}$$

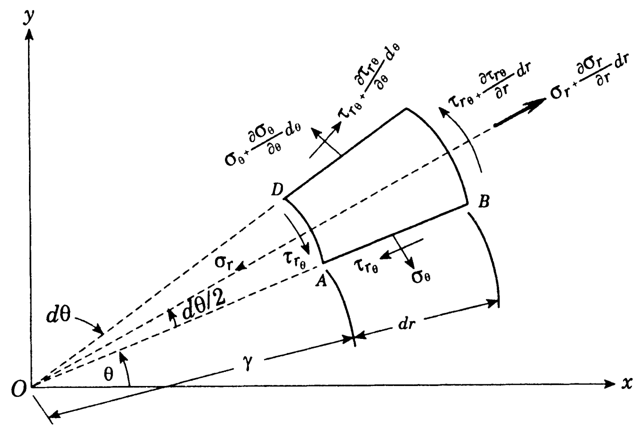
$$- \sigma_\theta.dr\frac{d\theta}{2} + B_r r d\theta dr = \rho\bar{a}_r.rd\theta dr$$

Dividing throughout by $r.d\theta.dr$, we get

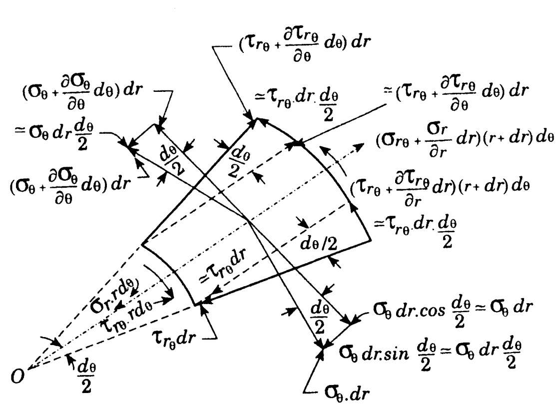
$$\frac{\partial \sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + B_r = \rho.\bar{a}_r$$

In the absence of body and inertia forces, we get

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} = 0$$



(*a*) Stress gradients in polar coordinates.



(*b*) Resolution of forces along r and θ directions

Fig. 1.23 Stress components on an elementary area in polar coordinates.

Now consider the equilibrium of forces in the θ-direction,

$$\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}.d\theta\right) dr - \sigma_\theta.dr + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r}.dr\right)(r + dr)\,d\theta$$

$$- \tau_{r\theta}.rd\theta + \left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta\right) dr . \frac{d\theta}{2} + \tau_{\theta r}.dr . \frac{d\theta}{2}$$

$$+ B_\theta.rd\theta.dr = \rho.\bar{a}_\theta.rd\theta dr$$

where $\bar{a}_\theta$ = acceleration in the θ-direction.

Simplifying and neglecting small quantities, we get

$$\frac{\partial \sigma_\theta}{\partial \theta}.d\theta dr + \frac{\partial \tau_{r\theta}}{\partial r}.rd\theta.dr + \tau_{r\theta}.dr.d\theta + \tau_{\theta r}.dr.\frac{d\theta}{2}$$

$$+ \tau_{\theta r}.dr.\frac{d\theta}{2} + B_\theta.rd\theta dr = \rho.\bar{a}_\theta.rd\theta dr$$

Dividing throughout by $rd\theta.dr$, we get

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta r}}{\partial r} + \frac{\tau_{r\theta}}{r} + \frac{\tau_{\theta r}}{r} + B_\theta = \rho\bar{a}_\theta$$

In the absence of body and inertia forces and noting that $\tau_{\theta r} = \tau_{r\theta}$, we get

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial_\theta} + \frac{2\tau_{r\theta}}{r} = 0$$

Hence the equilibrium equations in polar coordinates become,

$$\left.\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} = 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0 \end{aligned}\right\} \qquad \text{...(1.67)}$$

1.22 Equilibrium Equations in Cylindrical Coordinates

Consider an infinitesimal parallelopiped in cylindrical co-ordinates subjected to generalised state of stress, as shown in Fig. 1.24. Considering the equilibrium of forces in the r-direction by using Newton's second law of motion, we get

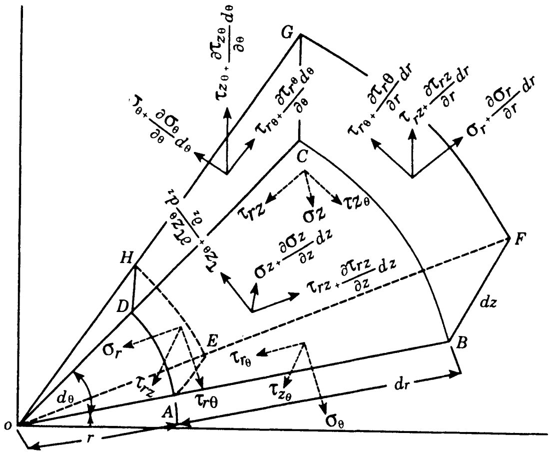


Fig. 1.24 Stress components on the faces of an elementary parallelopiped in cylindrical coordinates.

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} . dr\right)(r + dr)\, d\theta dz - \sigma_r . r d\theta . dz$$

$$+ \left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta} . d\theta\right) drdz - \tau_{\theta r} . drdz$$

$$+ \left(\tau_{zr} + \frac{\partial \tau_{zr}}{\partial z} . dz\right) r d\theta . dr - \tau_{zc} . r d\theta . dr$$

$$-\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} . d\theta\right) . drdz . \frac{d\theta}{2} - \sigma_\theta . dr . dz \frac{d\theta}{2}$$

$$+ B_r . r d\theta . dr . dz = \rho . \bar{a}_r . r d\theta . dr . dz$$

where $\bar{a}_r$ = acceleration in the r-direction.

Simplifying and neglecting small quantities, we get

$$\frac{\partial \sigma_r}{\partial r} . dr . r d\theta . dz + \sigma_r . dr . d\theta . dz + \frac{\partial \tau_{\theta r}}{\partial \theta} . d\theta . dr . dz$$

$$+ \frac{\partial \tau_{zr}}{\partial z} . dz . r d\theta . dr - \sigma_\theta . dr . dz . d\theta$$

$$+ B_r . r d\theta drdz = \rho . \bar{a}_r . r d\theta . dr . dz$$

Dividing throughout by $r d\theta . dr . dz$, we get

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \sigma_r + \frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\sigma_\theta}{r} + B_r = \rho . \bar{a}_r$$

In the absence of body and inertia forces, we get

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} (\sigma_r - \sigma_\theta) + \frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} = 0$$

Now consider the equilibrium of forces in the θ-direction

$$\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} . d\theta\right) drdz - \sigma_\theta.drdz + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} \; dr\right)(r + dr).d\theta.dz$$

$$- \tau_{r\theta}.rd\theta.dz + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} . d\theta\right) drdz. \frac{d\theta}{2} + \tau_{r\theta}.drdz . \frac{\partial \theta}{2}$$

$$+ \left(\tau_{\theta z} + \frac{\partial \tau_{\theta z}}{\partial z} dz\right) .rd\theta.dr - \tau_{\theta z}.rd\theta.dr + B_\theta.rd\theta drdz$$

$$= \rho.\bar{a}_\theta.rd\theta.dr.dz$$

Simplifying and neglecting small quantities, we get

$$\frac{\partial \sigma_\theta}{\partial \theta} .drd\theta dz + \frac{\partial \tau_{r\theta}}{\partial r} .dr.rd\theta dz + \tau_{r\theta}.drd\theta dz$$

$$+ \tau_{r\theta} drdzd\theta + \frac{\partial \tau_{\theta z}}{\partial z} dz.rd\theta.dr + B_\theta.rd\theta.dr.dz$$

$$= \rho.\bar{a}_\theta.rd\theta.dr.dz$$

where $\bar{a}_\theta$ = acceleration in the θ-direction

Dividing throughout by $rd\theta.dr.dz$, we get

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + \frac{\partial \tau_{\theta z}}{\partial z} + B_\theta = \rho.\bar{a}_\theta$$

In the absence of body and inertia forces, we get

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + \frac{\partial \tau_{\theta z}}{\partial z} = 0$$

Similarly in the z-direction, we get

$$\left(\sigma_z + \frac{\partial \sigma_z}{\partial z} . dz\right) rd\theta.dr - \sigma_z.rd\theta.dr + \left(\tau_{\theta z} + \frac{\partial \tau_{\theta z}}{\partial \theta} d\theta\right) .drdz$$

$$- \tau_{\theta z} . drdz + \left(\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r} dr\right)(r + dr)\, d\theta dz$$

$$- \tau_{rz}.rd\theta.dz + B_z.rd\theta.dr.dz = \rho.\bar{a}_z.rd\theta.drdz$$

Simplifying and neglecting small quantities, we get

$$\frac{\partial \sigma_z}{\partial z} dz.rd\theta.dr + \frac{\partial \tau_{\theta z}}{\partial \theta} .d\theta.dr.dz + \tau_{rz}.drd\theta dz$$

$$+ \frac{\partial \tau_{rz}}{\partial r} dr.rd\theta.dz + B_z.rd\theta drdz = \rho.\bar{a}_z.rd\theta.dr.dz$$

where $\bar{a}_z$ = acceleration in the z-direction

Dividing throughout by $rd\theta.dr.dz$, we get

$$\frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{1}{r}\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r} + B_z = \rho.\bar{a}_z$$

In the absence of body and inertia forces, we get

$$\frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\frac{\tau_{\theta z}}{\partial \theta} + \frac{1}{r}\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r} = 0$$

Hence the equilibrium equations in cylindrical coordinates in the absence of body and inertia forces become,

$$\left.\begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \\ &\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \\ &\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \end{aligned}\right\} \quad \text{...(1.68)}$$

For axial symmetry, and neglecting inertia forces, we get

$$\left.\begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0 \\ &\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + B_\theta = 0 \\ &\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + B_z = 0 \end{aligned}\right\} \quad \text{...(1.69)}$$

Example 1.12 *The components of stress at a point are given by*

$$\sigma_r = 2r^2 \cos\theta, \quad \sigma_\theta = 3r^2 z + 5z\cos\theta$$

$$\sigma_z = 2rz^2 + r^2, \qquad \tau_{r\theta} = r\theta, \tau_{\theta z} = \theta z, \tau_{rz} = rz.$$

Determine the body force distribution, for the body to be in equilibrium at the point (2, π/6, 3).

Solution. Equilibrium equation in the r-direction is given by :

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0$$

$$\therefore \quad 4r\cos\theta + \frac{1}{r}.r + r + \frac{1}{r}(2r^2\cos\theta - 3r^2 z - 5z\cos\theta) + B_r = 0$$

$$4r\cos\theta + 1 + r + 2r\cos\theta - 3rz - \frac{5z}{r}\cos\theta + B_r = 0$$

At the point $\left(2, \frac{\pi}{6}, 3\right)$, we get

$$4\times 2\times\cos\frac{\pi}{6} + 1 + 2 + 2\times 2\times\cos\frac{\pi}{6} - 3\times 2\times 3 - \frac{5\times 3}{2}\cos\frac{\pi}{6} + B_r = 0$$

$$8 \times \frac{\sqrt{3}}{2} + 1 + 2 + 4 \times \frac{\sqrt{3}}{2} - 18 - \frac{15}{2} \times \frac{\sqrt{3}}{2} + B_r = 0$$

$$4\sqrt{3} + 3 + 2\sqrt{3} - 18 - 3.75\sqrt{3} + B_r = 0$$

$$2.25 \times 1.732 + 3 - 18 + B_r = 0$$

$$\therefore \quad B_r = 18 - 3 - 3.897$$

or $$B_r = 11.103$$

Equilibrium equation in the θ-direction is :

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + B_\theta = 0$$

$$\therefore \quad \theta + \frac{1}{r}(-5z \sin \theta) + \theta + 2\theta + B_\theta = 0$$

$$\frac{\pi}{6} + \frac{1}{2}\left(-5 \times 3 \times \frac{1}{2}\right) + \frac{\pi}{6} + \frac{\pi}{3} + B_\theta = 0$$

$$\therefore \quad B_\theta = \frac{15}{4} - \frac{2\pi}{3} = 3.75 - 2.09 = 1.66$$

Similarly in the z-direction, we have

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + B_z = 0$$

$$r + \frac{1}{r} . z + 4rz + z + B_z = 0$$

$$3 + \tfrac{3}{2} + 4 \times 2 \times 3 + 3 + B_z = 0$$

$$\therefore \quad B_z = -3 - 1.5 - 24 - 3 = -31.5$$

∴ Body force field required for equilibrium is :

$$\overline{B} = 11.103\hat{i} + 1.66\hat{j} - 31.5\hat{k}$$

1.23 Equilibrium Equations in Spherical Co-ordinates

Consider an infinitesimal parallelopiped in the spherical co-ordinate system r, θ, ϕ as shown in Fig. 1.25. Volume of the elementary parallelopiped is : $r \sin \phi . r d\theta . dr d\phi = r^2 \sin \phi . dr d\theta d\phi$. The equilibrium of forces in the r-direction gives :

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right)(r + dr) \sin \phi . d\theta . (r + dr)\, d\phi - \sigma_r . r^2 \sin \phi \, d\theta d\phi$$

$$+ \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} d\theta\right) r d\phi dr - \tau_{r\theta} . r d\phi dr$$

$$+ \left(\tau_{\phi r} + \frac{\partial \tau_{\phi r}}{\partial \phi} d\phi\right) r \sin (\phi + d\phi)\, d\theta dr$$

$$- \tau_{\phi r} . r \sin \phi \, d\theta dr - \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta \right) r d\phi \, dr. \frac{d\theta}{2} \sin \phi$$

$$- \sigma_\theta \, r d\phi dr. \frac{d\theta}{2} \sin \phi - \left(\sigma_\phi + \frac{\partial \sigma_\phi}{\partial \phi} d\phi \right) r \sin \phi \, d\theta dr \frac{d\phi}{2}$$

$$- \sigma_\phi \, r \sin \phi \, d\theta dr. \frac{d\phi}{2} + B_r.r^2 \sin \phi \, dr d\theta d\phi$$

$$= \rho.\bar{a}_r.r^2 \sin \phi \, dr d\theta d\phi$$

where $\bar{a}_r$ = acceleration in the r-direction

Simplifying and neglecting small quantities, we get

$$\frac{\partial \sigma_r}{\partial r} \sin \phi + \frac{1}{r} \Bigg(2\sigma_r \sin \phi - \sigma_\theta - \sin \phi - \sigma_\phi \sin \phi$$

$$+ \tau_{\phi r} \cos \phi + \frac{\partial r_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\phi r}}{\partial \phi} \sin \phi \Bigg) + B_r \sin \phi \, \rho \bar{a}_r \sin \phi$$

In the absence of body and inertia forces, we get

$$r \frac{\partial \sigma_r}{\partial r} + (2\sigma_r - \sigma_\theta - \sigma_\phi + \tau_{\phi r} \cot \phi) + \frac{1}{\sin \phi} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\phi r}}{\partial \phi} = 0$$

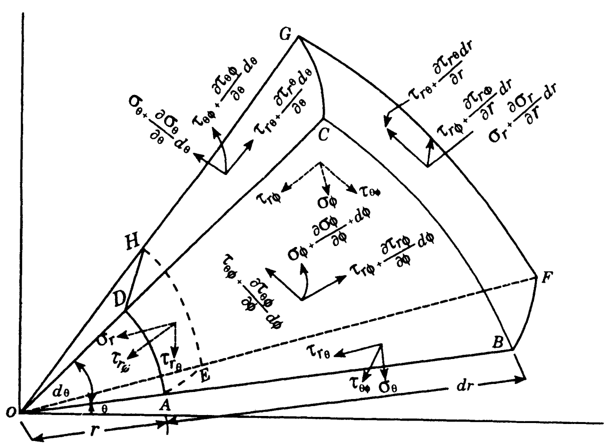


Fig. 1.25 Stress components on the faces of an elementary parallelopiped in spherical co-ordinates.

or
$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r \sin \phi} . \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\phi r}}{r \partial \phi} + \left(\frac{2\sigma_r - \sigma_\theta - \sigma_\phi + \cot \phi . \tau_{\phi r}}{r} \right) = 0$$

Equilibrium of forces in the θ-direction gives :

$$\left(\sigma_\theta + \frac{\partial\sigma_\theta}{\partial\theta} d\theta\right) r.d\phi.dr - \sigma_\theta\, r d\phi dr$$

$$+ \left(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial_r} dr\right)(r + dr)\,[\sin(\phi + d\phi) + \sin\phi]\,.\,\frac{d\theta}{2}\,.\,(r + dr)\, d\phi$$

$$- \tau_{r\theta}\, r\,[\sin(\phi + d\phi) + \sin\phi]\,.\,\frac{d\theta}{2}\,.r d\phi$$

$$+ \left(\tau_{\theta\phi} + \frac{\partial\tau_{\theta\phi}}{\partial\phi}\,.\, d\phi\right)\left(r + \frac{dr}{2}\right)\sin(\phi + d\phi)\, d\theta\, dr$$

$$- \tau_{\theta\phi}\left(r + \frac{dr}{2}\right)\sin\phi\,.d\theta.dr$$

$$+ \left(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial\theta} d\theta\right)\left(r + \frac{dr}{2}\right) d\phi\, dr.\,\frac{d\theta}{2}\sin\phi$$

$$+ \tau_{r\theta}\,.\left(r + \frac{dr}{2}\right) d\phi\, dr\,.\,\frac{d\theta}{2}\sin\phi$$

$$+ \left(\tau_{\theta\phi} + \frac{\partial\tau_{\theta\phi}}{\partial\theta} d\theta\right)\left(r + \frac{dr}{2}\right) d\phi\, dr \cos\phi\,.\,\frac{d\theta}{2}$$

$$+ \tau_{\theta\phi}\,.\left(r + \frac{dr}{2}\right) d\phi.dr.\cos.\phi.\,\frac{d\theta}{2}$$

$$+ B_\theta.r^2 \sin\phi\, dr\, d\theta\, d\phi = \rho.\bar{a}_\theta\, r^2 \sin\phi.dr.d\theta.d\phi$$

where $\bar{a}_\theta$ = acceleration in the θ-direction.

Simplifying and neglecting small quantities, we get

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{\theta\phi}}{r\partial\phi} + \frac{1}{r}(3\tau_{r\theta} + 2\tau_{\theta\phi}\,.\cot\phi) + B_\theta = \rho\bar{a}_\theta$$

Neglecting body and inertia forces, we get

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r\sin\phi}\,.\,\frac{\partial\sigma_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{3\tau_{r\theta} + 2\tau_{\theta\phi}\cot\phi}{r} = 0$$

Similarly equilibrium of forces in the φ-direction gives :

$$\left(\sigma_\phi + \frac{\partial\sigma_\phi}{\partial\phi} d\phi\right)\left(r + \frac{dr}{2}\right)\sin(\phi + d\phi)\, d\theta\, dr$$

$$- \sigma_\phi\left(r + \frac{dr}{2}\right)\sin\phi.d\theta.dr$$

$$+\left(\tau_{\phi r}+\frac{\partial \tau_{\phi r}}{\partial r}dr\right)(r+dr)\sin\phi\, d\theta\,(r+dr)\,d\phi$$

$$-\tau_{\phi r}\, r\sin\phi\, d\theta . r d\phi+\left(\tau_{\theta\phi}+\frac{\partial \tau_{\theta\phi}}{\partial \theta}d\theta\right)\left(r+\frac{dr}{2}\right)d\phi\, dr$$

$$-\tau_{\theta\phi}\cdot\left(r+\frac{dr}{2}\right)d\phi . dr$$

$$+\left(\tau_{\phi r}+\frac{\partial \tau_{\phi r}}{\partial \phi}d\phi\right)\left(r+\frac{dr}{2}\right)\sin\phi\, d\theta\, dr\,\frac{d\phi}{2}$$

$$+\tau_{\phi r}\cdot\left(r+\frac{dr}{2}\right)\sin\phi\, d\theta\, dr\,\frac{d\phi}{2}-\sigma_\theta.\cos\phi\, r dr d\theta\, d\phi$$

$$+B_\phi . r^2\sin\phi . dr d\theta\, d\phi=\rho.\bar{a}_\phi . r^2\sin\phi . dr d\theta\, d\phi$$

where $\bar{a}_\phi$ = acceleration in the ϕ-direction.

Simplifying and neglecting small quantities, we get

$$\frac{\partial \tau_{\phi r}}{\partial r}+\frac{1}{r\sin\phi}\frac{\partial \tau_{\theta\phi}}{\partial \theta}+\frac{1}{r}\frac{\partial \sigma_\phi}{\partial \phi}+\frac{1}{r}[3\tau_{\phi r}+(\sigma_\phi-\sigma_\theta)\cot\phi]+B_\phi=\rho\bar{a}_\phi$$

Neglecting body and inertia forces, we get

$$\frac{\partial \tau_{\phi r}}{\partial r}+\frac{1}{r\sin\phi}\cdot\frac{\partial \tau_{\theta\phi}}{\partial \theta}+\frac{1}{r}\frac{\partial \sigma_\phi}{\partial \phi}+\frac{1}{r}[3\tau_{\phi r}+(\sigma_\phi-\sigma_\theta)\cot\phi]=0$$

Hence the equilibrium equations in spherical coordinates become :

$$\frac{\partial \sigma_r}{\partial r}+\frac{1}{r\sin\phi}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{1}{r}\frac{\partial \tau_{\phi r}}{\partial \phi}+\frac{1}{r}(2\sigma_r-\sigma_\theta-\sigma_\phi+\cot\phi\,.\,\tau_{\phi r})=0$$

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r\sin\phi}\frac{\partial \sigma_{\theta\phi}}{\partial \theta}+\frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \phi}+\frac{1}{r}(3\tau_{r\theta}+2\cot\phi . \tau_{\theta\phi})=0 \quad ...(1.70)$$

$$\frac{\partial \tau_{\phi r}}{\partial r}+\frac{1}{r\sin\phi}\frac{\partial \tau_{\theta\phi}}{\partial \theta}+\frac{1}{r}\frac{\partial \sigma_\phi}{\partial \phi}+\frac{1}{r}[3\tau_{\phi r}+(\sigma_\phi-\sigma_\theta)\cot\phi]=0$$

1.24 Two-Dimensional Analysis

For plane stress problems, $\sigma_z=\tau_{xz}=\tau_{yz}=0$ and the stress tensor becomes :

$$\tau_{ij}=\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}$$

$$\therefore \quad \left.\begin{aligned} I_1 &= \sigma_x + \sigma_y \\ I_2 &= \sigma_x\sigma_y - \tau^2_{xy} \\ I_3 &= \sigma_x\sigma_y - \tau^2_{xy} \end{aligned}\right\} \qquad ...(1.71)$$

The principal stresses are the roots of the equation :

$$\sigma^2 - (\sigma_x + \sigma_y)\sigma + (\sigma_x.\sigma_y - \tau^2_{xy}) = 0 \qquad ...(1.72)$$

$$\therefore \quad \sigma_{1,\,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x + \sigma_y)^2 - 4(\sigma_x\sigma_y - \tau^2_{xy})}$$

or

$$\sigma_{1,\,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}} \qquad ..(1.73)$$

When the stresses are to be transformed into the new co-ordinate system $ox'y'$ by a rotation of angle θ about the old co-ordinate system oxy in the anticlockwise direction, as shows in Fig. 1.26, then the direction cosine matrix is

	x	y
x'	$\cos\theta$	$\sin\theta$
y'	$-\sin\theta$	$\cos\theta$

Now using the transformation law

$$\tau_{i'j'} = a_{il}\, a_{jm}\, \tau_{lm}$$

We get

$$\tau_{im} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_x\cos\theta + \tau_{xy}\sin\theta & \tau_{xy}\cos\theta + \sigma_y\sin\theta \\ -\sigma_x\sin\theta + \tau_{xy}\cos\theta & -\tau_{xy}\sin\theta + \sigma_y\cos\theta \end{bmatrix}$$

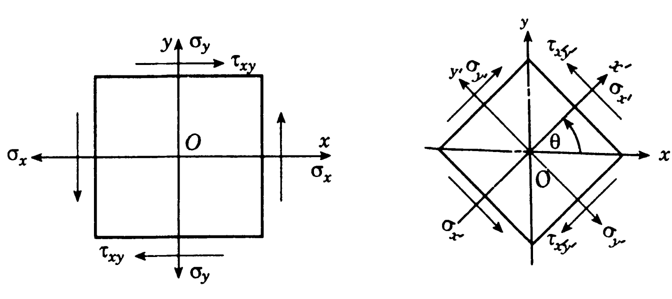


Fig. 1.26 Stresses in two dimensions.

$$\tau_{i'j} = a_{jm}\,[\tau_{mi}]^T$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_x \cos\theta + \tau_{xy}\sin\theta & -\sigma_x \sin\theta + \tau_{xy}\cos\theta \\ \sigma_y \sin\theta + \tau_{xy}\cos\theta & -\tau_{xy}\sin\theta + \sigma_y\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} (\sigma_x \cos^2\theta + \sigma_y \sin^2\theta \pm 2\tau_{xy}\sin\theta\cos\theta) & \begin{pmatrix} -\sigma_x\sin\theta\cos\theta + \tau_{xy}\cos^2\theta \\ +\sigma_y\sin\theta\cos\theta - \tau_{xy}\sin^2\theta \end{pmatrix} \\ \begin{pmatrix} -\sigma_x\sin\theta\cos\theta - \tau_{xy}\sin^2\theta \\ +\sigma_y\sin\theta\cos\theta + \tau_{xy}\cos^2\theta \end{pmatrix} & (\sigma_x\sin^2\theta - \tau_{xy}\sin\theta\cos\theta - \tau_{xy}\sin\theta\cos\theta + \sigma_y\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} (\sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta) & (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \\ (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) & \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \end{bmatrix}$$

$$\therefore \quad \left.\begin{aligned} \sigma_{x'} &= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau_{xy}\sin 2\theta \\ \sigma_{y'} &= \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - \tau_{xy}\sin 2\theta \\ \tau_{x'y'} &= \frac{(\sigma_y - \sigma_x)}{2}\sin 2\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned}\right\} \quad \text{...(1.74)}$$

1.24.1 Two-Dimensional Mohr's Circle

For the state of stress shown in Fig. 1.27, the normal and shear stresses are :

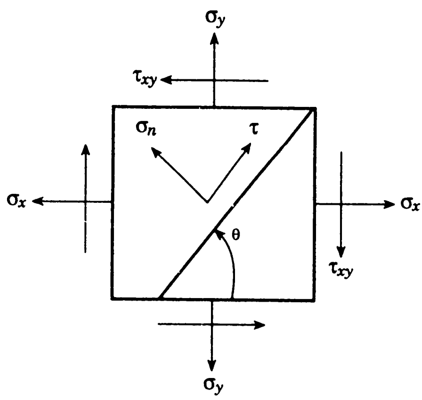


Fig. 1.27

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_y - \sigma_x)\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{...(1.75)}$$

$$\tau = \frac{1}{2}(\sigma_y - \sigma_x)\sin 2\theta - \tau_{xy}\cos 2\theta \quad \text{...(1.76)}$$

The principal stresses are :

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}} \quad \text{...(1.77)}$$

The principal planes are :

$$\tan 2\theta = \frac{-2\tau_{xy}}{\sigma_y - \sigma_x} \qquad \text{...(1.78)}$$

The Mohr's circle has been drawn in Fig. 1.28.

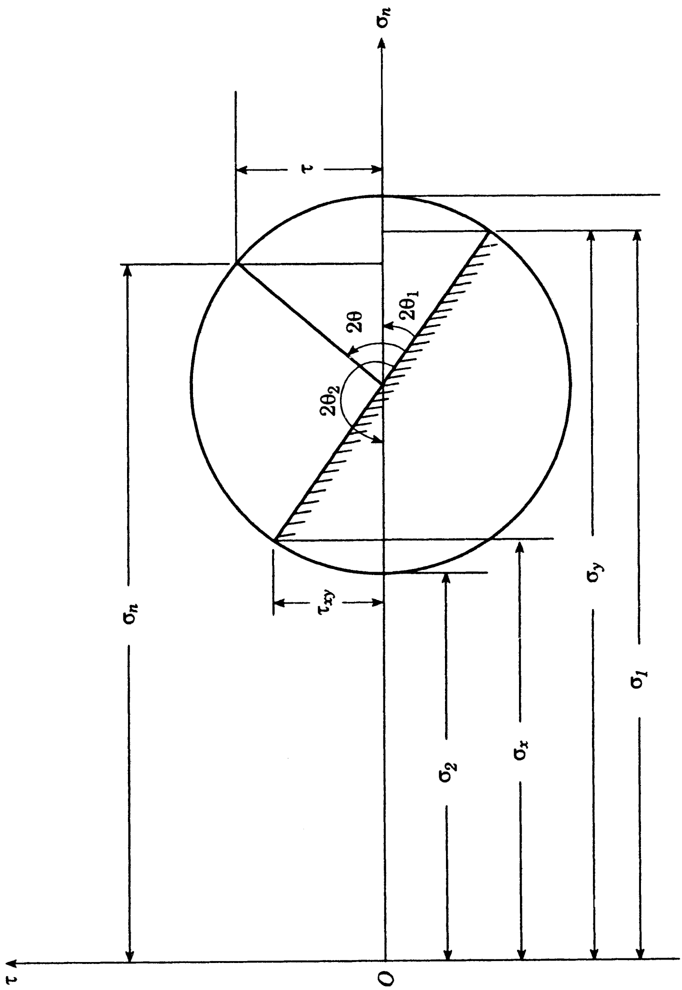


Fig. 1.28

Example 1.13 *The state of plane stress at a point is given by :*

$$\sigma_x = 60,\ \sigma_y = 40\ ;\ \tau_{xy} = 20\ kPa.$$

Determine (a) principal stresses, and (b) normal and shearing stresses on a plane inclined at 30° with the x-axis.

Solution. (*a*) Principal stresses are given by

$$\sigma_{1,\,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$= \frac{60 + (-40)}{2} \pm \frac{1}{2}\sqrt{(60 + 40)^2 + 4 \times 20^2}$$

$$= 10 \pm \frac{1}{2}\sqrt{10000 + 1600}$$

$$= 10 \pm \frac{1}{2}\sqrt{11600}$$

$$= 10 \pm \frac{1}{2}(107.7) = 10 \pm 53.85$$

$$= 63.85, -43.85 \text{ kPa}$$

(*b*) The direction cosines are

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\tau_{i'j'} = a_{il}\, a_{jm}\, \tau_{lm}$$

$$\tau_{im} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 60 & 20 \\ 20 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 30\sqrt{3} + 10 & 10\sqrt{3} - 20 \\ -30 + 10\sqrt{3} & -10 - 20\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 61.96 & -2.68 \\ -12.68 & -44.64 \end{bmatrix}$$

$$\tau_{i'j'} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 61.96 & -12.68 \\ 2.68 & -44.64 \end{bmatrix}$$

$$= \begin{bmatrix} 30.98\sqrt{3} - 1.34 & -6.34\sqrt{3} - 22.32 \\ -30.98 - 1.34\sqrt{3} & 6.34 - 22.32\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 52.31 & -33.30 \\ -33.30 & -32.32 \end{bmatrix}$$

$\therefore$ $\sigma_{x'} = 52.31$ kPa

$\sigma_{y'} = -32.32$ kPa

$\tau_{x'y'} = -33.30$ kPa

Example 1.14 *The stress components at a point in a body subjected to two dimensional state of stress are given by :*

$$\sigma_x = 2x^2y + 3xy$$
$$\sigma_y = 2x^3 + 5xy^2$$
$$\tau_{xy} = 4x^2y^2$$

Determine whether the equations of equilibrium are satisfied at the point (– 2, 3) or not.

Solution. The equations of equilibrium in the absence of body forces in two dimensions are given by :

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

Substituting the stress components, we get

$$(4xy + 3y) + 8x^2y$$

and $$8xy^2 + 10xy$$

At the point (– 2, 3), we get

$$4(-2)(3) + 3 \times 3 + 8 \times 4 \times 3$$
$$= -24 + 9 + 96$$
$$= 81$$

and $$8(-2)(9) + 10\,(-2)\,3$$
$$= -144 - 60$$
$$= -204$$

Therefore we find that the equations of equilibrium are not satisfied.

Example 1.15 *Calculate the principal stresses for the stress components given below :*

$$\begin{bmatrix} 3 & 5 & 6 \\ 5 & 4 & 2 \\ 6 & 2 & 5 \end{bmatrix} MPa.$$

Solution. $I_1 = 3 + 4 + 5 = 12$

$$I_2 = 12 + 20 + 15 - 25 - 4 - 36 = -18$$
$$I_3 = 3 \times 16 - 5 \times 13 + 6 \times (-14) = -101$$
$$f(\sigma) = \sigma^3 - 12\sigma^2 - 18\sigma + 101$$

Comparing with cubic equation

$$y^3 + py^2 + qy + r = 0$$

we have $p = -12,\ q = -18,\ r = 101$

$$a = q - \frac{p^2}{3} = -18 - 48 = -66$$

$$b = \frac{2}{27}p^3 - \frac{pq}{3} + r = \frac{2}{27}(-1728) - \frac{1}{3}(-12)(-18) + 101$$

$$= -128 - 72 + 101 = -99$$

$$\cos\theta = \frac{-b}{2\left(-\frac{a^3}{27}\right)^{1/2}} = \frac{99}{2(103.189)} = 0.4797$$

$$\theta = 61.334°$$

$$g = 2\sqrt{\frac{66}{3}} = 9.381$$

$$\sigma_1 = y_1 = g\cos\frac{\theta}{3} - \frac{p}{3}$$

$$= 9.381 \cos 20.445° + 4 = 12.79 \text{ MPa}$$

$$\sigma_2 = y_2 = g\cos\left(120° + \frac{\theta}{3}\right) - \frac{p}{3}$$

$$= 9.381 \cos 140.445° + 4 = -3.23 \text{ MPa}$$

$$\sigma_3 = y_3 = g\cos\left(240° + \frac{\theta}{5}\right) - \frac{p}{3}$$

$$= 9.381 \cos 260.445° + 4 = 2.44 \text{ MPa}$$

EXERCISES

1.1 (*a*) Define, body force, surface force and give examples.

(*b*) Differentiate clearly between internal and external forces.

1.2 The Cartesian components of stress at a piont are :

$$\sigma_x = 1000,\ \sigma_y = -600,\ \sigma_z = 0,\ \tau_{xy} = 200,\ \tau_{xz} = 0$$

and $\tau_{yz} = -400$ N/mm^2.

Determine the normal and shear stresses on a plane whose normal is given by $\bar{n} = 0.112\hat{i} + 0.35\hat{j} + 1.93\hat{k}$.

[**Ans.** – 305.67 N/mm^2 521.95 N/mm^2]

1.3 The state of stress at a point for a given reference axes *xyz* is given by : $\sigma_x = 200$, $\sigma_y = 0$, $\sigma_z = 500$, $\tau_{xy} = 100$, $\tau_{yz} = \tau_{xz} = 0$ N/m^2.

If a new set of axes $x'y'z'$ is formed by rotating *xyz* axes through 60° about the *z*-axis in the anticlockwise direction, determine the components of stress for the new axes. Prove that the invariants remain unchanged. [**Ans.** $\sigma_{x'} = 136.6$, $\sigma_{y'} = 63.4$, $\sigma_{z'} = 500$, $\tau_{x'y'} = -136.6$, $\tau_{y'z'} = \tau_{x'z'} = 0$ N/m^2]

1.4 The components of stress at a point are given by :

$$\sigma_x = 3xy^2z + 2x, \qquad \tau_{xy} = 0$$

$$\sigma_y = 5xyz + 3y \qquad \tau_{yz} = \tau_{xz} = 3xy^2z + 2xy$$

$$\sigma_z = x^2y + y^2z.$$

Determine whether these components of stress satisfy the equilibrium equations or not at the point (1,-1,2). If not then determine the suitable body force required at this point so that these stress components become under equilibrium.

[**Ans.** No, $\overline{B} = -11\hat{i} - 16\hat{j} + 5\hat{k}$]

1.5 The components of stress at a point are given by the following array of terms :

$$\begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 2 \end{bmatrix} \text{kPa}$$

Determine the normal and shear stresses on a plane whose direction cosines are $\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0$.

Also determine the direction of the shear stress.

[**Ans.** 5.552 kPa ; 4.164 kPa ; 0.3156, – 0.2232, 0.9229]

1.6 The components of stress at a point in a body are : $\sigma_x = 0$, $\sigma_y = 300$, $\sigma_z = 100$, $\tau_{xy} = \tau_{xz} = 0$, $\tau_{yz} = 100\sqrt{3}$ N/m^2.

Determine the principal stresses and the orientation of the principal axes. Check that the principal directions are mutually perpendicular. Also determine the normal and shearing stresses on the octahedral plane.

[**Ans.** $\sigma_1 = 400$ N/m^2, $a_{nx_1} = \sqrt{3}/2$, $a_{ny_1} = -1/2$, $a_{nz_1} = 0$; $\sigma_2 = \sigma_3 = 0$, $(\sigma_n)_{oct} = 248.80$ N/m^2 . $\tau_{oct} = 193.96$ N/m^2]

1.7 The Cartesian components of stress at a point are :

$\sigma_x = 15$, $\sigma_y = \sigma_z = 8$, $\tau_{xy} = 6$, $\tau_{yz} = 4$, $\tau_{xz} = 4$ kPa.

Using three dimensional Mohr's circle, determine the normal and shearing stresses on the plane whose direction cosines are

$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. [**Ans.** 19.667, 4.026 kPa]

1.8 The state of stress at a point for a given reference axes xyz is given by the following array of terms :

$$\begin{bmatrix} 150 & 80 & -60 \\ 80 & -120 & 50 \\ -60 & 50 & 80 \end{bmatrix} \text{kN/m}^2.$$

(*a*) Determine the stress invariants.

(*b*) If a new set of axes $x'y'z'$ is formed by rotating the xyz about the z-axis in anticlockwise direction by 45°, determine the stress components in the new co-ordinate system.

[**Ans.** (*a*) $I_1 = 110$, $I_2 = -28100$, $I_3 = -2{,}375{,}000$.
(*b*) $\sigma_x = 95$, $\sigma_y = -65$, $\sigma_z = 80$, $\tau_{xy} = -135$, $\tau_{yz} = 77.79$, $\tau_{xz} = -7.07$ kN/m^2]

1.9 The components of stress at a point are :

$$\sigma_x = 1, \sigma_y = -2, \sigma_z = 2, \tau_{xy} = 2, \tau_{yz} = 3, \tau_{xz} = 3 \text{ kPa}.$$

Determine :

(*a*) The principal stresses at the point.

(*b*) Deviatoric and spherical stress tensors.

[**Ans.** (*a*) $\sigma_1 = 6.13, \sigma_2 = -1.425, \sigma_3 = -3.650$ kPa.

(*b*) $\sigma_{x'} = 2/3, \sigma_{y'} = -7/3, \sigma_{z'} = 5/3, \tau'_{xy} = 2, \tau'_{yz} = 3, \tau'_{xz} = 3$ kPa.

$\sigma_{x''} = 1/3 = \sigma_{y''} = \sigma_{z''}, \tau_{xy''} = \tau_{yz''} = \tau_{xz''} = 0$ kPa]

1.10 It is desired to drill an oil hole through a member subjected to stress field given by :

$\sigma_x = 1050, \sigma_y = -350, \sigma_z = -700, \tau_{xy} = -210, \tau_{yz} = 0, \tau_{xz} = 70$ kN/m^2.

This oil hole should be drilled in a line parallel to the line of maximum normal tension. Indicate on a one cm cube where the centre line of the oil hole should be located.

[**Ans.** 1083.51 kN/m^2, 8.625°, 98.32°, 87.77°]

1.11 (*a*) Under what conditions the surfaces given by stress quadric of Cauchy become the surface of a sphere. What then are the principal stress axes ?

(*b*) Describe the state of stress corresponding to the case where the surface of stress

(*i*) is a sphere,

(*ii*) is a surface or revolution.

1.12 The components of stress acting on a cube of 10 × 4 × 5 cm size are given by :

$$\sigma_x = 70, \sigma_y = -70, \sigma_z = 70, \tau_{xy} = 0, \tau_{yz} = -35, \tau_{xz} = 35 \text{ kPa}.$$

Determine the normal stress on the diagonal plane and the shear stress in the direction of the *x*-axis.

1.13 The components of stress at a point in cylindrical co-ordinates are :

$$\sigma_r = r^2\theta + r, \sigma_\theta = r^2 z + \theta^2, \sigma_z = r^2 z^2 + \theta z, \tau_{r\theta} = r^2\theta,$$

$$\tau_{\theta z} = \theta z + \theta^2, \tau_{rz} = rz^2.$$

Determine the body force distribution at the point (3, π/3, 5) for the body to be in equilibrium.

[**Ans.** $\overline{B} = -56.349\hat{i} - 14.317\hat{j} - 139.921\hat{k}$]

1.14 The components of stress at a point in spherical co-ordinate are given by :

$\sigma_r = r^2\theta + \theta\phi, \sigma_\theta = \theta\phi^2 + r\theta, \sigma_\phi = \theta\phi + r\phi, \tau_{r\theta} = r\theta, \tau_{\theta\phi} = \theta\phi, \tau_{r\phi} = r\phi.$

Determine the body force distribution required at the point (– 2, π/4, π/2) so that the body is in equilibrium.

[**Ans.** $\overline{B} = 6.711\hat{i} - 2.517\hat{j} - 21.428\hat{k}$]

1.15 For the following state of stress, compute the stress vectors on planes with unit normals (2/3, 2/3, 1/3) and ($1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14}$).

$\sigma_x = 4, \sigma_y = 2, \sigma_z = -2, \tau_{xy} = 3, \tau_{yz} = 8, \tau_{zx} = -2$ kPa.

Also compute the normal and shearing stresses on these planes.

1.16 The Cartesian components of stress at a point are given as below :

$$\sigma_x = 100,\ \sigma_y = 50,\ \sigma_z = 40,\ \tau_{xy} = 20,\ \tau_{yz} = -40,\ \tau_{zx} = -60 \text{ kN/m}^2.$$

Determine the normal and shear stresses on a plane whose direction cosines are $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$. Derive the expressions used.

1.17 Derive equilibrium equations in three-dimensional Cartesian co-ordinate system. State the assumptions which you make in the derivation.

1.18 (*a*) Show that in the case of a two dimensional stress system, the magnitudes of the two principal stresses are given by the solution of the quadratic equation given by :

$$(\sigma - \sigma_x)(\sigma - \sigma_y) = \tau^2_{xy}$$

(*b*) Explain why stress is considered as a tensor quantity.

(*c*) Show that in a two-dimensional stress system, the two principal stresses are orthogonal.

1.19 The stress tensor at a point is given by

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} \text{MN/m}^2.$$

Determine the normal and shear stresses on a plane whose normal has direction cosines $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

1.20 On several planes through a point P in a body the stress vectors relative to axes (x_1, x_2, x_3) are given as shown below :

Unit normal to plane	*Stress vector on plane*
1, 0, 0	$\hat{i} + 2\hat{j} + 3\hat{k}$
$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$	$2\sqrt{3}\hat{i} + 2\sqrt{3}\hat{j}$
0, 1, 0	$2\hat{i} + 2\hat{j} + 2\hat{k}$

Determine the components of the stress array referred to (x_1, x_2, x_3) axes.

1.21 The stress tensor at a point P is given by the following array

$$\begin{bmatrix} 36 & 27 & 10 \\ 27 & -36 & 0 \\ 10 & 0 & 18 \end{bmatrix} \text{MN/m}^2$$

Determine (*a*) the components of the stress vector acting on a plane whose unit normal passing through point P has direction cosines $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$.

(*b*) the magnitude of the stress vector, and

(*c*) the angle between the stress vector and the unit normal of part (*a*).

1.22 The stress components at a point are $\sigma_x = 3, \sigma_y = 2, \sigma_z = -3, \tau_{xy} = 2, \tau_{yz} = 8, \tau_{zx} = -4$ MPa. Compute the stress vectors on planes with unit normals $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ and $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$. Also compute the normal and shearing stresses on these planes.

1.23 The stress tensor at a point is defined by the array :

$$\begin{bmatrix} 3 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix} \text{MN/m}^2$$

Determine the stress invariants, principal stresses and principal directions.

1.24 The stress components at a certain point of a body are given by :

$$\sigma_x = 8, \sigma_y = 6, \sigma_z = 2, \tau_{xy} = 2, \tau_{yz} = 1, \tau_{xz} = 4 \text{ MN/m}^2.$$

Determine :

(*a*) the stress vector on a plane normal to the vector $\hat{i} + 2\hat{j} + \hat{k}$,

(*b*) the principal stresses, and

(*c*) the octahedral shear stress.

1.25 The three principal stresses at a point in a body are $\sigma_1 = 6, \sigma_2 = 3, \sigma_3 = -2$ MN/m^2.

Determine :

(*a*) the octahedral shear stress.

(*b*) the maximum shear stress.

(*c*) the direction cosines of the normal to one of the planes on which the maximum shear stresses act.

1.26 The stress components at a point in a body are given by the array :

$$\begin{bmatrix} 10 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7 & -3 \\ \sqrt{2} & -3 & 7 \end{bmatrix} \text{kPa.}$$

Determine :

(*a*) the cubic equation for the principal stresses.

(*b*) the maximum stress at the point.

(*c*) the direction of the principal axis for minimum stress.

1.27 Show that the Lame's ellipsoid and the stress-director surface together completely define the state of stress at a point.

1.28 The state of stress at a point is given by the following array :

$$\begin{bmatrix} 20 & -40 & -60 \\ -40 & 40 & 20 \\ -60 & 20 & -20 \end{bmatrix} \text{MPa.}$$

Determine the three principal stresses and associated principal directions.

Ans.
$$\begin{bmatrix} 97.39\text{MPa}, & 6.44\text{MPa}, & -63.83\text{MPa}\ ; \\ a_{nx_1} = 0.6574 & a_{nx_2} = 0.4488 & a_{nx_3} = 0.6053 \\ a_{ny_1} = -0.6116 & a_{ny_2} = 0.7871 & a_{ny_3} = 0.0807 \\ a_{nz_1} = -0.4402 & a_{nz_2} = -0.4232 & a_{nz_3} = 0.7919 \end{bmatrix}$$

1.29 The state of plane stress at a point is given by :

$$\sigma_x = 35 \text{ MPa}, \sigma_y = 50 \text{ MPa and } \tau_{xy} = -35 \text{ MPa}.$$

Determine the principal stresses, the maximum shearing stress and the octahedral shearing stress.

1.30 The state of stress at a point is given by :

$$\sigma_x = 70 \text{ MPa}, \sigma_y = 10 \text{ MPa}, \sigma_z = -20 \text{ MPa},$$

$$\tau_{xy} = -40 \text{ MPa}, \tau_{yz} = 20 \text{ MPa} = \tau_{xz}.$$

Determine the principal stresses, maximum shear stress and the maximum principal stress direction.

1.31 For what body forces will the following stress field describe a state of equilibrium ?

$$\sigma_x = -2x^2 + 3y^2 - 5z$$

$$\sigma_y = -2y^2$$

$$\sigma_y = 3x + y + 3z - 5$$

$$\tau_{xy} = z + 4xy - 7$$

$$\tau_{xz} = -3x + y + 1$$

$$\tau_{yz} = 0.$$

1.32 At a point in a material subjected to a three dimensional stress system the Cartesian stress components are :

$$\sigma_x = 100, \sigma_y = 80, \sigma_z = 150,$$

$$\tau_{xy} = 40, \tau_{yz} = -30, \tau_{zx} = 50 \text{ MN/m}^2$$

Determine the normal, shear and resultant stresses on a plane whose normal makes angles of 52° with the x-axis and 68° with the y-axis.

[**Ans.** $\sigma_x = 169.7 \text{ MN/m}^2, \sigma_n = 166.8 \text{ MN/m}^2, \tau = 31 \text{ MN/m}^2$]

1.33 The state of stress at a point in a body is given by the following equations. If equilibrium is to be achieved what equations must the body force stresses B_x, B_y and B_z satisfy ?

$$\sigma_x = ax + by^2, \quad \tau_{xy} = gz$$

$$\sigma_y = cy^2 + dz^3, \quad \tau_{yz} = hy + kz$$

$$\sigma_z = ex + fz^2, \quad \tau_{xz} = lx^2 + mz^2.$$

1.34 The state of stress at a point is given by :

$$\begin{bmatrix} (3x^2 + 3y^2 - z) & (z - 6xy - \frac{3}{4}) & (x + y - \frac{3}{2}) \\ (z - 6xy - \frac{3}{4}) & 3y^2 & 0 \\ (x + y - \frac{3}{2}) & 0 & (3x + y - z + \frac{5}{4}) \end{bmatrix}$$

Show that, if the body forces are neglected, equilibrium exists.

1.35 The state of plane stress at a point is given by :

$$\sigma_x = x^3 y - 2axy + by$$

$$\sigma_y = xy^3 - 2x^3 y$$

$$\tau_{xy} = \frac{3}{2}x^2y^2 + ay^2 + \frac{x^4}{2} + C$$

Show that in the absence of body forces, equilibrium exists.

1.36 At a point in a material the stresses are :

$$\sigma_x = 40, \sigma_y = 80, \sigma_z = 150$$

$$\tau_{xy} = 70, \tau_{yz} = -20, \tau_{zx} = 30 \text{ MN/m}^2$$

Calculate the shear stress on a plane whose normal makes an angle of 48° with the x-axis and 71° with the y-axis.

1.37 At a point in a material a resultant stress of value 15 MN/m^2 is acting in a direction making angles of 43°, 75° and 50° 53′ with the co-ordinate axes x, y and z.

(*a*) Find the normal and shear stresses on an oblique plane whose normal makes angles of 67° 13′, 30° and 71° 34′, respectively with the same co-ordinate axes.

(*b*) If $\tau_{xy} = 1.5$ MN/m^2, $\tau_{yz} = -2$ MN/m^2 and $\tau_{zx} = 3$ MN/m^2, determine σ_x, σ_y and σ_z.

1.38 Three principal stresses of 300, 200 and – 150 MN/m^2 act in directions x, y, z respectively. Determine the normal, shear and resultant stresses which act on a plane whose normal is inclined at 30° to the z-axis and the projection of the normal on the xy-plane being inclined at 55° to the xz-plane.

1.39 The stress components at a point in a stressed body are given by :

$$\sigma_x = 100, \sigma_y = 140, \sigma_z = 170$$

$$\tau_{xy} = 50, \tau_{yz} = -20, \tau_{zx} = 100 \text{ MN/m}^2.$$

Determine the values of principal stresses and principal directions.

1.40 The principal stresses at a point are :

$$\sigma_1 = 50, \sigma_2 = 30, \sigma_3 = -10 \text{ MN/m}^2.$$

Calculate the resultant stress on a plane whose normal has direction cosines,

$$a_{nx} = 0.73,\ a_{ny} = 0.46 \text{ and } a_{nz} = 0.506$$

1.41 The following state of strain exists at a point. Taking $E = 207$ GN/m^2 and $\nu = 0.3$, determine the state of stress.

$$\varepsilon_x = 10 \times 10^{-4},\ \varepsilon_y = 5 \times 10^{-4},\ \varepsilon_z = 7 \times 10^{-4},$$

$$\gamma_{xy} = 2 \times 10^{-4},\ \gamma_{yz} = 10 \times 10^{-4},\ \gamma_{zx} = 8 \times 10^{-4}$$

1.42 The following state of stress exists at a point. If $E = 210$ GN/m^2 and $\nu = 0.3$, determine the state of strain.

$$\sigma_x = 225,\ \sigma_y = 75,\ \sigma_z = 150$$

$$\tau_{xy} = 110,\ \tau_{yz} = 50,\ \tau_{zx} = 70 \text{ MN/m}^2$$

1.43 The stress components at a point are :

$$\sigma_x = -40,\ \sigma_y = 80,\ \sigma_z = 120$$

$$\tau_{xy} = 70,\ \tau_{yz} = 50,\ \tau_{zx} = 30 \text{ MN/m}^2$$

Calculate the stress components after rotating the axes through 30° about the z-axis.