1

Compensation of Feed Back Control Systems

1.1. INTRODUCTION

The design objective of the components of a control system is to meet the desired performance specification requirements which are concerning time response, accuracy and relative stability.

These specifications are generally given in terms of numerical values or in the form of qualitative data. During the design process these requirements may have to be marginally adjusted to some tolerable extent, because, it may not be conveniently possible to satisfy all the given specifications at one time. For example, if the steady state error has to be given preference, then the time response specification are to be modified to some extent. The ultimate purpose of the design process is to have optimal system for given purpose.

Once the preliminary analysis is done, the next step towards designing a system is to select one of many possible alternatives that meets the desired specifications.

To get the desired performance characteristic of a control system, it becomes necessary to insert a compensation network in the system. The design of a compensator is carried out in time domain or frequency domain, as decided by the specifications given. The root locus approach is used if the time response specifications are given. Bode plot approach is used if frequency domain specifications are given.

The time response specifications are :

(1) rise time, t_r

(2) settling time, t_s

(3) damping ratio, ζ

(4) steady state error, e_{ss}

(5) % maximum overshoot, % M_p

The frequency domain specification are :

(1) gain margin, *G.M.*

(2) phase margin, *P.M.*

(3) band width, *B.W.*

(4) resonance frequency, ω_r

The gain adjustment of a control system is the preliminary step to achieve its desired performance. However, increasing the gain of a control system result in poor stability. Therefore, while increasing the gain and keeping stability of the system within limit, it is essential to introduce a suitable compensating device in the system to compensate for the deficit performance. The design and selection of a compensating network is in the form of a transfer function, which must be reliazed physically.
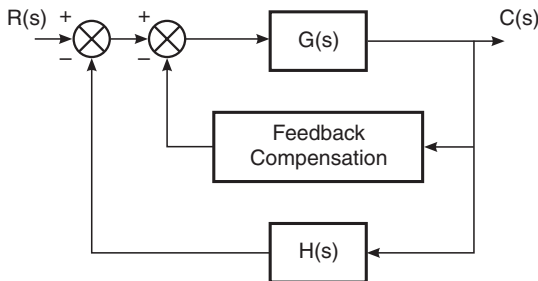
The compensating network is inserted in a control system in a manner as shown in Fig. 1.1.
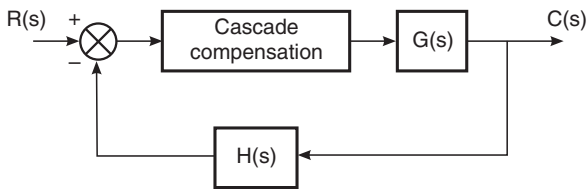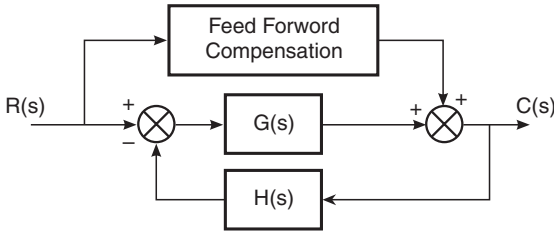

(a) Cascade Compensation
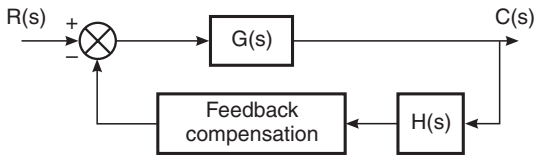
(b) Feedback Compensation (i)

(c) Feedback Compensation (ii)

(d) Forward path Compensation

Fig. 1.1. Compensation networks.

1.2. ROOT LOCUS APPROACH

Addition of a zero :

The root locus plot for the examples given below illustrate the effect of addition of zero to the open loop transfer function with reference to time response and steady state error performance.

The root locus plot for the open loop transfer function given below in shown in Fig. 1.2.

$$G(s)\ H(s)\ =\ \frac{K}{s(s+1)\ (s+5)} \qquad \qquad ...(1.1)$$

For a specified value of damping ratio say = 0.65.

Following data is obtained from the root locus plot

Closed loop dominant poles, s_1, $s_2 = -\ 0.4 \pm j\ 0.47$

Gain factor at s_1, s_2 , $K = 2.3$

Closed loop third pole, $s_3 = -\ 5.11$

Gain factor at $s_3 = 2.3$

Actual damping, $\xi\omega_n = 0.4$

Natural frequency of oscillations,

$$\omega_n\ =\ \frac{\omega_d}{\sqrt{1 - \xi^2}}\ =\ \frac{0.47}{\sqrt{1 - 0.65^2}}$$

$$=\ 0.61\ \text{rad/s}$$

Settling time, $t_s = \dfrac{4}{\xi\omega_n}\ =\ \dfrac{4}{0.4} = 10\ \text{sec.}$

Ramp error coefficient

$$K_v = \lim_{s \to 0}\ sG\ (s)\ H(s)$$

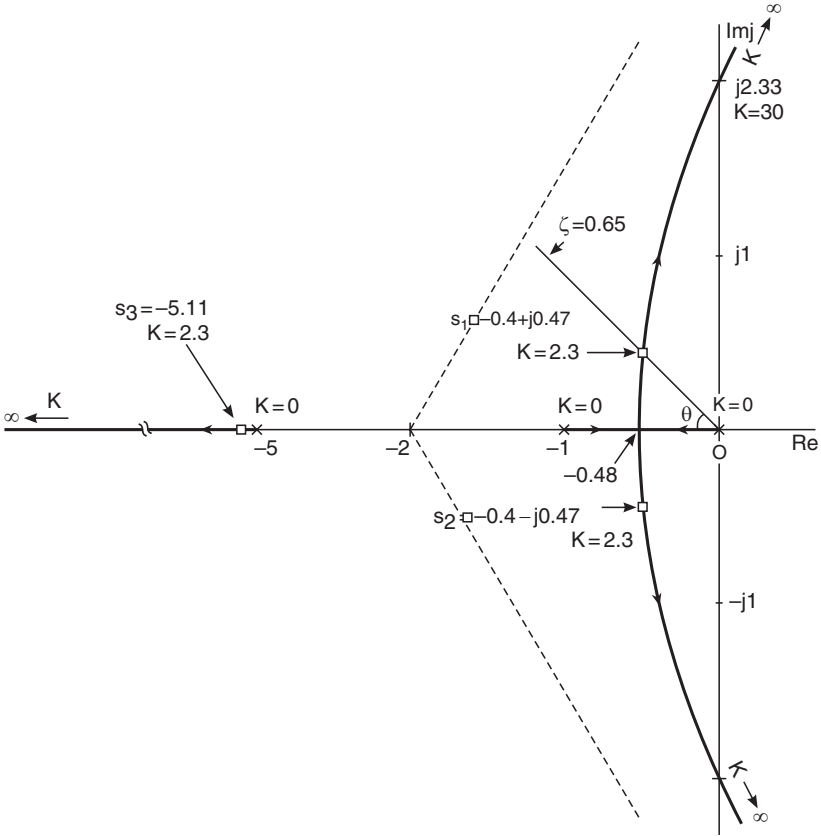$$=\ \lim_{s \to 0} \frac{s\ K}{s(s+1)\ (s+5)} = 0.2\ K$$

Fig. 1.2. Root locus plot for $G(s)\ H(s) = \dfrac{K}{s(s+1)\ (s+5)}$.

For unit ramp input the steady state error

$$e_{ss} = \frac{1}{K_v}$$

$$= \frac{1}{0.2K} = \frac{1}{0.2 \times 2.3} = 2.17$$

From the above example it is noted that the settling time $t_s = 10$ sec. gives a slow time response. The response can be made fast by reducing settling time. This can be done by reshaping the root locus such that the root locus plot is adjusted to shift towards left side by adding a compensating zero to the given open loop transfer function.

On addition of a zero at $s = -1.5$, the transfer function (1.1) takes the form

$$G(s)\ H(s)\ =\ \frac{K(s+1.5)}{s(s+1)\ (s+5)} \qquad\qquad ...(1.2)$$

The root locus plot for the above transfer function is shown in Fig. 1.3.

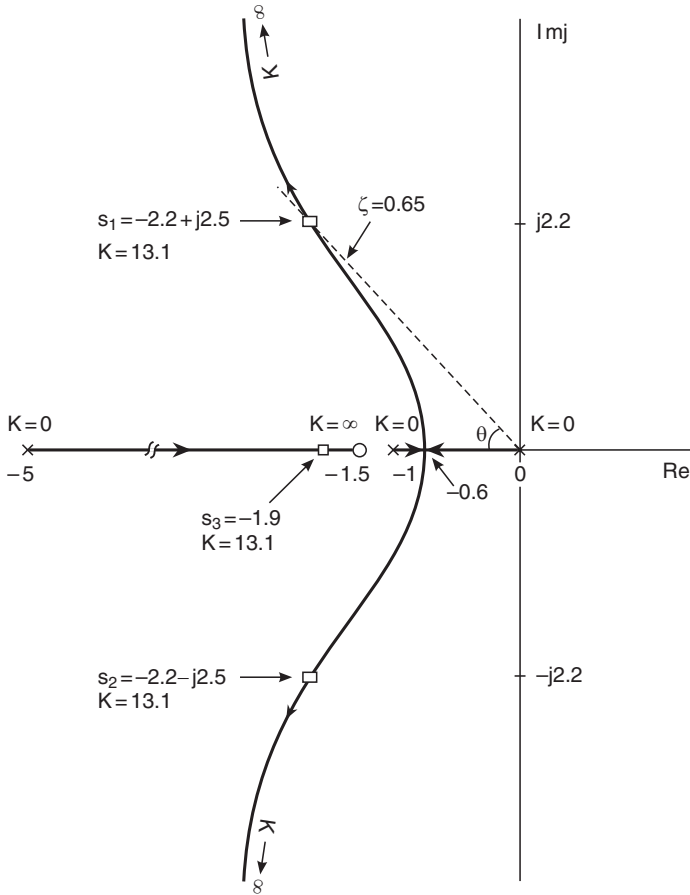

Fig. 1.3. Root locus plot for $G(s)\ H(s) = \dfrac{K(s+1.5)}{s(s+1)\ (s+5)}$.

For a specified damping ratio $\zeta = 0.65$ following data is obtained from the root locus plot (Fig. 1.3).

Closed loop dominate poles, s_1, $s_2 = -\ 2.2 \pm j\ 2.6$

Gain factor at s_1, s_2, $K = 13.1$

Closed loop third pole, $s_3 = -\ 1.9$

Gain factor at s_3, $K = 13.1$

Actual damping, $\zeta\omega_n = 2.2$

Natural frequency of oscillations,

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{2.6}{\sqrt{1 - 0.65^2}}$$

$$= 3.38 \text{ rad/s}$$

Setling time, $\quad t_s = \dfrac{4}{\zeta\omega_n} = \dfrac{4}{2.2} = 1.8 \text{ sec.}$

Ramp error coefficient

$$K_v = \lim_{s \to 0} s\, G(s)\, H(s)$$

$$= \lim_{s \to 0} \frac{s\, K(s + 1.5)}{s(s + 1)\,(s + 5)} = 0.3\, K$$

For unit ramp input the steady state error

$$e_{ss} = \frac{1}{K_v}$$

$$= \frac{1}{0.3K} = \frac{1}{0.3 \times 13.1}$$

$$= 0.254$$

On comparing the performance of the original system *i.e.*

$$G(s)\, H(s) = \frac{K}{s(s + 1)\,(s + 5)}$$

and the compensated system

$$G(s)\, H(s) = \frac{K\,(s + 1.5)}{s\,(s + 1)\,(s + 5)}$$

Following conclusions are drawn:

The addition of the zero at $s = -1.5$ to the original open loop transfer function bends the root locus plot towards left, shifting away from imaginary ($j\omega$) axis, thus making system more stable.

The real part of the dominant closed-loop pole increases, which decreases the settling time (t_s).

The natural (undamped) frequency ω_n increases which spreads the orginal band width.

The shifting of the third (real) root towards right helps in decreasing the overshoot.

The gain factor K has to be increased to a higher value which affects K_v to some extent decreasing steady state error.

From the foregoing conculsion it is noted that the addition of a zero to the open loop transfer function improves transient response of a control system.

However, it is not possible to add an isolated zero to a transfer function because of physical non-realizability.

Therefore, in order to realize the compensating network a pair of pole-zero has to be incorporated. The suitable location of the pole should be far away from the zero. This condition gives the zero a dominance with respect to pole because the zero is located near to the imaginary axis of s-plane.

The compensated open loop transfer function with the aforesaid provision having a zero and pole located say at $s = -1.5$ and $s = -10$ respectively is given below :

$$G(s)\ H(s)\ =\ \frac{K\ (s+1.5)}{s\ (s+1)\ (s+5)\ (s+10)} \qquad ...(1.3)$$

The root locus plot for the above open loop transfer function (1.3) is shown in Fig. 1.4.

For a specified value of damping ratio $\zeta = 0.65$, following data is obtained from the root locus plot (Fig. 1.4).

Closed loop dominant poles, $s_1, s_2 = -1.9 \pm j\ 2.2$

Gain factor at $s_1\ s_2$, $K = 102$

Closed loop third/forth pole, $s_3 = 1.895$, $s_4 = 11.35$

Gain factor at s_4/s_3, $K = 102$

Actual damping, $\zeta\omega_n = 1.9$

Natural frequency of oscillations,

$$\omega_n\ =\ \frac{\omega_d}{\sqrt{1-\xi^2}}\ =\ \frac{2.2}{\sqrt{1-0.65^2}}\ =\ 2.9\ \text{rad/sec}$$

Settling time, $\ t_s\ =\ \dfrac{4}{\xi\omega_n}\ =\ \dfrac{4}{1.9}\ =\ 2.1\ \text{sec.}$

Ramp error coefficient,

$$K_v\ =\ \lim_{s\to 0}\ s\ G(s)\ H(s)$$

$$=\ \lim_{s\to 0}\ \frac{s\ K\ (s+1.5)}{s(s+1)\ (s+5)\ (s+10)}\ =\ 0.03\ K$$

For unit ramp input steady state error

$$e_{ss}\ =\ \frac{1}{K_v}$$

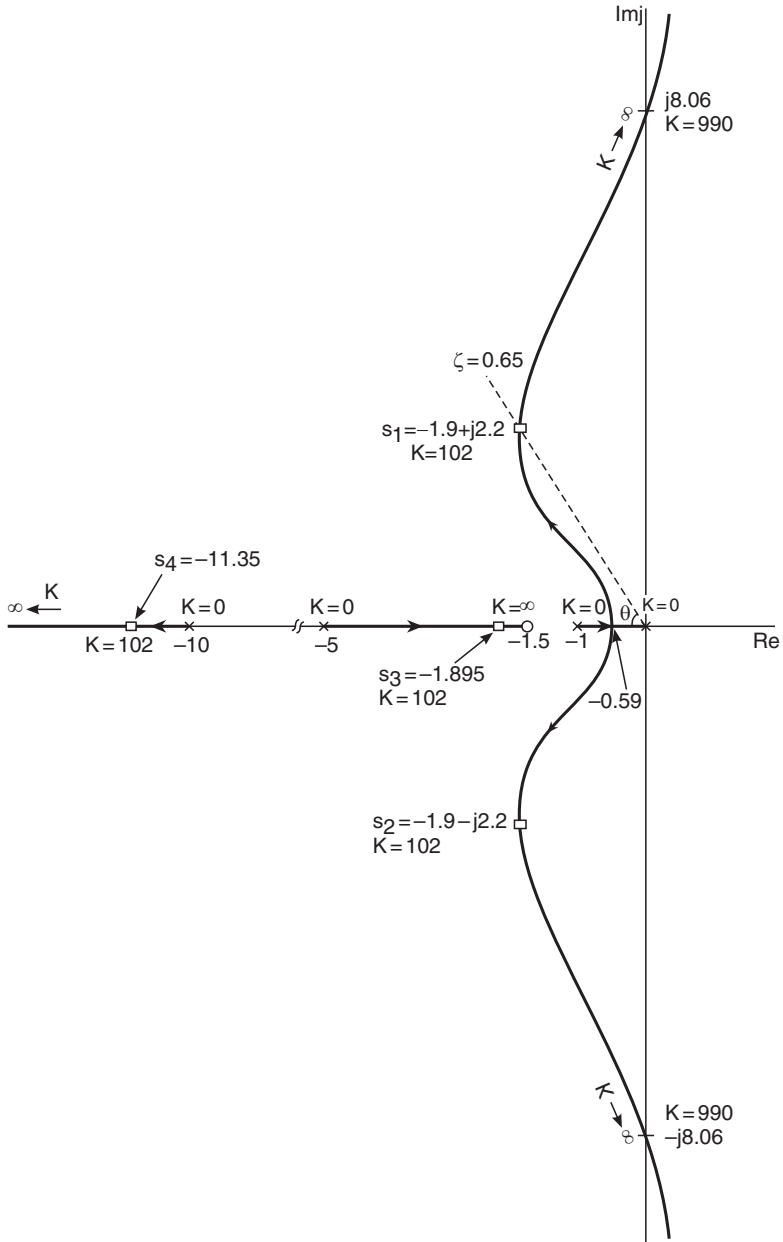$$=\ \frac{1}{0.03K}\ =\ \frac{1}{0.03\times 102}\ =\ 0.326$$

Fig. 1.4. Root locus pole for $G(s)\ H(s) = \dfrac{K\ (s+1.5)}{s\ (s+1)\ (s+5)\ (s+10)}$.

From the data determined as above, it si noted that by intersecting a zero dominance compensating network in cascade with the open loop transfer function of a control system, improvement in the time response is achieved. However, some improvement in steady state error performance is noted due to increase in the gain demand.

As the zero of the compensating network is nearer to the s- plane imaginary axis as compared to the pole, a phase lead to the orginal open-loop transfer function is added. Therefore, such a network is known as phase lead compensating network.

Addition of a pole :

The steady state error e_{ss} can be reduced by increasing the value of ramp (velocity) error coefficient K_v. This is possible if a pole at the origin is added to the open loop transfer function of a control system, thereby increasing the type of a system as explained below :

Let,

$$G(s)\,H(s) \;=\; \frac{K}{s\,(s+1)\,(s+5)} \qquad\qquad ...(1.4)$$

$$K_v \;=\; \lim_{s \to 0}\; G(s)\,H(s)$$

$$=\; \lim_{s \to 0} s\,\frac{K}{s(s+1)\,(s+5)} \;=\; 0.2\,K$$

Adding a pole at the origin to the transfer function (1.4) takes the following form :

$$G(s)\,H(s) \;=\; \frac{K}{s^2(s+1)\,(s+5)} \qquad\qquad ...(1.5)$$

$$K_v \;=\; \lim_{s \to 0}\; G(s)\,H(s)$$

$$=\; \lim_{s \to 0} \frac{s\,K}{s^2(s+1)\,(s+5)} \;=\; \infty$$

Thus, the steady state error is

$$e_{ss} \;=\; \grave{}\,\frac{1}{K_v} = \frac{1}{\infty} \;=\; 0$$

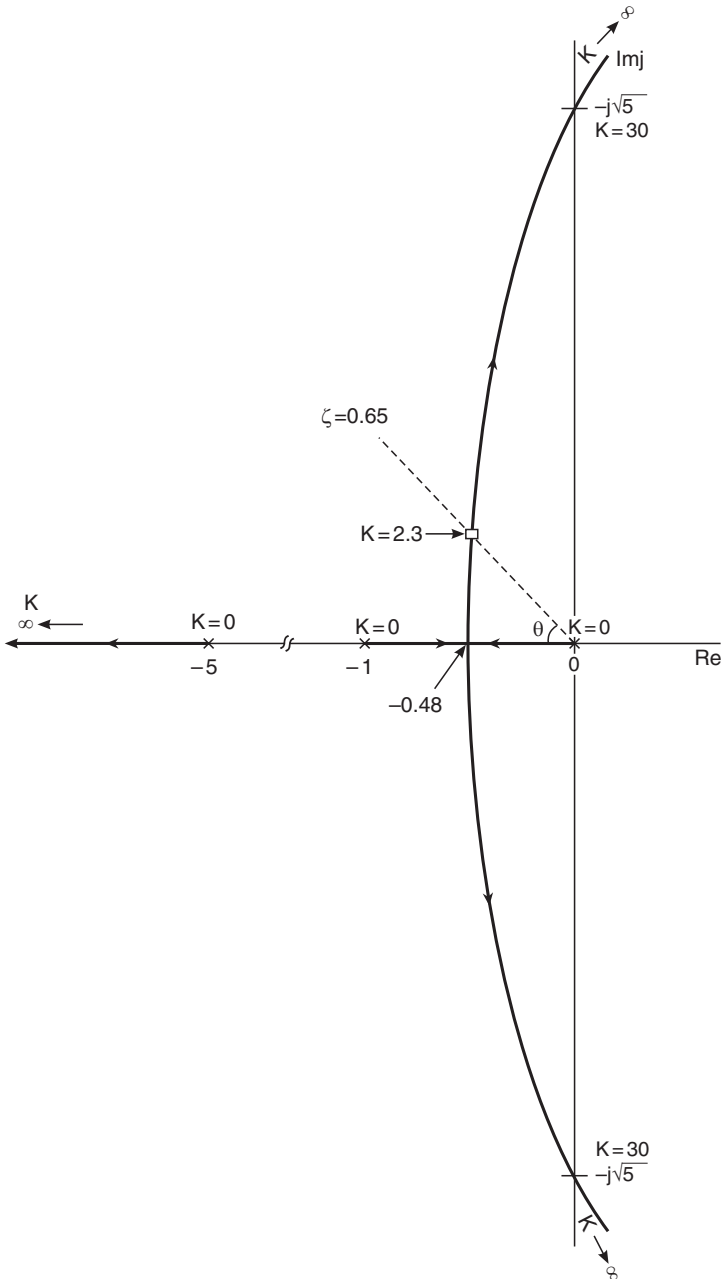The root locus plot for the open loop transfer function (1.5) is shown in Fig. 1.6.

Fig. 1.5. Root locus plot : $G(s) \, H(s) = \dfrac{K}{s(s+1) \, (s+5)}$.

Fig. 1.6. Root locus plot : $G(s) \, H(s) = \dfrac{K}{s^2(s+1)\,(s+5)}$.

Although, the addition of pole at the orgin makes the strady stay error $e_{ss} = 0$, but as seen from the root locus plot (Fig. 1.6), the system is unstable for any value of gain K.

In order to overcome the limitation of instability caused by addition of a pole at the orgin, a zero is also added. The suitable location of this zero is near the origin in order to reshape the root locus plot such that the system shifts towards stable operation for the desired value of gain K, suitable for a specified steady state error.

The open loop transfer function with the modification as mentioned above (say addition of a zero at $s = -0.1$) is given below :

$$G(s) \, H(s) \; = \; \frac{K\,(s+0.1)}{s^2(s+1)\,(s+5)} \qquad\qquad \text{...(1.6)}$$

The zero, thus added still maintains $K_v = \infty$. The root locus plot for the transfer function (1.6) is shown in Fig. 1.7.
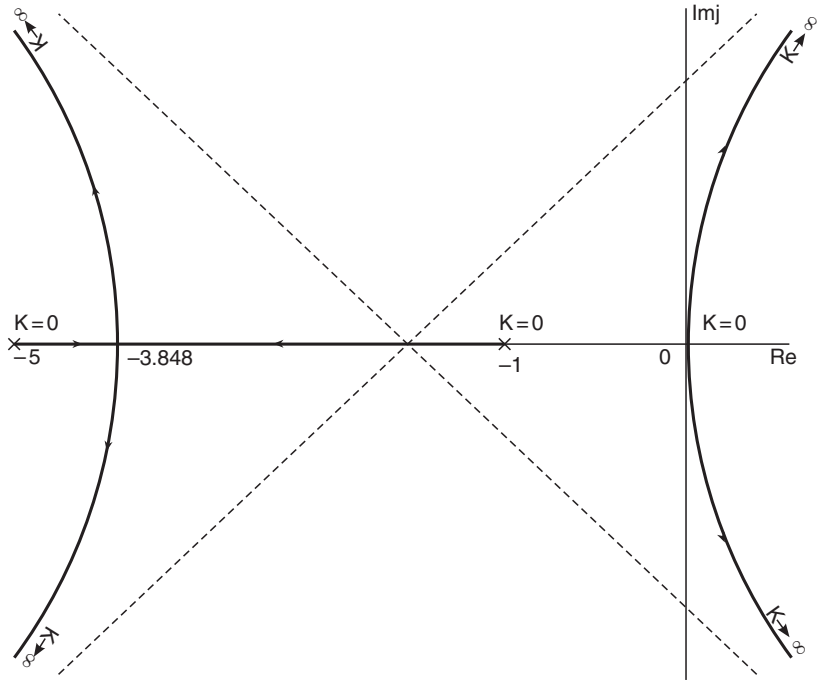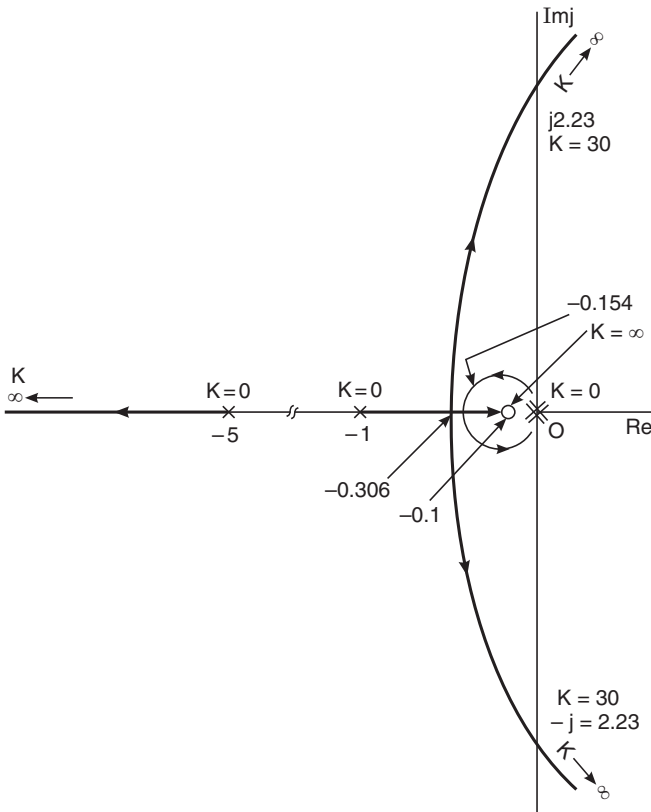
Fig. 1.7. Root locus plot, $G(s)\ H(s) = \dfrac{K\ (s + 0.1)}{s^2(s + 1)\ (s + 5)}$.

On comparing root locus plot Fig. 1.6 and 1.7 it is noted that the addition of a pole at the orgin alongwith a zero near the orgin to a transfer function makes that system stable and at the same time, $K_v = \infty$. The steady state error attains a zero value.

The addition of a pole to the open loop transfer function at the origin raises the type of the transfer function. However, if the type of transfer function is to remain same, in addition to the pole very near to the origin, zero is placed suitably away from this pole, so that the stability is not scarified and also the steady state error e_{ss} remarkbly improved.

Let a pole at $s = - 0.01$ and zero at $s = - 0.1$ be added to the transfer function (1.4) to take the following form :

$$G(s)\ H(s) = \frac{K\ (s + 0.1)}{s(s + 0.01)\ (s + 1)\ (s + 5)} \qquad \qquad ...(1.7)$$

The root locus plot for the transfer function (1.7) is shown in Fig. 1.8.
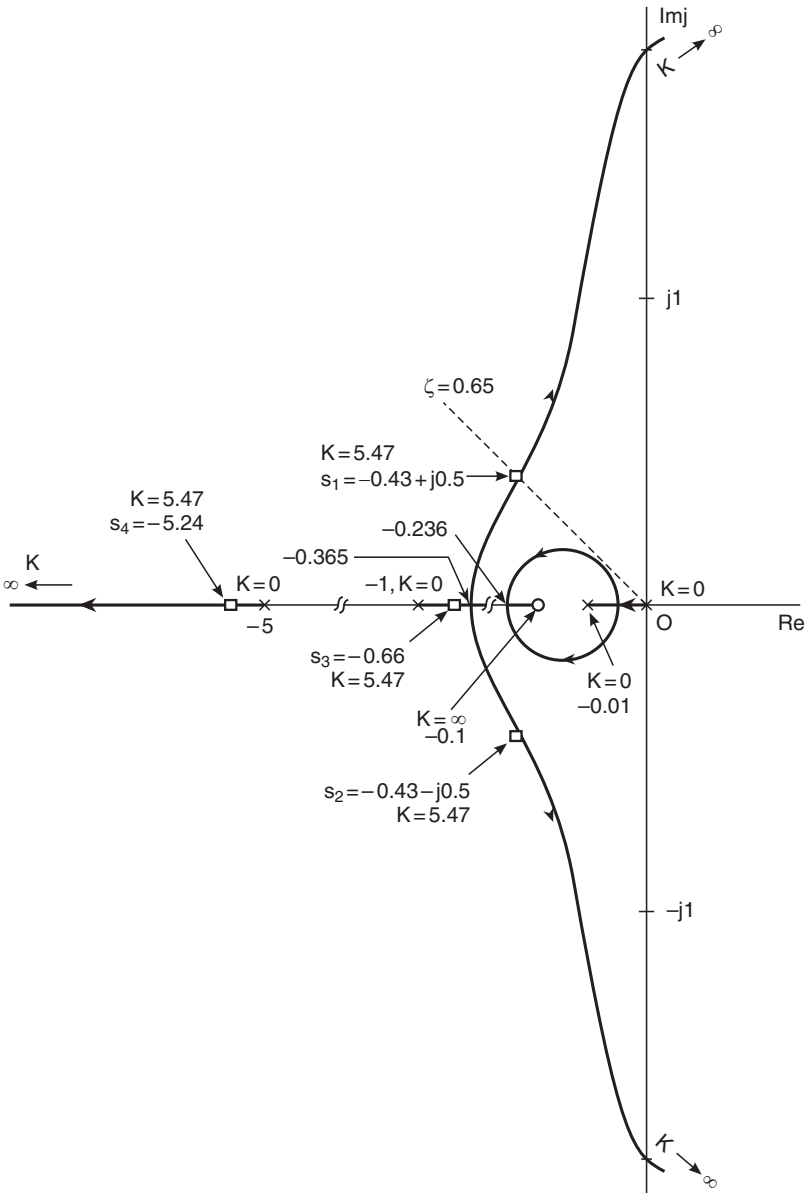

Fig. 1.8. Root locus plot, $G(s)\, H(s) = \dfrac{K\,(s + 0.1)}{s(s + 0.01)\,(s + 1)\,(s + 5)}$.

For a specified value of damping ratio $\xi = 0.65$ following data is obtained from the root locus plot (Fig. 1.8).

Closed loop dominant poles, s_1, $s_2 = -0.43 \pm j\,0.5$

Gain factor at s_1, $s_2 = K = 5.47$

Closed loop third pole, $s_3 = -5.24$

Closed loop fourth pole, $s_4 = -0.66$

Actual damping, $\zeta\omega_n = 0.5$

Natural frequency of oscillations,

$$\omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}} = \frac{0.43}{\sqrt{1-0.65^2}} = 0.56 \text{ rad/s}$$

Settling time, $t_s = \dfrac{4}{\xi\omega_n} = \dfrac{4}{0.43} = 9.3 \text{ sec.}$

Ramp error coefficient

$$K_v = \lim_{s \to 0} s\,G(s)\,H(s)$$

$$= \lim_{s \to 0} \frac{s\,K(s+0.1)}{s(s+0.01)(s+1)(s+5)} = 2\,K$$

Steady state error (for unit ramp input)

$$e_{ss} = \frac{1}{K_v}$$

$$= \frac{1}{2K}$$

$$= \frac{1}{2 \times 5.47} = 0.091$$

On comparing the performance data of the original system *i.e.*

$$G(s)\,H(s) = \frac{K}{s(s+1)\,(s+5)}$$

and the compensated transfer function *i.e.*

$$G(s)\,H(s) = \frac{K\,(s+0.1)}{s(s+0.01)\,(s+1)\,(s+5)}$$

it is concluded that, the location of the dominant closed loop pole is not much affected as such the transient response almost remains the same and much improvement in steady state error performance is observed.

The insertion of a compenseting network having pole near to imaginary axis introduces a phase lag, Therefore, this network is knowns as phase lag compensation network.

1.3 COMPENSATOR NETWORK DESIGN : ROOT LOCUS APPROACH.

The addition of a "zero" to an open-loop transfer function of a control system improves its time response. However, it is not possible to add an isolated zero/pole to a transfer function. Therefore, in order to realize the transfer function of a compensation network a pair of pole and zero is suitably incorporated as per the requirements. For phase lead compensator, the suitable location of the pole is selected away from the zero in the L.H.S of the s-plane. This condition gives the zero a dominance with respect to the associated pole, because the zero is placed near the s-plane imaginary axis. Thus, the original root locus shifts towards L.H.S of the s-plane. The use of this compensator network gives a phase angle lead. Hence, known as phase lead compensation.

The addition of a pole to an open-loop transfer function such that the said pole is placed near L.H.S of the s-plane imaginary axis and the zero suitably away from the pole introduces a phase angle lag. This type of compensator network is known as phase lag compensation. The use of phase lag compensator increases the steady state error coefficient, thereby improves (reduces) the steady state error. The phase angle contribution due to the phase lag compensator is very small as such the shape of the original root locus almost remains the same. Therefore, to a fair approximation the time response is not affected.

In general, the transfer function of a compensator is given by

$$G_c(s) = \frac{s + a}{s + b}$$

Depending on the s-plane location of pole $(- b)$/zero $(- a)$ the transfer function $G_c(s)$ is decided as that of lead or lag type.

(1) Lead compensation

For the case $a < b$, the pole-zero s-plane location of the lead compensator is shown in Fig. 1.9.

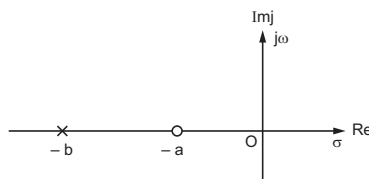Fig 1.9 pole-zero plot : Lead compensation

The zero at $s = - a$ of the lead compensator is placed near to the imaginary axis ($j\omega$) with respect to pole $s = - b$. Thus, the effect

of the compensator zero is more as compared to pole for modifying the performance of a given control system. The compensation thus introduced in cascade with the open-loop transfer function of a control system gives a phase lead.

As $a < b$ the K_c gain less than unity is introduced. Thus, to adjust the gain to its original value, the transfer function of a phase lead compensator takes the form as below :

$$G_c(s) = \frac{K_c \, (s + a)}{(s + b)} \; , a < b$$

The insertion of a phase lead compensator improves the time response

(2) Lag Compensation

For the case $a > b$ the s-plane pole-zero location is shown in Fig. 1.10. The corresponding transfer function is given below :

$$G_c(s) = \frac{(s + a)}{(s + b)} \; , a > b$$
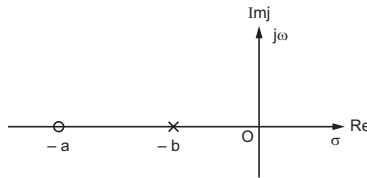
Fig 1.10 Pole-zero plot : Lag compensation

The pole of the compensator at $s = - b$ is placed near to the s-plane imaginary axis ($j\omega$) with respect to the at $s = - a$. The effect of the compensator pole is more as compared to the zero. The compensation network connected in cascade with open-loop transfer function gives a phase lag.

As $a > b$, the steady state error coefficient is increased which improves (reduces) the steady state error. The gain of the lag compensator is considered as unity.

(3) Lag-Lead compensation

For the case wherein both time response as well as steady state error are to be improved Lag-Lead compensation is used.

The transfer function of a lag-lead compensator is given below :

$$G_c(s) = \frac{K_c \, (s + a_1) \, (s + a_2)}{\underset{\text{lead}}{(s + b_1)} \; \underset{\text{lag}}{(s + b_2)}}$$

$a_1 < b_1$ and $a_2 > b_2$

(A) Phase lead compensator design

The design of a phase lead compensator is done as per the following methods :

Method No. 1

Graphical method :

(*a*) Locate the desired closed-loop "s_d" (*i.e.,* the desired root of the characteristic equation) at the location P on the *s*-plane.

(*b*) Bisect $\angle OPA$, the bisector intersects the *s*-plane real axis at point *C*.

(*c*) Draw line *PE* such that $\angle EPC = \dfrac{\theta_c}{2}$ where θ_c is the required phase angle lead.
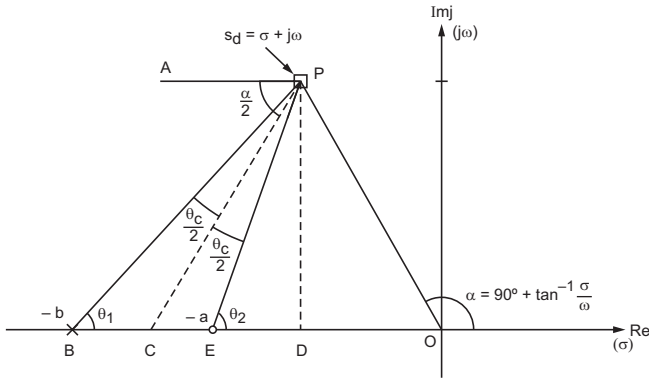
Fig 1.11 Phase lead compensation : Graphical method

Draw the lines *OP* and *PA*. The line *PA* is parallel to *s*-plane real axis.

(*d*) Draw line *PB* such that $\angle BPC = \dfrac{\theta_c}{2}$.

(*e*) Draw perpendicular from the point *P* to intersect *s*-plane real axis at point *D*.

(*f*) Locate the compensator zero ($s = -a$) at point E and the corresponding pole ($s = -b$) at point *B*.

With reference to Fig 1.11

$$\angle OPA = \propto, \therefore \angle BPA = \frac{\alpha}{2} - \frac{\theta_c}{2}$$

$$\because \qquad \angle BPA = \angle PBC = \theta_1$$

$$\therefore \qquad \theta_1 = \frac{\alpha}{2} - \frac{\theta_c}{2} \qquad\qquad ...(i)$$

Also, $\qquad \angle EPA = \angle PED = \theta_2$

$$\therefore \qquad \theta_2 = \frac{\alpha}{2} + \frac{\theta_c}{2} \qquad\qquad\qquad ...(ii)$$

The required phase lead (*i.e.,* phase angle contributed by the phase lead compensator) is $\angle BPE$

Subtracting Eqn. (*i*) from (*ii*)

$$\theta_c = \theta_2 - \theta_1$$

The length *DB* and *DE* are determined as below :

$$\tan \theta_1 = \frac{\omega}{DB} \text{ and } \tan \theta_2 = \frac{\omega}{DE}$$

Place the compensator zero "*a*" at location *E*.

$$\therefore \qquad\qquad -a = -(\sigma + DE)$$

Place the compensator pole "*b*" at location *B*

$$\therefore \qquad\qquad -b = -(\sigma + BD)$$

The transfer function of the phase lead compensator is,

$$G_c(s) = \frac{K_c (s + a)}{(s + b)}, \, a < b$$

The parameters *a* and *b* are determined as above. The open-loop transfer function of a unity feedback control system using phase lead compensation is given by:

$$G(s)G_c(s) = \frac{G(s)K_c (s + a)}{(s + b)}$$

Consider,

$$G(s) = \frac{K(p + z_1) \dots (s + z_n)}{s(s + p_1) \dots (s + p_m)}, \, n < m$$

$$\therefore \qquad G(s) \, G_c(s) = \frac{KK_c(s + z_1) \dots (s + z_m) \, (s + a)}{s(s + p_1) \dots (s + p_m) \, (s + b)}$$

Determination of the gain KK_c and *K*.

To determine the value of the compensated gain KK_c (gain factor) apply root locus magnitude condition at the desired closed-loop pole $s = s_d$

$$\therefore \qquad\qquad \left| G(s) \, G_c(s) \right|_{s \, = \, sd} = 1$$

The uncompensated (original) gain K is determined by applying root locus magnitude condition at the original operating point $s = s_1$ which lies on the uncompensated root locus plot *i.e.,* prior to the addition of phase lead compensator.

$$\therefore \qquad \left|G(s)\right|_{s\,=\,s_1} = 1$$

The gain KK_c and K are determined as above

$$\therefore \qquad K_c = \frac{KK_c}{K}$$

(2) Method No. 2

Place the compensator zero $s = -a$ coinciding with one of the real open-loop pole of the given control system.
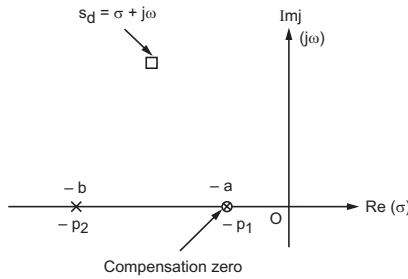
Fig 1.12. Placement of compensator zero : method no. 2.

Thus, select $a = p_1$ where p_1 is the said open-loop pole.

Consider,

$$\therefore \qquad G_c(s) = \frac{K_c(s + p_1)}{(s + b)} \quad p_1 < b$$

The phase angle contribution by the phase lead compensator

The phase angle θ_c contributed by the phase lead compensator at the desired closed-loop pole s_d is

$$\theta_c = \angle \left.\frac{K_c(s + p_1)}{(s + b)}\right|_{s\,=\,s_d\,=\,\sigma\,+\,j\omega}$$

Having given, the values of θ_c, $s_d = \sigma + j\omega$ and $a = p_1$, the value of compensator pole "b" can be determined. The open-loop pole is cancelled by the compensator zero the order of the characteristic equation remains unchanged.

(3) Method No. 3

Place the zero "a" of the compensator on the s-plane real axis below the desired closed-loop pole s_d as shown in Fig 1.13.
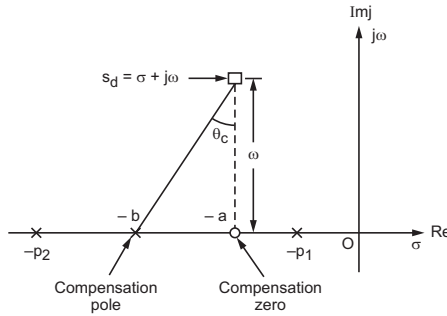
Fig 1.13. Placement of compensator zero : method no. 3.

The phase angle lead contributed by the compensator is θ_c.

$$\tan \theta_c = \frac{b - a}{\omega}$$

where $\qquad\qquad \theta_c = \angle G_c(s)\Big|_{s = s_d = \sigma + j\omega}$

SOLVED EXAMPLES : PHASE LEAD COMPENSATION

Example 1.3.1. *The forward path transfer function of a unity feedback control system is given below :*

$$G(s) = \frac{K}{s(s + 9)}$$

It is desired to have maximum overshoot as 9.4 % and the natural frequency of oscillation $\omega = 12$ rad./sec. the input is unit step.

Design a phase lead compensator.

Solution. (1) the desired closed loop pole "s_d".

$$\% \, M_p = e^{\dfrac{-\xi\pi}{\sqrt{-1-\xi^2}}}$$

$$\therefore \qquad \frac{9.4}{100} = e^{\dfrac{-\xi\pi}{\sqrt{-1-\xi^2}}}$$

From the above relation the value of the damping ratio is calculated as $\xi = 0.6$. The desired closed loop pole is given by

$$s_d = -\xi\omega_n + j\omega_n\sqrt{1 - \xi^2}$$

$\because \qquad\qquad \xi = 0.6$ and $\omega_n = 12$ rad./sec.

$\therefore \qquad\qquad s_d = 0.6 \times 12 + j12\sqrt{1 - 0.6^2}$

or $\qquad\qquad\qquad s_d = -7.2 + j9.6$

(2) Uncompensated root locus plot

As per the root locus plot procedure, the root locus plot for the open-loop transfer function

$$G(s) \ = \ \frac{K}{s(s+9)}$$

is shown in Fig 1.14
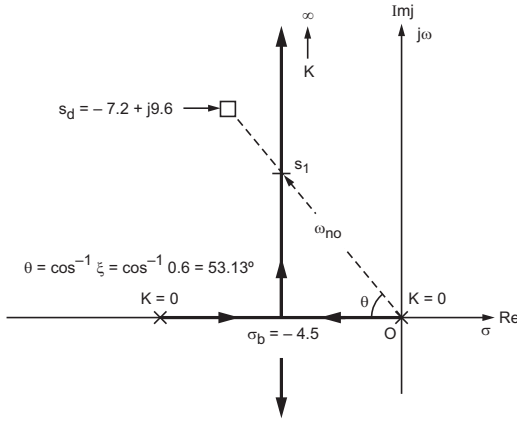
Fig 1.14 Root locus Example 1.3.1

The break away point is $\sigma_b = -4.5$ and asymptoe angles are ±90°. The damping ratio line angle is $\theta = \cos^{-1} \xi = \cos^{-1} 0.6 = 53.13°$. The damping ratio line intersects the uncompensated root locus at s_1 the length os_1 equals ω_{no}, where ω_{no} is the original (uncompensated) natural frequency of osicillations.

$$\because \qquad\qquad \cos \theta \ = \ \frac{4.5}{\omega_{no}}$$

$$\therefore \qquad\qquad \omega_{no} \ = \ \frac{4.5}{\cos \theta} \ = \ \frac{4.5}{0.6} = 7.5 \text{ rad./sec.}$$

The second order characteristic equation is given as

$s^2 + 2\,\xi\omega_{no}s + \omega_{no}^2 = 0$

Since, characteristic equation for the given open-loop transfunction is

$$s^2 + 9s + K \ = \ 0$$

On comparing coefficients.

$$K \ = \ \omega_{no}^2 = 7.5^2 = 56.25$$

(3) Compensated case

The desired closed-loop pole $s_d = -7.2 + j9.6$ does not lie on the uncompensated root locus. Hence, determine the phase angle θ_c to be contributed by compensator. The lead angle θ_c is calculated as below :

Applying root locus angle condition at
$$s_d = -7.2 + j9.6$$

$$\therefore \quad \theta_c + \angle G(s)\bigg|_{\substack{ = -180° \\ s = s_d = -7.2 + j\,9.6}}$$

or $\quad \theta_c + \angle \dfrac{K}{s(s+9)}\bigg|_{\substack{ = -180° \\ s = s_d = -7.2 + j\,9.6}}$

or $\quad \theta_c + \angle \dfrac{K}{(-7.2 + j\,9.6)\,(-7.2 + j\,9.6 + 9)} = -180°$

or $\quad \theta_c - \left(\tan^{-1} \dfrac{9.6}{-7.2} + \tan^{-1} \dfrac{9.6}{1.8} \right) = -180°$

$\therefore \quad \theta_c - \left[(180° - 53.3°) + 79.37° \right] = -180°$

Simplifying, $\theta_c = 26.24°$

(4) Determination of the parameters "a" and "b" of phase lead compensator.

Graphical Method :

The desired closed-loop pole $(s_d = -7.2 + j\,9.6)$ is shown at P in Fig 1.15.

$$\angle OPA = \angle POM$$

$$\therefore \quad \angle OPA = 90° + \tan^{-1}\left(\frac{OD}{PD} \right) = 90° + \tan^{-1} \frac{7.2}{9.6}$$
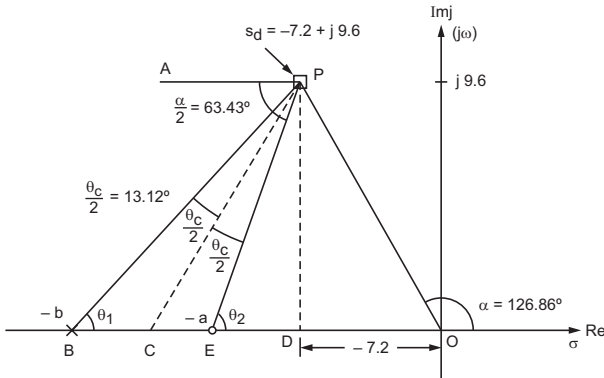
$$= 126.86°$$

Fig 1.15. Graphical method : Example 1.3.1

The line PC bisects $\angle\ OPA$.

$$\therefore \qquad \angle\ APC = \frac{126.86°}{2} = 63.43°$$

The required phase lead is 26.24° is contributed by the compensator. Draw lines PB and PE making angle $\theta_{c/2}$ with line PC and cut the s-plane real axis at B and E respectively.

$$\angle\ APB = \angle\ PBC$$

$$\angle\ APB = 63.43° - \frac{\theta_c}{2} = 63.43° - \frac{26.24°}{2} = 50.31°$$

$$\therefore \qquad \theta_1 = 50.31°$$

Also, $\qquad \angle\ APE = 63.43° + \dfrac{\theta_c}{2} = 63.43° + \dfrac{26.24°}{2} = 76.55°$

$$\therefore \qquad \theta_2 = 76.55°$$

$$\tan \theta_1 = \frac{PD}{BD} = \frac{9.6}{BD} ,$$

$$\therefore \qquad BD = \frac{9.6}{\tan \theta_1} = \frac{9.6}{\tan 50.31°} = 8$$

The parameter "b" of the lead compensator $w.r.t$ Fig 1.15 is

$$b = (BD + DO) = 8 + 7.2 = 15.2$$

Also, $\qquad \tan \theta_2 = \dfrac{PD}{ED} = \dfrac{9.6}{ED} ,$

$$\therefore \qquad ED = \frac{9.6}{\tan \theta_2} = \frac{9.6}{\tan 76.55°} = 2.3$$

The parameter "a" of the lead compensator $w.r.t$ Fig 1.15 is

$$a = (ED + DO) = 2.3 + 7.2 = 9.5$$

$$\therefore \qquad G_c(s) = \frac{K_c\ (s + 9.5)}{(s + 15.2)}$$

(5) Determination of compensator gain K_c.

The open-loop transfer function using phase lead compensator is given below :

$$G(s)\ G_c(s) = \frac{KK_c\ (s + 9.5)}{s(s + 9)\ (s + 15.2)}$$

The desired closed-loop pole $s_d = -\ 7.2 + j\ 9.6$ has to lie on the compensated root locus. Hence, the magnitude condition of the root locus is to be satisfied at $s_d = -\ 7.2 + j\ 9.6$.

$$\therefore \qquad \left| G(s)\ G_c(s) \right|_{s\ =\ s_d\ =\ -\ 7.2\ +\ j\ 9.6} = 1$$

$$\therefore \quad \left| \frac{KK_c(s+9.5)}{s(s+9)\,(s+15.2)} \right| \quad \begin{matrix} = 1 \\ s = s_d = -7.5 + j\,9.6 \end{matrix}$$

Rearranging,

$$KK_c = \left| \frac{s(s+9)\,(s+15.2)}{(s+9.5)} \right| \qquad s = s_d = -7.2 + j\,9.6$$

$$\therefore \ KK_c = \left| \frac{(-7.2 + j\,9.6)\,(-7.2 + j\,9.6 + 9)\,(-7.2 + j\,9.6 + 15.2)}{(-7.2 + j\,9.6 + 9.5)} \right|$$

$$= \left| \frac{(-7.2 + j\,9.6)\,(1.8 + j\,9.6)\,(8 + j\,9.6)}{(2.3 + j\,9.6)} \right|$$

$$= \sqrt{\frac{(51.84 + 92.16)\,(3.24 + 92.16)\,(64 + 92.16)}{(5.29 + 92.16)}}$$

Simplifying,

$$KK_c = 148.33$$

or

$$K_c = \frac{148.33}{K}$$

$$\because \qquad\qquad K = 56.25 \text{ (determined earlier)}$$

$$\therefore \qquad\qquad K_c = \frac{148.33}{56.25} = 2.63$$

The compensated open-loop transfer function is given below:

$$G(s)\,G_c(s) = \frac{56.25 \times 2.63\,(s+9.5)}{s(s+9)\,(s+15.2)} \quad \textbf{Ans.}$$

Example 1.3.2. *Determine the transfer function of a phase lead compensator for a unity feedback control system having the following open-loop transfer functions :*

$$G(s) = \frac{K}{(s+2)\,(s+4)}$$

Following specifications are to be satisfied

(i) % M_p = 16.3 %.

(ii) Reduction in settling time 60%.

(iii) The order of the characteristic to remain the same.

Solution. (1) **Uncompensated case**

As per the root locus plot procedure, the break away point is determined as $\sigma_b = -3$. The angle of asymptoes ±90°. The root locus plot is shown Fig. 1.16.
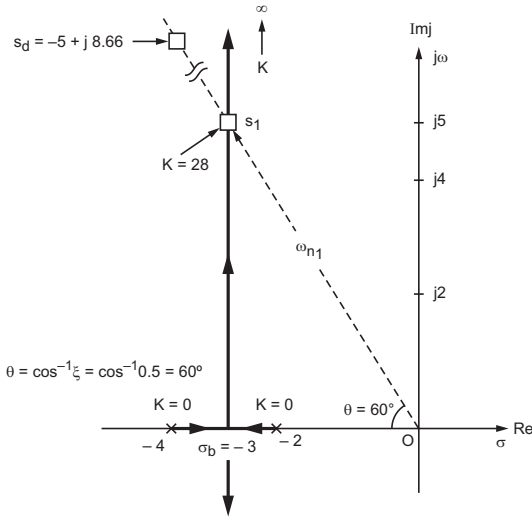
Fig 1.16 Root locus for Example

$$\% \ M_p \ = \ e^{\dfrac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

$$\therefore \qquad \dfrac{16.4}{100} \ = \ e^{\dfrac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

Solving above equation, the damping ration $\xi = 0.5$. The angle between the s-plane real axis and the dampratio line is $\theta = \cos^{-1}\xi = \cos^{-1}0.5 = 60°$ (Fig 1.16)

Intersection of the damping ratio line with root locus

The damping ratio line equation is give by

$$\dfrac{\omega}{\sigma} \ = \ -\tan\theta = -\tan 60° = -\sqrt{3}$$

$$\therefore \qquad\qquad \omega \ = \ -\sigma\sqrt{3}$$

$$\because \qquad\qquad \sigma \ = \ -3, \ \therefore \ \omega = 3\sqrt{3} \ \ \text{rad./sec.}$$

The intersection point is $s_1 = \sigma + j\omega = -3 + j\,3\sqrt{3}$.

The value of the gain K (uncompensated) is determined applying magnitude condition of root locus at $s = s_1 = -3 + j\,3\sqrt{3}$.

$$\therefore \qquad \left| G(s) \right|_{\substack{=1 \\ s=s_1=-3+j\,3\sqrt{3}}}$$

$$\therefore \quad \left| \frac{K}{(s+2)(s+4)} \right|_{s=s_1=-3+j\,3\sqrt{3}} = 1$$

$$\therefore \qquad\qquad K = \left| (s+2)(s+4) \right| \qquad s=s_1=-3+j\,3\sqrt{3}$$

$$= \left| (-3+j\,3\sqrt{3}+2)(-3+j\,3\sqrt{3}+4) \right| = 28$$

(2) Compensated case

Settling time : $t_{s1} = \dfrac{4}{\xi\omega_{n1}}$ $\because$ $(\xi\omega_{n1} = 3)$

$$\therefore \qquad\qquad t_{s1} = \frac{4}{3} \text{ sec.}$$

$$t_{s2} = 0.6\, ts_1 \quad \text{(given)}$$

$$\therefore \qquad\qquad t_{s2} = 0.6 \times \frac{4}{3} = 0.8 \text{ sec.}$$

Also $\qquad\qquad t_{s2} = \dfrac{4}{\xi\omega n_2},$

$$\therefore \qquad\qquad \omega_{n2} = \frac{4}{\xi t_{s2}} = \frac{4}{0.5 \times 0.8} = 10 \text{ rad./sec.}$$

Hence, the desired closed-loop pole "s_d" for the compensated case is

$$s_d = -\xi\omega_{n2} + j\omega_{n2}\sqrt{1-\xi^2}$$

$$= -0.5 \times 10 + j\,10\sqrt{1-0.5^2}$$

Simplifying,

$$s_d = -5 + j\,8.66$$

The desired closed-loop pole $s_d = -5 + j\,8.66$ does not lie on the root locus (Fig. 1.16). Hence, the phase angle θ_c contributed by the phase lead compensator is determined as below, by applying root locus condition at $s_d = -0.5 + j\,8.66$ *i.e.,*

$$\theta_c + \angle \left. \frac{K}{(s+2)(s+4)} \right|_{=s_d=-5+j\,8.66} = -180°$$

$$\therefore\ \theta_c + \angle \frac{K}{(-5+j\,8.66+2)(-5+j\,8.66+4)} = -180°$$

$$\therefore\ \ \theta_c - \left[\left(180° - \tan^{-1}\frac{8.66}{3} \right) + \left(180° - \tan^{-1}\frac{8.66}{1} \right) \right] = -180°$$

Simplifying,

$\theta_c - [(180° - 70.89) + (180° - 83.41°)] = -180°$

$\therefore$ $\qquad\qquad\qquad \theta_c = 25.7°$

The phase lead $\theta_c = 25.7°$ is to be provided by the lead compensator.

The transfer function of a phase lead compensator is given below :

$$G_c(s) = \frac{K_c\,(s + a)}{(s + b)}\,, a < b$$

As the order of the characteristic equation is to remain unchanged, the zero of the compensator *i.e.,* "*a*" is placed at the open-loop pole $s = -2$.

The phase lead compensator transfer function takes the form as below :

$$G_c(s) = \frac{K_c\,(s + 2)}{(s + b)}$$

The angle θ_c to be contributed by the lead compensator at $s_d = -5 + j\,8.66$ is given by :

$$\theta_c = \angle \left.\frac{K_c\,(s + 2)}{(s + b)}\right| \qquad s = s_d = -5 + j\,8.66$$

$\because$ $\qquad\qquad \theta_c = 25.7°$ (determined earlier)

$\therefore$ $\qquad 25.7° = \angle \left.\dfrac{K_c(s + 2)}{(s + b)}\right| \qquad s = s_d = -5 + j\,8.66$

or $\qquad\qquad 25.7° = \angle \dfrac{(-5 + j\,8.66 + 2)}{-5 + j\,8.66 + b}$

Rearranging,

$$25.7° = \angle \frac{-3 + j\,8.66}{(b - 5) + j\,8.66}$$

$$= \left(180° - \tan^{-1}\frac{8.66}{3}\right) - \tan^{-1}\frac{8.66}{(b - 5)}$$

$$= (180° - 70.89°) - \tan^{-1}\frac{8.66}{(b - 5)}$$

Simplifying,

$$\tan^{-1}\frac{8.66}{(b - 5)} = 83.41°$$

$$\therefore \qquad \frac{8.66}{(b-5)} = \tan 83.41°$$

Solving,

$$b = 6$$

Thus, the transfer function of the phase lead compensate is :

$$G_c(s) = \frac{K_c\,(s+2)}{(s+6)}$$

The open-loop transfer function using phase lead compensator is given below :

$$G(s)\,G_c(s) = \frac{KK_c\,(s+2)}{(s+2)\,(s+4)\,(s+6)}$$

$$= \frac{KK_c}{(s+4)\,(s+6)}$$

The desired close-loop pole lies on the compensated root locus, satisfying the root locus magnitude condition at $s_d = -5 + j\,8.66$

$$\therefore \qquad \left|G(s)\,G_c(s)\right| \quad\begin{matrix} = 1 \\ s = s_d = -5 + j\,8.66 \end{matrix}$$

Therefore,

$$\left|\frac{KK_c}{(s+4)\,(s+6)}\right| \quad\begin{matrix} = 1 \\ s = s_d = -5 + j\,8.66 \end{matrix}$$

Rearranging,

$$KK_c = \left|(s+4)\,(s+6)\right| \qquad\qquad s = s_d = -5 + j\,8.66$$

$$= |(-5 + j\,8.66 + 4)\,(-5 + j\,8.66 + 6)|$$

$$= |(-1 + j\,8.66)\,(1 + j\,8.66)|$$

$$= \sqrt{(1^2 + 8.66^2)\,(1^2 + 8.66^2)}$$

$$\therefore \qquad KK_c = 76$$

$$\because \qquad K = 28 \text{ (determined earlier)}$$

$$\therefore \qquad K_c = \frac{76}{28} = 2.71$$

and $\qquad G(s)\,G_c(s) = \dfrac{28 \times 2.71}{(s+4)\,(s+6)}$ **Ans.**

Example 1.3.3. *The open-loop transfer function of a unity feed back control system operating with a damping ratio x = 0.5 is given below :*

$$G(s) \ = \ \frac{K}{s(s + 2)\,(s + 5)}$$

The settling time is t_s = 5.63 sec.

Design a suitable compensator to reduce the settling time by a factor of 0.6.

Solution. As the settling time is to be reduced, a phase lead network is selected.

(1) **Uncompensated system**

$$t_{s1} \ = \ \frac{4}{\xi\omega_{n1}} \ , \ \because \ \xi = 0.5 \text{ and } t_{s1} = 5.63 \text{ sec}$$

$\therefore$ $\qquad\quad 5.63 \ = \ \dfrac{4}{0.5\,\omega_{n1}}$

Simplifying,

$$\omega_{n1} \ = \ \frac{4}{0.5 \times 5.63} \ = 1.42 \text{ rad./sec.}$$

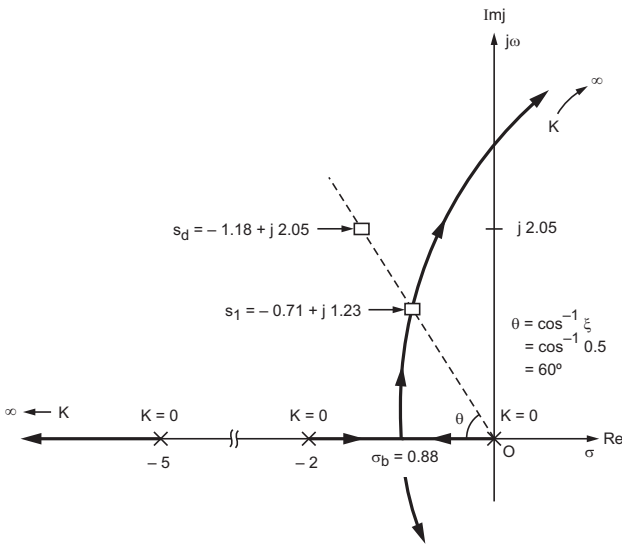The general shape of the root locus is shown in Fig 1.17.

Fig. 1.17 Root locus (uncompensated) : Example 1.3.3

As the value of ω_n is determined from the given settling time and $\xi = 0.5$, the uncompensated closed-loop pole is given by

$$s_1 = -\xi\omega_{n_1} + j\omega_{n_1}\sqrt{1-\xi^2}$$

$$= -0.5 \times 1.42 + j\,1.42\sqrt{1-0.5^2}$$

$$\therefore \qquad s_1 = -0.71 + j\,1.23$$

The value of uncompensated gain K is determined by applying magnitude condition to the original (uncompensated) root locus at $s_1 = -0.71 + j\,1.23$.

$$\therefore \quad \left|\frac{K}{s(s+2)\,(s+5)}\right| \qquad \begin{array}{l} = 1 \\ s = s_1 = -0.71 + j\,1.23 \end{array}$$

Rearranging,

$$K = \left| s(s+2)\,(s+5) \right| \qquad s = s_1 = -0.71 + j\,1.23$$

$$= \left| (-0.71 + j\,1.23)\,(-0.71 + j\,1.23 + 2)\,(-0.71 + j\,1.23 + 5) \right|$$

$$= \left| (0.71 + j\,1.23)\,(1.29 + j\,1.23)\,(4.29 + j\,1.23) \right|$$

$$= \sqrt{(0.71^2 + 1.23^2)\,(1.29^2 + 1.23^2)\,(4.29^2 + 1.23^2)}$$

$$\therefore \qquad K = 11.27$$

(2) Compensated Case

$$t_{s2} = 0.6\,t_{s1} \quad \text{(given)}$$
$$\because \qquad t_{s1} = 5.63 \text{ sec}$$
$$t_{s2} = 0.6 \times 5.63 = 3.378 \text{ sec.}$$

As the value of damping ratio $\xi = 0.5$ remains same for the compensated as well as uncompensated case

$$t_{s2} = \frac{4}{\xi\omega_{n2}}$$

$$\therefore \qquad \omega_{n2} = \frac{4}{\xi t_{s2}} = \frac{4}{0.5 \times 3.378} = 2.368 \text{ rad./sec.}$$

On compensation the desired closed-loop pole is

$$s_d = -\zeta\omega_{n2} + j\omega_{n2}\sqrt{1-\xi^2}$$

$$= -1.18 + j\,2.368\sqrt{1-0.5^2}$$

Simplifying,

$$s_d = -1.18 + j\,2.05$$

The closed-loop pole does not lie on the uncompensated root locus plot. Therefore, at $s_d = -1.18 + j\, 2.05$, the angle θ_c is to be contributed by the phase lead compensator plus $\angle\, G(s)\,|_{s_d}$ satisfies the angle condition of root locus. Thus,

$$\theta_c + \angle \frac{K}{s(s+2)\,(s+5)}\bigg|_{s\,=\,s_d\,=\,-1.18\,+\,j\,2.05} = -180°$$

or $\quad \theta_c + \angle \dfrac{K}{(-1.18 + j\,2.05)\,(-1.18 + j\,2.05 + 2)\,(-1.18 + j\,2.05 + 5)}$

$$= -180°$$

$$\therefore\; \theta_c + \left[-\left(180° - \tan^{-1} \frac{2.05}{1.18}\right) - \tan^{-1}\frac{2.05}{0.82} - \tan^{-1}\frac{2.05}{3.82}\right]$$

$$= -180°$$

or $\qquad \theta_c + [-(180° - \tan^{-1} 1.73) - \tan^{-1} 2.5 - \tan^{-1} 0.536] = -180°$

$\therefore\; \theta_c + [-(180° - 59.97°) - 68.19° - 28.19°\} = -180°$

Simplifying,

$$\theta_c = 36.41°$$

(2) Phase lead compensator

The transfer function of a phase lead compensator is given below :

$$G_c(s) = \frac{K_c\,(s+a)}{(s+b)}, \; a < b$$

The phase lead compensator provides a lead angle θ_c at $s_d = -1.18 + j\, 2.05$

$$\therefore \qquad \theta_c = \angle \frac{K_c\,(s+a)}{(s+b)}\bigg| \qquad s = s_d = -1.18 + j\,2.05$$

$\because \qquad \theta_c = 36.41°$ (calculated earlier)

$$\therefore \quad 36.41° = \angle\frac{K_c\,(s+a)}{(s+b)}\bigg| \qquad s = s_d = -1.18 + j\,2.05$$

Select $a = 2$ such that the open-loop pole at $s = -2$ is cancelled and the order of the characteristic equation remains unchanged.

$$\therefore \qquad 36.41° = \angle\frac{K_c\,(s+2)}{(s+b)}\bigg| \qquad s = s_d = -1.18 + j\,2.05$$

$$= \angle\frac{(-1.18 + j\,2.05 + 2)}{(b - 1.18) + j\,2.05}$$

$$= \tan^{-1}\frac{2.05}{0.82} - \tan^{-1}\frac{2.05}{b - 1.18}$$

$$\therefore \qquad 36.41° = 68.19 - \tan^{-1} \frac{2.05}{b - 1.18}$$

Rearranging,

$$\tan^{-1} \frac{2.05}{b - 1.18} = 31.78°$$

$$\therefore \qquad \tan 31.78° = \frac{2.05}{b - 1.18}$$

$$\therefore \qquad 0.619 = \frac{2.05}{b - 1.18}$$

Solving,

$$b = 4.49$$

Compensated gain KK_c

The value of the compensated gain KK_c at $s = s_d = -1.18 + j\,2.05$ is determined using root locus magnitude condition to compensated open-loop transfer function.

The compensated open-loop transfer function is

$$G(s)\,G_c(s) = \frac{KK_c(s+2)}{s(s+2)\,(s+5)\,(s+4.49)} = \frac{KK_c}{s(s+5)\,(s+4.49)}$$

Rearranging,

$$KK_c = \left| s(s+5)\,(s+4.49) \right| \qquad s = s_d = -1.18 + j\,2.05$$

$$= \left| (-1.18 + j\,2.05)\,(-1.18 + j\,2.05 + 5)\,(-1.18 + j\,2.05 + 4.49) \right|$$

$$= \left| (-1.18 + j\,2.05)\,(3.82 + j\,2.05)\,(3.31 + j\,2.05) \right|$$

$$= \sqrt{(1.18^2 + 2.05^2)\,(3.82^2 + 2.05^2)\,(3.31^2 + 2.05^2)}$$

Simplifying,

$$KK_c = 39.92$$
$$\because \qquad K = 11.27 \text{ (determined earlier)}$$

$$\therefore \qquad K_c = \frac{39.92}{11.27} = 3.54$$

Thus,

$$G(s)\,G_c(s) = \frac{11.27 \times 3.54}{s(s+5)\,(s+4.49)} \qquad \textbf{Ans.}$$

Example 1.3.4. *Design a phase lead compensator such that a unity feedback control system having following forward path transfer function*

$$G(s) \; = \; \frac{K}{s^2 + 1}$$

becomes stable with decamping ratio $\xi \; = \; 0.5$ *and settling time* $t_s = 2.5$ *sec. The uncompensated system operates at the closed loop pole* $s = \pm j\, 5$

Solution. (1) Uncompensated case

The characteristic equation is given below :

$$1 + G(s) \; = \; 0$$

$\therefore \qquad 1 + \dfrac{K}{s^2 + 1} \; = \; 0$

Simplifying,

$$s^2 + 1 + K \; = \; 0$$

For $K = 0$, $\quad s = \pm j\, 1$: The staring point of the root locus

For $K = 4$, $\quad s = \pm j\, \sqrt{5}$: Closed loop poles

Therefore, the value of the gain factor for the uncompensated system (at $s = \pm j\, \sqrt{5}$) is $K = 4$.

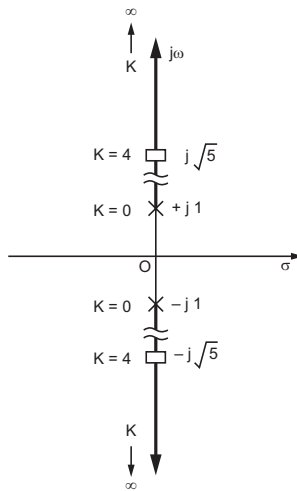The uncompensated root locus is shown in Fig 1.18 (*a*)

Fig 1.18 (*a*) uncompensated root locus : Example 1.3.4

(2) Compensated case

Settling time t_s = 2.5 sec. ξ = 0.5 (given)

$\because$ $\qquad\qquad\qquad t_s = \dfrac{4}{\xi\omega_n}, \quad \therefore \quad 2.5 = \dfrac{4}{\xi\omega_n}$ and $\xi\omega_n = 1.6$

$\because$ $\qquad\qquad\qquad \xi = 0.5, \quad \omega_n = \dfrac{1.6}{0.5} = 3.2$ rad./sec.

The desired closed-loop pole is

$$s_d = -\xi\omega_n + j\,\omega_n\sqrt{1-\xi^2}$$

$$= -1.6 + j\,3.2\sqrt{1-0.5^2}$$

Simplifying,

$$s_d = -1.6 + j\,2.77$$

The desired closed-loop pole $s_d = -1.6 + j\,2.77$ has to lie on the compensated root locus. The angle θ_c contributed by the phase lead compensated is determined below :

Applying root locus angle condition at $s_d = -1.6 + j\,2.77$ *i.e.*,

$$\theta_c + \angle\,G(s)\Big|_{\substack{= -180° \\ s=s_d=-1.6+j\,2.77}}$$

$\therefore$ $\qquad\qquad \theta_c + \angle\,\dfrac{K}{s^2+1}\Bigg|_{\substack{= -180° \\ s=s_d=-1.6+j\,2.77}}$

or $\qquad \theta_c + \angle\,\dfrac{K}{(s+j1)(s-j1)}\Bigg|_{\substack{= -180° \\ s=s_d=-1.6+j\,2.77}}$

$\therefore\ \theta_c + \angle\,\dfrac{K}{(-1.6+j\,2.77+j1)(-1.6+j\,2.77-j1)} = -180°$

$\therefore\ \theta_c + \left[-\left(180° - \tan^{-1}\dfrac{3.77}{1.6}\right) - \left(180° - \tan^{-1}\dfrac{1.77}{1.6}\right)\right]$

or $\qquad \theta_c + [-(180° - 67°) - (180° - 47.88°)] = -180°$

Simplifying,

$$\theta_c = 65.12°$$

Phase lead compensator

The transfer function of a phase lead compensator is given below :

$$G_c(s) = \dfrac{K_c(s+a)}{(s+b)}, a < b$$

On the *s*-plane red axis, select the parameter "*a*" just below the desired closed-loop poles

$$s_d = -1.6 + j\,2.77$$

$\therefore \qquad\qquad a = 1.6$, Thus,

$$G_c(s) = \frac{K_c(s + 1.6)}{(s + b)}$$

$$\theta_c = \angle G_c(s)\Big|_{s = s_d = -1.6 + j\,2.77}$$

$$= \angle \frac{K_c(s + 1.6)}{(s + b)}\Big|_{s = s_d = -1.6 + j\,2.77}$$

$$= \angle \frac{-1.6 + j\,2.77 + 1.6}{-1.6 + j\,2.77 + b}$$

$$= \angle \frac{j\,2.77}{(b - 1.6) + j\,2.77}$$

$$= 90° - \tan^{-1} \frac{2.77}{b - 1.6}$$

$\because \qquad\qquad \theta_c = 65.12°$ (calculated earlier)

$\therefore \qquad 65.12° = 90° - \tan^{-1} \dfrac{2.77}{b - 1.6}$

Simplifying,

$$\tan^{-1} \frac{2.77}{b - 1.6} = 24.88°$$

$\therefore \qquad \tan 24.88° = \dfrac{2.77}{b - 1.6}$

or $\qquad\qquad \dfrac{2.77}{b - 1.6} = 0.463$

Simplifying,

$$b = 7.58$$

Thus, the phase lead compensated transfer function is

$$G(s)\,G_c(s) = \frac{KK_c\,(s + 1.6)}{(s + j\,1)\,(s - j\,1)\,(s + 7.58)}$$

Apply root locus magnitude condition at $s_d = -1.6 + j\,2.77$ to determine the value of KK_c

$$\therefore \left|\frac{KK_c\,(s + 1.6)}{(s + j\,1)\,(s - j\,1)\,(s + 7.58)}\right| \begin{array}{l} = 1 \\ s = s_d = -1.6 + j\,2.77 \end{array}$$

Rearranging,

$$KK_c = \left|\frac{(s + j\,1)\,(s - j\,1)\,(s + 7.58)}{(s + 1.6)}\right| \qquad s = s_d = -1.6 + j\,2.77$$

$$= \left| \frac{(-1.6 + j\,2.77 + j\,1)\,(-1.6 + j\,2.77 - j\,1)\,(-1.6 + j\,2.77 + 7.58}{(-1.6 + j\,2.77 + 1.6)} \right|$$

$$= \left| \frac{(-1.6 + j\,3.77)\,(-1.6 + j\,1.77)\,(5.98 + j\,2.77)}{j\,2.77} \right| = 23.23$$
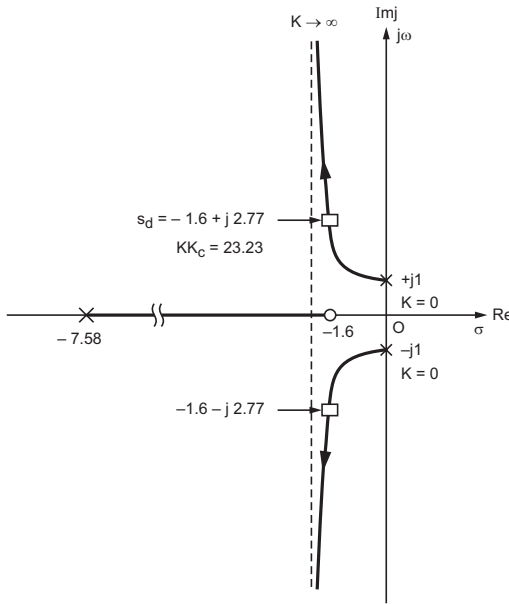
Fig 1.18 (b) Compensated root locus : Example 1.3.4

$$\therefore \qquad KK_c = 23.23$$

$$\because \qquad\quad K = 4 \text{ (determined earlier)}$$

$$\therefore \qquad\quad K_c = \frac{23.23}{4} = 5.8$$

The compensated transfer function is

$$G(s)\,G_c(s) = \frac{4 \times 5.8\,(s + 1.6)}{(s^2 + 1)\,(s + 7.58)} \quad \textbf{Ans.}$$

(B) Phase lag compensator design

The phase lag compensator is used in cascade with the forward path (open-loop) transfer function of a feedback control system where in the steady stat error is to be improved (i.e., reduced).

The phase lag compensator, thus introduced does not appreciably alter the original (uncompensated) root locus near to the desired closed-loop pole. This is possible by selecting a phase lag compensator contributing a phase lag angle in the range of 0° to − 3°.

The distances of the pole and zero of the lag compensator are measured small as compared with the distance of closed loop pole. Thus, the lag compensator does not affect the root locus shape much.

The s-plane complex part of the root locus slightly shifts towards the R.H.S. but this shift is insignificant.

Design Steps : Phase Lag compensator

(a) Locate the desired closed-loop pole on the root locus of the uncompensated system.

(b) Determine value of gain K (gain factor) for the uncompensated system at the desired closed-loop pole.

(c) Calculate the value of the new error coefficient say K_n to meet the steady state to meet the steady state error requirement.

(d) The transfer function of a lag compensator is given below :

$$G_c(s) = \frac{(s + a)}{(s + b)}, \quad a > b$$

The gain is considered as unity. The steady state error coefficient is increased which reduces steady state error.

Determination of parameters "a" (zero) and "b" (pole) of a phase lag compensator

(i) Type "O" transfer function :

The original steady state error e_{sso} for the uncompensated case in terms of positional error coefficient K_{po} is given by

$$e_{sso} = \frac{1}{1 + K_{p_o}}$$

The steady state error e_{ssn} due to the use of phase lag compensator interms of positional error coefficient K_{p_n} is given by

$$e_{ssn} = \frac{1}{1 + K_{p_n}}$$

$$\therefore \qquad \frac{e_{sso}}{e_{ssn}} = \frac{1 + K_{p_n}}{1 + K_{p_o}}$$

It can be shown that

$$\frac{e_{sso}}{e_{ssn}} = \frac{a}{b}$$

$$\therefore \qquad \frac{a}{b} = \frac{1 + K_{p_n}}{1 + K_{p_o}}$$

Having given the values K_{p_n}, K_{p_o} and selecting appropriate value of parameter "b", the parameter "a" is determined.

(*ii*) Type "1" transfer function :

The original steady state error e_{sso} in terms velocity error coefficient K_{vo} is given below,

$$e_{sso} \;=\; \frac{1}{K_{vo}}$$

It can be shown that,

$$\frac{e_{sso}}{e_{ssn}} \;=\; \frac{a}{b}$$

$$\therefore \qquad\qquad \frac{a}{b} \;=\; \frac{K_{vn}}{K_{vo}}$$

Having given values of K_{vn}, K_{vo} and selecting appropriate value of parameter "*b*", the parameter "*a*" is determined.

Solved Examples : Lag compensator

Example 1.3.5. *Design a compensator for a unity feed back control system having open-loop transfer function as*

$$G(s) \;=\; \frac{K}{s(s + 8)}$$

The required specifications are :

Damping ratio ξ = *0.5 and the steady state error due to unit ramp input is reduced by a factor of 6.25.*

Solution. As the steady state error is to be reduced a phase lag compensator is needed.

(1) Uncompensated case

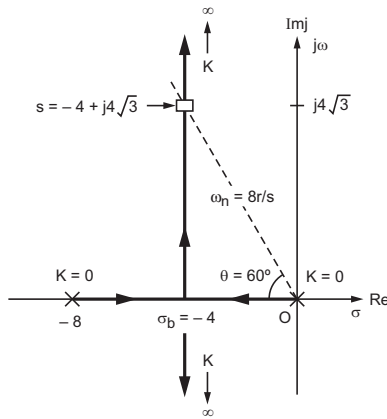As per procedure, the root locus is plotted as shown in Fig 1.19.

Fig 1.19 Root locus plot : Example 1.3.5

The characteristics equation is
$$1 + G(s) = 0$$

$\therefore \quad 1 + \dfrac{K}{s(s + 8)} = 0$

Simplifying,
$$K = -(s^2 + 8s)$$

The break away point is obtained by the relation given below :

$$\dfrac{dK}{ds} = 0$$

$\therefore \qquad \dfrac{dK}{ds} = -(2s + 8)$

or $\qquad -(2s + 8) = 0$

Solving, $s = -4$. Hence, the break away point of the root locus is located as $\sigma_b = -4$.

Angle of asymptotes is given by,

$$\angle A_{sym} = \dfrac{(2k + 1)}{(P - Z)} \times 180°, \; k = 0, 1$$

$$= \dfrac{(2 \times 0 + 1)}{2 - 0} = 90°$$

and $\qquad = \dfrac{(2 \times 1 + 1)}{2 - 0} \times 180° = -90°$

(2) Phase lag compensator

$$\sigma_b = -\xi\omega_n, \; \because \; \xi = 0.5 \text{ (given)},$$

$$\omega_n = \dfrac{4}{0.5} = 8 \text{ rad./sec.}$$

The desired closed-loop pole is given by

$$s_d = -\xi\omega_n + j\,\omega_n\sqrt{1 - \xi^2}$$

$$= -0.5 \times 8 + j\,8\sqrt{1 - 0.5^2}$$

Simplifying,
$$s_d = -4 + j\,4\sqrt{3}$$

The steady state error is to be reduced by a factor of 6.25

$\therefore \qquad \dfrac{K_{vn}}{K_{vo}} = 6.25$

The transfer function of a phase lag compensator is given below:

$$G_c(s) = \frac{(s + a)}{(s + b)}, \quad a > b$$

$$\because \qquad \frac{K_{vn}}{K_{vo}} = \frac{a}{b}$$

and $\qquad\qquad \dfrac{a}{b} = 6.25 \quad$ (given)

Select, $\qquad\quad b = 0.05, \quad \therefore \quad a = 0.05 \times 6.25 = 0.3125$

$$\therefore \qquad\qquad G_c(s) = \frac{(s + 0.3125)}{(s + 0.05)}$$

The angle contributed to the uncompensated root locus due the use of phase lag compensator at $s = s_d = -4 + j\,4\sqrt{3}$ is determined below :

$$\angle G_c(s) = \angle \frac{(s + 0.3125)}{(s + 0.05)} \bigg|_{\,s\,=\,s_d\,=\,-4\,+\,j\,4\sqrt{3}}$$

$$= \angle \frac{(-4 + j\,4\sqrt{3} + 0.3124)}{(-4 + j\,4\sqrt{3} + 0.05)}$$

$$= \angle \frac{-3.68 + j\,4\sqrt{3}}{-3.95 + j\,4\sqrt{3}}$$

$$= (180° - 62.02°) - (180° - 60.25°)$$

$$= -1.77°$$

As the angle – 1.77° falls within a range of 0° to – 3°. Hence, the phase lag compensator design is accepted.

The open-loop transfer function is given below :

$$G(s)\,G_c(s) = \frac{K(s + 0.3125)}{s(s + 8)\,(s + 0.05)}$$

The value of gain K (uncompensated) at $s = s_d = -4 + j\,4\sqrt{3}$ for the second order characteristic equation is $K = \omega_n^2 = 8 \times 8 = 64$

$$\therefore \qquad G(s)\,G_c(s) = \frac{64(s + 0.3125)}{s(s + 8)\,(s + 0.05)} \qquad \textbf{Ans.}$$

Example 1.3.6. *The forward path transfer function of a unity feedback control system is given below :*

$$G(s) = \frac{K}{s(s + 2)\,(s + 8)}$$

Design a phase lag compensator such that

(a) Max-overshoop 16.4 % for a unit step input.

(*b*) *The steady state error 0.16 rad. for unit ramp input.*

Solution.

The general shape of the root locus plot for the transfer function

$$G(s) \ = \ \frac{K}{s(s+2)\,(s+8)}$$
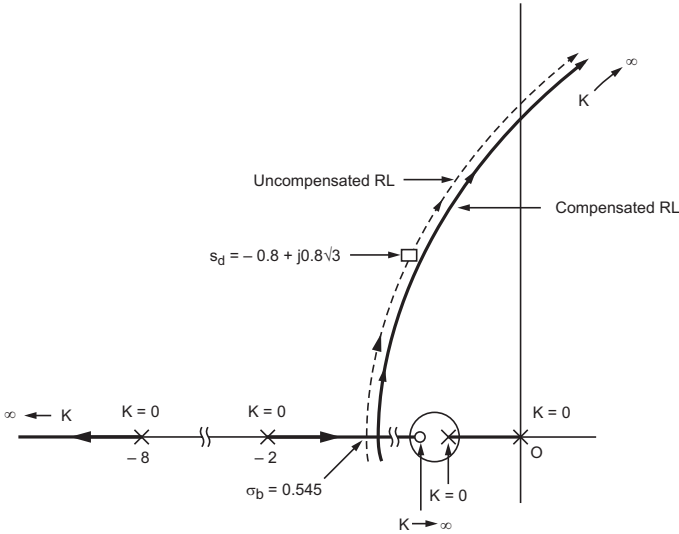
is shown in Fig 1.20

Fig 1.20 Root locus for example 1.3.6

(1) Uncompensated case

The uncompensated close-loop pole is obtained by the intersection of the root locus (complex part) with the damping ratio line.

Analytical method:

Equation for the complex part of root locus :

The equation for the complex part is derived hereunder.

Consider a point $s = \sigma + j\omega$ on the complex part of the root locus.

$$\therefore \qquad G(\sigma + j\omega) \ = \ \frac{K}{(\sigma + j\omega)\,(\sigma + j\omega + 2)\,(\sigma + j\omega + 8)}$$

Rearranging,

$$G(\sigma + j\omega) \ = \ \frac{K}{(\sigma + j\omega)\,(\sigma + 2 + j\omega)\,(\sigma + 8 + \ j\omega)}$$

For the point $s = \sigma + j\omega$, to lie on the root locus the phase angle condition is to be satisfied. Therefore,

$$\angle G(\sigma + j\omega) \ = \ -180°$$

$$\therefore \angle \frac{K}{(\sigma + j\omega)(\sigma + 2 + j\omega)(\sigma + 8 + j\omega)} = -180°$$

Thus,

$$-\tan^{-1}\frac{\omega}{\sigma} - \tan^{-1}\frac{\omega}{\sigma + 2} - \tan^{-1}\frac{\omega}{\sigma + 8} = -180°$$

Simplifying,

$$\tan^{-1}\left(\frac{\dfrac{\omega}{\sigma + 2} + \dfrac{\omega}{\sigma + 8}}{1 - \dfrac{\omega^2}{(\sigma + 2)(\sigma + 8)}}\right) = 180° - \tan^{-1}\frac{\omega}{\sigma}$$

or $\quad \tan^{-1}\left[\dfrac{\omega(2\sigma + 10)}{\sigma^2 + 10\sigma + 16 - \omega^2}\right] = 180° - \tan^{-1}\dfrac{\omega}{\sigma}$

$$\because \quad \tan^{-1} 180° = 0$$

$$\therefore \quad \frac{\omega(2\sigma + 10)}{\sigma^2 + 10\sigma + 16 - \omega^2} = \frac{\omega}{\sigma}$$

Rearranging the above equation for the complex part of the root locus relating σ and ω in s-plane is obtained as below :

$$\omega^2 = 3\sigma^2 + 20\sigma + 16$$

The break away point of the root locus on real s-plane real axis is determined by substituting $\omega = 0$

Therefore Solving :

$$3\sigma^2 + 20\sigma + 16 = 0$$

The acceptable solution is

$$\sigma = -0.545$$

Hence, breakaway point $\sigma_b = -0.545$

Equation for the damping ratio line

The damping ratio line makes an angle θ with the s-plane real axis as shown in Fig 1.20, where $\theta = \cos^{-1}\xi$.

The value of the damping ratio ξ is determined using relation below :

$$M_p = e^{\dfrac{-\xi\pi}{\sqrt{1 - \xi^2}}}$$

$$\therefore \qquad \frac{16.4}{100} = e^{\dfrac{-\xi\pi}{\sqrt{1 - \xi^2}}}$$

Solving above equation $\xi = 0.5$

$$\therefore \qquad\qquad \theta = \cos^{-1} 0.5 = 60°$$

The damping ratio line has a slope as :

$$\frac{\omega}{\sigma} = -\tan\theta \ (-\text{ve slope}) \quad \because \theta = 60°, \tan\theta = \sqrt{3}$$

Therefore, the equation for the damping ratio line is given below:

$$\omega = -\sqrt{3}\ (\sigma)$$

As already derived the equation for the complex part of the root locus is

$$\omega^2 = 3\sigma^2 + 20\sigma + 16 = 0$$

Substitute $\omega = -\sqrt{3}\ (\sigma)$ in the above equation. Thus,

$$3\sigma^2 = 3\sigma^2 + 20\sigma + 16$$

Solving

$$\sigma = -0.8$$

and $$\omega = -\sqrt{3}\ (\sigma) = -\sqrt{3}\ (-0.8) = 0.8\sqrt{3}$$

Therefore, the intersection point is located at

$$s_d = -0.8 + j\ 0.8\sqrt{3} \quad \text{(closed-loop pole)}$$

Determination of the value of gain K at

$$\boldsymbol{s_d = -0.8 + j\ 0.8\sqrt{3}}$$

The value of K at $s = s_d = -0.8 + j\ 0.8\sqrt{3}$ is determined using root locus magnitude condition i.*e.*,

$$\left|\frac{K}{s(s+2)(s+8)}\right| = 1 \qquad s = s_d = -0.8 + j\ 0.8\sqrt{3}$$

Rearranging,

$$K = \left|(s+2)(s+8)\right| \qquad s = s_d = -0.8 + j\ 0.8\sqrt{3}$$

$$= \left|(-0.8 + j\ 0.8\sqrt{3})(-0.8 + j\ 0.8\sqrt{3} + 2)(-0.8 + j\ 0.8\sqrt{3} + 8)\right|$$

$$= \left|(-0.8 + j\ 0.8\sqrt{3})(1.2 + j\ 0.8\sqrt{3})(7.2 + j\ 0.8\sqrt{3})\right|$$

$$= \sqrt{(0.64 + 1.92)(1.44 + 1.92)(51.84 + 1.92)}$$

Simplifying,

$$K = 2.15$$

(2) Phase Lag Compensator.

The original (uncompensated) velocity error coefficient is determined below :

$$K_{vo} = \lim_{s \to o} s\ G(s)$$

$$= \lim_{s \to o} \frac{sK}{s(s+2)(s+8)}$$

$$= \frac{K}{16}$$

$\because$ $\qquad\qquad K = 21.5, \because K_{vo} = \dfrac{21.5}{16} = 1.34 \text{ sec}^{-1}.$

On phase lag compensation the new velocity error coefficient is required as below :

$$K_{vn} = \frac{1}{e_{ssn}} = \frac{1}{0.16}$$

$$= 6.25 \text{ sec}^{-1}, (e_{ssn} = 0.16 \text{ sec}^{-1} \text{ given})$$

The transfer function of a phase lag compensator is given by

$$G_c(s) = \frac{(s+a)}{(s+b)}, \quad a > b$$

$$\frac{a}{b} = \frac{K_{vn}}{K_{vo}}$$

$\therefore$ $\qquad\qquad \dfrac{a}{b} = \dfrac{6.25}{1.34} = 4.66$

Select the parameter $b = 0.02$.

The parameter a $= 0.02 \times 4.66 = 0.0932$.

$\therefore$ $\qquad\qquad G_c(s) = \dfrac{(s+0.0932)}{(s+0.02)}$

The phase angle contributed by the phase lag compensator at $s = s_d = -0.8 + j\,0.8\sqrt{3}$ is calculated below :

$$\angle G_c(s) = \left.\angle\frac{(s+0.0932)}{(s+0.02)}\right|_{s = s_d = -0.8 + j\,0.8\sqrt{3}}$$

$$= \angle\frac{(-0.8 + j\,0.8\sqrt{3} + 0.0932)}{(-0.8 + j\,0.8\sqrt{3} + 0.02)}$$

$$= \angle\frac{(-0.7068 + j1.385)}{(-0.78 + j1.385)}$$

$$= \left(180° - \tan^{-1}\frac{1.385}{0.7068}\right) - \left(180° - \frac{1.385}{0.78}\right)$$

$$= (180° - 62.95°) - (180° - 60.6°)$$

$$= -2.35°$$

As the phase angle contributed by the lag compensator at $s_d =$ $-0.8 + j\,0.8\,\sqrt{3}$ lies in the range of $-0°$ to $-3°$, the design is accepted.

The open-loop transfer function cascaded with lag compensator is given below :

$$G(s)\ G_c(s)\ =\ \frac{21.5\ (s + 0.0932)}{s(s + 2)\ (s + 8)\ (s + 0.02)}\quad \textbf{Ans.}$$

Note : The distances of the pole/zero of the lag compensator are measured to be very small form the s-plane origin, as such the root locus shape around the aforesaid pole/zero is also very small as compared to the original root locus plot.

However, in Fig 1.20 the root locus around lag compensator pole/zero is plotted to a large scale for clarity.

Example 1.3.7. *The open-loop transfer function of a unity feed-back control system is given below :*

$$G(s)\ =\ \frac{K}{s(s + 4)}$$

The damping ratio $\xi = 0.4$. *It is specified that the velocity error coefficient be* $K_v = 25\ sec^{-1}$. *Design a cascade phase lag compensator.*

Solution. The root locus plot for the open-loop transfer function

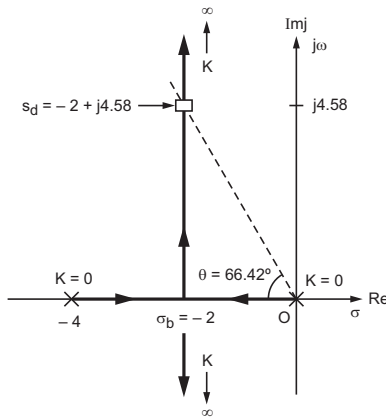$$G(s)\ =\ \frac{K}{s(s + 4)}$$

is shown in Fig 1.21.

Fig 1.21 Root locus plot for Example 1.3.7

The break away point can be determined as $\sigma_b = -2$ and angle of asymptotes $\pm 90°$.

(1) Uncompensated case.

The intersection of the damping ratio line with the root locus (Fig 1.21) is determined below :

The equation for the damping ratio line is

$$\frac{\omega}{\sigma_b} = -\tan\theta \qquad (-\text{ve slope})$$

$$\because \qquad \theta = \cos^{-1}\xi = \cos^{-1} 0.4 = 66.42°$$

$$\therefore \qquad \frac{\omega}{-2} = -\tan 66.42°$$

or $\qquad \omega = (-2)(-2.29) = 4.58$

The damping ratio line intersects the root-locus at "s_d"

$$\therefore \qquad s_d = \sigma_b + j\omega$$
$$= -2 + j\,4.58$$

Applying root locus magnitude condition at $s_d = -2 + j\,4.58$, the value of gain K is determined.

$$\therefore \qquad \left|\frac{K}{s(s+4)}\right| \qquad \begin{matrix} = 1 \\ s = s_d = -2 + j\,4.58 \end{matrix}$$

Rearranging,

$$K = \left| s(s+4) \right| \qquad s = s_d = -2 + j\,4.58$$

$$= \left|(-2 + j\,4.58)(-2 + j\,4.58 + 4)\right|$$

$$= \left|(-2 + j\,4.58)(2 + j\,4.58)\right|$$

$$= \sqrt{(2^2 + 4.58^2)(2^2 + 4.58^2)}$$

$$= 24.97$$

Uncompensated velocity error coefficient is determined below :

$$K_{vo} = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{sK}{s(s+4)}$$

$$= \frac{24.97}{4} = 6.24 \text{ sec}^{-1}.$$

(2) Compensated case

The new velocity error coefficient as given $K_{vn} = 25$ sec^{-1}. The transfer function of a phase lag compensator is given by

$$G_c(s) = \frac{(s+a)}{(s+b)}, \qquad a > b$$

$$\therefore \qquad \frac{K_{vn}}{K_{vo}} \;=\; \frac{a}{b}\,, \quad \therefore \quad \frac{a}{b} \;=\; \frac{25}{6.24} \;=\; 4 \text{ sec}^{-1}.$$

Select $b = 0.075$, $\therefore$ $a = 0.075 \times 4 = 0.3$ and

$$G_c(s) \;=\; \frac{(s + 0.3)}{(s + 0.075)}$$

The angle contributed by the phase lag compensator $s_d = -2 + j\,4.58$ is determined below :

$$\angle G_c(s_d) = \; \angle\frac{(s + 0.3)}{(s + 0.075)}\Bigg| \qquad s = s_d = -2 + j\,4.58$$

$$= \; \angle\frac{(-2 + j\,4.58 + 0.3)}{(-2 + j\,4.58 + 0.075)}$$

$$= \; \angle\frac{-1.7 + j\,4.58}{-1.925 + j\,4.58}$$

$$= \left(180° - \tan^{-1}\frac{4.58}{1.7}\right) - \left(180° - \tan^{-1}\frac{4.58}{1.925}\right)$$

$$= \; -2.43°$$

As the angle contribution is in the range $-0°$ to $-3°$, the design is accepted.

$$\therefore \qquad G_c(s) \;=\; \frac{(s + 0.3)}{(s + 0.075)} \qquad \textbf{Ans.}$$

(C) lag-lead compensator design

The phase lead compensation improves the time response of a feedback control system whereas lag compensation improves the steady state error. However, if both the aforesaid requirements are to be fulfilled then the compensator consists of a cascade combination of lag and lead compensation known as lag-lead compensation.

In this process lead compensator is designed to meet the desired transient response specifications then the lag compensator parameters are determined as per the steady state error requirement.

The procedure for designing lead as well as lag compensator already described earlier.

Solved Examples

Examples 1.3.8. *Design a suitable compensator having given that the damping ratio* $\xi = \dfrac{1}{\sqrt{2}}$ *, settling time, t_s = 1.0 sec and the desired velocity error coefficient K_v = 20 sec^{-1}.*

The open-loop transfer function of the unity feedback transfer function is given below :

Solution. $G(s) = \dfrac{K}{s(s+8)}$

(1) Uncompensated case

The characteristic equation is given by

$$1 + G(s) = 0$$

$\therefore$ $1 + \dfrac{K}{s(s+8)} = 0$

Simplifying,

$$s^2 + 8s + K = 0$$

Rearranging,

$$K = -(s^2 + 8s)$$

The root locus plot for $G(s) = \dfrac{K}{s(s+8)}$ is drawn in Fig 1.22 as per calculations below :
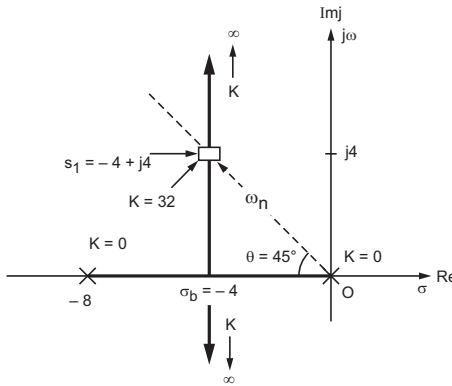
Fig 1.22   Root locus plot : Example 1.3.8

The break away point is determined using the equation $(dk/ds) = 0$

$\because$ $K = -(s^2 + 8s)$

$\therefore$ $\dfrac{dk}{ds} = -(2s + 8)$

$\therefore$ $(2s + 8) = 0$ and $s = -\dfrac{8}{2} = -4$

The break away point : $\sigma_b = -4$

The angle of asymptotes can be calculated as $\pm 90°$.

Intersection of the root locus with damping ratio line.

As the root locus shown in Fig 1.22, the equation of the damping ratio line is

$$\frac{\omega}{\sigma} = -\tan\theta \qquad (-\text{ ve slope})$$

$$\because \qquad \theta = \cos^{-1}\xi = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right),$$

$$\therefore \qquad \cos\theta = \frac{1}{\sqrt{2}} \text{ and } \theta = 45°$$

$$\therefore \qquad \tan\theta = \tan 45° = 1$$

Thus, $\dfrac{\omega}{\sigma} = -1$, the damping ratio line equation is $\omega = -\sigma$

$$\because \qquad \sigma_b = -4. \text{ Hence, } \omega = -(-4) = 4 \text{ rad./sec.}$$

Therefore, the intersection point of the root locus and the damping ratio line is obtained below

$$\because \qquad s_1 = \sigma_b + j\omega$$
$$\therefore \qquad s_1 = -4 + j4$$

Determination of the value of K at $s_1 = -4 + j4$

The point $s_1 = -4 + j4$ is located on the root locus. The root locus magnitude condition is satisfied *i.e.,*

$$\left|\frac{K}{s(s+8)}\right|_{s=s_1=-4+j4} = 1$$

or
$$K = \left|s(s+8)\right|_{s=s_1=-4+j4}$$

$$\therefore \qquad K = \left|(-4+j4)(-4+j4+8)\right|$$
$$= \left|(-4+j4)(4+j4)\right|$$

Hence,
$$K = \sqrt{(4^2+4^2)(4^2+4^2)} = 32$$

$$K_{vo} = sG_{(s)} ; \atop \to 0$$

$$\therefore \qquad K_{vo} \atop s\to 0 = \frac{s\times 32}{s(s+8)} = 4 \text{ sec}^{-1}$$

(2) Compensated Case

Settling time : $t_s = 1.0$ sec   (given)

$$\because \qquad t_s = \frac{4}{\xi\omega_n}, \quad \therefore \quad \xi\omega_n = 4$$

$$\because \qquad \xi = \frac{1}{\sqrt{2}} \quad \therefore \quad \omega_n = \frac{4}{1/\sqrt{2}} = 4\sqrt{2} \text{ rad./sec.}$$

The desired closed-loop pole : $s_d = -\xi\omega_n + \omega_n\sqrt{1-\xi^2}$

$\therefore$ $$s_d = -4 + 4\sqrt{2}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}$$

Simplifying, $s_d = -4 + j4$

As the desired pole $s_d = -4 + j4$ and the point $s_1 = -4 + j4$ (*i.e.,* uncompensated closed-loop pole) are concident, the phase angle to be contributed by the phase lead compensator is nil. Hence, the phase lead compensator is not required.

Phase Lag compensator

The transfer function of a phase lag compensator is given below:

$$G_c(s) = \frac{(s+a)}{(s+b)}, \quad a > b$$

Also $$\frac{a}{b} = \frac{K_{vm}}{K_{vo}}, K_{vo} = 4.0 \text{ sec}^{-1} \text{ (calculated earlier)}$$

and $$K_{vm} = 20 \text{ sec}^{-1} \quad \text{(given)}$$

$\therefore$ $$\frac{a}{b} = \frac{20}{4} = 5$$

Select, $b = 0.1, \quad \therefore \quad a = 5 \times 0.1 = 0.5$

The angle contributed by the phase lag compensator at $s_d = -4 + j4$ is determined below

$$\angle G_c(s) = \angle\frac{s+a}{s+b}\bigg|_{s = s_d = -4 + j4}$$

$$\angle\frac{s+a}{s+b}\bigg|_{s = s_d = -4 + j4} = \frac{-4 + j4 + 0.5}{-4 + j4 + 0.1}$$

$$= \angle\frac{-3.5 + j4}{-3.9 + j4}$$

$$= \left(180° - \tan^{-1}\frac{4}{3.5}\right) - \left(180° - \tan^{-1}\frac{4}{3.9}\right)$$

$$= -3.09°$$

As the angle contribution by the phase lag compensator is not within the range 0° to $-3°$, the design is not accepted.

However, select $b = 0.05, \therefore a = 5 \times 0.05 = 0.25$

$$\therefore \quad \angle \frac{s+a}{s+b} \bigg|_{s=s_d=-4+j4}$$

$$= \angle \frac{-4+j4+0.25}{-4+j4+0.05}$$

$$= \angle \frac{-3.75+j4}{-3.95+j4}$$

$$= \left(180° - \tan^{-1}\frac{4}{3.75}\right) - \left(180° - \tan^{-1}\frac{4}{3.95}\right)$$

$$= -1.45°$$

which is within the range 0° to – 3°, the design is accepted.

$$G(s)\ G_c(s)\ =\ \frac{32(s+0.25)}{s(s+8)\ (s+0.05)} \quad \textbf{Ans.}$$

Example 1.3.9. *The open-loop transfer function of a unity feedback control system is given below :*

$$G(s)\ =\ \frac{K}{s(s+1)\ (s+4)}$$

where K = 1.68 and the damping ratio ξ *= 0.707*

Design a lag-lead compensator such that ω_n *= 1 rad./sec and the steady state error* e_{ss} *= 0.3 rad. for a unit ramp input.*

Solution : The desired closed-loop pole is given by

$$s_d\ =\ -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$

$$\therefore \qquad \xi\ =\ 0.707$$

and $\qquad \omega_n\ =\ 1\ \text{rad./sec}$

$$\therefore \qquad s_d\ =\ -0.707\times 1 + j1\sqrt{1-0.707^2}$$

Simplifying, $\quad s_d\ =\ -0.707 + j\ 0.707$

The general shape of the root locus for the transfer function

$$G(s)\ =\ \frac{K}{s(s+1)\ (s+4)}$$

can be plotted as per the root locus plot procedure determining the data as below :

Break away point : $\sigma_b = -0.44$

Angle of asymptotes : 60°; 180° and 240°

Intersection of asympotes on real axis at σ = 1.33

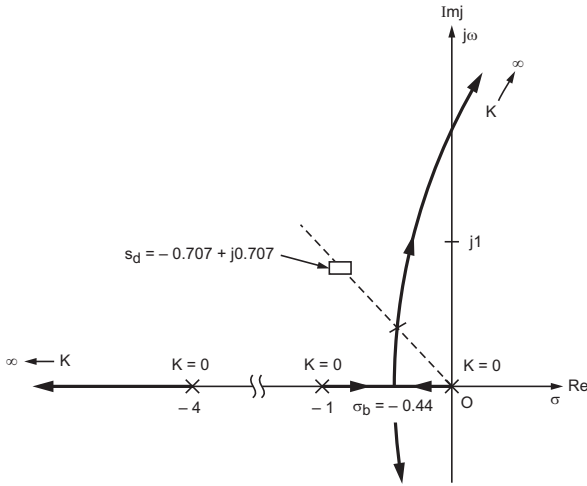The root locus plot is shown in Fig 1.23

Fig 1.23 Root locus for $G(s) = \dfrac{K}{s(s+1)(s+4)}$

As the desired closed-loop pole $s_d = 0.707 + j\,0.707$ does not lie on the uncompensated root locus a phase lead compensator required. The magnitude condition of root locus has to satisfied at

$$s_d = -0.707 + j\,0.707 \ i.e.,$$

$$\theta_c + \angle \left.\frac{K}{s(s+1)(s+4)}\right| \qquad = -180^{\circ}$$
$$s = s_d = -0.707 + j\,0.707$$

(where θ_c is the angle contributed by the phase lead compensator)

$$\therefore \ \theta_c + \angle \frac{K}{(-0.707 + j0.707)(-0.707 + j0.707 + 1)(-0.707 + j0.707 + 4)} = -180^{\circ}$$

Simplifying,

$$\theta_c + \angle \frac{K}{(-0.707 + j0.707)(0.293 + j0.707)(3.293 + j0.707)} = -180^{\circ}$$

or $\quad \theta_c + \left[-\left(180^{\circ} - \tan^{-1}\dfrac{0.707}{0.707}\right) - \tan^{-1}\dfrac{0.707}{0.293} - \tan^{-1}\dfrac{0.707}{3.293} \right]$

$$= -180^{\circ}$$

or $\quad \theta_c + [-(180^{\circ} - 45^{\circ}) - (67.46^{\circ}) - (12.07^{\circ})] = -180^{\circ}$

Simplifying,

$$\theta_c = 34.53^{\circ}$$

Lead Compensator :

The transfer function of a lead compensator is given below:

$$G_c(s) = \frac{K_c\,(s+a)}{(s+b_1)}, \quad a_1 < b_1$$

As determined above the angle to be contributed by the lead compensator is $\theta_c = 34.53°$

$$\therefore \qquad \theta_c = \angle G(s)\Big|_{s\,=\,s_d\,=\,-\,0.707\,+\,j\,0.707}$$

$$\therefore \qquad 34.53° = \angle\frac{(s+a_1)}{(s+b_1)}\Big|, \quad \begin{array}{l} a_1 < b_1 \\ s = s_d = -\,0.707 + j0.707 \end{array}$$

Place a_1 (zero) of the lead compensator at the open-loop pole $s = -1$. This placement cancels the open-loop pole and the order of the characteristics equation remains unchanged.

$$\therefore \qquad 34.53° = \angle\frac{K_c\,(s+1)}{(s+b_1)}\Big|_{s\,=\,s_d\,=\,-\,7.7\,+\,j\,0.707}$$

$$\therefore \qquad 34.53° = \angle\frac{K_c(-\,0.707 + j0.707 + 1)}{(-\,0.707 + j0.707 + b)}$$

Simplifying,

$$34.53° = \angle\frac{K_c\,(0.293 + j0.707)}{(b - 0.707) + j0.707}$$

$$= \tan^{-1}\frac{0.707}{0.293} - \tan^{-1}\frac{0.707}{b_1 - 0.707}$$

$$= 67.46° - \tan^{-1}\frac{0.707}{b_1 - 0.707}$$

Rearranging,

$$32.93° = \tan^{-1}\frac{0.707}{(b_1 - 0.707)}$$

$$\therefore \qquad \tan 32.93° = \frac{0.707}{(b_1 - 0.707)}$$

$$\therefore \qquad 0.647 = \frac{0.707}{b_1 - 0.707}$$

Solving,

$$b_1 = 1.8$$

Thus, $\qquad G_c(s) = \dfrac{K_c\,(s+1)}{(s+1.8)}$

and $\qquad$ $G(s)\,G_c(s) \;=\; \dfrac{KK_c\,(s+1)}{s(s+1)\,(s+4)\,(s+1.8)} = \dfrac{KK_c}{s(s+4)\,(s+1.8)}$

Determination the value : KK_c

The closed-pole $s_d = -\,0.707 + j\,0.707$ being located on the phase lead compensated root locus. Hence, the root locus magnitude condition is satisfied *i.e.,*

$$\left|\begin{matrix} G(s)\,G_c(s)\\ lead \end{matrix}\right| \qquad\quad = 1$$
$$s = s_d = -\,0.707 + j0.707$$

Therefore,

$$\left|\frac{KK_c}{s(s+1.8)\,(s+4)}\right| \qquad\quad = 1$$
$$s = s_d = -0.707 + j0.707$$

Rearranging,

$$KK_c \;=\; \left|\,s(s+1.8)\,(s+4)\,\right|$$
$$s = s_d = -0.707 + j0.707$$

$\therefore\quad KK_c \;=\; |\,(-\,0.707 + j0.707)\,(-\,0.707$
$$+ j\,0.707 + 1.8)\,(-\,0.707 + j\,0.707 + 4)\,|$$
$$=\; |\,(-\,0.707 + j0.707)\,(1.093$$
$$+ j0.707)\,(3.293 + j\,0.707)\,|$$
$$=\; \left|\left(-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)\left(1.093 + j\frac{1}{\sqrt{2}}\right)\left(3.293 + j\frac{1}{\sqrt{2}}\right)\right|$$
$$=\; \sqrt{\frac{1}{2} + \frac{1}{2}}\,\sqrt{1.194 + \frac{1}{2}}\,\sqrt{10.84 + \frac{1}{2}}$$
$$=\; 1 \times \sqrt{1.694}\,\sqrt{11.34}$$

$\therefore\quad KK_c = 4.38$

$\because\qquad K = 1.68$ (given), $\quad \therefore\quad K_c = \dfrac{4.38}{1.68} = 2.6$

The open-loop transfer function with lead compensator is given by

$$G(s)\,G_c(s)_{\substack{lead}} \;=\; \frac{KK_c}{s(s+1.8)\,(s+4)}$$

$\therefore\qquad\qquad K_{vo} \;=\; \lim_{s \to 0}\,\frac{s\,KK_c}{s(s+1.8)\,(s+4)}, \quad \because\; KK_c = 4.38$

$\therefore\qquad\qquad K_{vo} \;=\; \lim_{s \to 0}\,\frac{4.38}{(s+1.8)\,(s+4)}$

or
$$K_{vo} = \frac{4.38}{1.8 \times 4} = 0.6 \text{ sec}^{-1}.$$

$$K_{vn} = \frac{1}{e_{ss} \ (given)} = \frac{1}{0.3} \text{sec}^{-1}$$

$$\therefore \qquad \frac{K_{vn}}{K_{vo}} = \frac{1/0.3}{0.6} = 5.55$$

Lag Compensator :

The transfer function of a lag compensator is given by

$$G_c(s) = \frac{(s + a_2)}{(s + b_2)}, \quad a_2 > b_2$$
$$\underset{lag}{}$$

$$\therefore \qquad \frac{a_2}{b_2} = \frac{K_{vn}}{K_{vo}}, \quad \therefore \quad \frac{a_2}{b_2} = 5.55$$

Select (say) $b_2 = 0.02$, $\therefore$ $a_2 = 5.55 \times 0.02 = 0.111$. Determine the angle contributed by the lag compensator at $s_d = -0.707 + j0.707$ *i.e.,*

$$\theta_c = \angle \frac{s + a_2}{s + b_2} \bigg| \qquad s = s_d = -0.707 + j0.707$$

$$\because \qquad a_2 = 0.111 \text{ and } b_2 = 0.02$$

$$\therefore \qquad \theta_c = \angle \frac{(-0.707 + j0.707 + 0.111)}{(-0.707 + j0.707 + 0.02)}$$

$$= \angle \frac{(-0.596 + j0.707)}{(-0.687 + j0.707)}$$

$$= \left(180° - \tan^{-1} \frac{0.707}{0.596}\right) - \left(180° - \tan^{-1} \frac{0.707}{-0.687}\right)$$

$$= (180° - \tan^{-1} 1.186) - (180° - \tan^{-1} 1.029)$$

$$= (180° - 49.86°) - (180° - 45.81°)$$

Simplifying,

$$\theta_c = -4.06^o$$

which is not in the range 0° to -3°. Hence, the design is not accepted.

However,

select (say) $b_2 = 0.012$, $\therefore a_2 = 5.55 \times 0.012 = .066$

$$\therefore \qquad G_c(s) = \angle \frac{s + 0.066}{s + 0.012} \bigg|_{s = s_d = -0.707 + j0.707}$$
$$\underset{lag}{}$$

$$= \angle \frac{(-0.707 + j0.707 + 0.066)}{(-0.707 + j0.707 + 0.012)}$$

$$= \angle \frac{(-0.641 + j0.707)}{(-0.695 + j0.707)}$$

$$= \left(180° - \tan^{-1}\frac{0.707}{0.641}\right) - \left(180° \tan^{-1}\frac{0.707}{0.695}\right)$$

$$= (180° - \tan^{-1}1.102) - (180° - \tan^{-1}1.017)$$

$$= (180° - 47.77°) - (180° - 45.48°)$$

Simplifying,

$$\theta_c = -2.29°$$

As the angle contributed by the lag compensator $\theta_c = -2.29°$ is within the range 0° to – 3°, the design is accepted.

$$\underset{lead \quad lag}{G(s)\, G_c(s)\, G_c(s)} = \frac{4.38\,(s + 0.066)}{s(s + 4)\,(s + 1.8)\,(s + 0.012)} \quad \textbf{Ans.}$$

Example 1.3.10. *A unity feedback control system operating at the damping ratio* ξ *= 0.6 and having velocity error coefficient* K_v *= 0.95 sec^{-1}.*

The open-loop transfer function is given below

$$G(s) = \frac{K}{s(s + 2)\,(s + 6)}$$

Design a suitable compensator such that

(*i*) *Settling time* t_s *= 4 sec.*

(*ii*) *Steady state error improved by a factor of 5*

Solution. The settling times (t_s), as well as the steady state error (e_{ss}) are to be adjusted. Hence, a lag-lead compensator is to be used.

The phase lead-compensator for settling time and phase lag compensator for steady state error.

Part (*i*)

$$t_s = 4.0 \text{ sec. and } \xi = 0.6 \quad \text{(given)}$$

$$\because \qquad t_s = \frac{4}{\xi\omega_n}, \quad \therefore \quad 4 = \frac{4}{\xi\omega_n} \text{ and } \xi\omega_n = 1$$

$$\because \qquad \xi\omega_n = 1 \quad \& \quad \xi = 0.6,$$

$$\therefore \qquad \omega_n = \frac{1}{\xi} = \frac{1}{0.6} = \frac{5}{3} \text{ rad./sec.}$$

The desired closed-loop pole is given by

$$s_d = -\xi\omega_n + j\omega_n\sqrt{1 - \xi^2}$$

$$= -0.6 \times \frac{5}{3} + j\frac{5}{3}\sqrt{1 - 0.6^2}$$

Simplifying,

$$s_d = -1 + j1.33$$

The given value of the velocity error coefficient is 0.95 sec^{-1}. Determine the gain K as below :

$$K_v = \lim_{s \to 0} sG(s)$$

$$= \lim_{s \to 0} \frac{s. \, K}{s(s + 2) \, (s + 6)}$$

$$= \frac{K}{2 \times 6} = \frac{K}{12}$$

$\because$ $\qquad K_v = 0.95 \text{ sec}^{-1}, \quad K = 0.95 \times 12 = 11.4$

As per the root locus plot procedure as applied to open-loop transfer function

$$G(s) = \frac{K}{s(s + 2) \, (s + 6)} \, ,$$

the following data can be determined,

Break away point $\sigma_b = -0.9$

Angle of asymptotes, 60°, 180°, 240°

Asymptotes intersect on the real axis at $s = -\dfrac{8}{3}$

with reference to above data, the general shape of the root locus is shown in Fig 1.24
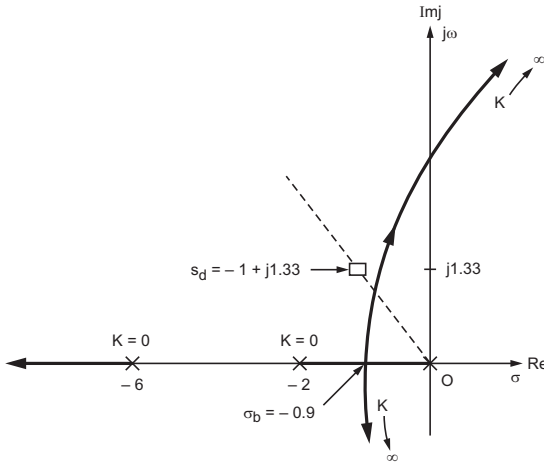

Fig 1.24   Root locus : Example 1.24

It is noted that the desired closed-loop pole $s_d = -1 + j1.33$ does not lie on the root locus.

Therefore, the phase angle θ_c to be contributed by the phase lead compensator is determined by applying angle condition of the root locus *i.e.,*

$$\theta_c + \angle \left. \frac{K}{s(s+2)(s+6)} \right|_{\substack{\\ s = s_d = -1 + j1.33}} = 180°$$

$$\therefore \ \theta_c + \angle \frac{K}{(-1+j1.33)(-1+j1.33+2)(-1+j1.33+6)} = -180°$$

Simplifying,

$$\theta_c + \angle \frac{K}{(-1+j1.33)(1+j1.33)(5+j1.33)} = -180°$$

$$\therefore \quad \theta_c + \left[\begin{array}{l} -\left(180° - \tan^{-1}\dfrac{1.33}{1}\right) - \\[2mm] \left(\tan^{-1}\dfrac{1.33}{1}\right) - \left(\tan^{-1}\dfrac{1.33}{5}\right) \end{array}\right] = -180$$

or $\theta_c + [-(180° - 53.06°) - 53.06° - 14.89°] = -180°$

Simplifying,

$$\theta_c = 14.89°,$$

The transfer function of a phase lead compensator is given below :

$$\underset{lead}{G_c(s)} = \frac{K_c(s+a_1)}{(s+b_1)}, \quad a_1 < b_1$$

Select, a_1 on the s-plane real axis just below the closed-loop pole $s_d = -1 + j1.33$. The angle contribution by the phase lead compensator is

$$\theta_c = \left. \angle \frac{K_c(s+1)}{(s+b_1)} \right|_{s = s_d = -1 + j1.33}$$

$$\therefore \qquad 14.89° = \left. \angle \frac{K_c(s+1)}{(s+b_1)} \right|_{s = s_d = -1 + j1.33}$$

$$\therefore \qquad 14.89° = \angle \frac{(-1+j1.33+1)}{(b_1 - 1) + j1.33}$$

Simplifying,

$$14.89° = \angle \frac{j1.33}{(b_1 - 1) + j1.33}$$

or $\tan 14.89° = 90° - \tan^{-1}\dfrac{1.33}{b_1 - 1}$

or $\qquad 75.11° = \tan^{-1} \dfrac{1.33}{b_1 - 1}$

$\therefore \qquad \tan 75.11° = \dfrac{1.33}{b_1 - 1}$

or $\qquad 3.76 = \dfrac{1.33}{b_1 - 1}, \quad$ Solving, $b_1 = 1.353$

$\therefore \qquad G(s)\,G_c(s) \atop lead = \dfrac{KK_c\,(s+1)}{s(s+2)\,(s+6)\,(s+1.353)}$

The value of KK_c is determined by applying root locus magnitude condition at $s_d = -1 + j1.33$ to lead compensated open-loop transfer function *i.e.*,

$$\left| G(s)\,G_c(s) \atop lead \right|_{s = s_d = -1 + j1.33} = 1$$

$\therefore \quad \left| \dfrac{KK_c\,(s+1)}{s(s+2)\,(s+6)\,(s+1.353)} \right|_{s = s_d = -1 + j1.33} = 1$

Rearranging,

$$KK_c = \left| \dfrac{s(s+2)\,(s+6)\,(s+1.353)}{(s+1)} \right|_{s = s_d = -1 + j1.33}$$

or $\qquad KK_c = \left| \dfrac{(-1+j1.33)\,(-1+j1.33+2)\,(-1+j1.33+6)\,(-1+j1.33+1.353)}{(-1+j1.33+1)} \right|$

$$= \left| \dfrac{(-1+j1.33)\,(1+j1.33)\,(5+j1.33)\,(0.353+j1.33)}{j1.33} \right|$$

$$= \dfrac{\sqrt{(1^2+1.33^2)\,(1^2+1.33^2)\,(5^2+1.33^2)\,(0.353^2+1.33^2)}}{\sqrt{1.33^2}}$$

Simplifying,

$\qquad\qquad KK_c = 14.8 \quad \because \quad K = 11.4$ (determined earlier)

$\therefore \qquad\qquad K_c = \dfrac{14.8}{11.4} = 1.298$

On using phase lead compensator :

$$K_{vo} = \lim_{s \to 0} \dfrac{s\,KK_c\,(s+1)}{s(s+2)\,(s+6)\,(s+1.353)}$$

$$= \lim_{s \to 0} \frac{s \times 14.8\,(s+1)}{s(s+2)\,(s+6)\,(s+1.353)}$$

or $\qquad K_{vo} \;=\; \dfrac{14.8 \times 1}{2 \times 6 \times 1.353} \;=\; 0.91\ \mathrm{sec^{-1}}.$

$$K_{vn} \;=\; K_v\ (\text{uncompensated}) \times 5$$

$\therefore \qquad\qquad K_{vn} \;=\; 0.95 \times 5 = 4.75\ \mathrm{sec^{-1}}.$

The transfer function of a phase lag compensator is given by,

$$\underset{lag}{G_c(s)} \;=\; \frac{s + a_2}{s + b_2}, \quad a_2 > b_2$$

Also, $\qquad \dfrac{a_2}{b_2} \;=\; \dfrac{K_{vn}}{K_{vo}} \;=\; \dfrac{4.75}{0.91} = 5.22$

$$K_{vo} \;=\; \text{error coefficient w.r.t. lag compensator input}$$

Select $\qquad\qquad b_2 \;=\; 0.02$ (say very near to s-plane origin)

$\therefore \qquad\qquad\quad a_2 \;=\; 5.22 \times 0.02 = 0.1044 \simeq 0.1$

$\therefore \qquad\quad \underset{lag}{G_c(s)} \;=\; \dfrac{(s+0.1)}{(s+0.02)}$

The angle contributed by the phase lag compensator at $s_d = -1 + j1.33$ is determined below :

$$\left|\angle G_c(s)\right|_{s\,=\,s_d\,=\,-1\,+\,j1.33} = \left|\angle \frac{s+0.1}{s+0.02}\right| \quad s = s_d = -1 + j1.33$$

$$= \angle \frac{(-1 + j1.33 + 0.1)}{(-1 + j1.33 + 0.02)}$$

$$= \angle \frac{(-0.9 + j1.33)}{(-0.98 + j1.33)}$$

$$= \left(180° - \tan^{-1}\frac{1.33}{0.9}\right) - \left(180° - \tan^{-1}\frac{1.33}{0.98}\right)$$

$$= (180° - 55.9°) - (180° - 53.6°)$$

$$= -2.3°$$

Since, the angle contributed by the phase lag compensator is within the range 0° to − 2.3°. The design is accepted

$$\therefore\ G(s)\,\underset{lead}{G_c(s)}\,\underset{lag}{G_c(s)} = \frac{14.81\,(s+1)\,(s+0.1)}{s(s+2)\,(s+6)\,(s+1.353)\,(s+0.02)}\ \textbf{Ans.}$$

1.4. PHASE LEAD COMPENSATION NETWORK

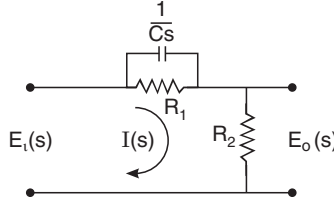Fig. 1.25 shows a phase lead network. The transfer function of a phase lead network is determined below :

Fig. 1.25. Phase lead network.

$$E_i(s) = \left[\frac{R_1 \dfrac{1}{Cs}}{R_1 + \dfrac{1}{Cs}} \right] I(s) + R_2 I(s) \qquad ...(1.8)$$

or $$E_i(s) = \frac{[(R_1 + R_2) + R_1 R_2 Cs]}{R_1 Cs + 1} I(s) \qquad ...(1.9)$$

and $$E_o(s) = R_2 I(s) \qquad ...(1.10)$$

∴ Transfer function is given by

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 (R_1 Cs + 1)}{(R_1 + R_2) + R_1 R_2 Cs} \qquad ...(1.11)$$

$$= \frac{R_2}{R_1 + R_2} \times \frac{1 + R_1 Cs}{1 + \dfrac{R_2}{R_1 + R_2} R_1 Cs} \qquad ...(1.12)$$

Put $$\frac{R_2}{R_1 + R_2} = \alpha \text{ and } R_1 C = T$$

$$\frac{E_o(s)}{E_i(s)} = \frac{\alpha (1 + sT)}{(1 + s \alpha T)} \qquad ...(1.13)$$

In sinusoidal form

$$G_c(j\omega) = \frac{\alpha(1 + j\omega T)}{(1 + j\omega \alpha T)} \qquad ...(1.14)$$

The pole-zero configuration and Bode plot for the above transfer function is shown in Fig. 1.26.
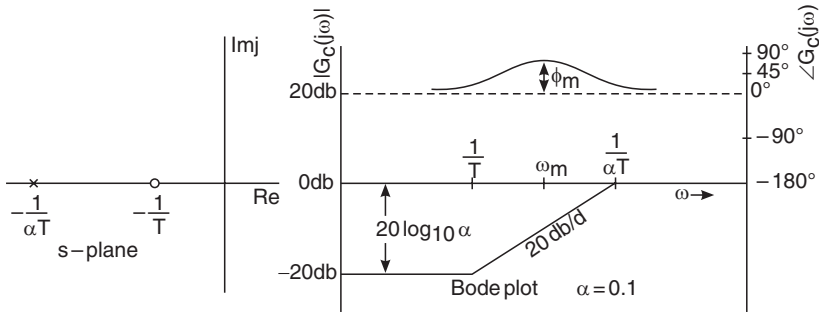
Fig. 1.26. Pole zero configuration and Bode plot for phase-lead network.

$$\omega = \frac{1}{T} \text{ Lower corner frequency}$$

$$\omega = \frac{1}{\alpha T} \text{ upper corner frequency}$$

The maximum phase load occurs at mid-frequency between upper and lower corner frequency.

$$\therefore \qquad \log_{10} \omega_m = \frac{1}{2}\left[\log_{10}\left(\frac{1}{T}\right) + \log_{10}\left(\frac{1}{\alpha T}\right)\right] \qquad ...(1.15)$$

$$\therefore \qquad \omega_m = \frac{1}{(\sqrt{\alpha})\,T} \qquad\qquad\qquad\qquad ...(1.16)$$

The phase angle $\angle G_c(j\omega)$ can be calculated as

$$\angle G_c(j\omega) = \tan^{-1}(\omega\,T) - \tan^{-1}(\omega\,\alpha T) \qquad\qquad ...(1.17)$$

At $\omega = \omega_m = \dfrac{1}{(\sqrt{\alpha})\,T}$. The phase angle is ϕ_m

$$\therefore \qquad \phi_m = \tan^{-1}\omega_m\,T - \tan^{-1}\omega_m\,\alpha T \qquad\qquad\qquad ...(1.18)$$

$$= \tan^{-1}\left\{\frac{1}{(\sqrt{\alpha})\,T}\cdot T\right\} - \tan^{-1}\left\{\frac{1}{(\sqrt{\alpha})\,T}\cdot\alpha T\right\} \quad ...(1.19)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{\alpha}}\right) - \tan^{-1}(\sqrt{\alpha})$$

$$= \tan^{-1}\frac{\dfrac{1}{\sqrt{\alpha}} - \sqrt{\alpha}}{1 + \dfrac{1}{\sqrt{\alpha}}\cdot\sqrt{\alpha}} = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right) \qquad\qquad ...(1.20)$$

$$\therefore \qquad \tan \phi_m = \frac{1 - \alpha}{2\sqrt{\alpha}} \qquad\qquad ...(1.21)$$

$$\therefore \qquad \sin \phi_m = \frac{1 - \alpha}{1 + \alpha} \qquad\qquad ...(1.22)$$

Allowing an amplification $\dfrac{1}{\alpha}$, the magnitude of phase-lead compensation network transfer function at $\omega = \omega_m$ is determined below :

$$\because \qquad\qquad \omega_m = \frac{1}{(\sqrt{\alpha})\,T}$$

$$\therefore \qquad G_c(j\omega)\bigg|_{\omega_m} = \left| \frac{1 + j\omega_m T}{1 + j\omega_m\,\alpha T} \right| \qquad\qquad ...(1.23)$$

$$= \left| \frac{1 + j\dfrac{1}{(\sqrt{\ }\,\alpha)\,T}.T}{1 + j\dfrac{1}{(\sqrt{\alpha})\,T}.\alpha T} \right| \qquad\qquad ...(1.24)$$

$$= \left| \frac{1 + \dfrac{j}{\sqrt{\alpha}}}{1 + j\sqrt{\alpha}} \right| \qquad\qquad ...(1.25)$$

$$= \left| \frac{1}{\sqrt{\alpha}}.\frac{(\sqrt{\alpha} + j1)}{(1 + j\alpha)} \right| \qquad\qquad ...(1.26)$$

$$= \frac{1}{\sqrt{\alpha}}\frac{\sqrt{\alpha^2 + 1}}{\sqrt{1 + \alpha^2}} \qquad\qquad ...(1.27)$$

$$= \frac{1}{\sqrt{\alpha}} \qquad\qquad ...(1.28)$$

In decibel, $20 \log_{10} |G_c(j\omega_m)| = 20 \log_{10} \dfrac{1}{\sqrt{\alpha}} \qquad\qquad ...(1.29)$

1.5. COMPENSATION NETWORK DESIGN : BODE PLOT APPROACH

Phase Lead Network

(1) Determine the open loop gain K to satisfy steady state error specification.

(2) Sketch the bode plot for the uncompensated system and determine the phase margin for the open loop gain K as determined above

(3) If the phase margin as above is less than the desired value, determine the phase-lead angle to be contributed by phase-lead network.

(4) The phase-lead network modifies uncompensated $|G(s) H(s)|_{db}$ plot by shifting gain cross-over frequency ω_1 to a higher value, therefore, a margin of say 5° is added to the phase-lead angle. The phase-lead angle ϕ_m is contributed at ω_m [the mid-corner frequency of the phase lead network transfer function $G_c(j\omega)$].

(5) Determine the parameter α as below :

$$\sin \phi_m = \frac{1 - \alpha}{1 + \alpha}$$

At $\omega = \omega_m$

$$|G_c(j\omega_m)| = \frac{1}{\sqrt{\alpha}}$$

or $\qquad |G_c(j\omega_m)|_{db} = 20 \log_{10} \frac{1}{\sqrt{\alpha}}$

(6) Locate $\omega = \omega_m$ such that

$$|G(j\omega_m) H(j\omega_m)| = 20 \log_{10} \frac{1}{\sqrt{\alpha}}$$

The new gain cross-over frequency is thus ω_m

$\therefore \qquad\qquad \omega_m = \frac{1}{(\sqrt{\alpha}) \, T}$

The zero of the lead compensation network is located at

$$\frac{1}{T} = \omega_m \sqrt{\alpha}$$

and the pole $\qquad \dfrac{1}{\alpha T} = \dfrac{\omega_m \sqrt{\alpha}}{\alpha} = \dfrac{\omega_m}{\sqrt{\alpha}}$

Plot the Bode plot for the compensated open-loop transfer function $G_c(j\omega) \, G(j\omega)$ and verify the given specifications.

Example 1.11. *The open-loop transfer function of a unity feed back control system is given below :*

$$G(s) = \frac{2000\ K}{s(s + 20)}$$

It is desired that $e_{ss} = 1\%$ for unit ramp input and phase margin > 45°. Determine the parameters of a phase lead network to be used for this purpose.

Solution. $K_v = \lim_{s \to 0} s\ G(s)$

$$= \lim_{s \to 0} \frac{s\ 2000\ K}{s(s + 20)} = 100\ K$$

$$e_{ss} = \frac{1}{100}, \quad \because \quad e_{ss} = \frac{1}{K_v}$$

$\therefore$ $\dfrac{1}{100} = \dfrac{1}{100\ K} \quad \therefore \quad K = 1$

Hence, $G(s) = \dfrac{2000}{s(s + 20)}$

or $G(s) = \dfrac{100}{s(0.05s + 1)}$

The transfer function is type 1. The initial slope of the bode plot is – 20 db/decade intersecting 0 db axis at $\omega = 100$. The corner frequency is $\omega = (1/0.05) = 20$ rad/sec which is due to pole of the transfer function. The slope of the bode plot beyond this corner frequency is $(-20 - 20) = -40$ db/decade. The magnitude part of the bode plot is shown in Fig. 1.27. It is noted that the gain cross-over frequency $\omega_1 = 42.9$ rad/sec.

$$\angle G_c(j\omega_1) = -90° - \tan^{-1} 0.05\ \omega_1$$
$$= -90° - \tan^{-1} 0.05 \times 42.9$$
$$= -90° - \tan^{-1} 2.145$$
$$= -90° - 65° = -155°$$

$\therefore$ Phase margin (uncompensated)
$$= 180° - 155° = 25°$$

As the desired phase margin in 45°, a phase lead $\phi_m = (45° - 25°)$ + allowance say 5° = 25° is to be supplied by the phase lead network.

The α parameter of the lead network is calculated as

$$\alpha = \frac{1 - \sin \phi_m}{1 + \sin \phi_m} = \frac{1 - \sin 25°}{1 + \sin 25°} = \frac{1 - 0.42}{1 + 0.42} = 0.4$$
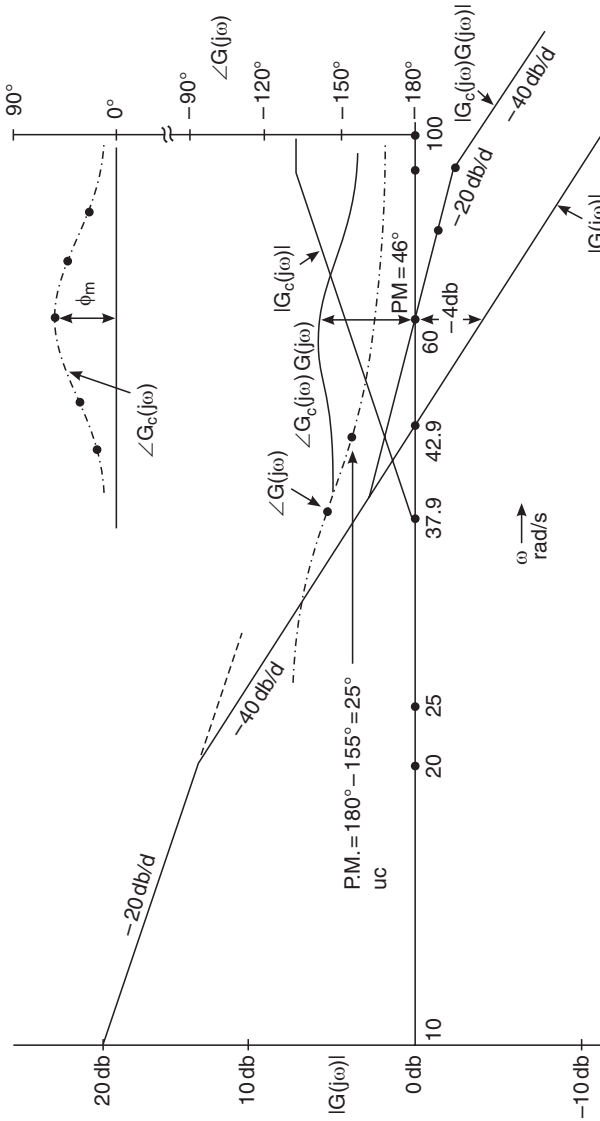
Fig. 1.27 Bode plot for example 1.3.

The maximum phase lead is contributed at the mid-corner frequency ω_m of the phase lead network and the magnitude of the phase lead network at ω_m is given by

$$| G_c\,(j\omega_m) |_{db} = 20\,\log_{10}\,\frac{1}{\sqrt{\alpha}}$$

$$= 20\,\log_{10}\,\frac{1}{\sqrt{0.4}} = 4.0\ db$$

At $\omega = 60$ rad/sec, the magnitude $|G(j\omega)| = -4.0$ db which is to be made 0 db with the help of phase lead network, hence $\omega_m = 60$ rad/sec is selected.

The parameters T and α of the phase lead network are determined below :

Lower corner frequency $\dfrac{1}{T} = \omega_m \sqrt{\alpha} = 60\sqrt{0.4} = 37.9$ rad/sec.

Upper corner frequency $\dfrac{1}{T} = \dfrac{\omega_n}{\sqrt{\alpha}} = \dfrac{60}{\sqrt{0.4}} = 94.9$ rad/sec.

The phase lead compensation network is given below :

$$T = \frac{1}{37.9} = 0.026; \ \alpha \, T = \frac{1}{94.9} = 0.01$$

$$G_c(j\omega) = \frac{\alpha(1 + j\,\omega T)}{(1 + j\,\omega \propto T)} = \frac{0.4(1 + j\,0.026\,\omega)}{(1 + j\,0.01\,\omega)}$$

The compensated transfer function is given below :

$$G_c(j\omega)\,G(j\omega) = \frac{(1 + j\,0.026\,\omega)}{(1 + j\,0.01\,\omega)} \times \frac{100}{j\omega(j\,0.05\,\omega + 1)}$$

The attenuation due to phase lead network is taken into account

increasing gain by a factor of $\left(\dfrac{1}{0.4}\right)$.

The bode plot for the compensated system is shown in Fig. 1.27 and the desired phase margin 46° is obtained.

1.6. PHASE LAG COMPENSATION NETWORK

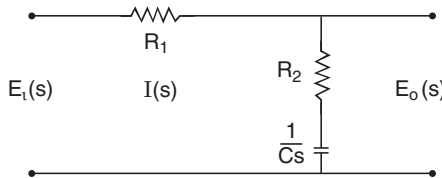Fig. 1.28 shows a phase lag compensation network. The transfer function of a phase-lag network is determined below :

Fig. 1.28. Phase-lag network.

$$E_i(s) = \left(R_1 + R_2 + \frac{1}{Cs}\right) I(s) \qquad \qquad ...(1.30)$$

$$E_o(s) = \left(R_2 + \frac{1}{Cs}\right) I(s) \qquad \qquad ...(1.31)$$

$$\therefore \qquad \frac{E_o\ (s)}{E_i\ (s)} \ = \ \frac{R_2 + \dfrac{1}{Cs}}{R_1 + R_2 + \dfrac{1}{Cs}} \qquad\qquad ...(1.32)$$

$$= \ \frac{1 + R_2\ Cs}{1 + (R_1 + R_2)\ Cs} \qquad\qquad ...(1.33)$$

$$= \ \frac{1 + R_2\ Cs}{1 + \dfrac{(R_1 + R_2)}{R_2}\ R_2\ Cs} \qquad\qquad ...(1.34)$$

Put $R_2\ C = T$

$$\frac{R_1 + R_2}{R_2} \ = \ \beta$$

$$\therefore \qquad \frac{E_o\ (s)}{E_i\ (s)} \ = \ \frac{1 + sT}{1 + s\,\beta T} \qquad\qquad ...(1.35)$$

In sinusoidal form

$$\frac{E_o\ (j\omega)}{E_i\ (j\omega)} \ = \ \frac{1 + j\,\omega T}{1 + j\,\omega\beta T} \qquad\qquad ...(1.36)$$

or $\qquad\qquad G_c(j\omega) \ = \ \dfrac{1 + j\,\omega T}{1 + j\,\omega\beta T} \qquad\qquad ...(1.37)$

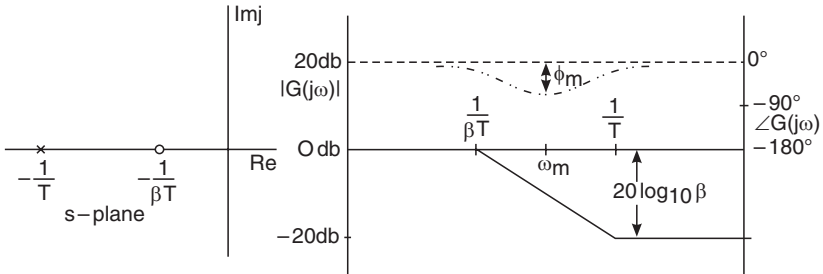The pole-zero configuration and bode plot for the above transfer function is shown in Fig. 1.29.

Fig. 1.29. Pole-zero configuration and Bode plot for lag network.

$$\omega \ = \ \frac{1}{\beta T} \ \text{ lower corner frequency}$$

$$\omega \ = \ \frac{1}{T} \ \text{ upper corner frequency}$$

The maximum phase lag occurs at mid frequency ω_m between lower and upper corner frequency

$$\log_{10} \omega_m = \frac{1}{2}\left[\log_{10}\left(\frac{1}{\beta T}\right) + \log_{10}\left(\frac{1}{T}\right)\right] \qquad ...(1.38)$$

$$\therefore \qquad \omega_m = \frac{1}{(\sqrt{\beta})\,T} \qquad\qquad ...(1.39)$$

The phase angle $\angle G_c\,(j\omega)$ can be calculated as

$$\angle G_c\,(j\omega) = \tan^{-1}(\omega T) - \tan^{-1}(\omega\beta\,T) \qquad ...(1.40)$$

at $\omega = \omega_m = \dfrac{1}{(\sqrt{\beta})\,T}$, the phase angle is ϕ_m

$$\therefore \qquad \phi_m = \tan^{-1}\omega_m\,T - \tan^{-1}\omega_m\,\beta\,T \qquad ...(1.41)$$

$$= \tan^{-1}\frac{1}{(\sqrt{\beta})\,T}\;T - \tan^{-1}\frac{1}{(\sqrt{\beta})\,T}\;\beta T \quad ...(1.42)$$

$$= \tan^{-1}\frac{1}{\sqrt{\beta}} - \tan^{-1}\sqrt{\beta} \qquad\qquad ...(1.43)$$

or $\qquad\qquad \tan\phi_m = \dfrac{1-\beta}{2\sqrt{\beta}} \qquad\qquad\qquad ...(1.44)$

and $\qquad\qquad \sin\phi_m = \dfrac{1-\beta}{1+\beta} \qquad\qquad\qquad ...(1.45)$

1.7. PHASE LAG NETWORK DESIGN : BODE PLOT APPROACH

(1) Determine the open loop gain factor K_{uc} for the uncompensated system to meet steady state error specification and plot Bode plot.

(2) Determine the phase margin from the Bode plot, if the phase margin is not adequate (too small) phase lag compensation is required.

(3) Add 5° tolerance to the phase margin specified.

(4) Locate the frequency at which the desired phase margin with tolerance as above is measured on the uncompensted phase plot. Select the frequency located above as new gain crossover frequency ω_1 (Fig. 1.30)

(5) The phase-lag compensation provides attenuation at low frequency and as such the gain cross-over frequency is shifted towards a lower value.

(6) The zero of the phase-lag compensation network transfer

function $G_c(s)$ is selected at $\dfrac{1}{10}$ th of the new gain cross-over frequency.

(7) At the selected zero of the lag compensation network measure the $|G_c(j\omega)|$ gain (db) of the uncompensated system from the

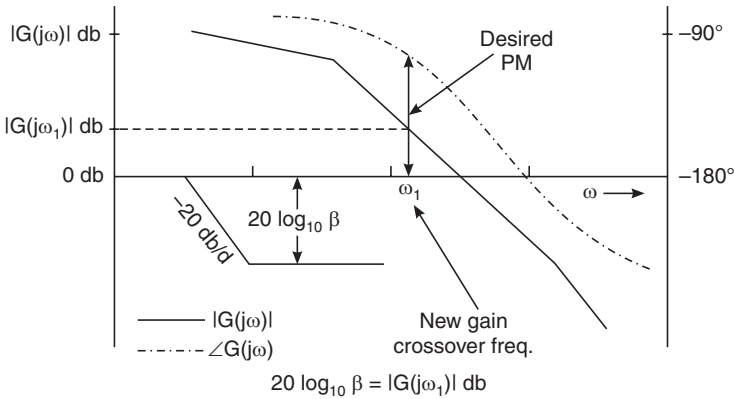magnitude plot. Determine parameter β using relation below,

Fig. 1.30. Determination of β parameter.

$$20 \log_{10} \beta = |G_c(j\omega_1)| \qquad\qquad ...(1.46)$$

The zero of lag network $\dfrac{1}{T}$

The pole of lag network $\dfrac{1}{\beta T}$ } $\qquad ...(1.47)$

Example 1.12. *Design a phase lag compensation for a feed back control system having following open loop transfer function*

$$G(s)\ H(s) = \frac{K}{s(1 + 0.005s)\ (1 + 0.001s)}$$

to having given $K_v = 800\ sec^{-1}$ and phase margin around 40°.

Solution. $K_v = \lim_{s \to 0}\ s\ G(s)\ H(s)$

$$= \lim_{s \to 0} s\ \frac{K}{s(1 + 0.005s)\ (1 + 0.001s)} = K$$

∴ $800 = K$ or $K = 800$

$$G(s)\ H(s) = \frac{800}{s(1 + 0.005s)\ (1 + 0.001s)}$$

and $G(j\omega) = \dfrac{800}{j\omega(1 + j\ 0.005\ \omega)\ (1 + j\ 0.001\ \omega)}$

The transfer function is type 1, the initial plot of the Bode plot is – 20 db/decade intersecting 0 db axis at $\omega = K = 800$. The two corner

frequencies due to pole are $\omega = \dfrac{1}{0.005} = 200$ rad/sec and $\omega = \dfrac{1}{0.001}$ = 1000 rad/sec.
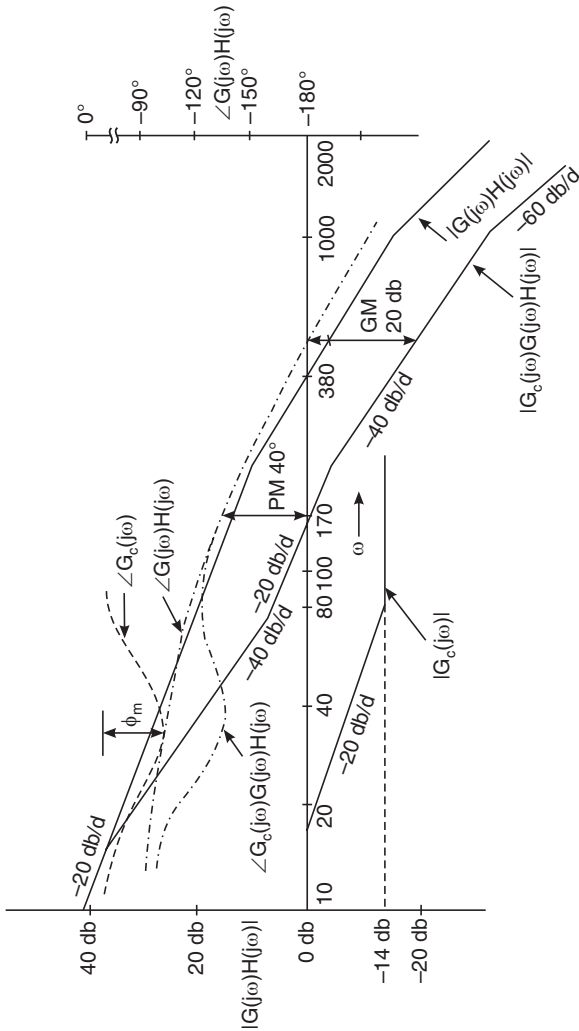
Fig. 1.31. Bode plot for example 1.12.

$$\angle G(j\omega)\ H(j\omega) = -90 - \tan^{-1} 0.005\ \omega - \tan^{-1} 0.001\ \omega$$

ω rad../s	170	200	400	450
$\angle G(j\omega)\quad H(j\omega)$	$-139.9°$	-146.3	-175.2	$-180°$

The Bode plot for the uncompensated transfer function $\angle G(j\omega)$ $H(j\omega)$ is shown in Fig. 1.31.

The $\angle G(j\omega)\ H(j\omega) = -139.9°$ occurs at $\omega = 170$ rad/sec which gives phase margin as $180° - 139.9° \simeq 40°$. Therefore the gain cross-over frequency for the compensated system is selected at $\omega = 170$ rad/sec. The gain at $\omega = 170$ rad/sec is noted as 14 db which is to be made odb using a phase lag network.

$\therefore \qquad 20 \lag_{10} \beta = 14$

$\therefore \qquad\qquad\qquad \beta = 5$

Upper corner frequency $\dfrac{1}{T}$ of the lag network is selected lower than the new gain crossover frequency

say $\qquad\qquad\qquad \dfrac{1}{T} = 80$ rad/sec ; $T = \dfrac{1}{80} = 0.0125$

The lower corner frequency of the lag network is selected say one fifth of the upper corner frequency

$\therefore \qquad\qquad \dfrac{1}{\beta T} = \dfrac{1}{5} \times 80 = 16$ rad/sec ; $\beta T = \dfrac{1}{16} = 0.0625$

Thus, the transfer function of the lag network is

$$G(j\omega) = \dfrac{(1 + j\, 0.0125\, \omega)}{(1 + j\, 0.0625\, \omega)}$$

The compensated open loop transfer function is given below :

$$G_c(j\omega)G(j\omega) = \dfrac{(1 + j\, 0.0125\, \omega)}{(1 + j\, 0.0625\, \omega)} \times \dfrac{800}{j\omega(1 + j\, 0.005\, \omega)\,(1 + j\, 0.001\, \omega)}$$

The Bode plot for the compensated system $G_c(j\omega)\ G(j\omega)\ H(j\omega)$ is shown in Fig. 1.31 and the phase margin = 40° is noted.

1.8.   LAG-LEAD COMPENSATION NETWORK

The phase lead compensation decreases the rise time and overshoot with increase in the band width of a feedback control system and that the phase lag compensation decreases the steady state error with affects stability margin and decrease in band width results in longer rise time thus, the phase lead and phase lag compensation have in effect advantages and disadvantages. In order to take the advantage of the situation a combination of phase-lead and phase-lag network termed as phase lag-lead compensation network is used which contains improvement features of both the schemes.

The phase lag-lead network is shown in Fig. 1.32.

The transfer function of the lag-lead network can be determined as

$$\dfrac{E_o\ (s)}{E_i\ (s)} = \dfrac{(1 + R_1 C_1 s)\,(1 + R_2 C_2 s)}{R_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)\, s + 1} \qquad ...(1.48)$$

The transfer function given above is rewritten as

$$\dfrac{E_o\ (s)}{E_i\ (s)} = \dfrac{(1 + sT_1)\,(1 + sT_2)}{(1 + s\alpha T_1)\,(1 + s\,\beta T_2)} \qquad\qquad ...(1.49)$$
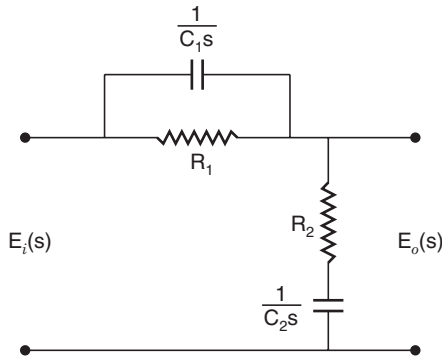
Fig. 1.32. Lag-lead network.

It can be shown that

$$\left.\begin{array}{l} R_1 C_1 = T_1 \\ R_2 C_2 = T_2 \\ \alpha \beta \, T_1 T_2 = R_1 C_1 R_2 C_2 \end{array}\right\} \qquad ...(1.50)$$

The phase lead part of the lag-lead network is given by $(1 + sT_1)/ (1 + s\,\alpha T_1)$ and lag $(1 + sT_2)/(1 + s\beta T_2)$.

1.9. LAG-LEAD NETWORK DESIGN : BODE PLOT APPROACH

The transfer function of a lag-lead network is given by

$$G_c(s) \;=\; \frac{(1 + sT_1)\,(1 + sT_2)}{\underset{\text{Lead}}{(1 + s\alpha T_1)}\,\underset{\text{Lag}}{(1 + s\,\beta T_2)}}$$

where
$$\alpha\beta \;=\; 1$$

and
$$\frac{1}{\beta T_2} < \frac{1}{T_2} < \frac{1}{T_1} < \frac{1}{\alpha T_1} \;.$$

The frequency response characteristic (Bode plot) for the lag-lead compensation network is shown in Fig. 1.33.

The first step in the design of lag-lead compensation network is the determination of the parameters of the lead part as follows.

Determine the gain K to suit the steady state error requirement.

Plot the Bode plot of the uncompensated transfer function and determine the phase margin therefrom. Any shortage of the phase margin in to be compensated by the lead part of the compensation network.

The α parameter is calculated as

$$\alpha \;=\; \frac{1 - \sin \phi_m}{1 + \sin \phi_m}$$

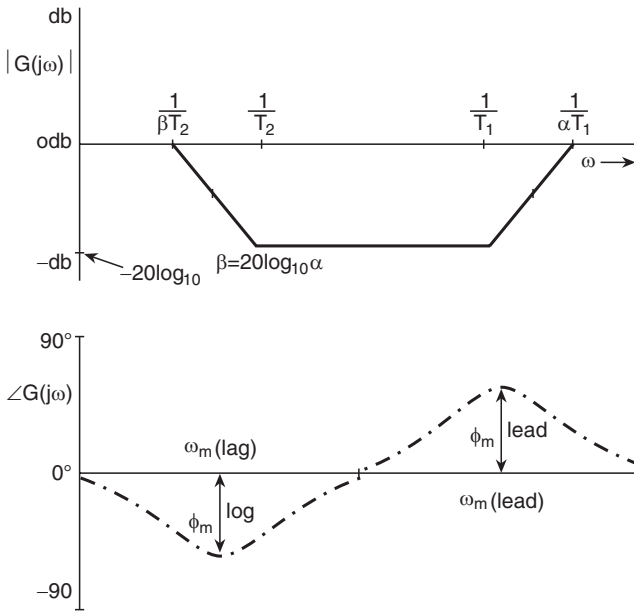where ϕ_m is the phase lead needed.

Fig. 1.33. Bode plot for lag-lead network.

The magnitude of the lead part of the compensation network at the mid-corner frequency ω_m is calculated as $20 \log_{10} \left(\dfrac{1}{\sqrt{\alpha}} \right)$. This gain is nullified by the negative gain (db) of the uncompensated system at a frequency ω_c select $\omega_m = \omega_c$ as the new gain crossover frequency, ω_m being the mid-corner frequency of the phase lead part of compensated transfer function.

The zero $\left(\dfrac{1}{T_1} \right)$ and pole $\dfrac{1}{\alpha T_1}$ with regard to the phase lead part are determined as below :

$$\frac{1}{T_1} \ = \ \omega_m \ \sqrt{\alpha}$$

and

$$\frac{1}{\alpha T_1} \ = \ \frac{\omega_m}{\sqrt{\alpha}}$$

The Bode plot of the system transfer function with phase lead part of the lag-lead compensation network is ploted for the verification of desired phase margin.

The gain crossover frequency of the uncompensated system is increased as a result of the use of phase lead network. This may reduce

the gain margin. To obtain the specified gain margin phase lag part of the compensation network is designed as below :

Select the pole

$$\frac{1}{\beta T_2} \ = \ 0.1 \ \omega_m$$

For a phase lag-lead network $\alpha\beta = 1$, hence $\beta = \dfrac{1}{\alpha}$. This determines the location of zero $= \dfrac{1}{T_2}$ of the phase lag part of the compensation network.

Example 1.13. *Draw the Bode plot of the phase lag-lead compensated system and check for the gain margin and phase margin.*

Given that $e_{ss} = 0.04$ for a unit ramp input and the phase margin around 42°, design a lag-lead compensator for the system having open-loop transfer function given below :

$$G(s) \ H(s) \ = \ \frac{K}{s(0.5s + 1) \, (0.0166s + 1)} \, .$$

Solution. $e_{ss} \ = \ \dfrac{1}{K_v}$

$\therefore$ $K_v \ = \ \dfrac{1}{e_{ss}} \ = \ \dfrac{1}{0.04} = 25 \ \sec^{-1}$

$K_v \ = \ \lim\limits_{s \to 0} \ s \ G(s) \ H(s)$

$= \ \lim\limits_{s \to 0} \ s \ \dfrac{K}{s(0.5 \, s + 1) \, (0.0166 \, s + 1)}$

$\therefore$ $25 \ = \ \lim\limits_{s \to 0} s \dfrac{K}{s(0.5 \, s + 1) \, (0.0166 \, s + 1)}$

$\therefore$ $K \ = \ 25$

Hence, $G(s) \ H(s) \ = \ \dfrac{25}{s(0.5 \, s + 1) \, (0.0166 \, s + 1)}$

which satisfies steady state error requirement.

The sinusoidal form of above transfer function is given below,

$$G(j\omega) \ H(j\omega) \ = \ \frac{K}{j\omega(\, j \, 0.5 \, \omega + 1) \, (\, j \, 0.0166 \, \omega + 1)}$$

As the system type 1, the initial slope of the Bode plot is – 20 db/ decade, intersecting 0 *db* axis at $K = \omega = 25$ rad/sec.

The two corner frequencies due to open-loop zeros being

$$\omega = \frac{1}{0.5} = 2 \text{ rad/sec.}$$

$$\omega = \frac{1}{0.0166} = 60.2 \text{ rad/sec.}$$

$$\angle G(j\omega) \, H(j\omega) = -90° - \tan^{-1} 0.5 \, \omega - \tan^{-1} 0.0166 \, \omega$$

Therefore,

ω rad/sec	2	5	8	10	40
$\angle G(j\omega) \, H(j\omega)°$	-136.9	-163	-173.5	-178	-210

From the above data Bode plot for the uncompensated is drawn as shown in Fig. 1.34.

From the bode plot it is noted that, the gain crossover frequency for the uncompensated system is $\omega_1 = 7.2$ rad/sec and $\angle G(j\ 7.2)$ $H(j\ 7.2) = 171°$. Hence,

$$\underset{\text{(Uncompensated)}}{P.M.} = 180° - 170° = 9°$$

As the desired phase margin is 42°, the phase lead to be contributed by the lead compensation network is

$$\phi_m = 42° - 9° + \underset{\text{(allowance)}}{7°} = 40°$$

The α parameter of the lead compensator is determined below,

$$\alpha = \frac{1 - \sin \phi_m}{1 + \sin \phi_m} = \frac{1 - \sin 40°}{1 + \sin 40°} = 0.218$$
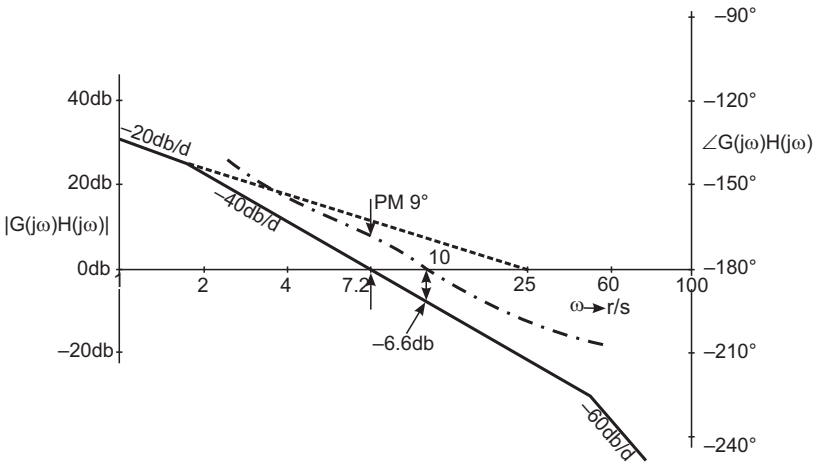

Fig. 1.34. Bode plot (uncompensated) for example 1.13.

The lead compensation network provides maximum phase lead ϕ at a frequency $\omega_m = \dfrac{1}{\sqrt{\alpha}\,T}$ and the magnitude of the said network at this frequency can be determined as follows.

The transfer function of a phase lead compensation network in sinusoidal from is given below,

$$G_c(j\omega) = \frac{\alpha(1 + j\,\omega T)}{(1 + j\,\omega\,\alpha T)}$$

At $\omega = \omega_m = \dfrac{1}{(\sqrt{\alpha})\,T}$

$$|\,G_c(j\omega_m)\,| = \left| \frac{\alpha(1 + j\,\omega_m T)}{(1 + j\,\omega_m\,\alpha T)} \right| = \left| \frac{\alpha\left[1 + j\dfrac{1}{(\sqrt{\alpha})\,T}\,T\right]}{\left[1 + j\dfrac{1}{(\sqrt{\alpha})\,T}\,\alpha T\right]} \right|$$

$$= \left| \frac{\alpha\left(1 + j\dfrac{1}{\sqrt{\alpha}}\right)}{(1 + j\sqrt{\alpha})} \right| = \frac{\alpha\sqrt{1 + \dfrac{1}{\alpha}}}{\sqrt{1 + \alpha}} = \sqrt{\alpha}$$

$\therefore \quad |\,G_c(j\omega_m)\,|_{db} = 20 \log_{10} \sqrt{\alpha} \quad \because \quad \alpha = 0.218$

$|\,G_c(j\omega_m)\,|_{db} = 20 \log_{10} \sqrt{0.218} = -6.6$ db.

From the Bode plot (Fig. 1.34), $-6.6\ db$ gain occurs at $\omega = 10$ rad/sec and this frequency is selected as ω_m.

$\therefore \qquad\qquad \omega_m = 10$ rad/sec.

Lead Network

(*i*) locate zero at $\dfrac{1}{T_1} = \omega_m \sqrt{\alpha} = 10 \sqrt{0.218} = 4.67$

(*ii*) locate pole at $\dfrac{1}{\alpha T_1} = \dfrac{\omega_m}{\sqrt{\alpha}} = \dfrac{10}{\sqrt{0.218}} = 21.4$

The lead part of the lag-lead compensation network is given below,

$$\underset{\text{(Lead)}}{G_c(j\omega)} = \frac{1 + j\,\omega T_1}{1 + j\,\omega\,\alpha T_1} = \frac{\left(1 + \dfrac{j\,\omega}{4.67}\right)}{\left(1 + \dfrac{j\,\omega}{21.4}\right)} = \frac{(1 + j\,0.214\,\omega)}{(1 + j\,0.046\,\omega)}$$

Lag Network

(*i*) Locate zero at : $0.1\ \omega_m = 0.1 \times 10 = 1$

$\therefore \qquad\quad T_2 = 1$

$\because \qquad\quad \alpha\beta = 1$ and $\alpha = 0.218$

$$\beta = \frac{1}{0.218} = 4.58$$

(*ii*) Locate pole at : $\dfrac{1}{\beta T_2} = \dfrac{1}{4.5 \times 1} = 0.218$

The lag part of the lag-lead compensation network is given below,

$$G_c(j\omega)_{\text{(Lag)}} = \frac{(1 + j\ \omega T_2)}{(1 + j\ \omega\beta T_2)} = \frac{(1 + j\ \omega)}{\left(1 + \dfrac{j\ \omega}{0.218}\right)} = \frac{(1 + j\ \omega)}{\left(1 + j\ 4.58\ \omega\right)}$$

The transfer function of the lag-lead compensation network in given by,

$$G_c(j\omega)_{\text{(Lag-lead)}} = \frac{25}{j\omega(j\ 0.5\ \omega + 1)\ (j\ 0.0166\ \omega + 1)}$$

The lag-lead compensated open-loop transfer function is given by,

$$G_c(j\omega)\ G(j\omega)\ H(j\omega) = \frac{(1 + j\ \omega)\ (1 + j\ 0.214\ \omega)}{\left(1 + j\ 4.58\ \omega\right)(1 + j\ 0.046\ \omega)}$$

$$\times\ \frac{25}{j\omega(j\ 0.5\ \omega + 1)\ (j\ 0.0166\ \omega + 1)}$$

The initial slope of the Bode plot is -20 *db*/decade, intersecting *odb* axis at $K = \omega = 25$ rad/sec.

The corner frequencies are,

$$\omega = \frac{1}{4.58} = 0.214 \text{ rad/sec, due to pole.}$$

$$\omega = 1 \text{ rad/sec, due to zero.}$$

$$\omega = \frac{1}{0.5} = 2 \text{ rad/sec, due to pole.}$$

$$\omega = \frac{1}{0.214} = 4.67 \text{ rad/sec, due to zero.}$$

$$\omega = \frac{1}{0.046} = 21.7 \text{ rad/sec, due to pole.}$$

$$\omega = \frac{1}{0.0166} = 60.2 \text{ rad/sec, due to pole.}$$

$$\angle G_c(j\omega)\ G(j\omega)\ H(j\omega) = -90 - \tan^{-1} 0.458\ \omega - \tan^{-1} 0.046\ \omega$$
$$- \tan^{-1} 0.5\ \omega - \tan^{-1} 0.0166\ \omega$$
$$+ \tan^{-1} \omega + \tan^{-1} 0.214\ \omega$$

For some values of frequency,

$\angle G_c(j\omega)\ G(j\omega)\ H(j\omega)$ can be determined as below,

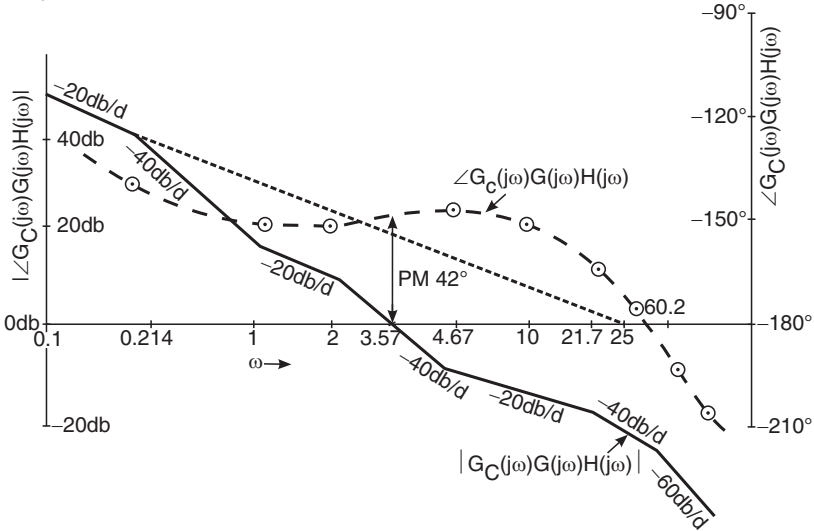| ω rad/sec	1	3.57	4	20	32	40	60
$\angle G_c(j\omega)\ G(j\omega)\ H(j\omega)^\circ$	– 140	– 138	– 137.9	– 160	– 180	– 190	– 208

Fig. 1.35. Bode plot (compensated) for example 1.13.

As per the data calculated as above, the Bode plot for the compensated system is drawn as shown in Fig. 1.35.

From the Bode plot (Fig. 1.35), the desired phase margin = 42° is noted at a gain crossover frequency ω = 3.57 rad/sec. **Ans.**

1.10. PROPORTIONAL PLUS DERIVATIVE PLUS INTEGRAL (PID) CONTROL
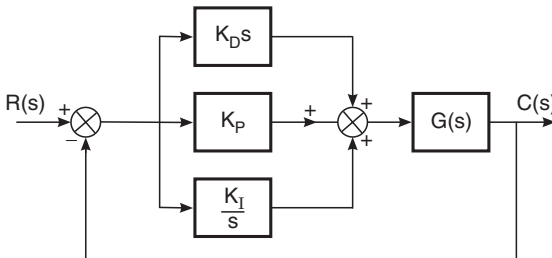
The PID controller is shown in Fig. 1.36.

Fig. 1.36. PID control.

The plant transfer function is $G(s)$. The PID controller transfer function is

$$G_c(s) = K_P + K_D\,s + \frac{K_I}{s} \qquad\qquad ...(1.51)$$

or $$G_c(s) = \frac{K_I + K_P s + K_D s^2}{s} \qquad\qquad ...(1.52)$$

The controller and plant are in cascade and the combined open loop transfer function is $G_c(s)\,G(s)$.
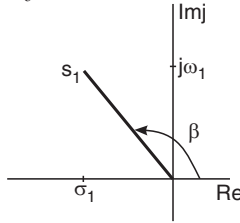

Fig. 1.37. Dominant pole location for PID control.

At any desired dominant pole location say (Fig. 1.37)

$$s_1 = \sigma_1 + j\,\omega_1$$

$$|s_1| = \sqrt{\sigma_1^2 + \omega_1^2}$$

$$\angle s_1 = \beta$$

$$\beta = 90° + \tan^{-1}\frac{\sigma_1}{\omega_1}$$

Say at s_1, $G(s_1) = |G(s_1)|\ \angle\phi$ $\qquad\qquad ...(1.53)$

Let $\quad G_c(s_1)\,G(s_1) = |A|\ \angle\delta$ $\qquad\qquad ...(1.54)$

From Eqns. (1.53) and (1.54)

$$G_c(s_1) = \frac{|A|\angle\delta}{|G(s_1)|\angle\phi}$$

or $$G_c(s_1) = \frac{1}{M}\ \angle\theta \qquad\qquad ...(1.55)$$

where $$M = \frac{|G(s_1)|}{|A|} \qquad\qquad ...(1.56)$$

and $$\theta = \delta - \phi$$

For PID controller, at $s = s_1$

$$G_c(s_1) = \frac{K_I + K_P s_1 + K_D s_1^2}{s_1} \qquad\qquad ...(1.57)$$

Equating Eqns. (1.55) and (1.57)

$$K_I + K_P \, |s_1| \angle \beta + K_D \, |s_1|^2 \angle 2\beta = |s_1| \, \angle \beta \times \frac{1}{M} \, \angle \theta$$

In complex notation form

$$K_I + K_P + |s_1|(\cos\beta + j\sin\beta) + K_D \, |s_1^2| \, (\cos 2\beta + j\sin 2\beta)$$

$$= \frac{|s_1|}{M} \{\cos(\theta + \beta) + j\sin(\theta + \beta)\}$$

Equating real and imaginary parts separately :

$$K_I + K_P |s_1| \cos\beta + K_D |s_1^2| \cos^2 2\beta = \frac{|s_1|}{M} \cos(\theta + \beta) \qquad ...(1.58)$$

and $\quad K_P \, |s_1| \sin\beta + K_D \, |s_1^2| \, \sin 2\beta = \dfrac{|s_1|}{M} \sin(\theta + \beta) \qquad ...(1.59)$

The coefficients K_I K_P and K_D are to be determined from above two equations. However, the value of K_I is either specified or determined from steady state error requirement.

The coefficient K_P and K_D are determined in terms of K_I as given below :

Multiply Eqn. (1.58) by $\sin 2\beta$ and Eqn. (1.59) by $\cos 2\beta$

$\therefore \quad K_I \sin 2\beta + K_P \, |s_1| \cos\beta \sin 2\beta + K_D \, |s_1|^2 \cos 2\beta \sin 2\beta$

$$= \frac{|s_1|}{M} \cos(\theta + \beta) \sin 2\beta \qquad\qquad ...(1.60)$$

$K_P \, |s_1| \sin\beta \cos 2\beta + K_D \, |s_1|^2 \sin 2\beta \cos 2\beta$

$$= \frac{|s_1|}{M} \sin(\theta + \beta) \cos 2\beta \qquad\qquad ...(1.61)$$

Subtract Eqn. (1.61) from Eqn. (1.60)

$K_I \sin 2\beta + K_P |s_1| \, (\cos\beta \sin 2\beta - \sin\beta \cos 2\beta)$

$$= \frac{|s_1|}{M} \{\cos(\theta + \beta) \sin 2\beta$$

$$- \sin(\theta + \beta) \cos 2\beta\} \qquad\qquad ...(1.62)$$

or $\quad K_I \sin 2\beta + K_P \, |s_1| \, (\sin\beta) = \dfrac{|s_1|}{M} \sin(\beta - \theta) \qquad\qquad ...(1.63)$

or $\quad 2K_I \sin\beta \cos\beta + K_P \, |s_1| \sin\beta = \dfrac{|s_1|}{M} \sin(\beta - \theta) \qquad\qquad ...(1.64)$

or $\quad K_P \, |s_1| \sin\beta = \dfrac{|s_1|}{M} \sin(\beta - \theta) - 2 K_I \cos\beta \sin\beta \qquad\qquad ...(1.65)$

$$\therefore \qquad\qquad K_P = \boxed{K_P = \frac{\sin(\beta - \theta)}{M \sin\beta} - \frac{2 K_I \cos\beta}{|s_1|}} \qquad ...(1.66)$$

Substituting the value of K_P in Eqn. (1.59)

$$\left\{ \frac{\sin (\beta - \theta)}{M \sin \beta} - \frac{2 K_I \cos \beta}{|s_1|} \right\} \ |s_1| \ \sin \beta + K_D \ |s_1|^2 \sin 2\beta$$

$$= \frac{|s_1|}{M} \ \sin (\theta + \beta)$$

or $\quad 2K_D \ |s_1|^2 \sin \beta \cos \beta = \dfrac{|s_1|}{M} \ \sin (\theta + \beta) - \dfrac{|s_1|}{M} \ \sin (\beta - \theta)$

$$+ 2K_I \cos \beta \sin \beta \qquad\qquad ...(1.67)$$

or $\quad 2K_D \ |s_1|^2 \sin \beta \cos \beta = \dfrac{|s_1|}{M} \ [\sin (\theta + \beta) - \sin (\beta - \theta)]$

$$+ 2K_I \cos \beta \sin \beta \qquad\qquad ...(1.68)$$

or $\quad 2K_D \ |s_1|^2 \sin \beta \cos \beta = \dfrac{|s_1|}{M} \ \{\sin \theta \cos \beta + \cos \theta \sin \beta$

$$- (\sin \beta \cos \theta - \sin \theta \cos \beta)\} + 2K_I \cos \beta \sin \beta \qquad ...(1.69)$$

$$2K_D \ |s_1|^2 \sin \beta \cos \beta = \frac{|s_1|}{M} \ 2 \sin \theta \cos \beta$$

$$+ 2K_I \cos \beta \sin \beta \qquad\qquad ...(1.70)$$

or $\quad K_D = \dfrac{2 \, |s_1| \sin \theta \cos \beta}{M \, |s_1|^2 \, 2 \sin \beta \cos \beta} + \dfrac{2 \, K_I \cos \beta \sin \beta}{2 \, |s_1|^2 \sin \beta \cos \beta} \qquad ...(1.71)$

$$\therefore \qquad \boxed{K_D = \frac{\sin \theta}{M \, |s_1| \sin \beta} + \frac{2K_I}{|s_1|^2}} \qquad\qquad ...(1.72)$$

1.11. PID CONTROLLER DESIGN : ROOT LOCUS APPROACH

The open-loop transfer function of a feed-back control system incorporating PID controller is given by $G_c(s) \, G(s)$. The characteristic equation is

$$1 + G_c(s) \, G(s) = 0$$

Having given the desired dominant root location on the root locus plot at say $s = s_1$

$$G_c(s_1) \, G(s_1) = - 1 \qquad\qquad ...(1.73)$$

Put $|G_c(s_1) \, G(s_1)| = A$

$$|A| = |\,G_c(s_1) \, G(s_1)\,| = 1 \ \left.\begin{array}{l}\text{a point on}\\[4pt]\text{root locus}\end{array}\right\} \ \ ...(1.74)$$

and $\qquad \angle G(s_1) \, G(s_1) = \pm 180°$

$$\because \qquad\qquad \phi = \angle G(s_1)$$

$$\therefore \qquad\qquad \theta = \delta - \phi = 180° - \angle G(s_1) \qquad\qquad ...(1.75)$$

where $\qquad\qquad \angle \theta = \angle G_c(s_1), \angle \delta = \angle G_c(s_1) \, G(s_1)$ and $\angle \phi = \angle G(s_1)$

From Eqn. (1.56), $M = \dfrac{|G(s_1)|}{|A|} = \dfrac{|G(s_1)|}{1}, \because \quad |A| = 1$

$\therefore \qquad\qquad M = |G(s_1)|$ \hfill ...(1.76)

From Eqn. (1.66)

$$K_P = \frac{\sin(\beta - \theta)}{M \sin\beta} - \frac{2 K_I \cos\beta}{|s_1|}$$

$\because \qquad\qquad \theta = 180° - \angle G(s_1)$

$\therefore \qquad K_P = \dfrac{\sin[\beta - \{180° - \angle G(s_1)\}]}{|G(s_1)| \sin\beta} - \dfrac{2 K_I \cos\beta}{|s_1|}$

$\because \qquad \angle G(s_1) = \phi$

$\therefore \qquad K_P = \dfrac{\sin(\beta + \phi - 180°)}{|G(s_1)| \sin\beta} - \dfrac{2K_I \cos\beta}{|s_1|}$

or
$$\boxed{K_P = -\frac{\sin(\beta + \phi)}{|G(s_1)| \sin\beta} - \frac{2K_I \cos\beta}{|s_1|}}$$ \hfill ...(1.77)

From Eqn. (1.72)

$$K_D = \frac{\sin\theta}{M |s_1| \sin\beta} + \frac{K_I}{|s_1|^2}$$

$\because \qquad\qquad \theta = 180° - \angle G(s_1) = 180° - \phi$

and $\qquad\qquad M = |G(s_1)|$

$\therefore \qquad K_D = \dfrac{\sin(180° - \phi)}{|G(s_1)| |s_1| \sin\beta} + \dfrac{K_I}{|s_1|^2}$

or
$$\boxed{K_D = \frac{\sin\phi}{|G(s_1)| |s_1| \sin\beta} + \frac{K_I}{|s_1|^2}}$$ \hfill ...(1.78)

Example 1.14. *The forward path transfer function of a unity feed back control system is given below :*

$$G(s) = \frac{100}{(s + 1)(s + 5)(s + 10)}$$

$e_{ss} \leq 0.075$ for unit ramp input

The closed loop dominant poles are to be located at $-1.5 \pm j\,2$. Design a PID controller.

Solution. $K_v = \dfrac{1}{e_{ss}} = \dfrac{1}{0.075} = 13.3$

The forward path transfer function using PID controller is given below :

$$G_c(s)\, G(s) = \frac{(K_I + K_P s\ K_D s^2)}{s} \times \frac{100}{(s+1)\,(s+5)\,(s+10)}$$

$$K_v = \lim_{s \to 0}\ s\, G_c(s)\, G(s)$$

$$= \lim_{s \to 0} s\frac{(K_I + K_P s + K_D s^2)}{s} \times \frac{100}{(s+1)\,(s+5)\,(s+10)}$$

$$= \frac{100 K_I}{1 \times 5 \times 10}$$

$$\therefore \quad 13.3 = \frac{100\ K_I}{50}\ ; K_I = 6.65$$

$$s_1 = -1.5 + j\,2\ \text{(given)}$$

$$G(s_1) = \frac{100}{(-1.5 + j\,2 + 1)\,(-1.5 + j\,2 + 5)\,(-1.5 + j\,2 + 10)}$$

$$= \frac{100}{(-0.5 + j\,2)\,(3.5 + j\,2)\,(8.5 + j\,2)}$$

$$= \frac{100}{\sqrt{0.5^2 + 2^2}\ \sqrt{3.5^2 + 2^2}\ \sqrt{8.5 + 2^2}}$$

$$\angle\tan^{-1}\frac{2}{-0.5} - \tan^{-1}\frac{2}{3.5} - \tan^{-1}\frac{2}{8.5}\ = 1.38\angle 33.02°$$

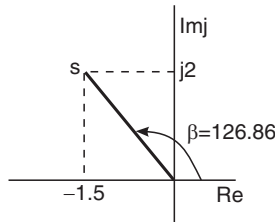$$\therefore \qquad \phi = 33.02°.$$

Fig 1.38. Diagram for example 1.14.

$$s_1 = -1.5 + j\,2$$

or $$\qquad s_1 = \sqrt{1.5^2 + 2^2}\ \angle\beta = 2.5\ \angle\beta$$

$$\beta = 90° + \tan^{-1}\frac{1.5}{2}\ = 126.86°$$

$$K_P = \frac{-\sin(\beta + \phi)}{|G(s_1)|\sin\beta} - \frac{2K_I\cos\beta}{|s_1|}$$

$$= \frac{-\sin(126.86° + 33.02)}{1.38 \sin 126.86°} - \frac{2 \times 6.65 \cos 126.86}{2.5}$$

$$= -\frac{0.34}{1.38 \times 0.8} - \frac{2 \times 6.65\,(-0.59)}{2.5}$$

or $\quad K_P = 2.83$

$$K_D = \frac{\sin \phi}{|s_1||G(s_1)|\sin \beta} + \frac{K_1}{|s_1|^2}$$

$$= \frac{\sin 33.02°}{2.5 \times 1.38 \times \sin 126.86} + \frac{6.65}{2.5^2}$$

$$= \frac{0.54}{2.5 \times 1.38 \times 0.8} + \frac{6.65}{6.25} = 0.19 + 1.06 = 1.25$$

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s$$

or $\quad G_c(s) = 2.83 + \dfrac{6.65}{s} + 1.25\,s$ **Ans.**

Example 1.15. *Design a PD controller such that the dominant root of the characteristic equation (closed-loop pole) is located at s(– 1.2 + j 1.2) in s-plane*

The feed back control system has unity feed back and the forward path transfer function is given below :

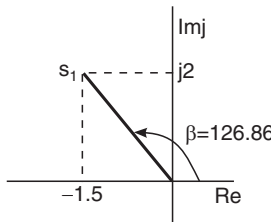$$G(s) = \frac{30}{s(s+1)(s+3)} \ .$$

Solution.

Fig. 1.39. Diagram for example 1.15.

$$s_1 = -1.2 + j\,1.2$$

$$= \sqrt{1.2^2 + 1.2^2}\ \angle 90° + \tan^{-1}\frac{1.2}{1.2}$$

$$= 1.69\ \angle 135°$$

$\therefore \qquad \beta = 135°$

$$G(s_1) = \frac{30}{(-1.2 + j\,1.2)\,(-1.2 + j\,1.2 + 1)\,(-1.2 + j\,1.2 + 3)}$$

$$= \frac{30}{(-1.2 + j\,1.2)\,(-0.2 + j\,1.2)\,(1.8 + j\,1.2)}$$

$$= \frac{30}{\sqrt{1.2^2 + 1.2^2}\ \sqrt{0.2^2 + 1.2^2}\ \sqrt{1.8^2 + 1.2^2}}$$

$$\times \left(\angle - \tan^{-1} \frac{1.2}{-1.2} \right.$$

$$\left. - \tan^{-1} \frac{1.2}{-0.2} - \tan^{-1} \frac{1.2}{1.8} \right)$$

$$= \frac{30}{1.2\sqrt{2}\ \sqrt{1.48}\ \sqrt{4.68}}\quad \angle 92.5°$$

$$= 6.71\ \angle 92.5°\quad \therefore\quad \phi = 92.5°$$

$$K_P = \frac{-\sin(\beta + \phi)}{|G(s_1)|\sin\beta} + \frac{2\,K_1\cos\beta}{|s_1|}$$

$\because \qquad\qquad K_I = 0$, being *PD* control

$\therefore \qquad\qquad K_P = -\dfrac{\sin(\beta + \phi)}{|G(s_1)|\sin\beta}$

$$= -\frac{\sin(135° + 92.5°)}{6.71 \times \sin 135°} = \frac{0.737}{4.74} = 0.155$$

$$K_D = \frac{\sin\phi}{|s_1|\,|G(s_1)|\sin\beta} - \frac{K_I}{|s_1|^2}$$

$\because \qquad\qquad K_I = 0$

$\therefore \qquad\qquad K_D = \dfrac{\sin\phi}{|s_1|\,|G(s_1)|\sin\beta} = \dfrac{\sin 92.5°}{1.2\sqrt{2} \times 6.71 \sin 135°}$

$$= \frac{0.99}{8.02} = 0.123$$

$$G_c(s) = K_P + sK_D = (0.155 + 0.123\ s)\ \textbf{Ans.}$$

Example 1.16. *A unity feed back control system has the following time domain specifications.*

$$\omega_n = 2.5\ rad/sec\ ;\ \xi = 0.8.$$

The forward path transfer function is given below :

$$G(s) = \frac{4.5}{(s + 3)\,(s + 5)}$$

Design a PI controller.

Solution. The desired root location is given by

$$s_1 = -\xi\,\omega_n \pm j\,\omega_n\,\sqrt{1-\xi^2}$$

$$= -0.8 \times 2.5 \pm j\,2.5\,\sqrt{1-0.8^2}$$

$$= (-2 \pm j\,1.5)$$

$$s_1 = \sqrt{2^2 + 1.5^2}\ \angle 90° + \tan^{-1}\frac{2}{1.5} = 2.5\,\angle 143.1°$$

$\therefore$ $\quad |s_1| = 2.5$

and $\quad\quad \beta = 143.1°$

$$G(s_1) = \frac{4.5}{(s_1 + 3)(s_1 + 5)}$$

$$= \frac{4.5}{(-2 + j\,1.5 + 3)(-2 + j\,1.5 + 5)}$$

$$= \frac{4.5}{(1 + j\,1.5)(3 + j\,1.5)}$$

$$= \frac{4.5}{\sqrt{3.25}\,\sqrt{11.29}}\ \angle - \tan^{-1}\frac{1.5}{1} - \tan^{-1}\frac{1.5}{3}$$

$$= \frac{4.5}{6.05}\ \angle - 56.3° - 26.56° = 0.74\,\angle - 82.86°$$

$\therefore$ $\quad |G(s_1)| = 0.74$ and $\phi = -82.86°$

$$K_D = \frac{\sin\phi}{|s_1|\,|G(s_1)|\,\sin\beta} - \frac{K_I}{|s_1^2|}$$

$\because$ $\quad\quad K_D = 0 \quad$ for PI controller

$\therefore$ $\quad\quad K_I = \frac{-|s_1|\sin\phi}{|G(s_1)|\,\sin\beta}$

$$= \frac{-2.5\sin(-82.86)}{0.74\sin 143.1°} = \frac{-2.5\,(-0.99)}{0.74 \times 0.59} = 5.6$$

$$K_P = \frac{-\sin(\beta + \phi)}{|G(s_1)|\,\sin\beta} - \frac{2\,K_I\cos\beta}{|s_1|}$$

$$= \frac{-\sin[143.1 + (-82.86)]°}{0.74\sin 143.1°} - \frac{2 \times 5.6\cos 143.1°}{2.5}$$

$$= \frac{-0.869}{0.74 \times 0.59} - \frac{2 \times 5.6\,(-0.8)}{2.5} = -1.99 + 3.58$$

or $\quad\quad K_P = 1.59$

$$G_c(s) = K_P + \frac{K_I}{s}$$

or $$G_c(s) = \left(1.59 + \frac{5.6}{s}\right) \quad \textbf{Ans.}$$

1.12 PID CONTROLLER DESIGN : FREQUENCY RESPONSE APPROACH

Having given the specified phase and gain crossover frequency, the frequency response approach is followed to design a PID controller.

For a desired gain crossover frequency ω_1.

$$s_1 = j\,\omega_1$$
$$|s_1| = |\omega_1| \angle 90°$$
$$\therefore \qquad \beta = 90°$$

with the use of *PID* controller, at ω_1

$$|G_c(j\omega_1)\,G(j\omega_1)| = 1 \qquad\qquad ...(1.79)$$

From Eqn. (1.54)

$$|G_c(s_1)\,G(s_1)| = |A| \qquad\qquad ...(1.80)$$

$$\therefore \qquad\qquad s_1 = j\omega_1,\ |G_c(j\omega_1)\,G(j\omega_1)| = |A| \qquad ...(1.81)$$

From Eqn. (1.79)

$$|G_c(j\omega_1)\,G(j\omega_1)| = 1$$
$$\therefore \qquad\qquad |A| = 1$$

From Eqn. (1.56)

$$M = \frac{|G(s_1)|}{|A|}$$

$$\therefore \qquad\qquad s_1 = j\,\omega \text{ and } |A| = 1$$

$$\therefore \qquad\qquad M = \frac{|G(j\,\omega_1)|}{1} \qquad\qquad ...(1.82)$$

$$\delta = \angle G_c(j\,\omega_1)\,G(j\,\omega_1) \Big\}$$

and $$\phi = \angle G(j\,\omega_1) \qquad\qquad\qquad ...(1.83)$$

$$\therefore \qquad\qquad \theta = (\delta - \phi)$$

From, Eqn. (1.66)

$$K_P = \frac{\sin(\beta - \theta)}{M \sin \beta} - \frac{2\,K_I \cos\beta}{|s_1|}$$

at $$s_1 = j\omega_1,\ M = |G(j\omega_1)|$$

$$K_P = \frac{\sin(\beta - \theta)}{|G(j\,\omega_1)|\sin\beta} - \frac{2\,K_I \cos\beta}{|j\,\omega_1|} \qquad ...(1.84)$$

$$\therefore \qquad\qquad \beta = 90°, \cos\beta = 0$$

$$\therefore \qquad K_P = \frac{\sin(90 - \theta)}{|G(j\,\omega_1)|\sin\beta} \qquad\qquad \text{...(1.85)}$$

$$\text{or} \qquad K_P = \frac{\cos\theta}{|G(j\,\omega_1)|} \qquad\qquad \text{...(1.86)}$$

From Eqn. (1.72)

$$K_D = \frac{\sin\theta}{M\,|s_1|\sin\beta} - \frac{2\,K_I}{|s_1|^2}$$

At $s_1 = j\omega_1$

$$K_D = \frac{\sin\theta}{M\,|j\,\omega_1|\sin\beta} - \frac{2\,K_I}{|j\,\omega_1|^2} \qquad\qquad \text{...(1.87)}$$

At $s_1 = j\omega_1$, $M = |G(j\omega_1)|$ and $\beta = 90°$

$$\therefore \qquad K_D = \frac{\sin\theta}{|G(j\,\omega_1)|\,\omega_1 \times 1} - \frac{2\,K_I}{\omega_1^2} \qquad\qquad \text{...(1.88)}$$

Example 1.17. *A unity feed back control system having an open-loop transfer function*

$$G(s) = \frac{100}{(s+1)\,(s+5)\,(s+10)}$$

required $e_{ss} \le 0.08$ for a unit ramp input and phase margin > 50° at a frequency of 6 rad/sec. Design a PID controller.

Solution. $\quad K_v = \dfrac{1}{e_{ss}} = \dfrac{1}{0.08} = 12.5$

The open-loop transfer function of the system using *PID* controller is given by

$$G_c(s)\ G(s) = \frac{K_I + K_P s + K_D s^2)}{s} \times \frac{100}{(s+1)\,(s+5)\,(s+10)}$$

$$\therefore \qquad K_v = \lim_{s \to 0}\ sG_c(s)\ G(s)$$

$$\therefore \qquad 12.5 = \lim_{s \to 0} \frac{s\,(K_I + K_P s + K_D s^2)}{s} \times \frac{100}{(s+1)\,(s+5)\,(s+10)}$$

$$= \frac{100\,K_I}{1 \times 5 \times 10}$$

$$\therefore \qquad K_I = 6.25$$

The transfer function $G(j\,\omega)\ H(j\,\omega)$ at gain crossover frequency $\omega_1 = 6$ rad-sec is determined below :

$$G(j\,6) = \frac{100}{(j\,6+1)\,(j\,6+5)\,(j\,6+10)}$$

$$\angle - \tan^{-1}\frac{6}{1} - \tan^{-1}\frac{6}{5} - \tan^{-1}\frac{10}{6}$$

$$= \frac{100}{\sqrt{31}\sqrt{61}\sqrt{136}} \angle - 194.15°$$

$$= 0.18 \angle - 194.15°$$

As the required phase margin $\phi_m = 50°$ the phase angle θ to be contributed by *PID* controller is determined below :

$$\phi_m = 180° + \angle G(j\omega_1) + \theta$$

or $\quad\quad 50° = 180° + (-194.15°) + \theta$

$\therefore \quad\quad\quad \theta = 64.15°$

$$K_P = \frac{\cos\theta}{|G(j\,\omega_1)|}$$

$$= \frac{\cos 64.15}{0.18} = \frac{0.436}{0.18}$$

or $\quad\quad\quad K_P = 2.42$

$$K_D = \frac{\sin\theta}{\omega_1\,|G(j\,\omega_1)|} + \frac{K_I}{\omega_1^2}$$

$$= \frac{\sin 64.15°}{6\times 0.18} + \frac{6.25}{6^2} = 0.83 + 0.17$$

or $\quad\quad\quad K_D = 1.0$

$\quad\quad K_I = 6.25 \; ; K_P = 2.42 \; ; K_D = 1.0$

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s$$

or $\quad\quad G_c(s) = 2.42 + \frac{6.25}{s} + 1\,s$ **Ans.**

Example 1.18. *The open loop transfer function of a unity feed back control system is given below :*

$$G(s) = \frac{10}{s(s+1)(s+3)}$$

Design a PD controller such that P.M. = 35° at $\omega = 2$ rad/sec.

Solution.

$$G(j\omega_1) = G(j2) = \frac{10}{j2(j2+1)(j2+3)}$$

$$= \frac{10}{2\sqrt{2^2+1^2}\sqrt{2^2+3^2}} \angle -90 - \tan^{-1}2 - \tan^{-1}\frac{2}{3}$$

$$= 0.62 \angle - 187.12°$$

$$\phi_m = 180° + \angle G(j\omega_1) + \theta$$

or
$$\theta = \phi_m - [(180° + \angle G(j\omega_1)]$$
$$= 35° - [(180° + (-187.12)] = 35° + 7.12° = 42.12°$$

$$K_P = \frac{\cos\theta}{|G(j\,\omega_1)|} = \frac{\cos 42.12°}{0.62} = 1.19$$

$$K_D = \frac{\sin\theta}{\omega_1\,|G(j\,\omega_1)|} = \frac{\sin 42.12°}{2\times 0.62} = 0.54$$

$$G_c(s) = K_P + K_D s$$
or
$$G_c(s) = 1.19 + 0.54\,s \qquad \textbf{Ans.}$$

Example 1.19. *Design a PI controller having giving unity feed back forward path transfer function as*

$$G(s) = \frac{150}{(s+1)\,(s+5)}$$

such that phase margin = 35° at $\omega_1 = 8$ rad/sec.

Solution. $G(j\omega) = \dfrac{150}{(j\omega + 1)\,(j\omega + 5)}$

$$G(j\omega_1) = G(j\,8) = \frac{150}{(j8+1)\,(j8+5)}$$

$$= \frac{150}{\sqrt{8^2+1^2}\,\sqrt{8^2+5^2}}\ \angle - \tan^{-1}\frac{8}{1} - \tan^{-1}\frac{8}{5}$$

$$= \frac{150}{\sqrt{65}\,\sqrt{89}}\ \angle - 82.8° - 57.9°$$

$$= 1.97\ \angle - 140.7°$$

$$\phi_m = 180 + \angle G(j\omega_1) + \theta$$

$$35° = 180 + (-140.7°) + \theta$$

∴
$$\theta = -4.3°$$

From eqn. 1.86

$$K_P = \frac{\cos\theta}{|G(j\,\omega_1)|}$$

$$= \frac{\cos(-4.3°)}{1.97} = \frac{0.997}{1.97}$$

or
$$K_P = 0.5$$

$$K_D = \frac{\sin\theta}{\omega_1\,|G(j\,\omega_1)|} + \frac{K_I}{\omega_1^{\,2}}$$

$$\because \qquad K_D = 0 \quad \text{(PI Controller)}$$

$$\therefore \qquad 0 = \frac{\sin \theta}{\omega_1 \,|\, G(j\,\omega_1)\,|} + \frac{K_I}{\omega_1{}^2}$$

or
$$K_I = \frac{-\omega_1 \sin \theta}{|\, G(j\,\omega_1)\,|} = \frac{-8 \sin (-4.3°)}{1.97}$$

$$= \frac{-8 \times (-0.074)}{1.97} = \frac{0.592}{1.97}$$

or
$$K_I = 0.3$$

$$G_c(s) = K_P + \frac{K_I}{s}$$

or
$$G_c(s) = 0.5 + \frac{0.3}{s} \qquad \textbf{Ans.}$$

PROBLEMS

1.1. The open-loop transfer function of a control system is given by

$$G(s)\,H(s) = \frac{K}{s(s+2.5)\,(s+40)}$$

Design a phase-lead compensation network to meet the following specifications.

$P.M. \simeq 40°$, $G.M. = 20$ db and $e_{ss} \leq 0.5$ rad.

1.2. Apply Bode plot method to design a lag compensation network for a unity feed back system having

$$G(s) = \frac{K}{s(s+2.5)\,(s+25)} \quad \text{such that}$$

$P.M. = 35°$, $G.M. \geq 12.5$ db and $e_{ss} = 0.05$ rad. for unit ramp input.

1.3. The open-loop transfer function of a unity feed-back control system is given below,

$$G(s) = \frac{K}{(s+10)}$$

The maximum overshoot is 24.3% (unit step input). Maintaining the same damping ratio, design a compensator such that the settling time is reduced by a factor of 2.0.

1.4. A unity feed-back control system has the following transfer function

$$G(s) = \frac{K}{s(s^2 + 6s + 10)}$$

The specifications are

(*i*) Damping ratio $\xi = 0.6$

(*ii*) $\omega_n = 3.5$ rad/sec.

(*iii*) $e_{ss} = 0.25$ rad for unit ramp input.

Design a lag-lead compensation network using root locus method.

1.5. The forward path transfer function of a unity feed-back control system is given below :

$$G(s) \; = \; \frac{K}{s(s^2 + 6s + 10)}$$

It is desired that

(*i*) % overshoot for a unit step input $\leq 20\%$

(*ii*) $K_v = 6$ sec^{-1}.

Apply root locus method to design a lag compensation network.

1.6. The open-loop transfer function of feed-back control system is given

$$G(s) \, H(s) \; = \; \frac{K}{s(s + 5)}$$

The specification are as follows :

(*i*) % overshoot for step input = 25%

(*ii*) e_{ss} for unit ramp input : ≤ 0.25 rad.

(*iii*) peak time : $t_p = 1.71$ sec.

Design a phase-lead compensation network.

1.7. Design a *PID* controller having given

$$G(s) \, H(s) \; = \; \frac{5}{(s + 1) \, (0.1s + 1) \, (0.02s + 1)}$$

The specifications are given as :

The steady state error for a unit ramp input ≤ 0.1 rad. and phase margin $\simeq 50°$ at a frequency of 6 rad/sec.

1.8. A unity feed-back control system has the following time domain specifications :

$\omega_n = 4$ rad/sec ; $\xi = 0.6$

The forward path transfer function is given below :

$$G(s) \; = \; \frac{5}{(s + 2) \, (s + 5)}$$

Design a *PI* controller.

1.9. The forward path transfer function of a unity feed-back control system is given below :

$$G(s) \; = \; \frac{100}{(s + 1.5)\,(s + 4)\,(s + 8)}$$

(i) $e_{ss} \leq 0.1$ for a unit ramp input

(ii) The closed-loop dominant poles are to be located at
  $s = (-1 \pm j\,2.5)$.

Design a *PID* controller.

1.10. Design a *PD* controller such that the dominant roots of the characteristic equation are located at $s = (-2 \pm j\,2)$.

The feed-back control system has a unity feed back and the forward path transfer function is given below :

$$G(s) \; = \; \frac{25}{s(s + 3)\,(s + 4)} \; .$$