1

Laplace Transforms

The Laplace transforms are so widely used in the study of control systems that this chapter is devoted to the discussion and review of Laplace transforms. These transforms have become popular because of the following advantages :

(*i*) This transformation transforms the transcendental and exponential functions to simple algebraic functions.

(*ii*) This transformtion transforms the operations of differentiation and integration to multiplication and division respectively.

(*iii*) In thè solution of differential equations, arbitrary constants do not occur.

(*iv*) We can effectively make use of step and impulse response which is very relevant in control systems

1·1. Definition

The Laplace transform of a function $f(t)$ is denoted by $Lf(t)$ and is a function of s normally written as

$$F(s)=L\,f(t) \qquad \ldots(1)$$

The correspondence between $f(t)$ and $F(s)$ is unique and is established by the following relation :

$$F(s)=\int_0^\infty f(t)\, e^{-st}\, dt \qquad \ldots(2)$$

Example 1·1. *Find the Laplace transforms of* :

(*a*) *a costant*, K

(*b*) $f(t)=t^2$

(*c*) $f(t)=e^{-\alpha t}$

Solution. (*a*) $f(t)=K$

$$F(s)=\int_0^\infty K\, e^{-st}\, dt$$

$$=\left|\frac{K}{-s}\, e^{-st}\right|_0^\infty=\frac{K}{s}$$

(*b*) $f(t)=t^2$

$$F(s)=\int_0^\infty t^2\, e^{-st}\, dt$$

By repeated application of the rules of integration by parts,

$$F(s)=t^2\left|\frac{e^{-st}}{-s}\right|_0^{\infty}-(2t)\left|\frac{e^{-st}}{s^2}\right|_0^{\infty}+(2)\left|\frac{e^{-st}}{-s^3}\right|_0^{\infty}$$

Making use of l'Hospital's rule,

$$F(s)=\frac{2}{s^3}$$

As a matter of fact, it can be shown that

$$L(t^n)=\frac{n!}{s^{n+1}}$$

(c) $f(t)=e^{-\alpha t}$

$$F(s)=\int_0^{\infty} e^{-\alpha t}\, e^{-st}\, dt=\left|\frac{e^{-(s+\alpha)}}{-(s+\alpha)}\right|_0^{\infty}=\frac{1}{s+\alpha}$$

Example 1·2. *Find the Laplace transform for*

$$f(t)=\cos \omega t.$$

Solution. Evaluation of Laplace transform, by direct integration, in this case is quite tedious and an easy way to find this transform is to recognize :

$$\cos \omega t=Re[e^{j\omega t}]$$

$$\therefore \quad L[\cos \omega t]=Re[L(e^{j\omega t})]$$

Now, $L(e^{j\omega t})=\dfrac{1}{s-j\omega}$ [example 1·1(c)]

Hence, $L[\cos \omega t]=Re\left[\dfrac{1}{s-j\omega}\right]=\dfrac{s}{s^2+\omega^2}$.

It is not always convenient to derive Laplace transforms by definition. Mostly we make use of same of the properties of Laplace transforms. These properties are listed below.

1·2. Properties of Laplace Transforms

1·2·1. Laplace transform of the sum of two functions is equal the sum of the Laplace transforms of the two functions

$$L[f_1(t)\pm f_2(t)]=Lf_1(t)\pm Lf_2(t) \qquad ...(3)$$

1·2·2. Laplace transform of a function multiplied by any constant is constant times the Laplace transform of the function.

$$L[cf(t)]=cL\,f(t) \qquad ...(4)$$

1·2·3. Laplace transform of the derivative of a function is

$$L\left[\frac{d}{dt}\right]f(t)=s\,F(s)-f'(0) \qquad ...(5)$$

Similarly,

$$L\left[\frac{d^2}{dt^2} f(t)\right]=s^2F(s)-sf(0)-f'(0) \qquad \text{...(6)}$$

In general,

$$L\left[\frac{d^n}{dt^n} f(t)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0) \qquad \text{...(7)}$$

1·2·4. Laplace transform of the integral of a function is

$$L\left[\int f(t)\,dt\right]=\frac{F(s)}{s}+\frac{f^{-1}(0)}{s} \qquad \text{...(8)}$$

where, $f^{-1}(0)=\left|\int f(t)\,dt\right|_{t=0}$

1·2·5. Shifting Theorems

$$L\left[f(t-T)\right]=e^{-sT}F(s) \qquad \text{...(9)}$$

$$L\left[e^{-\alpha t}f(t)\right]=F(s+\alpha) \qquad \text{...(10)}$$

Equation (9) applies to functions shifted in time domain and (10) applies to functions shifted in frequency domain.

1·2·6. Scaling Theorem

$$L\left[f(at)\right]=\frac{1}{a}F\left(\frac{s}{a}\right) \qquad \text{...(11)}$$

1·2·7. Laplace transform of a function multiplied by t is

$$L\left[tf(t)\right]=-\frac{d}{ds}F(s) \qquad \text{...(12)}$$

1·2·8. Laplace transform of a function divided by t is

$$L\left[\frac{f(t)}{t}\right]=\int_s^{\infty} F(s)\,ds \qquad \text{...(13)}$$

1·2·9. Initial Value Theorem

If $f(t)$ and $f'(t)$ Laplace transformable,

$$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s) \qquad \text{...(14)}$$

1·2·10. Final Value Theorem

If the indicated limits exist,

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s) \qquad \text{...(15)}$$

The proof of the above properties is not given here.

Any standard text on Mathematics could be used for the same. We solve below a number of examples to illustrate the use of these properties.

Example 1·3. *Find Laplace transforms for :*

(*a*) $f(t)=7e^{-3t}+2\cos t$

(*b*) $f(t)=K\sinh \alpha t$

(*c*) $f(t)=t^2 e^{-\alpha t}$.

Solution. (*a*) $f(t)=7e^{-3t}+2\cos t$

$$F(s)=L[7e^{-3t}+L[2\cos t]$$
$$=7L(e^{-3t})+2L(\cos t)$$
$$=7.\frac{1}{s+3}+2.\frac{s}{s^2+1}$$
$$=\frac{9s^2+6s+7}{s^3+3s^2+s+3}$$

(*b*) $$f(t)=K\sinh \alpha t=\frac{K}{2}[e^{\alpha t}-e^{-\alpha t}]$$

$$F(s)=\frac{K}{2}[L(e^{\alpha t})-L(e^{-\alpha t})$$
$$=\frac{k}{2}\left[\frac{1}{s-\alpha}-\frac{1}{s+\alpha}\right]=\frac{k\alpha}{s^2-\alpha^2}$$

(*c*) $$L(t^2)=\frac{2}{s^3}$$

$\therefore$ $$L(t^2e^{-\alpha t})=\frac{2}{(s+\alpha)^3}$$

Example 1·4. *Determine the Laplace transforms for :*

(*a*) $f(t)=t^2\sin \omega t$

(*b*) $f(t)=\frac{1}{t}\sin \omega t$.

Solution. (*a*) $L[\sin \omega t]=\frac{\omega}{s^2+\omega^2}$

The above Laplace transform can be evaluated by following steps similar to example 1·2.

$\therefore$ $$L[t\sin \omega t]=-\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right]$$

and $$L[t^2\sin \omega t]=\frac{d^2}{ds^2}\left[\frac{\omega}{s^2+\omega^2}\right]$$
$$=\frac{d}{ds}\left[\frac{d}{ds}\left\{\omega(s^2+\omega^2)^{-1}\right\}\right]$$

$$=\frac{d}{ds}[(-2\omega)(s)(s^2+\omega^2)^{-2}]$$

$$=(-2\omega)\left[\frac{d}{ds}\left\{s(s^2+\omega^2)^{-2}\right\}\right]$$

$$=(-2\omega)[(s)(-2)(s^2+\omega^2)^{-3}(2s)+(s^2+\omega^2)^{-2}]$$

$$=\frac{2\omega\,(3s^2-\omega^2)}{(s^2+\omega^2)^3}.$$

(b) $$L\left(\frac{1}{t}\sin\omega t\right)=\int_s^\infty \frac{\omega}{s^2+\omega^2}$$

$$=\left|\tan^{-1}\left(\frac{s}{\omega}\right)\right|_s^\infty$$

$$=\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{\omega}\right)=\tan^{-1}\left(\frac{\omega}{s}\right)$$

Example 1·5. *Determine initial and final values of $f(t)$ for which the Laplace transform is :*

$$F(s)=\frac{10s\,(s+7)}{(s+1)(s+8)(s+10)}$$

Solution. From initial value theorem,

$$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$$

$$=\lim_{s\to\infty}\left[\frac{10s^2\,(s+7)}{(s+1)(s+8)(s+10)}\right]=10$$

From final value theorem,

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$$

$$=\lim_{s\to 0}\left\{\frac{10s^2\,(s+7)}{(s+1)(s+8)(s+10)}\right\}=0.$$

Example 1·6. *Evaluate the integrals :*

(a) $$\int_0^\infty t\,e^{-2t}\cos t\,dt$$

(b) $$\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt.$$

Solution. (a) We know that

$$L\,[\cos t]=\frac{s}{s^2+1}$$

$\therefore$ $$L\,[t\cos t]=-\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$

$$=\frac{s^2-1}{(s^2+1)^2}$$

Using equation (1), therefore,

$$\int_0^\infty t\, e^{-st} \cos t\, dt = \frac{s^2-1}{(s^2+1)^2}$$

Putting $s=2$,

$$\int_0^\infty t\, e^{-2t} \cos t\, dt = \left.\frac{s^3-1}{(s^2+1)^2}\right|_{s=2} = \frac{3}{25}.$$

(*b*) We know that

$$L[e^{-t}-e^{-3t}] = \frac{1}{s+1} - \frac{1}{s+3}$$

$$\therefore \quad L\left[\frac{e^{-t}-e^{-3t}}{t}\right] = \int_s^\infty \left[\frac{1}{s+1} - \frac{1}{s+3}\right] ds$$

$$= \Big|\ln (s+1) - \ln (s+3)\Big|_s^\infty$$

$$= \left|\ln\left(\frac{s+1}{s+3}\right)\right|_s^\infty = \ln\left(\frac{s+3}{s+1}\right)$$

By definition,

$$\int_0^\infty e^{-st}\left[\frac{e^{-t}-e^{-3t}}{t}\right] dt = \text{lr}\left(\frac{s+3}{s+1}\right)$$

Putting $s=0$

$$\int_0^\infty \frac{e^{-t}-^{-3t}}{t} dt = \left|\ln\left(\frac{s+3}{s+1}\right)\right|_{s=0} = \ln 3$$

Example 1·7. *Find out the Laplace transforms for* :

(*a*) *Step function*
(*b*) *Ramp function*
(*c*) *Parabolic function*
(*d*) *Impulse function.*

Solution. (*a*) A step function (also called Constant Position input) is defined as a function having zero value before $t=0$ and a constant value after $t=0$. Mathematically,

$$u_{-1}(t) = \begin{cases} 0\,;\ t \leqslant 0 \\ K\,;\ t \geqslant 0 \end{cases}$$

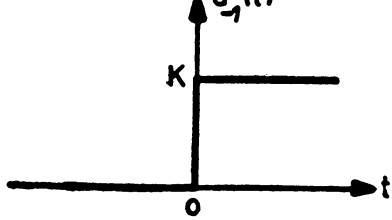


Fig. 1·1. A step function.

The function is shown in Fig. 1·1 and has a discontinuity at $t=0$. The function is of great importance physically, as the basic phenomenon of switching in systems corresponds to this function if the instant of switching is reckoned at $t=0$. The Laplace transform of this function is derived by definition.

$$F(s)=\int_0^{\infty} Ke^{-st}\,dt=\frac{K}{s} \qquad ...(16)$$

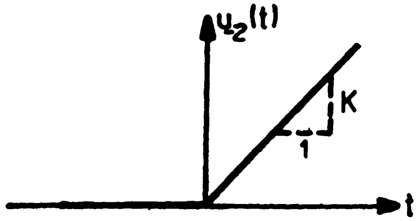


Fig. 1·2. A Ramp function.

(*b*) A ramp function (also called constant velocity input) is mathematically defined as :

$$u_{-2}(t)=\begin{cases}0 \; ; \; t \leqslant 0\\ Kt \; ; \; t \geqslant 0.\end{cases}$$

The function is shown in Fig. 1·2. Its Laplace transform is :

$$F(s)=\int_0^{\infty} Kte^{-st}\,dt=\frac{K}{s^2}. \qquad ...(17)$$

(*c*) A parabolic function (also called constant acceleration input) is denoted by $u_{-3}(t)$. The function is shown in Fig 1·3 and is mathematically described as :

$$u_{-3}(t)=\begin{cases}0 \; ; \; t \leqslant 0\\ \frac{1}{2}Kt^2 \; ; \; t \geqslant 0\end{cases}$$

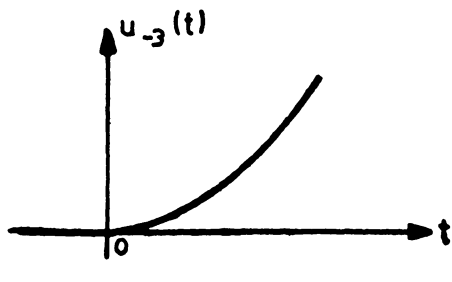


Fig. 1·3. A Parabolic function.

The Laplace transform $F(s)$ is :

$$F(s) = \int_0^{\infty} \tfrac{1}{2} K t^2 e^{-st}\, dt$$

$$= \tfrac{1}{2} K \frac{2}{s^3} = \frac{K}{s^3} \qquad \text{...(18)}$$

(*d*) An impulse function is denoted by $u_0(t)$ and it has a value zero everywhere except for $t=0$. Mathematically,

$$u_0(t) = \begin{cases} 0 & ; t \neq 0 \\ \neq 0 & ; t = 0 \end{cases}$$

This function is shown in Fig. 1·4. Practically, any pulse of of large amplitude and small width can be approximated as an

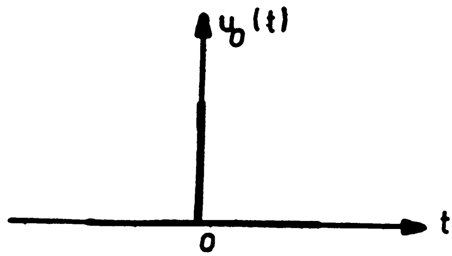


Fig. 1·4. An Impulse function

Impulse function. The area under an impulse is constant and is called the strength of the impulse.

$$\int_{0-}^{0} u_0(t)\, dt = K$$

The Laplace transform of an Impulse function is :

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$= \int_{0-}^{0+} u_0(t)\, dt = K \qquad \text{...(19)}$$

Example 1·8. *Find the Laplace transforms for :*

(*a*) $f(t) = \sin \omega\, (t - t_0)$

(*b*) $f(t) = \sin \omega\, (t - t_0)\, u_{-1}\, (t - t_0)$

(*c*) $f(t) = \sin \omega\, (t)\, u_{-1}\, (t - t_0)$

(*d*) $f(t) = \sin \omega\, (t - t_0)\, u_{-1}\, (t - t_0)$

Solution. This example is chosen to clearly understand the difference between four apparently similar functions.

(*a*) This function, shown in Fig. 1·5, just represents the sinusoidal function shifted right by t_0. Its Laplace transform is :

$$F(s) = L \sin \omega (t-t_0)$$

$$= L \sin \omega t \cos \omega t_0 - L \cos \omega t \sin \omega t_0$$

$$= \cos \omega t_0 \left[\frac{\omega}{s^2+\omega^2}\right] - \sin \omega t_0 \left[\frac{s}{s^2+\omega^2}\right]$$

$$= \frac{\omega \cos \omega t_0 - s \sin \omega t_0}{s^2+\omega^2}$$

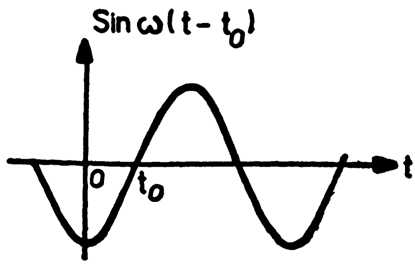


Fig. 1·5. $f(t) = \sin \omega (t-t_0)$

(*b*) For this function, the portion of Fig. 1·5 before $t=0$ is erased out by multiplying the function with unit step function. As this function is identical to (*a*) after $t=0$ and Laplace transform does not consider the function before $t=0$, its Laplace transform is the same as that for (*a*).

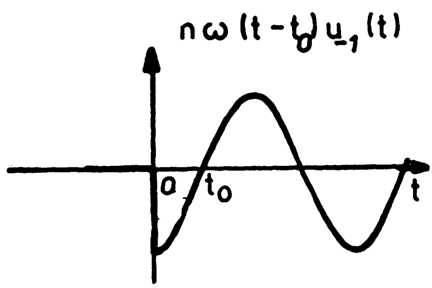


Fig. 1·6. $f(t) = \sin \omega (t-t_0) u_{-1}(t)$

(*c*) This function is shown in Fig. 1·7 and is just the ordinary sinusoidal function which has been erased before $t=t_0$ by multiplying it with $u_{-1}(t-t_0)$.

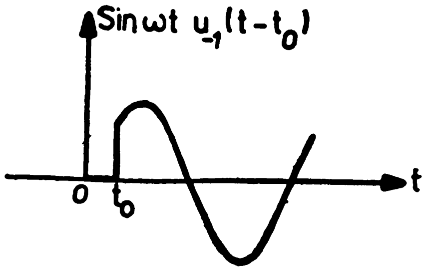


Fig. 1·7. $f(t)=\sin \omega t \; u_{-1}\,(t-t_0)$

The Laplace transform for this function is evaluated as :

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$= \int_0^{t_0} f(t)\, e^{-st}\, dt + \int_{t_0}^{\infty} f(t)\, e^{-st}\, dt$$

$$= 0 + \int_{t_0}^{\infty} \sin \omega\, t\, e^{-st}\, dt$$

Now, $\quad \sin \omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$

$$\therefore \quad F(s) = \frac{1}{2j}\int_{t_0}^{\infty} [e^{(-s+j\omega)t} - e^{(-s-j\omega)t}\, dt$$

$$= \frac{1}{2j}\left[\left\{\frac{e^{(-s+j\omega)t_0}}{(-s+j\omega)}\right\} - \left\{\frac{e^{(-s+j\omega)t_0}}{(-s-j\omega}\right\}\right]_{t_0}^{\infty}$$

$$= \frac{1}{2j}\left[\frac{e^{(-s+j\omega)t_0}}{s-j\omega} - \frac{e^{(-s-j\omega)t_0}}{s+j\omega}\right]$$

$$= e^{-t_0 s}\left[\frac{\omega \cos \omega t_0 + s \sin \omega t_0}{s^2+\omega^2}\right]$$

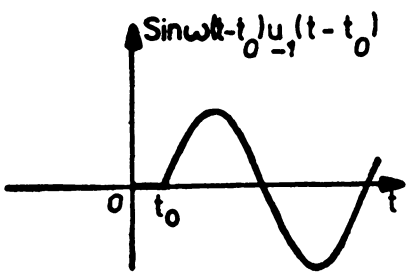


Fig. 1·8. $f(t)=\sin \omega\, (t-t_0)\, u_{-1}\,(t-t_0)$

(*d*) In this case the whole sinusoidal function is shifted as in (*a*) ; but the portion before $t=t_0$ does not exist because of the $u_{-1}(t-t_0)$. The function is shown in Fig. 1·8. Its Laplace transform can be derived by applying theorem to the Laplace transform for an ordinary sinusoid.

$$F(s)=e^{-t_0 s}\, L\,[\sin \omega t]$$

$$=e^{-t_0 s}\frac{\omega}{s^2+\omega^2}$$

1·3. Laplace Transforms for Waveforms

The general methods for determining the Laplace transforms for waveforms commonly used in control systems are discussed in this section.

1·3·1. Aperiodic Signals

For finding out the Laplace transform of aperiodic signals, two techniques are normally used :

(*a*) Decomposition of the given signal into sum of simpler signals like step, ramp, etc.

(*b*) Using the gating function.

We shall illustrate both the techniques with the help of an example.

Example 1·9. *Find out the Laplace transform of a waveform shown in Fig. 1·9.*

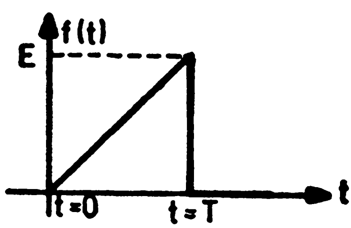


Fig. 1·9. Waveform for example 1·9.

Solution.

(*a*) **Decomposition Technique.** The waveform given above can be decomposed into simpler functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ as shown in Fig. 1·10

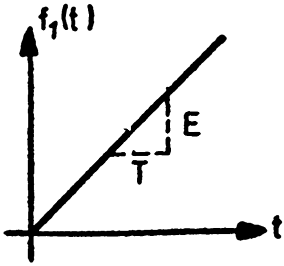
From Fig. 1·10,

$$f(t)=f_1(t)-f_2(t)-f_3(t)$$

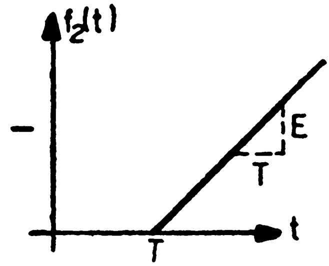
where $$f_1(t)=\frac{E}{T}t\, u_{-1}(t)$$

$$f_2(t)=\frac{E}{T}(t-T)\,u_{-1}(t-T)$$

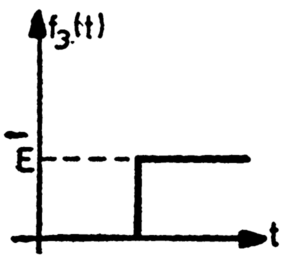
$$f_3(t)=E\,u_{-1}(t-T)$$



(a)



(b)



(c)

Fig. 1·10. Decomposition of $f(t)$.

Therefore, $F(s)=L\,[f_1(t)]-L\,[f_2(t)]-L\,[f_3(t)]$

$$=\frac{E}{T}\cdot\frac{1}{s^2}-\frac{E}{T}\cdot\frac{1}{s^2}e^{-Ts}-\frac{E}{s}e^{-Ts}$$

$$=\frac{E}{Ts^2}[1-(Ts+1)e^{-Ts}].$$

(*b*) **Gating Function Technique.** A gate function is a function having a constant amplitude for a finite interval of time and zero everywhere else. The property of this function is that the multiplication by gate of any function will make its zero outside the gate but will not change its nature within the gate. If we have a gate function shown in Fig. 1·11 and multiply it by $\frac{E}{T}\,t$, we shall obtain the function $f(t)$.

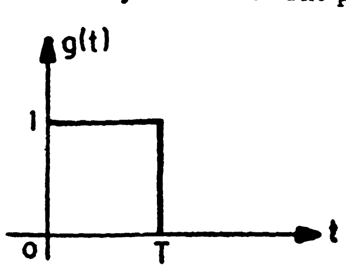


Fig. 1·11. A gate function.

Therefore, $$f(t)=\frac{E}{T}t\,g(t) \qquad ...(20)$$

Now, $$g(t)=u_{-1}(t)-u_{-1}(t-T)$$

Therefore, $$L\,[g(t)]=\frac{1}{s}(1-e^{-Ts})$$

Hence $$L\,[t\,g(t)]=-\frac{d}{ds}\left[\frac{1-e^{-Ts}}{s}\right]$$

$$=\frac{1-(Ts+1)\,e^{-Ts}}{s^2}$$

$$F(s)=\frac{E}{T}\left[\frac{1-(Ts+1)\,e^{-Ts}}{s^2}\right] \qquad ...(21)$$

1·3·2. Periodic Signals

To find out the Laplace transform of periodic signals, firstly only one period is considered and Laplace transform is derived as illustrated in sec. 1·3·1. Let this be $G(s)$.

Then, $$F(s)=\frac{1}{1-e^{-Ts}}\,G(s) \qquad ...(22)$$

where T is the period of repetition.

Example 1·10. *Derive the Laplace transform for a periodic sawtooth wave shown in Fig. 1·12.*

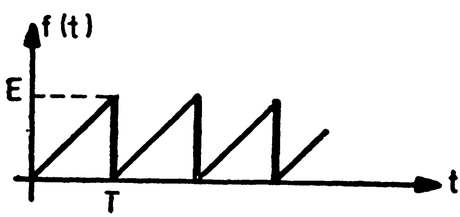


Fig. 1·12. Periodic sawtooth wave.

Solution. This is a periodic signal with repetition period T. One period of this signal is the same as the function of example 1·9. Hence, from Eq. (21),

$$G(s)=\frac{E}{Ts^2}[1-(Ts+1)\,e^{-Ts}]$$

Using Eq. (22), $$F(s)=\frac{E}{Ts^2}\left[\frac{1-(Ts+1)\,e^{-Ts}}{1-e^{-Ts}}\right].$$

1·4. Inverse Laplace Transforms

As the name suggests, Inverse Laplace transformation is an operation which makes it possible to find out $f(t)$ whose Laplace transform $F(s)$ is known. Mathematically,

$$f(t)=L^{-1}\,F(s) \qquad ...(23)$$

An integration relation for the evaluation of Eq. (23) exists but the integration required is quite tedious. Because the relationship $f(t)$ and $F(s)$ is unique, for simple $F(s)$ we can write $f(t)$ by memory. For example,

$$L^{-1}\frac{K}{s}=K$$

$$L^{-1}\frac{K}{s+\alpha}=Ke^{-\alpha t}$$

$$L^{-1}\frac{K\omega}{(s+\alpha)^2+\omega^2}=Ke^{-\alpha t}\sin\omega t$$

In standard mathematical texts, quite exhaustive tables listing inverse Laplace transforms are available and they can be used as reference.

For most of the problems, however, it is possible to decompose $F(s)$ into simpler constituents using partial fractions and then the inverse Laplace transform for each of the terms can be separately written.

Example 1·11. *Determine inverse Laplace transform for :*

(*a*) $F(s)=\dfrac{s^2+2}{s(s+1)(s+2)}$

(*b*) $F(s)=\dfrac{s}{(s+1)^3(s+2)}$

(*c*) $F(s)=\dfrac{s+1}{(s+2)(s^2+4)}$

Sol. (*a*) $F(s)=\dfrac{s^2+2}{s(s+1)(s+2)}=\dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{s+2}$

Now, $A=\dfrac{s^2+2}{(s+1)(s+2)}\Big|_{s=0}=1$

$B=\dfrac{s^2+2}{s(s+2)}\Big|_{s=-1}=-3$

$C=\dfrac{s^2+2}{s(s+1)}\Big|_{s=-2}=3$

Hence, $F(s)=\dfrac{1}{s}-\dfrac{3}{s+1}+\dfrac{3}{s+2}$

Therefore, $f(t)=1-3e^{-t}+3e^{-2t}$

(*b*) $F(s)=\dfrac{s}{(s+1)^3(s+2)}\equiv\dfrac{A}{(s+1)^3}+\dfrac{B}{(s+1)^2}+\dfrac{C}{s+1}+\dfrac{D}{s+2}$

Now, $A=F(s).\ (s+1)^3\,{}_{s=-1}=-1$

$$D=F(s).\,(s+2)\Big|_{s=-2}=2$$

$$B=\frac{1}{1\,!}\left[\frac{d}{ds}F(s)\,(s+1)^3\right]\Bigg|_{s=-1}=2$$

$$C=\frac{1}{2!}\left[\frac{d^2}{ds^2}F(s).(s+1)^3\right]\Bigg|_{s=-1}=-2$$

Hence, $F(s)=\dfrac{-1}{(s+1)^3}+\dfrac{2}{(s+1)^2}-\dfrac{2}{(s+1)}+\dfrac{2}{(s+2)}$.

Therefore, $f(t)=2e^{-2t}-2e^{-t}+2t\,e^{-t}-\frac{1}{2}t^2e^{-t}$

(*c*) $F(s)=\dfrac{s+1}{(s+2)(s^2+4)}\equiv\dfrac{A}{s+2}+\dfrac{Bs+C}{s^2+4}$

Now, $A=F(s).(s+2)\Big|_{s=-2}=-\frac{1}{8}$.

Then by comparing coefficients, we can find

$$B=\tfrac{1}{8}\text{ and }C=\tfrac{3}{4}$$

Hence, $F(s)=-\dfrac{1}{8}\quad\dfrac{1}{s+2}+\dfrac{1}{8}\cdot\dfrac{s}{s^2+4}+\dfrac{3}{8}\cdot\dfrac{2}{s^2+4}$

Therefore, $f(t)=\frac{1}{8}[\cos 2t+3\sin 2t-e^{-2t}]$.

1·4·1. Convolution Theorem

This theorem gives a relation for the inverse Laplace transformation of the product of two functions of s is terms of the inverse Laplace transforms of the individual functions.

If $F(s)=F_1(s).\,F_2(s)$

where $L^{-1}\;F_1(s)=f_1(t)$ and $L^{-1}\;F_2(s)=f_2(t)$ are known;

then $\quad L^{-1}\,F(s)=f(t)=f_1(t)\,*\,f_2(t)$ ...(24)

The notation (*) is used for convolution operation which is defined as

$$f(t)=\int_0^t f_1(\tau)\,f_2(t-\tau)d\tau \qquad ...(25)$$

$$=\int_0^t f_1(t-\tau).\,f_2(\tau)d\tau.$$

Example 1·12. *Determine the inverse Laplace transform for :*

$$F(s)=\frac{1}{(s^2+a^2)^2}$$

Solution. This example can be conveniently solved only by the use of convolution theorem, let $F(s)=F_1(s).\;F_2(s)$, where $F_1(s)=F_2(s)=\frac{1}{s^2+a^2}$

Therefore, $f_1(t)=f_2(t)=L^{-1}\left[\frac{1}{{}^2+a^2}\right]=\frac{1}{a}\sin at$

Hence, $f(t)=\frac{1}{a}\sin at * \frac{1}{a}\sin at$

$$=\frac{1}{a^2}\int_0^t \sin at \,.\, \sin a(t-\tau)d\tau$$

$$=\frac{1}{a^2}\left[\int_0^t \sin a\tau\ \{\sin at\ \cos a\tau-\cos at\ \sin a\tau\}\ d\tau\right]$$

$$=\frac{1}{a^2}\left[\left\{\sin at\int_0^t \sin a\tau \cos a\tau\ d\tau\right\}\right.$$

$$\left.-\left\{\cos at\int_0^t \sin^2 a\tau\ d\tau\right\}\right]$$

$$=\frac{1}{a^2}\left[\left\{\sin at\left(\frac{-\cos 2a\tau}{4a}\right)\right\}\Bigg|_0^t-\left\{\cos at\left(\frac{\tau}{2}-\frac{\sin 2a\tau}{4a}\right)\right\}\Bigg|_0^t\right]$$

$$=\frac{1}{a^2}\left[\left\{\sin at\left(\frac{1-\cos 2at}{4a}\right)\right\}-\left\{\cos at\left(\frac{t}{2}-\frac{\sin 2at}{4a}\right)\right\}\right]$$

$$=\frac{1}{a^2}\left[\frac{1}{2a}\sin at-\frac{t}{2}\cos at\right]$$

$$=\frac{1}{2a^3}(\sin at-at\cos at).$$

Example 1·13. *Determine the inverse Laplace transform for*

$$F(s)=\frac{1}{s^3(s^2+1)}.$$

Solution. This problem can be solved by repeated application of convolution theorem, but we shall illustrate a heuristic approach where the following property of the Laplace transforms will be used repeatedly.

$$L\left[\int_0^t f(t)\ dt\right]=\frac{F(s)}{s}$$

Now, $L^{-1}\left[\frac{1}{s^2+1}\right]=\sin t$

$\therefore$ $L^{-1}\left[\frac{1}{s(s^2+1)}\right]=\int_0^t \sin t\ dt=1-\cos t$

$$L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]=\int_0^t (1-\cos t)\,dt=t-\sin t$$

and, $$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]=\int_0^t (t-\sin t)\,dt=\frac{t^2}{2}+\cos t-1$$

1·5. Solution of Differential Equations

The most important reason for the popularity of Laplace transforms is the case with which it can solve differential equations. This emerges from the fact that the operations of differentiation and integration are translated to the algebraic operations of multiplication and division, thereby changing the differential equation into algebraic equation. The solution of the algebraic equation is obviously much easier than the solution of differential equations. The various steps of the solution are :

(*i*) Take the Laplace transforms of both the sides of the differential equation incorporating initial conditions simultaneously

(*ii*) Rewrite the transformed equation as

$$F(s)=\frac{P(s)}{Q(s)}$$

(*iii*) Resolve $F(s)$ into partial fractions and obtain $f(t)$.

Example 1·14. *If* $y=0$ *and* $\frac{dy}{dt}=1$ *for* $t=0$, *solve the differential equation.*

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=\cos 2t.$$

Solution. Taking Laplace transforms on both the sides,

$$[s^2Y(s)-s\,y(0)-y(0)]+4[s\,Y(s)-y(0)]+4\,Y(s)=\frac{s}{s^2+4}$$

Substituting the initial conditions and transposing,

$$Y(s)\,[s^2+4s+4]=\frac{s}{s^2+4}+1=\frac{s^2+s+4}{s^2+4}$$

or $$Y(s)=\frac{s^2+s+4}{(s^2+4)(s^2+4s+4)}$$

$$=\frac{s^2+s+4}{(s^2+4)(s+2)^2}$$

Making partial fractions,

$$Y(s)=\frac{3}{4}\cdot\frac{1}{(s+2)^2}+\frac{1}{4}\cdot\frac{1}{s^2+4}$$

or $$(t)=\frac{3}{4}te^{-2t}+\frac{1}{8}\sin 2t.$$

Example 1·15. *Solve for x(t) and y(t) if*

$$\frac{dx(t)}{dt}=2x(t)-3y(t) \qquad ...(26)$$

and

$$\frac{dy(t)}{dt}=y(t)-2x(t) \qquad ...(27)$$

Subject to initial conditions

$$x(0)=8 \text{ and } y(0)=3.$$

Solution. The Laplace transform of equation (26) is

$$sX(s)-x(0)=2X(s)-3Y(s)$$

or

$$X(s)\,[s-2]+Y(s)[3]=8 \qquad ...(28)$$

Similarly, the Laplace transform of equation (27) is

$$sY(s)-y(0)=Y(s)-2X(s)$$

or

$$X(s)[2]+Y(s)[s-1]=3 \qquad ...(29)$$

Solving the algebraic equations (28) and (29),

$$X(s)=\frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}}=\frac{8s-17}{(s+1)(s-4)}$$

$$Y(s)=\frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}}=\frac{3s-22}{(s+1)(s-4)}$$

Decomposing into partial fractions

$$X(s)=\frac{8s-17}{(s+1)(s-4)}=\frac{5}{s+1}+\frac{3}{s-4}$$

$$Y(s)=\frac{3s-22}{(s+1)(s-4)}=\frac{5}{s+1}-\frac{2}{s-4}$$

Hence,

$$x(t)=5e^{-t}+3e^{4t}$$

$$y(t)=5e^{-t}-2e^{4t}.$$

1·6. Applications

In this section, we discuss some of the applications of Laplace transforms to engineering problems.

Example 1·16. *A particle P of mass 2 grams moves on the x-axis and is attracted towards origin O with a force numerically equal to 8x. If it is initially at rest at x=10, find its position at any subsequent time assuming* (*a*) *no other forces act*

(*b*) *a damping force numerically equal to 8 times the instantaneous velocity acts.*

Solution. (*a*) By Newton's law of motion,

$$M\frac{d^2x}{dt^2}=f(t)$$

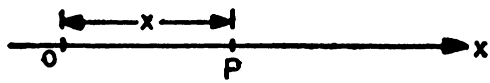


Fig. 1·13. Motion of a particle.

or $$2\frac{d^2x}{dt^2}=-8x$$

or $$\frac{d^2x}{dt^2}+4x=0$$

Using Laplace transforms,

$$s^2X(s)-x(0)-x'(0)+4X(s)=0$$

Now, $x(0)=10$ and $x'(0)=0$

Therefore, $s^2X(s)-10s+4X(s)=0$

or, $$X(s)[s^2+4]=10s$$

or, $$X(s)=\frac{10s}{s^2+4}$$

Hence, $x(t)=10\cos 2t$

The particle will keep on oscillating between $x=10$ and $x=-10$ with a cosinusodial variation. It will first reach origin when $2t=\frac{\pi}{2}$ or $t=\frac{\pi}{4}$.

(*b*) The force now is

$$-8x-8\frac{dx}{dt}$$

The equation of motion, therefore, is :

$$2\frac{d^2x}{dt^2}=-8x-8\frac{dx}{dt}$$

or $$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=0$$

Taking Laplace transforms and substituting initial conditions,

$$X(s)[s^2+4s+4]=10(s+4)$$

or $$X(s)=\frac{10(s+4)}{s^2+4s+4}=\frac{10(s+4)}{(s+2)^2}$$

$$=\frac{10}{s+2}+\frac{20}{(s+2)^2}$$

Therefore, $$x(t)=10\,e^{-2t}+20\,t\,e^{-2t}$$

or $$x(t)=10\,e^{-2t}\,(1+2t)$$

The motion is non-oscillatory. The particle will approach origin, but will never reach it.

Example 1·17. *Calculate the current $i(t)$ in the circuit given in Fig. 1·14 after clossing the switch if the voltage across the condenser C is 15 volts just before closing the switch.*

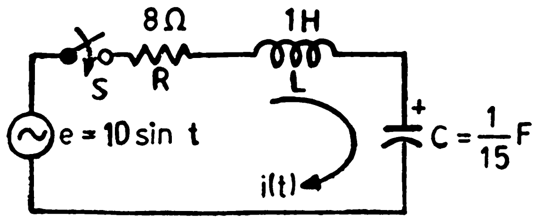


Fig. 1·14. A Electrical Circuit.

Solution. Applying Kirchhoff's law,

$$Ri+L\frac{di}{dt}+\frac{1}{C}\int i\,dt=e$$

Taking Laplace transforms,

$$RI(s)+L\,[sI(s)-i(0^+)]+\frac{1}{C}\left\{\frac{I(s)}{s}+\frac{i^{-1}(0^+)}{s}\right\}=E(s)$$

Now before closing the switch current $i(0^-)=0$ and the current through an inductor cannot change suddenly.

Hence, $$i(0^+)=i(0^-)=0$$

Also as the charge across a capacitor can't change all of a sudden : $$i^{-1}(0^+)=i^{-1}(0^-)=C.15$$

$$\therefore\quad I(s)\left[R+Ls+\frac{1}{Cs}\right]=E(s)-\frac{C.15}{C.s}$$

or $$I(s)\left[8+s+\frac{15}{s}\right]=\frac{10}{s^2+1}-\frac{15}{s}$$

$$=\frac{10s-15s^2-15}{s(s^2+1)}$$

or $$I(s)=\frac{(10s-15s^2-15)s}{s(s^2+1)(s^2+8s+15)}$$

$$=\frac{10s-15s^2-15}{(s^2+1)(s^2+8s+15)}$$

$$=\frac{10s-15s^2-15}{(s^2+1)(s+3)(s+5)}$$

$$=\frac{-9}{s+3}+\frac{110/13}{s+5}+\frac{1}{13}\cdot\frac{7s+4}{s^2+1}$$

Therefore, $$i(t)=\frac{110}{13}e^{-5t}-9e^{-3t}+\frac{7}{13}\cos t+\frac{4}{13}\sin t$$

Example 1·18. *A rectangular voltage pulse of unit height and duration T is applied to a series RC combination at t=0. Determine the voltage across the capacitance as a function of time.*

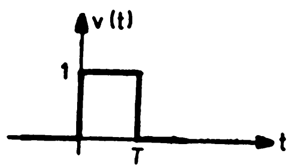


Fig. 1·15. A rectanular fulse.

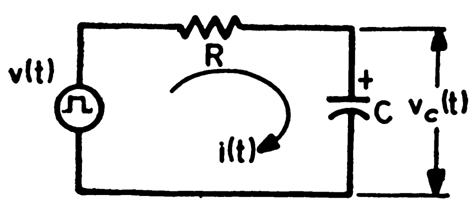


Fig. 1·16. An *RC* circuit.

Solution. The input voltage can be expressed as

$$v(t)=u_{-1}(t)-u_{-1}(t-T)$$

Therefore, $V(s)=\frac{1}{s}\,(1-e^{-Ts})$ ...(30)

Now the *KVL* equation for the *RC* circuit is

$$v(t)=Ri(t)+v_c(t)$$

Also $$i(t)=C\frac{dv_c(t)}{dt}$$

Hence, $$v(t)=RC\frac{dv_c(t)}{dt}+v_c(t)$$

Taking Laplace transforms,

$$V(s)=RC[s\,V_c(s)-v_c(0)]+V_c(s)$$

or $$V_c(s)[s\,RC+1]=V(s)$$

$$V_c s=\frac{V(s)}{1+s\,RC}$$

Substituting $V(s)$ from equation (30)

$$V_c(s)=\frac{1}{s}\cdot\frac{1-e^{-Ts}}{1+s\,RC}$$

Next, we are to find the inverse Laplace transform of $V_c(s)$. To make it easy we recollect that factor e^{-Ts} just amounts to a simple shift in t in the inverse transformation we proceed as follows :

$$\frac{1}{s(RC\,s+1)}=\frac{1}{s}-\frac{1}{s+1/RC}$$

Therefore, $$L^{-1}\left[\frac{1}{s(RC\,s+1)}\right)=1-e^{-t/RC}\Big]$$

Hence, $$v_c(t)=(1-e^{-\frac{t}{RC}})\,u_{-1}(t)-(1-e^{-\frac{t-T}{RC}})\,u_{-1}(t-T)$$

A plot of $v_c(t)$ a as function of t is shown in Fig. 1·17

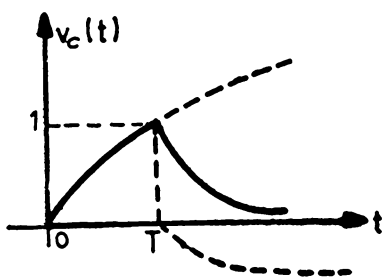


Fig. 1·17. Voltage across capacitor

PROBLEMS

1·1. Derive the Laplace transforms of following :

(a) $f(t)=A\sin(\omega t+\theta)$

(b) $f(t)=A\cos\alpha t\cos\beta t$

(c) $f(t)=e^{At}\cosh 5t$

(d) $f(t)=\sin 2\omega(t-t_0)u_{-1}(t-2t_0)$

1·2. Find out the Laplace transforms of the following :

(a) $f(t)=(t+2)^2\, e^t$

(b) $f(t)=\dfrac{\sinh t}{t}$

(c) $f(t)=\sin^3 t$

1·3. Show that

(a) $\displaystyle\int_0^\infty t\, e^{-3t} \sin t \; dt = \frac{3}{50}$

(b) $\displaystyle\int_0^\infty \frac{e^{-t} \sin t}{t} \; dt = \frac{\pi}{4}$

1·4. Verify the initial value theorem for

(a) $3-2 \cos t$

(b) $(2t+3)^2$

(c) $t+\sin 3t$

1·5. Verify the final value theorem for

(a) $1+e^{-t}(\sin t+\cos t)$

(b) $t^3 e^{-2t}$.

1·6. Find out the Laplace transforms for the aperiodic singnals shown in Figs. 1·18 and 1·19.

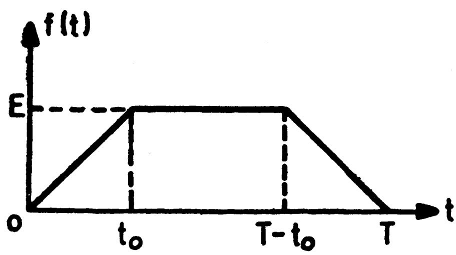


Fig. 1·18. A Trapezoidal wavefrom.

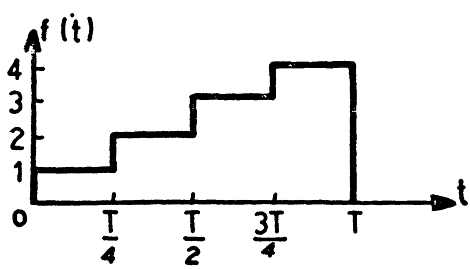


Fig 1·19. A staircase waveform.

1·7. Derive the Laplace transforms for the following periodic waveforms :

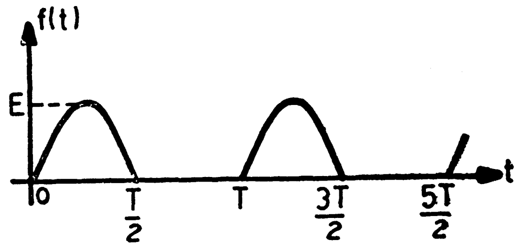


Fig. 1·20. Rectified sine wave.

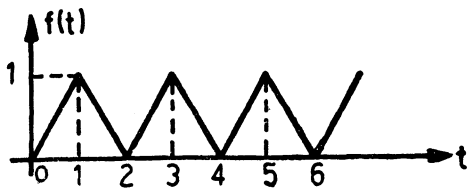


Fig. 1·21. Triangular wave.

1·8. Find out the Inverse Laplace transforms :

(*a*) $\dfrac{s+1}{(s+2)(2s+3)}$ (*b*) $\dfrac{s^3+s^2+2}{(s-1)^4}$

(*c*) $\dfrac{s-6}{s^2-4s+8}$ (*d*) $\dfrac{4s^2-16}{s^3(s+2)^2}$

(*e*) $\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}$ (*f*) $\dfrac{1}{(s^2+1)^3}$

(*g*) $\dfrac{3e^{-s/3}}{s^2(s^2+2)^2}$ (*h*) $\ln\left(\dfrac{s}{s^2+9}\right)$

1·9. Solve :

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=40\,e^{3t}$$

subject to the initial conditions

$$y(0)=0, \quad y'(0)=2$$

1·10. Solve for x and y :

$$2\ddot{x}+\ddot{y}=0$$

$$\ddot{x}+x-\dot{y}=4-7e^{2t}$$

with $x(0)=3, \quad \dot{x}(0)=-2$

$y(0)=2, \quad \dot{y}(0)=4$

1·11. The switch in the circuit of Fig. 1·22 is closed at $t=0$. Find v_t as a function of time.

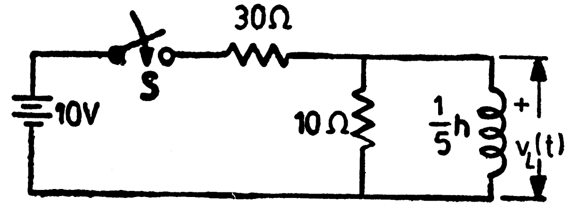


Fig. 1·22. An electrical circuit.

1·12. In the electrical circuit of Fig. 1·22.

$$E=500 \sin 10\,t$$
$$R_1=R_2=10\,\Omega$$
$$L=1\text{ H.}$$
$$C=0{\cdot}01\text{ F.}$$

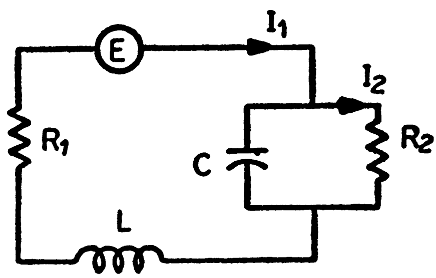


Fig. 1·23. Electrical ciruit for above problem.

If the charge on the capacitor and the currents I_1 and I_2 are zero at $t=0$; find the charge on the capacitor at any time $t>0$.