1

Set Theory

Among many branches of mathematics, theory of sets, which was introduced by the German mathematician Cantor occupies a unique place. It helps to reduce many mathematical concepts to their logical foundations in an elegant and systematic manner. In this chapter we discuss the basic features of set theory, permutation, combination, mathematical induction inclusion-exclusion and probability theory, multisets.

1.1 SET NOTATION AND BASIC DEFINITIONS

A collection of well-defined objects is said to be a **set**. The objects in the collection are called **members** or **elements** of the set, and they are said to **belong to** or **contained** in the set. The set inturn is said to **contain** or **composed of** its elements.

Sets are usually denoted by capital letters A, B, C, ...and elements by small letters a, b, c,.......

Let S be a set. We write $x \in S$ to mean "x is an element of S", or "x belongs to S". If x does not belong to S, we write $x \notin S$. We sometimes designate sets by displaying the elements in braces ; for example {2, 4, 6, 8} is a set. However, when the elements of a set share a common property called as the characteristic property, there is a compact notation called the **set builder notation** to describe the set.

For a given set S, we constantly use

$$\text{A} = \{a \in S : P(a)\}$$

to read "A is the set of elements of S for which the property P holds".

For *e.g*; the set of all numbers between 0 and 1 can be denoted by $\{x : 0 < x < 1\}$.

Note that the order in which the elements of a set are listed is not important. For instant, the set {3, 4} is same as the set {4, 3}.

We introduce here several sets (and their notations) that will be used throughout this book.

$$\begin{aligned} N &= \text{Set of all natural numbers} \\ &= \{1, 2, 3......\} \\ Z &= \text{Set of all integers} \\ &= \{......... -3, -2, -1, 0, 1, 2, 3.......\} \\ R &= \text{Set of real numbers} \\ Q &= \text{Set of all rational numbers} \\ &= \{a/b \ : \ a, b \in R, b \neq 0\} \\ C &= \text{Set of all complex numbers} \\ &= \{a + ib : a, b \in R, i = \sqrt{-1}\} \end{aligned}$$

Note : Think of real numbers as the set of points on a number line.

Definition 1.1.1. Let A and S be sets. A is said to be a *subset* of S if every element in A is an element of S, that is if $a \in A$ implies $a \in S$. We write $A \subseteq S$.

Examples

(*a*) If $A = \{1, 2, 3, 4\}$, $S = \{0, 1, 2, 3, 4, 5, 6\}$, then $A \subseteq S$.

(*b*) $N \subseteq Z \subseteq R \subseteq C$.

The statement $A \subseteq B$ does not rule out the possibility that $B \subseteq A$.

Definition 1.1.2. Let A, B be two sets. Then A and B are said to be *equal* (notation $A = B$) if and only if $A \subseteq B$ and $B \subseteq A$, that is, A and B have the same elements. If A and B are not equal, we write $A \neq B$. If $A \subseteq B$, $A \neq B$, we say A is a proper subset of B.

Null Set and Universal Set.

The **null set** or **empty set,** ϕ, is the set having no elements; it is a subset of every set.

Given any set A, both ϕ, A, are subsets of A, which we call **improper subsets** of A.

If all sets under discussion are subsets of a given set U, then U is said to be a **universal set.** Universal set is therefore a relative term.

A set consisting of a single element is called a **singleton set.**

Examples

(*a*) If $A = \{1, 2, 3\}$, $B = \{x$: x is a positive integer and $x^2 < 12\}$, then $A = B$.

(*b*) If $A = \{1, 2, 3, 4\}$, $B = \{2, 3\}$, $C = \{1, 2, 3, 4, 5\}$ then $B \subseteq A, B \subseteq C, A \subseteq C$, but $A \not\subseteq B$, $C \not\subseteq B$, $C \not\subseteq A$.

(*c*) Let A be a set and let $B = \{A, \{A\}\}$. Then since A and $\{A\}$ are elements of B, we have $A \in B$, $\{A\} \in B$. It then follows that $\{A\} \subseteq B$, $\{\{A\}\} \subseteq B$. However, it is not true that $A \subseteq B$.

Definition 1.1.3. A set A is called **finite** if it has n distinct elements, where $n \in N$. In this case n is called the **cardinality** of A and is denoted by $|A|$. A set that is not finite is called *infinite*. An empty set is said to have cardinality zero. Note that N, Z, R, Q, C are examples of infinite sets.

Consider the set of natural numbers $N = \{0, 1, 2, \ldots\ldots\}$. Although N is infinite, the elements in N are enumerable or countable, that is the elements can be arranged sequentially. Such a set is said to be **countably infinite** or **denumerable.**

Power set : For a set A, collection or family of all subsets of A is called the **power set** of A and is denoted by $P(A)$. If A has k elements, then $P(A)$ had 2^k elements. The power set of the null set ϕ has only the element ϕ. That is, $P(\varphi) = \varphi$.

Example. Let $A = \{1, 2, 3\}$. Then $P(A)$ consists of the sets ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ $\{1, 2, 3\} = A$.

Exercise 1.1.

1. Enumerate the elements in the following sets :

(*a*) $\{x \in R: x^2 - 3x + 2 = 0\}$

(*b*) $\{x \in C : x^2 + 1 = 0\}$

2. Describe the following sets using set-builder notation

(*a*) $\{3, 5, 7, 9, \ldots\ldots\ldots., 77, 79\}$

(*b*) even integers

3. Let $A = \{0, 2, 3\}$, $B = \{0, 2\}$, $C = \{1, 5, 9\}$. Determine which of the following statements are true. Give reasons.

(*a*) $3 \in A$ (*b*) $\{3\} \in A$

(c) $\{3\} \subseteq A$ (d) $B \subseteq A$

(e) $A \subseteq B$ (f) $\phi \subseteq C$

(g) $\phi \in A$ (h) $(B \cap A) \subseteq C$.

4. Let $U = \{x : 0 \leq x \leq 9, x \text{ is an integer}\}$

$A = \{x \in U : x \text{ is a multiple of } 3\}$

$B = \{x \in U : x^2 - 5 \geq 0\}$.

Determine

(a) $A \cup B$ (b) $A \cap B$

(c) B^C (d) $B^C \cup A^C$

1.2 VENN DIAGRAMS AND SET OPERATIONS

Given two sets we can combine them to form new sets. There is nothing particular about this number to be two. We can carry out the same procedure for any number of sets, finite or infinite. Consider the following figure.

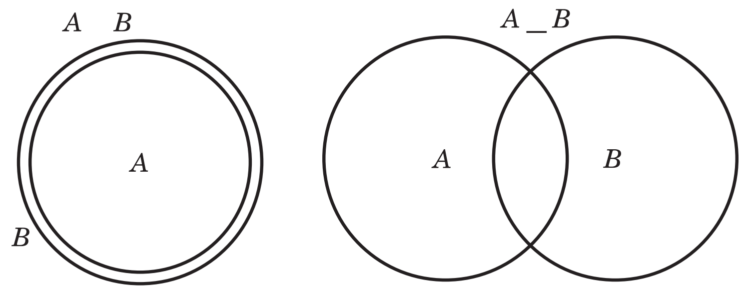

Fig. 1.1

The diagrams, as shown above, which are used to show relationships between sets are called **Venn diagrams.** In a Venn diagram the universal set U is represented by the interior of a rectangle and the sets by circle inside the rectangle.

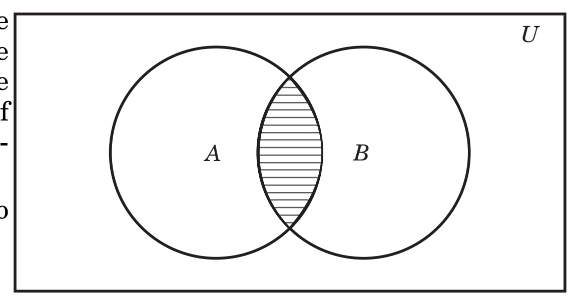

Fig. 1.2

Definition 1.2.1. The *intersection* of the two sets A and B written as $A \cap B$, is the set

$\{x : x \in A \text{ and } x \in B\}$.

If B is any subset of A, then $A \cap B = B$. By Venn diagram we illustrate the intersection of two sets A and B by Fig. 1.2.

Here A is the circle on the left, B that on the right, while their intersection is the shaded portion.

Examples

(i) Let $A = \{1, 3, 8\}$, $B = \{8, 22, 6\}$, then $A \cap B = \{8\}$.

(ii) If $A = \{a, b, c, e, f\}$, $B = \{b, e, f, r, s\}$ $C = \{a, t, u, v\}$, Find $A \cap B$, $A \cap C$, $B \cap C$.

Here $A \cap B = \{b, e, f\}$, $A \cap C = \{a\}$, $B \cap C = \phi$.

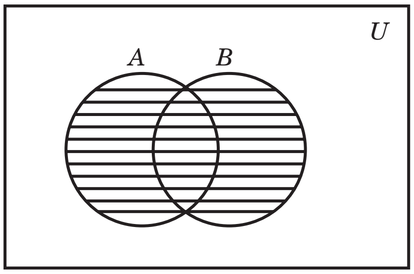

Fig. 1.3

Definition 1.2.2. The *union* of the two sets A and B, written as $A \cup B$, is the set

$$\{x : x \in A \quad \text{or} \quad x \in B\}$$

when we say x is in A or x is in B we mean x is in at least one of A or B, and may be in both. Whenever B is a subset of A, $A \cup B = A$. By Venn diagram we illustrate the union of two sets A and B by Fig. 1.3.

Here A is the circle on the left, B that on the right and $A \cup B$ is the shaded part.

Examples

(*i*) $A = \{a, b, c, e, f\}$, $B = \{b, d, r, s\}$, then $A \cup B = \{a, b, c, d, e, f, r, s\}$.

(*ii*) $A \cup \phi = A$, for any set A.

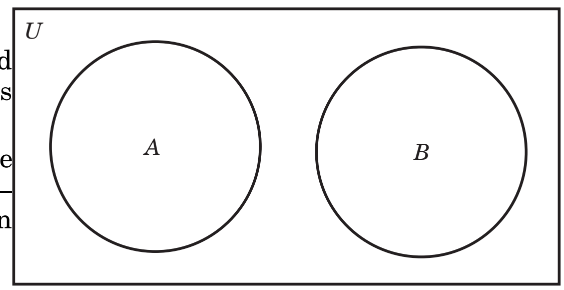

Fig. 1.4

Definition 1.2.3. Two sets A and B are said to be **disjoint** if $A \cap B = \phi$. Fig. 1.4 gives $A \cap B = \phi$.

Definition 1.2.4. Let A and B be sets. The **complement of** A **relative to** B, denoted by B – A, is defined to be the set of elements that are in B and not in A. That is

$$B - A = \{x : x \in B \text{ and } x \notin A\}.$$

If U is the universal set, then the **complement of** A **relative to** U, denoted by A^C is defined to be the set of all elements in U which are not in A. In symbol

$$A^C = \{x : x \in U \text{ and } x \notin A\}$$

Clearly $A^C = U - A$.

Definition 1.2.5. The **cartesian product** of two sets A and B denoted by $A \times B$ is defined to be the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. That is

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

The pairs (a_1, b_1), (a_2, b_2) are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

Examples

(*i*) If $A = \{1, 2, 3\}$, $B = \{1, 2\}$

$$A \times B = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1) (3, 2)\}$$
$$B \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$
$$A \times B \neq B \times A.$$

(*ii*) C the set of complex numbers is the cartesian product $R \times R = \{(x, y) : x, y \in R\}$, where R is the set of real numbers.

We can define the cartesian product of three (or more) sets similarly. For, example $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$, we denote $A \times A = A^2, A \times A \times A = A^3$, and for any positive integer n, $A^n = \{(a_1, a_2, \ldots\ldots\ldots, a_n) : a_i \in A, i = 1 \text{ to } n\}$.

Symmetic Difference : Symmetic Difference of any two sets A and B, denoted by $A \,\Delta\, B$ is $A \,\Delta\, B = (A - B) \cup (B - A)$.

Solved Problems

1. *Let A, B, C be sets such that* $A \cup B = A \cup C$ *and* $A \cap B = A \cap C$, *show that* $B = C$.

Proof. Let $x \in B$. Then $x \in A \cup B$. Since $A \cup B = A \cup C$, $x \in A \cup C$. That is $x \in A$ or $x \in C$. Suppose $x \in A$. Since $x \in B$, $x \in A \cap B$. But $A \cap B = A \cap C$, $x \in A \cap C$. So $x \in A$ and $x \in C$. Therefore $x \in B$ implies $x \in C$. Hence $B \subseteq C$. Similarly we prove $C \subseteq B$. Hence $B = C$.

2. *Let A, B be sets. Then* $A - B = A \cap B^C$.

Proof. Let $x \in A \cap B^C$. Then $x \in A$ and $x \in B^C \Leftrightarrow x \in A$ and $x \notin B \Leftrightarrow x \in A - B$

Since x is an arbitrary element, $A \cap B^C = A - B$.

3. *Show that if A and B are two sets, then* $A - B$, $B - A$, *and* $A \cap B$ *are pairwise disjoint.*

Proof. Let $x \in (A - B) \cap (B - A)$. Then $x \in A - B$ and $x \in B - A$. That is, ($x \in A$ and $x \notin B$) and ($x \in B$ and $x \notin A$). Clearly there is no such element. Hence $(A - B) \cap (B - A) = \phi$. Since $A - B = A \cap B^C$, $(A - B) \cap (A \cap B) = (A \cap B^c) \cap (A \cap B) = (A \cap (B \cap B^C)) = A \cap \phi = \phi$. Similarly $(B - A) \cap (A \cap B) = \phi$.

4. *Show that* $A - (B \cup C) = (A - B) \cap (A - C)$ *and* $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof. Let $x \in A - (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. That is $x \in A$ and ($x \notin B$ and $x \notin C$). Therefore ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$). That is $x \in A - B$ and $x \in A - C$. That is $x \in (A - B) \cap (A - C)$. Hence $A - (B \cup C) \subseteq (A - B) \cap (A - C)$. The other implications can be proved by retracing the steps. Therefore $A - (B \cup C) = (A - B) \cap (A - C)$. The proof of $A - (B \cap C) = (A - B) \cup (A - C)$ is left to the reader.

Note.

(*i*) $x \notin A \cup B$ if and only if $x \notin A$ and $x \notin B$

(*ii*) $x \notin A \cap B$ if and only if $x \notin A$ or $x \notin B$

(*iii*) $x \notin A - B$ if and only if $x \notin A$ or $x \in B$.

5. *For any three sets A, B, C,* $(A \cup B) = B$ *if and only if* $A \subseteq B$.

Proof. Let $A \cup B = B$. Let $x \in A$. Then $x \in A \cup B = B$. Hence $A \subseteq B$. Conversely, let $A \subseteq B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Since $A \subseteq B$, $x \in A$ implies $x \in B$. So $x \in B$. Therefore $A \cup B \subseteq B$. Always $B \subseteq A \cup B$. Hence $A \cup B = B$.

6. *For any three sets A, B, C,*

(*i*) $A - B = A - (A \cap B) = (A \cup B) - B$

(*ii*) $(A - B) - C = A - (B \cup C) = (A - C) - (B - C)$

(*iii*) $A - (B - C) = (A - B) \cup (A \cap C)$

(*iv*) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Proof. (*i*) $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow x \in A$ and $x \notin A \cap B$

$\Leftrightarrow x \in A - (A \cap B)$.

Hence $A - B = A - (A \cap B)$.

(*ii*) $x \in (A - B) - C \Leftrightarrow x \in (A - B)$ and $x \notin C$

$\Leftrightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$

$\Leftrightarrow x \in A$ and ($x \notin B$ and $x \notin C$)

$\Leftrightarrow x \in A$ and ($x \notin B \cup C$)

$\Leftrightarrow x \in A - (B \cup C)$.

Since x is arbitrary, $(A - B) - C = A - (B \cup C)$. Next we prove $A - (B \cup C) = (A - C) - (B - C)$. For, $x \in (A - C) - (B - C) \Leftrightarrow x \in (A - C)$ and $x \notin (B - C)$

$\Leftrightarrow (x \in A \text{ and } x \notin C) \text{ and } (x \notin B \text{ or } x \in C)$
$\Leftrightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$
$\Leftrightarrow x \in A \text{ and } (x \notin B \cup C)$
$\Leftrightarrow x \in A - (B \cup C)$.

Hence
$A - (B \cup C) = (A - C) - (B - C)$.

(*iii*) Let $x \in A - (B - C)$. Then
$x \in A$ and $x \notin B - C \Leftrightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \in C)$
$\Leftrightarrow x \in A - B \text{ or } x \in A \cap C$
$\Leftrightarrow x \in (A - B) \cup (A \cap C)$

Hence $A - (B - C) = (A - B) \cup (A \cap C)$.

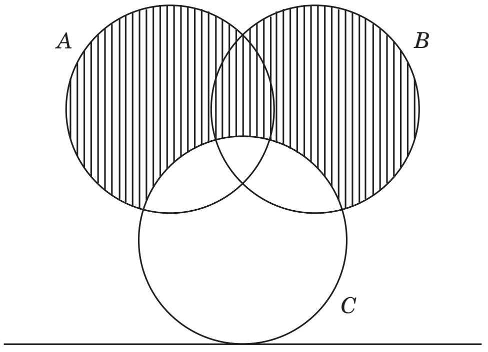

Fig. 1.5

(*iv*) $x \in (A \cap B) - (A \cup C)$
$\Leftrightarrow x \in A \cap B \text{ and } x \notin A \cap C$
$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ and } (x \notin A \text{ or } x \notin C)$
$\Leftrightarrow x \in A \text{ and } (x \in B \text{ and } x \notin C)$
$\Leftrightarrow x \in A \text{ and } x \in B - C$
$\Leftrightarrow x \in A \cap (B - C)$.

Hence, $(A \cap B) - (A \cap C) = A \cap (B - C)$.

7. *Prove by Venn diagram*
$$(A - C) \cup (B - C) = (A \cup B) - C.$$

Proof.

See Fig. 1.5

Region with vertical lines is $(A - C) \cup (B - C)$.

See Fig. 1.6

Region with horizontal lines is $(A \cup B) - C$.

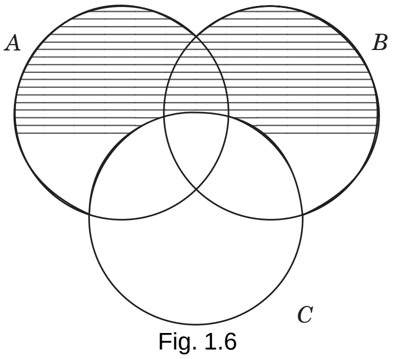

Fig. 1.6

8. *If A, B, C are sets, then*
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$
$\Leftrightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$
$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$
$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

9. *Let A, B, C be sets. Then*
$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof.

$(x, y) \in A \times (B \cap C) \Leftrightarrow x \in A \text{ and } y \in B \cap C$
$\Leftrightarrow x \in A \text{ and } (y \in B \text{ and } y \in C)$
$\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)$
$\Leftrightarrow (x, y) \in A \times B \text{ and } (x, y) \in A \times C$
$\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$.

Hence $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

10. *Let A and B be sets.* $A \subseteq B$ *if and only if* $B^C \subseteq A^C$.

Proof. *Let* $A \subseteq B$. *To show* $B^C \subseteq A^C$. *Let* $x \in B^C$. *Then* $x \notin B$. *Since* $A \subseteq B$, $x \notin A$. *So* $x \in A^C$. *Hence* $B^C \subseteq A^C$. *Conversely, let* $B^C \subseteq A^C$. *Let* $x \in A$. *That is* $x \notin A^C$, *since* $B^C \subseteq A^C$, $x \notin B^C$. *Hence* $x \in B$, *and hence* $A \subseteq B$.

Exercise 1.2.

1. Let
$$U = \{x : x \in Z,\quad 1 \le x \le 10\}$$
$$A = \{x : x \in U \text{ and } x \text{ is a prime number}\}$$
$$B = \{x : x \in U \text{ and } x \text{ is even}\}$$
$$C = \{x : x \in U \text{ and } x \text{ is a multiple of } 3\}$$

Find

(a) $A \cup B$ (b) $B \cup C$ (c) $B \cap C$

(d) $A \cap C$ (e) $A \cup B^C$ (f) $B^C \cap C^C$

(g) $A^C \cup (B \cap C)$ (h) $A - B$ (i) $B - A$

(j) $(B \cup C) - A$

2. Prove the following :

(i) $U^C = \phi$ and $\phi^C = U$ (ii) $A \cap A^C = \phi$ and $A \cup A^C = U$

(iii) $(A^C)^C = A$ (iv) If $A \subseteq B$, then $A \times C \subseteq B \times C$

(v) $A \cap B = \phi$ if and only if $A \subseteq B^C$ (vi) $(A \cap B^C) \cup (A^C \cap B) = \phi$ if and only if $A = B$.

3. If $A = \{1, 2, 3\}$, and $B = \{2, 3, 4\}$ write down the following sets

(i) $\{(x, y) : (x, y) \in A \times B \text{ and } x < y\}$

(ii) $\{(x, y) : (x, y) \in A \times B \text{ and } x \ge y\}$

4. Show that $A \times B = B \times A \Leftrightarrow A = \phi$ or $B = \phi$ or $A = B$.

5. Prove : (i) $A \cap B = A$ if and only if $A \subseteq B$.

(ii) $A \cap B = \phi$ if and only if $A - B = A$

(iii) $A \subseteq B$ if and only if $A - B = \phi$

(iv) $A = B$ if and only $A - B = B - A = \phi$.

6. Simplify (i) $((A \cap B) \cup C)^C \cap B^C)^C$

(ii) $((A \cup B) \cap A^C) \cup (B \cap A)^C$.

7. If $A = \{a, b\}$, $B = \{c, d, e\}$ and $C = \{00, 01, 10, 11\}$

(i) List all the elements of $A \times (B \times C)$

(ii) How many elements do $A \times (B \times C)^2$ have.

1.3 LAWS OF SET THEORY

1.	$A \cup B = B \cup A$	Commutative Laws
	$A \cap B = B \cap A$	
2.	$A \cup (B \cup C) = (A \cup B) \cup C$	
	$A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
3.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
4.	$A \cup \phi = A = \phi \cup A$	
	$A \cap U = A = U \cap A$	Identity Laws

5. $A \cup A^C = U$
   $A \cap A^C = \phi$ — Complement Laws
6. $A \cup A = A$
   $A \cap A = A$ — Impotent Laws
7. $A \cup U = U$
   $A \cap \phi = \phi$ — Null Laws
8. $A \cup (A \cap B) = A$
   $A \cap (A \cup B) = A$ — Absorption Laws
9. $(A \cup B)^C = A^C \cap B^C$
   $(A \cap B)^C = A^C \cup B^C$ — De Morgans Laws
10. $(A^C)^C = A$ — Involution law

Theorem 1.3.1. *Let A, B be sets. Then*

$$(A \cap B) \cup (A \cap B^C) = A$$

Proof. $(A \cap B) \cup (A \cap B^C) = A \cap (B \cup B^C)$ by distributive law

$= A \cap \phi$ by complement law

$= A$ by identity law.

Theorem 1.3.2. *Let A, B, C be sets. If* $A \subseteq B$, $B \cap C = \phi$. Then $A \cap C = \phi$.

Proof. Let $A \subseteq B$, $B \cap C = \phi$. Let $A \cap C \neq \phi$. Suppose $x \in A \cap C$. Then $x \in A$ and $x \in C$. Since $A \subseteq B$, $x \in A$ implies $x \in B$. Hence $x \in B \cap C$, which is a contradiction to the given hypothesis. Hence $A \cap C = \phi$.

1.4 PRINCIPLE OF INCLUSION AND EXCLUSION

Theorem 1.4.1. (*Principle of inclusion and exclusion*).

Let A_1 and A_2 *be sets with cardinalities* $|A_1|$ and $|A_2|$, *then*

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Proof. The sets A_1 and A_2 have some common elements. The number of common elements between A_1 and A_2 is $|A_1 \cap A_2|$. Each of these elements are counted twice in $|A_1| + |A_2|$ (Once in $|A_1|$ and once in $|A_2|$) although it should be counted as one element in $|A_1 \cup A_2|$. Therefore, the double count of these elements in $|A_1| + |A_2|$ should be adjusted by the subtraction of the term $|A_1 \cap A_2|$. Hence $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

Note. Extending the above result for three sets A_1, A_2, A_3, by using distributive law, we obtain,

$$\begin{aligned}
|A_1 \cup A_2 \cup A_3| &= |(A_1 \cup A_2) \cup A_3| \\
&= |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3| \\
&= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - |(A_1 \cap A_3) \cup (A_2 \cap A_3)| \\
&= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| \\
&\quad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|
\end{aligned}$$

We extend this result to a collection of n sets, for any +ve integer n.

Theorem 1.4.2. *Let* $A_1, A_2, \ldots, A_n$ *be a collection of n sets. Then*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1}$$
$$|A_1 \cap A_2 \cap \ldots \cap A_n| \qquad \ldots(1.4.1)$$

Proof. We prove the theorem by induction on n. First we assume $(A_1 \cup A_2 \cup \ldots \cup A_{n-1})$ and A_n as two sets. Then by Theorem 1.4.1, we have

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1 \cup A_2 \cup \ldots \cup A_{n-1}| + |A_n| - |A_n \cap (A_1 \cup A_2 \cup \ldots \cup A_{n-1}| \qquad \ldots(1.4.2)$$

Now

$$|A_n \cap (A_1 \cup A_2 \cup \ldots \cup A_{n-1})| = |(A_n \cap A_1) \cup (A_n \cap A_2) \cup \ldots \cup (A_n \cap A_{n-1})|$$

According to the induction hypothesis, for $n - 1$ sets $A_n \cap A_1, A_n \cap A_2, \ldots, A_n \cap A_{n-1}$, we have,

$$|(A_n \cap A_1) \cup (A_n \cap A_2) \cup \ldots\ldots (A_n \cap A_{n-1})|$$
$$= |A_n \cap A_1| + |A_n \cap A_2| + \ldots + |A_n \cap A_{n-1}|$$
$$- |(A_n \cap A_1) \cap (A_n \cap A_2)| - |(A_n \cap A_1) \cap (A_n \cap A_3)|$$
$$\ldots\ldots\ldots$$
$$+ |(A_n \cap A_1) \cap (A_n \cap A_2) \cap (A_n \cap A_3)|$$
$$\ldots\ldots\ldots\ldots$$
$$+ (-1)^{n-2} | (A_n \cap A_1) \cap (A_n \cap A_2) \ldots (A_n \cap A_{n-1})|$$
$$= |A_n \cap A_1| + |A_n \cap A_2| + \ldots + |A_n \cap A_{n-1}|$$
$$- |A_n \cap A_1 \cap A_2| - |A_n \cap A_1 \cap A_3| - \ldots.$$
$$+ |A_n \cap A_1 \cap A_2 \cap A_3| + \ldots.$$
$$+ (-1)^{n-2} |A_n \cap A_1 \cap A_2 \cap \ldots \cap A_{n-1}| \qquad \ldots(1.4.3)$$

Also, according to the induction hypothesis, for $n - 1$ sets, $A_1, A_2, \ldots A_{n-1}$, we have

$$|A_1 \cup A_2 \cup \ldots. \cup A_{n-1}| = |A_1| + |A_2| + \ldots + |A_{n-1}|$$
$$- |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots + (-1)^{n-2} |A_1 \cap A_2 \cap \ldots \cap A_{n-1}| \qquad \ldots(1.4.4)$$

Substituting (1.4.3) and (1.4.4) in (1.4.2), we get (1.4.1).

Example 1.4.3. *In an advertising survey conducted on 200 people, it was found that 140 drink tea, 80 drink coffee, and 40 drink both. Find how many drink atleast one beverage and how many drink neither* ?

Solution. Let A denote the set of tea drinkers, and B the set of coffee drinkers. Then

$|A \cup B| = |A| + |B| - |A \cap B| = 140 + 80 - 40 = 180.$

Hence 180 peoples drink atleast one beverage and 200 – 180 = 20 drink neither.

Note. We use $[x]$ to denote the largest integer that is smaller than or equal to x. So the number of multiples of x that do not exceed n is $[n/x]$. (See also problem 17 of page 296)

Example 1.4.4. *Find how many integer between 1 and 567 are divisible by 3, but not by 5.*

Solution. Integers between 1 and 567 divisible by 3 is $\left[\dfrac{567}{3}\right] = 189$. Integers between 1 and 567 divisible by both 3 and 5 is $\left[\dfrac{567}{3 \times 5}\right] = 37$. Hence, the number of integers between 1 and 567 divisible by 3 but not by 5 is 189 – 37 = 152.

Example 1.4.5. *How many integers are between 1 and 250 that are divisible by any of the integers 2, 3, 5 and 7.*

Solution. Let A_1, A_2, A_3, A_4 denote the set of integers between 1 and 250 that are divisible by 2, 3, 5 and 7, respectively.

$$|A_1| = \left[\frac{250}{2}\right] = 125 \qquad |A_1 \cap A_2| = \left[\frac{250}{2 \times 3}\right] = 41$$

$$|A_2| = \left[\frac{250}{3}\right] = 83 \qquad |A_1 \cap A_3| = \left[\frac{250}{2 \times 5}\right] = 25$$

$$|A_3| = \left[\frac{250}{5}\right] = 50 \qquad |A_1 \cap A_4| = \left[\frac{250}{2 \times 7}\right] = 17$$

$$|A_4| = \left[\frac{250}{7}\right] = 35 \qquad |A_2 \cap A_3| = \left[\frac{250}{3 \times 5}\right] = 16$$

$$|A_2 \cap A_4| = \left[\frac{250}{3 \times 7}\right] = 11, \quad |A_3 \cap A_4| = \left[\frac{250}{5 \times 7}\right] = 7$$

$$|A_1 \cap A_2 \cap A_3| = \left[\frac{250}{2 \times 3 \times 5}\right] = 8$$

$$|A_1 \cap A_2 \cap A_4| = \left[\frac{250}{3 \times 5 \times 7}\right] = 5$$

$$|A_1 \cap A_3 \cap A_4| = \left[\frac{250}{2 \times 5 \times 7}\right] = 3$$

$$|A_2 \cap A_3 \cap A_4| = \left[\frac{250}{3 \times 5 \times 7}\right] = 2$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = \left[\frac{250}{2 \times 3 \times 5 \times 7}\right] = 1$$

By Theorem 1.4.2,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1 = 193$$

Example 1.4.6. *How many words of length 7 can be formed from the letters {a, b, c, d, e} if the first letter must be 'a' or the last letter be 'b'.*

Solution. There are 5^6 words that begins with a, there are 5^6 words that end with b, and there are 5^5 words that both begin with 'a' and end with 'b'. By principle of inclusion and exclusion, there are $5^6 + 5^6 - 5^5 = 28,125$ words that begin with a or end with b.

Applications of Inclusion Exclusion

We can solve many problems using principle of inclusion - exclusion. The famous hatcheck problem can be solved using the principle of inclusion exclusion. This problem asks for the probability that no person is given the correct hat back by a hatcheck person who gives the hats back randomly.

There is an alternative form of the principle of inclusion-exclusion that is useful in counting problems.

This form is used to solve problems that is asked to find the number of elements in a set that have none of n properties $P_1, P_2, ... P_n$.

Let A_i be the subset containing the elements that have property P_i. The number of elements with all the properties $P_{i_1}, P_{i_2}, ... P_{i_k}$ are denoted by $N(P_{i_1}, P_{i_2}, ... P_{i_k})$. Note that $|A_{i_1} \cap A_{i_2} \cap ..... \cap A_{i_k}| = N(P_{i_1} P_{i_2} .... P_{i_k})$. If the number of elements with none of the properties $P_1, P_2, ... P_n$ is denoted by $N(P'_1, P'_2 ..... P'_n)$ and the number of elements in the set is N (say), then

$$N(P'_1 P'_2 P'_3 \; ..... \; P'_n) = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j)$$
$$- \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + ... + (-1)^n N(P_1 P_2 \; .... \; P_n)$$

Example 1.4.7. *How many solution does $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, x_3 are non-negative integers with $x_1 \le 3$, $x_2 \le 4$, $x_3 \le 6$?*

Solution. Let $P_1 : x_1 > 3$; $P_2 : x_2 > 4$; $P_3 : x_3 > 6$. The number of solutions satisfying the inequalities $x_1 \le 3$, $x_2 \le 4$, $x_3 \le 6$ is :

$$N(P'_1 P'_2 P'_3) = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2)$$
$$+ N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

Using the technique of Example 1.8.18, we have

N = total number of solutions $= C(3 + 11 - 1, 11) = 78$

$N(P_1)$ = (number of solutions with $x_1 \ge 4$) $= C(3 + 7 - 1, 7) = C(9, 7) = 36$

$N(P_2)$ = (number of solutions with $x_2 \ge 5$) $= C(3 + 6 - 1, 6) = C(8, 6) = 28$

$N(P_3)$ = (number of solutions with $x_3 \ge 7$) $= C(3 + 4 - 1, 4) = C(6, 4) = 15$

$N(P_1 P_2)$ = (number of solutions with $x_1 \ge 4$, $x_2 \ge 5$) $= C(3 + 2 - 1, 2) = C(4, 2) = 6$

$N(P_1 P_3)$ = (number of solutions with $x_1 \ge 4$, $x_3 \ge 7$) $= C(3 + 0 - 1, 0) = 1$

$N(P_2 P_3)$ = (number of solutions with $x_2 \ge 5$, $x_3 \ge 7$) $= 0$

$N(P_1 P_2 P_3)$ = (number of solutions with $x_1 \ge 4$, $x_2 \ge 5$, $x_3 \ge 7$) $= 0$

Hence $N(P'_1 P'_2 P'_3) = 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6$.

The sieve of Eratosthenes 1.4.8

The principle of inclusion and exclusion is used to find the number of primes not exceeding a specified positive integer. To find the *number of primes not exceeding* 100, first note that composite integers not exceeding 100 must have a prime factor not exceeding 10. Because the only primes less than 10 are 2, 3, 5, 7, the primes not exceeding 100 are these four primes and those positive integers greater than 1 and not exceeding 100 that are divisible by none of 2, 3, 5, 7.

Let P_1 : An integer is divisible by 2

P_2 : An integer is divisible by 3

P_3 : An integer is divisible by 5

P_4 : An integer is divisible by 7

Then by principle of inclusion-exclusion, the number of primes not exceeding 100 is $4 + N(P'_1P'_2P'_3P'_4)$

Since there are 99 positive integers greater than 1, not exceeding 100, we have

$$N(P'_1P'_2P'_3P'_4) = 99 - N(P_1) - N(P_2) - N(P_3) - N(P_4)$$
$$+ N(P_1 P_2) + N(P_1 P_3) + N(P_1 P_4) + N(P_2 P_3) + N(P_2 P_4) + N(P_3 P_4)$$
$$- N(P_1 P_2 P_3) - N(P_1 P_2 P_4) - N(P_1 P_3 P_4) - N(P_2 P_3 P_4) + N(P_1 P_2 P_3 P_4).$$

We know the number of integers not exceeding 100 (greater than 1) that are divisible by the primes in a subset of {2, 3, 5, 7} is $\left[\frac{100}{N}\right]$, where N is the product of primes in this subset. Hence

$$N(P'_1\, P'_2\, P'_3\, P'_4) = 99 - \left[\frac{100}{2}\right] - \left[\frac{100}{3}\right] - \left[\frac{100}{5}\right] -$$

$$\left[\frac{100}{7}\right] + \left[\frac{100}{2\times 3}\right] + \left[\frac{100}{2\times 5}\right] + \left[\frac{100}{2\times 7}\right] + \left[\frac{100}{3\times 5}\right] + \left[\frac{100}{3\times 7}\right] + \left[\frac{100}{5\times 7}\right]$$

$$-\left[\frac{100}{2\times 3\times 5}\right] - \left[\frac{100}{2\times 3\times 7}\right] - \left[\frac{100}{2\times 5\times 7}\right] - \left[\frac{100}{3\times 5\times 7}\right] + \left[\frac{100}{2\times 3\times 5\times 7}\right]$$

$$= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0$$

$$= 21$$

Hence, there are 4 + 21 = 26 primes not exceeding 100.

The principle of inclusion - exclusion is also used to find the number of onto functions from a set with m elements to a set with n elements.

Example 1.4.9 *How many onto functions are there from a set with six elements to a set with three elements.*

Solution. Let codomain have 3 elements say b_1, b_2, b_3. Let P_1, P_2, P_3 be the properties that b_1, b_2, b_3 are not in the range of the function. Note that a function is onto iff it has none of the properties. P_1, P_2 or P_3. A function corresponds to a choice of one of the 3 elements in the codomain for each of the 6 elements in the domain. Hence, by the product rule there 3.3.3.3.3.3 $= 3^6$ functions (by permutation technique). So $N = 3^6$. Note that $N(P_i)$ is the number of functions that do not have b_i in their range. Hence there are two choices for the value of the function at element of the domain. Therefore $N(P_i) = 2^6$. Further there are $C(3, 1)$ terms of this kind. Similarly $N(P_iP_j)$ is the number of functions that do not have b_i and b_j in their range. So $N(P_i\, P_j) = 1^6 = 1$. Further there are $C(3, 2)$ terms of this kind. Note that $N(P_1\, P_2\, P_3) = 0$, since this term is the number of functions that have none of b_1, b_2, b_3 in their range. Therefore, the number of onto functions from a set with six elements to one with three elements is

$3^6 - C(3, 1)\, 2^6 + C(3, 2)\, 1^6 = 729 - 192 + 3 = 540.$

Note 1. Let m and n be positive integers with $m \geq n$. Then, there are

$n^m - C(n, 1)\,(n-1)^m + C(n, 2)\,(n-2)^m - .... + (-1)^{n-1}\, C(n, n-1)\, 1^m = n\,!\, S_2(m, n)$

onto functions from a set with m elements to a set with n elements. Here $S_2(m, n)$ is a *stirling number of the second kind*.

Note 2. Number of assignment of five different jobs to four different employees, if every employee is assigned atleast one job is

$4^5 - C(4, 1)3^5 + C(4, 2)2^5 - C(4, 3)1^5 = 1024 - 972 + 192 - 4 = 240$ ways

Example 1.4.10. Hatcheck Problem

A new employee checks the hats of n people at a restaurant, forgetting to put claim check number on the hats. When customers return for their hats, the checkers gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat ?

Let us identify the hats with the integers 1 to n. Each way of giving back the hats is a permutation of $[n]$. For instance, the permutation 1, 2, 3 ... n represents the case that each person gets back their own hat. Let π be a permutation of $[n]$. If for all i, we have $\pi(i) \neq i$, then nobody gets their own hat back. We call such permutations *derangements*.

Then the probability asked is $\dfrac{\text{number of derangements of } [n]}{n!}$

Hence our goal is to compute the number of derangements of $[n]$; we denote the set of derangements by D_n and its cardinality by d_n.

Let S_n be the set of the $n!$ permutations of n elements and for each i with $1 \le i \le n$; let A_i be the subset of all permutations π such that $\pi(i) = i$. We have

$$D_n = A_1^c \cap A_2^c \cap \ldots \cap A_n^c = S_n - (A_1 \cup A_2 \cup \ldots \cup A_n).$$

Hence,
$$d_n = n! - |A_1 \cup A_2 \cup \ldots \cup A_n|,$$

by the principle of inclusion exclusion,

$$d_n = n! - \sum_{i=1}^{n} |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^n |A_1 \cap \ldots \cap A_n|.$$

The only thing left now is to count the sizes of the intersections.

- $|A_i| = |\{\pi \in S_n \mid \pi(i) = i\}| = (n-1)!$, since the element i is fixed and we can permute the remaining $n - 1$ in any way. Note that this is independent of the element i.
- $|A_i \cap A_j| = |\{\pi \in S_n \mid \pi(i) = i, \pi(j) = j\}| = (n-2)!$, since two elements are fixed and the remaining $n - 2$ can be permuted arbitrarily. Again, the result is the same for all pairs i, j.

In general,

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = (n-k)!, \text{ for } 1 \le i_1 < i_2 < \ldots < i_k \le n.$$

Therefore,
$$d_n = n! - \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Hence, the probability asked is $\sum_{i=0}^{n} \frac{(-1)^i}{i!}$. Does this probability have a limit as n tends to infinite? Recall the series expansion for the exponential function $e^x = \sum_{i \ge 0} \frac{x^i}{i!}$; this series converges for all real values of x. Hence, as $n \to \infty$, the probability that nobody gets their own hats gets closer to $e^{-1} \sim 0.37$; actually, the rate of convergence is really fast, since the absolute error is bounded by $1/(n+1)!$.

Example 1.4.11. Euler's ϕ function. Given a positive integer n, $\phi(n)$ is the number of integers smaller than n that are relatively prime to n (including 1). For instance.

$$\phi(2) = |\{1\}| = 1, \quad \phi(3) = |\{1, 2\}| = 2, \quad \phi(4) = |\{1, 3\}| = 2, \quad \phi(5) = |\{1, 2, 3, 4\}| = 4.$$

Note that if n is prime, then $\phi(n) = n - 1$, since all integers smaller than n are relatively prime to n. Our goal is to find a formula for $\phi(n)$ for any integer n. We assume that we have the decomposition of n into prime factors.

$$n = \prod_{i=1}^{r} p_i^{\alpha_i},$$

where r is the number of distinct prime factors of n, the p_i are the distinct prime factors, and α_i stands for their multiplicities.

The integers that are relatively prime with n are those that do not contain any of the p_i as a factor. This suggests to define $B_i = \{m : m < n, p_i \mid m\}$, that is, the set of integers smaller than n that are divisible by p_i.

Hence,
$$\phi(n) = |B_1^c \cap B_2^c \cap \ldots \cap B_n^c|.$$

By inclusion-exclusion,

$$\phi(n) = n - \sum_{i=1}^{r} |B_i| + \sum_{1 \le i < j \le r} |B_i \cap B_j| - \sum_{1 \le i < j < k \le r} |B_i \cap B_j \cap B_k| + \ldots + (-1)^r |B_1 \cap \ldots \cap B_r|.$$

Again, the problem reduces to computing the intersections of B_i's. It is not difficult to show that

$$|B_{i_1} \cap \ldots \cap B_{i_k}| = |\{m : m < n, p_{i_1} \ldots p_{i_k} | m\}| = \frac{n}{p_{i_1} \ldots p_{i_k}}.$$

Note that in this case the size of $B_{i_1} \cap \ldots \cap B_{i_k}$ not only depends on k but also on the specific sets we intersect. Putting this into the formula given by inclusion-exclusion, we have

$$\phi(n) = n - n\left(\sum_{1 \le i \le r} \frac{1}{p_i}\right) + n\left(\sum_{1 \le i < j \le r} \frac{1}{p_i p_j}\right) + \ldots + (-1)^r n \frac{1}{p_1 \ldots p_r},$$

which can be written more compactly as

$$\phi(n) = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_r}\right).$$

Example 1.4.12. *Five gentlemen A, B, C, D, E attend a party, where before joining the party, they leave their over coats in a clock room. After the party, the overcoats get mixed up and are returned to the gentlemen in a random manner. Using principles of inclusion and exclusion, find the probability that none receives his own coat.*

Solution. The probability is $\frac{D_5}{5!}$ where

$$D_5 = 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right] = 5!\ [0.3667]$$

So $$\frac{D_5}{5!} = 0.3667$$

Example 1.4.13. *Find the number of r-digit quintary sequences (made up to digits 0, 1, 2, 3, 4) that contain at least a 0, a1, and a2 by using principle of inclusion and exclusion.*

Solution. Let N be the number of r-digit quintary sequences. Then $N = 4^r$. Let P_1, P_2, P_3 respectively denote the number of r-digit quintary sequences that contain no 0's, 1's and no 2's. Then the number of r-digits quintary sequences that contain atleast a0, a1, and a2 is

$$\begin{aligned} N(P'_1P'_2P'_3) &= N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) \\ &\quad + N(P_1 P_3) + N(P_2 P_3) - N(P_1 P_2 P_3) \\ &= 4^r - 3^r - 3^r - 3^r + 2^r + 2^r + 2^r - 1 = 4^r - (3 \cdot 3^r) + 3.2^r - 1 \end{aligned}$$

Example 1.4.14. *A student wants to make up a schedule for a seven day period during which she will study one subject each day. She is taking four subjects : Mathematics, Economics, Physics, Chemistry. Find the number of schedules that devote atleast one day to each subject.*

Solution. There are 4^7 different schedules. Let A_1, A_2, A_3, A_4 denote the set of schedules in which Mathematics, Economics, Physics, Chemistry respectively is not included. Then $A_1 \cup A_2 \cup A_3 \cup A_4$ is the set of schedules in which one or more subjects is not included. Now

$$|A_1| = |A_2| = |A_3| = |A_4| = 3^7$$

$$|A_1 \cap A_2| = |A_1 \cap A_3| = \ldots = |A_3 \cap A_4| = 2^7$$

$$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = 1^7$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$$

Hence,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 4(3^7) - 6\ (2^7) + 4.$$

Hence schedules in which all subjects will be included = $4^7 - (4\ (3^7)) - 6(2^7) + 4)$

Exercise 1.4

1. Out of a total of 130 students, 60 are wearing hats to class, 51 are wearing scarves and 30 are wearing both hats and scarves. Out of 54 students who are wearing sweaters, 26 are wearing hats, 21 are wearing scarves, and 12 are wearing both hats and scarves. Every one wearing neither a hat nor a scarf is wearing gloves.

(*a*) How many students are wearing gloves ?

(*b*) How many students not wearing a sweater are wearing hats but not scarves ?

(*c*) How many students not wearing a sweater are wearing neither a hat nor a scarf.

2. Of a graduating class of 153 computer science branch students, 63 have taken English composition (EC), 54 have taken Technical writing (TW),62 have taken Report writing (RW), 43 have taken EC and TW, 46 have taken EC and RW, 45 have taken TW and RW and 37 have taken all the three courses. How many of the students have not taken any one of these three courses ? Use a Venn diagram and obtain the result.

3. Two programs are running simultaneously. First requires 1200 bytes of memory, second requires 900 bytes. If 200 bytes can be shared what is the total number of bytes that must be allowed to run both the programs.

1.5 PARTITIONS

A *partition* or quotient set of a nonempty set A is a collection P of nonempty subsets $A_1, A_2, \ldots,$ of A such that

(*i*) $A_1 \cup A_2 \cup \ldots = A$ (*ii*) $A_i \cap A_j = \phi$ for $i \neq j$

The sets in P are called the blocks or cells of the partition.

Example 1.5.1. *Let* $A = \{a, b, c\}$. *Find all the partitions of* A.

Solution . $\{\{a, b\}, \{c\}\}, \{\{a\}, \{b\}, \{c\}\}, \{\{a\}, \{b, c\}\}, \{\{a, c\}, \{b\}\}$ are the possible partitions of A.

Example 1.5.2. Let Z = set of all integers, Z_+ = set of all even integers, Z_- = set of all odd integers, then. $\{Z_-, Z_+\}$ is a partition of Z.

Example 1.5.3. Let A = {a, b, c, d, e, f, g, h}. Consider the following subsets of A.

$A_1 = \{a, b, c, d\}$ $A_2 = \{a, c, e, f, g, h\}$

$A_3 = \{a, c, e, g\}$ $A_4 = \{b, d\}$, $A_5 = \{f, h\}$.

Then $\{A_1, A_4\}$ is not a partition, since $A_1 \cap A_2 \neq \phi$. $\{A_1, A_5\}$ is not a partition, since $e \notin A_1$ and $e \notin A_5$. The collection $P = \{A_3, A_4, A_5\}$ is a partition of A.

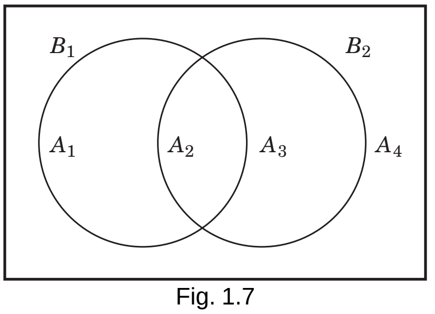

Fig. 1.7

Theorem 1.5.4. Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of a set A and B be any nonempty subset of A. Then $\{A_i \cap B\ :\ A_i \cap B \neq \phi\}$ is a partition of $A \cap B$.

Proof : By definition of partition $A_1 \cup A_2 \cup \ldots \cup A_n = A$ and $A_i \cap A_j = \phi$ for $i \neq j$. Let B be any subset of A. Since $A_1, A_2, \ldots A_n$ are subsets of A, $A_1 \cap B, A_2 \cap B, \ldots, A_n \cap B$ are subsets of $A \cap B$. By applying distributive law repeatedly, we get

$(A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B) = (A_1 \cup A_2 \cup \ldots \cup A_n) \cap B = (A \cap B)$ and

$(A_i \cap B) \cap (A_j \cap B) = A_i \cap B \cap A_j \cap B = (A_i \cap A_j) \cap B = \phi \cap B = \phi,$

since for $i \neq j$, $A_j \cap A_j = \phi$, Hence $\{A_i \cap B \;:\; A_i \cap B \neq \phi\}$ is a partition of $A \cap B$.

Exercise 1.5.

1. Find all the partitions of the set $A = \{0, 1, 2\}$.

1.6 MINSETS

Let B_1 and B_2 be subsets of a set A. Consider the Venn diagram (Fig. 1.7).

Let the sets A_1, A_2, A_3, A_4 be described with B_1 and B_2 as follows.

$$A_1 = B_1 \cap B_2^C \qquad A_2 = B_1 \cap B_2$$
$$A_3 = B_1^C \cap B_2 \qquad A_4 = B_1^C \cap B_2^C$$

Each of the $A'_i\, s$ is called a *minset* or *minterm generated* by B_1 and B_2.

Note that for given two sets B_1 and B_2 there are 2^2 minsets. If B_1, B_2, B_3 are three sets given, there are 2^3 minsets. In general for subsets $B_1, B_2, \ldots, B_n$ of A, there are 2^n minsets. Generally, we define minset as :

Definition 1.6.1. Let A be a set. Let $\{B_1, B_2, \ldots, B_n\}$ be set of subsets of A. A set of the form $D_1 \cap D_2 \cap \ldots \cap D_n$, where each D_i may be either B_i or B_i^C is called a *minset* or *minterm generated* by $B_1, B_2, \ldots, B_n$.

Example 1.6.2. *Let A ={1, 2, 3, 4, 5, 6, 7, 8, 9}. Find the minsets generated by*

$$B_1 = \{5, 6, 7\},\ B_2 = \{2, 4, 5, 9\},$$
$$B_3 = \{3, 4, 5, 6, 8, 9\}.$$

Solution : $B_1^C \cap B_2^C \cap B_3^C = \{1\}$, $B_1 \cap B_2 \cap B_3 = \{5\}$,

$B_1 \cap B_2^C \cap B_3 = \{6\}$, $\quad B_1^C \cap B_2 \cap B_3 = \{4, 9\}$,

$$B_1^C \cap B_2^C \cap B_3 = \{3, 8\},\ B_1 \cap B_2^C \cap B_3^C = \{7\},$$
$$B_1^C \cap B_2 \cap B_3^C = \{2\},\ \ B_1 \cap B_2 \cap B_3^C = \varphi$$

The above sets are the minsets generated by B_1, B_2, B_3

Example 1.6.3. *Consider the following set*

$A = \{1, 2, 3, 4, 5, 6\}$ *with subsets* $B_1 = \{1, 3, 5\}$, $B_2 = \{1, 2, 3\}$. *Then find the minsets generated by* B_1 and B_2.

Solution. Note that

$$B_1 \cap B_2 = \{1, 3\}$$
$$B_1^C = \{2, 4, 6\},\ B_2^C = \{4, 5, 6\}$$
$$B_1 \cap B_2^C = \{5\}$$
$$B_1^C \cap B_2 = \{2\}$$
$$B_1^C \cap B_2^C = \{4, 6\}$$

Then $B_1 \cap B_2$, $B_1 \cap B_2^C$, $B_1^C \cap B_2$, $B_1^C \cap B_2^C$ are the minsets generated by B_1 and B_2. Since the minsets are mutually disjoint and their union is the given set A, the minsets forms a partition of A.

Theorem 1.6.4. Let A be a set and let $B_1, B_2, \ldots., B_n$ be subsets of A. The set of nonempty minsets generated by $B_1, B_2, \ldots, B_n$ is a partition of A.

Proof follows by Theorem 1.5.4.

Definition 1.6.5. A set is said to be *minset normal (or canonical) form*, when it is expressed as the union of distinct nonempty minsets or it is empty.

1.7 DUALITY PRINCIPLE FOR SETS

Let S be any identity involving sets and operations - complement 'C', intersection $\cap$, and union $\cup$. If S^* is obtained from S by making the substitution $\cap$ for $\cup$, $\cup$ for $\cap$, ϕ for $\boldsymbol{U}$, $\boldsymbol{U}$ for ϕ, then the statement S^* is true, and is called the dual of the statement S.

Example. The dual of $A \cup (B \cap A) = A$ is $A \cap (B \cup A) = A$.

The dual of minset is maxset which is defined as

Definition 1.7.1 Let $\{B_1, B_2, ..., B_n\}$ be a set of subsets of a set A. A set of the form $D_1 \cup D_2 \cup \ldots \cup D_n$, where each D_i may be either B_i or B_i^C is called a *maxset generated* by $B_1, B_2, \ldots, B_n$.

Note that a set is said to be in max set normal form when it is expressed as the intersection of distinct nonempty maxsets or it is empty.

Example 1.7.2. *Let $A = \{1, 2, 3, 4, 5, 6,\}$. Let $B_1 = \{1, 3, 5\}$, $B_2 = \{1, 2, 3\}$. Find the max sets generated by B_1 and B_2.*

Solution. Note that $B_1^C = \{2, 4, 6\}$, $B_2^C = \{4, 5, 6\}$

$$B_1 \cup B_2^C = \{1, 3, 4, 5, 6\},\ B_1^C \cup B_2 = \{1, 2, 3, 4, 6\}.$$

$$B_1 \cup B_2 = \{1, 2, 3, 5\},\ B_1^C \cup B_2^C = \{2, 4, 5, 6\}$$

Hence $B_1 \cup B_2^C$, $B_1 \cup B_2$, $B_1^C \cup B_2$, $B_1^C \cup B_2^C$ are the maxsets generated by B_1 and B_2. Note that the set of maxsets does not constitute a partition of A.

Exercise 1.7

1. State the dual of

(a) $A \cup (B \cap A)^C = A$

(b) $(A \cup B^C)^C \cap B = A^C \cup B$

2. Let $A = \{ 1, 2, 3, ..., 9 \}$ and let $B_1 = \{5, 6, 7\}$, $B_2 = \{2, 4, 5, 9\}$, $B_3 = \{3, 4, 5, 6, 8, 9\}$.

(i) Find all max sets generated by B_1, B_2, B_3.

(ii) Illustrate via a Venn diagram all minsets obtained in part A.

(iii) Express the following sets in maxset normal form B_1^C, $B_1 \cup B_2$, $B_2^C \cup B_2^C$.

(iv) Find the dual of (i), (ii) and (iii).

3. Is the dual of Theorem 1.5.4 true ? Why ?

1.8 PERMUTATION AND COMBINATION (Generalized)

Permutation : Let A be a set. A permutation of A is an ordering of the elements of A.

Example 1.8.1. *Suppose there are r distinct colored balls together with n distinct numbered boxes are given, where a box can hold only one ball. In how many distinct ways we place the r balls in n distinct boxes.*

Solution. The first ball can be placed in any one of the n boxes, the second ball can be placed in any one of the remaining $(n - 1)$ boxes, the third ball can be placed in any one of the remaining $(n - 2)$ boxes, ... r^{th} ball can be placed in any one of the $(n - r + 1)$ boxes. So the total number of balls is given by rule of product as $n\ (n - 1)\ (n - 2)$ .... $(n - r + 1)$. For ready reference we give Rule of sum and Rule of product.

Rule of sum. If an event can occur in m ways and another event can occur in n ways there are $m + n$ ways in which exactly one event can occur.

Rule of product. If there are m outcomes for event E_1 and n possible outcomes for event E_2, then there are mn outcomes for the composite event $E_1\,E_2$. If there are m_i outcomes of an event E_i, $i = 1$ to k, then there are $m_1 \times m_2 \times \ldots \times m_k$, outcomes for a designated order of k events.

Remark 1.8.2. If A has n elements then the number of possible permutations of r elements taken from A is

$$P\,(n\,;\,r) = n\,(n-1)\,(n-2)\,\ldots\,(n-r+1) = \frac{n!}{(n-r)!}$$

Example 1.8.3. *Find the number of three letter words using the given six letters with out repeating any letter in a given word.*

Solution. The first letter can be chosen in six different ways and the second letter can be chosen in five different ways, the last letter can be chosen in four different ways. So

$$P\,(6\,;\,3) = 6\,.\,5\,.\,4 = 120$$

There are 120 three letter words possible for the given six letters.

Example 1.8.4. *In how many ways can 5 boys and 5 girls be seated around a table so that no boys are adjacent.*

Solution. We fix one seat for a boy, the remaining boys can sit in 4! ways on the alternate seats. The girls can sit in 5! ways on 5 seats. Thus boys and girls can sit in 5! $\times$ 4! ways.

Permutation with repetition

Theorem 1.8.5. The number of permutations of n objects of which n_1 are alike, n_2 are alike,.... n_r are alike is

$$\frac{n!}{n_1!\; n_2!\; ....\; n_r!}$$

Note. The permutation of n things, with repetition, taken r at a time is n^r.

Example 1.8.6. *In how many ways we place r colored balls into n numbered boxes if a box can hold any number.*

Solution. $P\,(n\,;\,r) = n^r$.

Example 1.8.7. *Find the number of permutations of the word ACCOUNTANT, and ENGINEERING.*

Solution. The total number of letters in the word ACCOUNTANT is 10 of which 2 are *A's,* 2 are *C's* and 2 are *T's*, and 2 are *N's* and other two letters are different. Therefore number of permutations is

$$\frac{10!}{2!\;2!\;2!\;2!}$$

Since word ENGINEERING has 11 letters of which 3 are *E's,* 3 are *N*'s, 2 are *G's* and 2 are *I's*, the number of permutations are

$$\frac{11!}{3!\;3!\;2!\;2!}$$

Example 1.8.8. *How many words can be formed from the letters of the word MONDAY if*

(a) *4 letters are used at a time* (b) *All letters are used together*

(c) *All letters are used but first is vowel*

Solution. There are six letters in MONDAY.

(a) We have to find the number of permutations of 6 letters taken 4 at a time. The required result is $P\,(6\,;\,4) = 6\,.\,5\,.\,4\,.\,3 = 360$.

(b) $P\,(6\,;\,6) = 6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1 = 720$

(*c*) Since O and A are the vowels in the word MONDAY, and in its permutation's first place will be filled in only two ways and the rest five places may be filled by 5 ways. Thus the total number of words $= 2!\, P(5\,;\,5) = 2!\, 5! = 240$.

Example 1.8.9. *In how many ways the letter of the word COMPUTING be arranged so that the letters N and G are not together in any arrangements.*

Solution. Denote the letters (N, G) by a single letter A. So we have A, C, O, M, P, U, T, I to be arranged. Since the above letters can be arranged in 8! ways and in each one of these arrangements the letters N and G can be permuted in 2! ways. So by rule of product the number of arrangement in which N and G occur together is $8! \times 2!$. In the word 'COMPUTING' there are 9 letters, so 9! arrangements are possible. The required number of arrangements in which N and G are not together is $9! - 8! \times 2! = 7 \times 8!$.

Example 1.8.10. *How many odd number of 3 digits can be formed with the figures 1, 2, 3, 4, 5 ?*

Solution. The 3 digit number to be odd depends on the last digit. So the last digit is either 1 or 3 or 5. The last digit can be chosen in 3 ways and the first and second digits can be chosen in 5 ways each. Hence the number of 3 digit number which is odd is $5 \times 5 \times 3 = 75$.

COMBINATION

A permutation of objects involves ordering, where as a combination does not take ordering into account.

Definition 1.8.10. Let A be a set having n elements then the number of combinations of n elements taken r at a time is given by

$$C\,(n\,;\,r) = nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}\quad,\quad \binom{n}{r} \text{ is the binomial coefficient.}$$

Example 1.8.11. *Find the number of strings of 0's and 1's having length n that contains exactly r 1's.*

Solution. We choose the number of strings of length n having r 1's in nC_r ways.

Example 1.8.12. *Out of 7 consonants and 4 vowels how many words can be made each containing 3 consonants and 2 vowels.*

Solution. The number of ways of choosing the three consonants is $7C_3$ and the number of ways of choosing the 2 vowels is $4C_2$.

Since each of the first groups can be associated with each of the second, the combined groups each containing 3 consonants and 2 vowels is $7C_3 \times 4C_2$. Further each of these groups contains 5 letters, which may be arranged among themselves in 5! ways. So the required number of words is $7C_3 \times 4C_2 \times 5!$.

Theorem 1.8. (Binomial Theorem).

$$(x+y)^n = \sum_{k=0}^{n} \left(\frac{n}{k}\right) n^{n-k}\, y^k$$

This follows from the indentity $(1+x)^n = \sum_{k=0}^{n} C(n, k)\, x^k$

Since $(x+y)^n = x^n\,(1 + (y/x))^n$, the coefficient of $x^n\,(y/x)^k$ in $(x+y)^n$ is $C(n, k)$.

Note. The quantity $\dfrac{(r_1 + r_2 + \ldots + r_n)!}{r_1!\, r_2! \ldots r_n!}$ is called a *multinomial coefficient* and is denoted

$$\binom{r_1 + r_2 + \ldots + r_n}{r_1, r_2, \ldots, r_n}$$

Given a set with $r_1 + r_2 + \ldots + r_n$ elements, the multinomial coefficient $\binom{r_1 + r_2 + \ldots + r_n}{r_1, r_2, \ldots, r_n}$ represents the number of ways to choose r_1 elements, then r_2 of the remaining elements, and so forth.

Multinomial coefficients also arise in the Multinomial Theorem, a generalization of the Binomial Theorem. The result is stated below, but not proved.

(Multinomial Theorem).

$$(x_1 + x_2 + \ldots + x_n)^r = \sum_{r_1 + r_2 + \ldots + r_n = r} \binom{r}{r_1, r_2, \ldots, r_n} x^{r_1} x^{r_2} \ldots x^{r_n}$$

Example 1.8.13. *What is the coefficient of* x^4 *in the expansion of* $(1 + x + 2x^2)^5$.

Solution.

$$(1 + x + 2x^2)^5 = \Sigma \frac{5!}{r!\, s!\, t!} (1^r . x^s (2x^2)^t)$$

x^4 occurs in 3 cases :

(i) $r = 3, s = 0, t = 2,$ coefficient $= \dfrac{5!}{3!\ 0!\ 2!} = 40$

(ii) $r = 2, s = 2, t = 1,$ coefficient $= \dfrac{5! \times 2}{2!\ 2!\ 0!} = 60$

(iii) $r = 1, s = 4, t = 0,$ coefficient $= \dfrac{5! \times 1}{4! \times 1! \times 0!} = 5$

Hence the answer is $40 + 60 + 5 = 105$.

Combinations with Repetition

Now we move from permutations with repetition to combinations with repetition. Let S be the set $\{A, B, C\}$. This set has three 2-combinations. That is, there are three ways to choose two distinct elements of S where order does not matter. The three 2-combinations of S are shown below:

$$\{A, B\} \quad \{A, C\} \quad \{B, C\}$$

Suppose that we are not required to choose distinct elements of S, but rather can choose the same element repeatedly. The resulting sets are called the *r-combinations with repetition* of the set S. Listed below are the six 2-combinations with repetition of S.

$$\{A, B\} \quad \{A, C\} \quad \{B, C\} \quad \{A, A\} \quad \{B, B\} \quad \{C, C\}$$

Strictly speaking, these are multisets (bags), not sets, since an element may appear multiple times.

Counting r-Combinations with Repetition

The following theorem gives a nice formula for the number of r-combinations with repetition of an n-element set.

Theorem 1.8.13. *The number of r-combinations with repetition of an n-element set is*

$$\binom{n + r - 1}{r}$$

in other words, the number of solutions to the equation $a_1 + a_2 + \ldots + a_n = r$, $a_i \geq 0$, $a_i \in N$, is

$$\binom{n + r - 1}{r}$$

In the example above, we found six ways to choose two elements from the set $S = \{A, B, C\}$ with repetition allowed. Sure enough, the theorem says that the number of 2-combinations of a 3-element set is $\binom{3+2-1}{2} = 6$.

For comparison, recall that the number of ordinary r-combinations of an n-element set is $\binom{n}{r}$. Every ordinary r-combination is also a valid r-combination with repetition. So, as one would expect, the number of r-combinations with repetition is greater if $r > 1$.

The proof of this theorem uses an important trick called "stars and bars".

Proof. Let S be a set with n elements that are ordered in some way. We will establish a bijection between r-combinations with repetition of the set S and strings of stars and bars.

Let R be a particular r-combination with repetition of S. Write down $n - 1$ bars. These $n - 1$ bars divide the line into n regions.

$$\underbrace{\quad | \quad | \quad | \quad \cdots \quad | \quad}_{n\,-\,1 \text{ bars define } n \text{ regions}}$$

Put one star in the i-th region for each time that the i-th element of S appears in R. This procedure maps each r-combination with repetition to a string with r stars and $n - 1$ bars.

(For example, let S be the set $\{A, B, C, D, E\}$, with elements ordered alphabetically. Let R be the 7-combination with repetition $\{A, B, B, B, D, E, E\}$. The stars-and-bars string corresponding to R is shown below.

$$\underset{A}{*} \;|\; \underset{B,\,B,\,B}{*\;*\;*} \;|\;|\; \underset{D}{*} \;|\; \underset{E,\,E}{*\;*}$$

The two bars with no stars between indicate that element C never appears in R.)

This mapping is a bijection because it has an inverse. That is, given any stars-and-bars string, we can construct the corresponding r-combination with repetition. The number of stars in the first region determines the number of times that the first element of S appears in the r-combination, the stars in the second region determine the number of times that the second element appears, etc.

Since, the mapping is a bijection, the number of r-combinations with repetition of an n-element set is equal to the number of strings containing $n - 1$ bars and r stars. The number of such strings is equal to the number of ways to choose r distinct positions for the stars in a string of $n + r - 1$ stars and bars. This is the number of ordinary r-combinations of a set with $n + r - 1$ elements, which is $\binom{n+r-1}{r}$.

Example 1.8.14. *Suppose that a sequence of n items has n_i identical objects of type i, for i = 1, ..., t, then the number of orderings of S is*

$$\frac{n!}{n_1!\, n_2! \ldots n_t!}$$

Proof.

There are $C(n, n_1)$ ways of choosing n_1 positions for object 1.

With these chosen, there are $C(n - n_1, n_2)$ ways for object 2 etc.

Hence, number of orderings is

$C(n, n_1)\, C(n - n_1, n_2)\, C(n - n_1 - n_2, n_3) \ldots C(n - n_1 - \ldots - n_{t-1}, n_t)$

$$= \frac{n!\,(n - n_1)!}{n_1!\,(n - n_1)!\, n_2!(n - n_1 - n_2)!} \cdots = \frac{n!}{n_1!\, n_2! \ldots n_t!}$$

Example 1.8.15. *Consider 3 books, a computer science, a mathematics and an electronics book. Suppose that the library has at least 6 copies of each. In how many ways can we select 6 books ?*

We could have

CS		M		E
XXX	|	XX	|	X

or we could have

CS		M		E	
	|	XXXX	|	XX	etc.

The ordering of 2 | 's and 5 X's make a selection

Total number of selections = $C(8, 2) = C(8, 6) = 28$.

Example 1.8.16. Prove $n\binom{n-1}{r-1} = \binom{n}{r}r$

Proof. Consider choosing a committee of size r and a leader, from n people. One way is to first pick the leader and then his $r-1$ subjects; this can be done in $n\binom{n-1}{r-1}$ ways. Another way is to pick the r committee members first and then pick a leader from among them; this can be done in $\binom{n}{r}r$ ways. Thus,

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Now an important theorem due to Pascal:

Theorem 1.8.17. (Pascal) Suppose $1 \le r \le n-1$. Then

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Algebraic proof.

$$\binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= (n-r)\frac{(n-1)!}{r!(n-r)!} + r\frac{(n-1)!}{r!(n-r)!} = n\,\frac{(n-1)!}{r!(n-r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Combinatorial proof. We use case analysis (a tree diagram) and the sum rule. Let S :: $= \{1, \ldots, n\}$. Let A be the set of r-element subsets of S. Let B be the set of r-element subsets of S that contain n. Let C be the set of r-element subsets of S that don't contain n.

Then $A = B \cup C$, and B and C are disjoint. So $|A| = |B| + |C|$, by the Sum Rule. But now we can get expressions for $|A|$, $|B|$ and $|C|$ as numbers of combinations:

$$|A| = \binom{n}{r}.$$

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$$|B| = \binom{n-1}{r-1}.$$

This is because, in addition to n, another $r-1$ elements must be chosen from $\{1, \ldots, n-1\}$.

$$|C| = \binom{n-1}{r}.$$

This is because r elements must be chosen from $\{1, ..., n-1\}$.

So (by the Sum Rule)

$$\binom{n}{r} = |A| = |B| + |C| = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Pascal's theorem has a nice pictorial representation: The row represents n, starting with 0 in the top row. Successive elements in the row represent r, starting with 0 at the left.

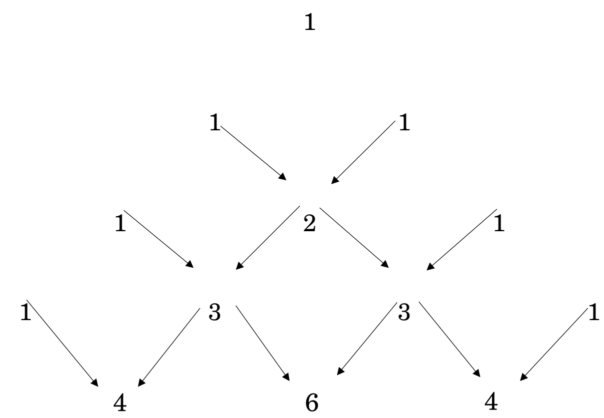


Figure 6: Pascal's triangle.

(Note that it's just the double-induction matrix "reshaped".)

This triangle has lots of nice properties. Experiment with it. For example, what happens if we add the coefficients in one row?

For the following two theorems we provide their combinatorial proofs, only. The corresponding algebraic proofs are easy inductive arguments.

Theorem 1.8.18. *Suppose n is any natural number. Then*

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Combinatorial proof. We use case analysis and the sum rule for disjoint unions. There are 2^n different subsets of a set of n elements. Decompose this collection based on the sizes of the subsets. That is, let A_r be the collection of subsets of size r. Then the set of all subsets is the disjoint union $U_r A_r$. These are $\binom{n}{r}$ subsets of size r, for $r = 0, 1, ..., n$. Hence the theorem follows.

Theorem 1.8.19. (Vandermonde). *Suppose* $0 \le r \le m, n$. *Then*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Combinatorial proof. We use case analysis with the sum and product rules. Suppose there are m red balls and n blue balls, all distinct. There are $\binom{m+n}{r}$ ways to choose r balls from the two sets combined. That's the LHS. Now decompose this collection of choices based on how many of the chosen balls are blue. For any, k, $0 \le k \le r$, there are $\binom{m}{r-k}$ ways to choose $r - k$ red balls, and $\binom{n}{k}$ ways to choose k blue balls. By the product rule, that makes $\binom{m}{r-k}\binom{n}{k}$ ways to choose r balls such that k of them are blue. Adding up these numbers for all k gives the RHS.

Restricted Combination

(*i*) The number of combinations on n things taken r at a time in which p particular things always occur is $n - pC_{r-p}$

(*ii*) The number of combinations on n things taken r at a time in which p particular things never occur is $n - pC_r$.

Example 1.8.20. *Find the value of r if* $18C_r = 18C_{r+2}$.

Solution. Since $nC_r = n\,C_{n-r}$, $18C_r = 18C_{18-r}$. Given $18C_r = 18C_{r+2}$. So $18C_{r+2} = 18C_{18-r}$. This implies $r + 2 = 18 - r$, which implies $r = 8$.

Example 1.8.21. *In how many ways can 14 books from a set of 20 be arranged on 3 shelves, if 6 books are to be placed on first shelf, 3 on the second, and 5 on the third.*

Solution. On the first shelf, 6 books can be arranged in $C\,(20\,;\,6)$ ways. In the second shelf, 3 books can be arranged in $C\,(14\,;\,3)$ ways. On the third shelf 5 books can be arranged in $C\,(11, 5)$ ways. Hence applying product rule there are $C\,(20, 6)\,C\,(14, 3)\,C\,(11, 5)$ ways to arrange 14 books in the required manner.

Theorem 1.8.22. If X is a set containing t elements the number of unordered, k element selection from X, repetitions allowed is

$$C\,(k + t - 1, t - 1) = C\,(k + t - 1, k).$$

Proof. Let $X = \{a_1, a_2, \ldots a_t\}$. Consider the $k + t - 1$ slots. Here $k + t - 1$ symbols consisting of k *x's* and $(t - 1)$ *y's*. Each placement of these symbols into the slot determine a selection. The number n_1 of x's up to the first y' represents the selection of n_1 a_1's ; the number n_2 of x's between the first and second y's represents the selection of $n_2\,a_2$'s so on. Since there are $C\,(k + t - 1, t - 1)$ ways to select the positions for the y's there are also $C\,(k + t - 1, t - 1)$ selections. This is the same as the number $C(k + t - 1, k)$ of ways to select the positions for the x's ; hence there are $C\,(k + t - 1, t - 1) = C\,(k + t - 1, k)$ unordered k-element selection from x, repetitions allowed.

Example 1.8.23. *Suppose that there are piles of red, blue, and green balls and that each pile contains at least eight balls. In how many ways we can select eight balls.*

Solution. The number of ways of selecting eight balls is

$$C(8 + 3 - 1, 3 - 1) = C\,(10, 2) = 45.$$

Example 1.8.24. *How many solutions does the equation*

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, x_3 *are non-negative integers ?*

Solution. To count the number of solutions, note that a solution corresponds to a way of selecting 11 items from a set with three elements, so that x_1 item of type 1, x_2 item of type 2, and x_3 item of type 3 are chosen. Hence the number of solutions is equal to the number of 11-combinations with repetition allowed from a set with three elements are $C\,(3 + 11 - 1, 11) = C\,(13, 11) = C\,(13, 2) = \dfrac{13 \times 12}{1 \times 2} = 78$ Solutions.

Note. For the problem if the constraints $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$ are included, then the solution to the equation subject to these constraints corresponds to a selection of 11 items with x_1 items of type one, x_2 items of type 2, x_3 items of type 3, in addition, there is atleast one item of type one, two items of type 2, 3 items of type 3. So choose one item of type 1, two item of type 2, 3 item of type 3. Then select five items. Hence, this can be done in $C\,(3 + 5 - 1, 5) = C\,(7, 5) = C\,(7, 2) = \dfrac{7 \times 6}{1 \times 2} = 21$ ways. There are 21 solutions.

Example 1.8.25. *How many ways are there to place ten indistinguishable balls into eight distinguishable bins ?*

Solution. The number of ways to place ten indistinguishable balls into eight bins equals the number of 10 combinations from a set with 8 elements when repetition is allowed. Hence $C\ (8 + 10 - 1, 101) = C\ (17, 10) = \frac{17!}{10!\ 7!} = 19,448$

Example 1.8.26.

1. *A college library has 40 text books on sociology and 50 text books dealing with anthropology. By the rule of sum, a student at this college can select among 40 + 50 = 90 text books in order to learn more about one or the other of these two subjects.*
2. *The drama club of a college is holding tryout for a play. With six men and 7 women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in* $6 \times 7 = 42$ *ways.*
3. *In a set of 20 administrators 15 are graduates and the rest are not. In how many ways a committee of 4 can be formed so that 3 are graduates and one is not a graduate.*

Solution. There are 15 graduates and 5 non-graduates 3 graduates from 15 graduates can be selected in $15\ C_3$ ways and at the same time one non-graduate can be selected in $5C_1$ ways. Hence the required answer is $15\ C_3 \times 5C_1$.

4. *How many digits between 1000 and* 10^4 *contain exactly one 8 and one 9.*

Solution. The maximum number of digits a number can contain is 4. Of the 4 positions, 8 and 9 can be placed in (4×3) ways. The remaining 2 positions can be placed by any of the remaining 8 digits (0, 1, 2 ..... 7) in (8×8) ways. Hence the required number is $4 \times 3 \times 8 \times 8$.

5. *How many 5 letter words can be formed when the first and last letters are*

(*a*) Consonants, (*b*) Vowels and (*c*) Vowels and the middle letters being consonants

Solution. (*a*) The first and last letters, which must be consonants can be chosen in (21×21) ways ; in the remaining 3 positions, any of the 26 letters can appear. Hence the required number is $(21 \times 21) \times (26 \times 26 \times 26)$

(*b*) If the first and last letters should be vowels, $(5 \times 5) \times (26 \times 26 \times 26)$ arrangements are possible.

(*c*) Vowels can be chosen in the first and last positions in (5×5) ways and consonants for the three middle positions in $(21 \times 21 \times 21)$ ways. Hence the required number is $(5 \times 5) \times (21 \times 21 \times 21)$.

6. How many 5-letter words can be formed from the word WARIWASSU ?

Solution. We have 2 W's, 2 A's, 2 S's, and the remaining letters appear once.

Case 1. Any two letters be same and the remaining 3 letters are different. It can be done in $3C_1 \times 5C_3$ ways. The number of permutations of each of these is $\frac{5!}{(5-3)\ !}$. Hence the required answer in this case is $3C_1 \times 5C_3 \times \frac{5!}{2!}$.

Case 2. Two pairs of same letter and one letter is different. Out of 3 pairs we can choose 2 pairs in $3C_2$ ways and the one letter from the remaining can be done in $4C_1$ ways and the number of permutations of each of these is $\frac{5!}{2! \times 2!}$, thus giving $3C_2 \times 4C_1 \times \frac{5!}{4}$.

Case 3. All 5 letters are different. Hence the number of arranging a words in $6P_5$ ways.

The required number is

$$3C_1 \times 5C_3 \times \frac{5!}{2!} + 3C_2 \times 4C_1 \times \frac{5!}{2! \times 2!} + 6P_5$$

Example 1.8.27. (Counting Poker Hands)

In the poker game five-card draw, each player is dealt a hand consisting of 5 cards from a deck of 52 cards. Each card in the deck has a suit (clubs ♣, hearts ♥, diamonds ◊, or spades ♠) and a value (A, 2, ... 10, J, Q, k).

The order in which cards are dealt does not matter. Two cards are a **pair** if they have the same values.

(*i*) How many different hands are possible in 5-card draw.

(*ii*) How many hands are there with four-of-a-kind? For example, 9 ♠, 4 ♦, 9 ♣, 9 ♠, 9 ◊ has a four-of-a kind, because there are four 9's.

(*iii*) How many 5 card hands consists of a pair and a triple (In poker, such a hand is called a 'full house') (e.g. : 7 ♠, 7 ◊, J ♣, J ♠, J ◊)

(*iv*) How many 5 card hands consists of 2 pairs.

(*v*) How many 5-card hands consists of cards from a single suit?

Solution. (*i*) A hand is just a 5-card subset of the 52 card deck. The possible hands in 5-card draw are exactly the 5-combinations of a 52-element set. There are $52C_5 = \dfrac{52!}{47!\,5!} = 2{,}596{,}960$ possible hands.

(*ii*) Choose one value to appear in all four suits from the set of 13 possible values. There are 13 choices of this value, and if we pick one we can't have any others (that would require 8 cards). So the choice of value gives 13 disjoint cases to count. Then we have to choose one more card from the remaining set of 48. This can be done in $48C_1 = 48$ ways. In total there are $13 \times 48 = 624$ hands with four-of-a kind.

(*iii*) We can choose the value that appears three times in 13 ways. Then we can pick any three of the four cards in the deck with this value, and can be done in $4C_3 = 4$ ways. There are then 12 remaining choices for the value that appears two times. We can pick any two of the four cards with this value, and can be done in $4C_2 = 6$ ways. Hence the total number of full house is $13 \times 4 \times 12 \times 6 = 3744$.

(*iv*) Choose the values for the two pairs in $13C_2$ ways and choose the 2 suits for the pair with the larger value in $4C_2$ ways. Then choose the suits for the pair with the smaller value in $4C_2$ ways and choose the remaining cards from the 4×11 cards that have different values from the pairs. The required number is $13C_2 \times 4C_2 \times 4C_2 \times 44 = 123{,}552$.

(*v*) For each of the 4 suits there are $13C_5$ 5-card hands. Hence there are a total of $4 \times 13C_5$ such hands.

Generation of permutations and combinations.

Suppose we want to write down the $n\,!$ permutations of n distinct objects. For $n = 3$, there are only six permutations. For $n = 4$, there are 24 permutations, so there is no difficulty in writing all the permutation. Now we give a procedure which generate all the permutations exhaustively with no repetitions in some order called lexicographic order. Suppose {1, 2, 3,....., n} be the n objects to be permuted. For two permutations $a_1\, a_2$ ...... a_n and b_1, b_2 ..... b_n, we say $a_1\, a_2$ ..... a_n comes before $b_1\, b_2$ .... b_n in the lexicographic order if, for some $1 \le m < n$, $a_1 = b_1$, $a_2 = b_2$, ..., $a_{m-1} = b_{m-1}$, and $a_m < b_m$. For example, the permutation 124635 comes before the permutation 125643, and the permutation 125463 comes after the permutation 125346.

Suppose we were given a permutation $a_1\, a_2$ ..... a_n. According to lexicographic order we compute the next permutation $b_1\, b_2$ ..... b_n as follows.

1. $a_i = b_i \;\; 1 \le i \le m - 1$, and $a_m < b_m$ for the largest possible m.
2. b_m is the smallest element among a_{m+1}, a_{m+2}, ... a_n that is larger than a_m.

3. $b_{m+1} < b_{m+2} < ..... < b_n$

For example, the permutation following 124653 in the lexicographic order is 125346. For a given permutation $a_1 a_2 ..... a_n$ we note that the largest possible m for which (1) is satisfied is the largest possible m for which a_m is less than at least one of $a_{m+1}, a_{m+2}, ... a_n$. Also, it is possible that $a_m < a_{m+1}$. Therefore, if we examine the permutation $a_1 a_2 .... a_n$ element by element from right to left, the first time we observe a decrement, we know the value of m and can determine $b_m b_{m+1} .... b_n$ according to (2) and (3). For example, suppose we were given the permutation 124653. When we scan the permutation from right to left element by element, according to (1), we determine that the next permutation is of the form 12 *xxxx*. In other words the subscript $m = 3$. According to (2), we can further determine that the next permutation is of the form 125 *xxx*. Finally, according to (3), we determine that the next permutation is 125346. (Actually, we can make use of the fact that $a_{m+1} > a_{m+2} > ...... > a_n$ to carry out steps 2 and 3 in a rather simple manner).

This observation leads immediately to a systematic procedure for generating the n ! permutations of n objects by starting with the permutations 1234.....n and stopping at the permutation n......4321.

Suppose we want to generate all subsets of size k of the set {1, 2, 3,....., n}. Introduce a lexicographic order of the subsets, let us agree first that each subset will be represented by a sequence with the elements in the subset arranged in increasing order. We can then arrange the sequences according to the lexicographic order. For example, the subsets of size 4 of {1, 2, 3, 4, 5, 6} are represented and ordered as

1234
1235
1236
1245
1246
1256
1345
1346
1356
1456
2345
2346
2356
2456
3456

Along exactly the same line as our procedure to generate permutations, let us observe how we can design a procedure to generate all subsets of size k of the set {1, 2, ..., n}. Let $a_1 a_2 .... a_k$ be a subset with size k.

It can be shown that the next subset with size k $b_1 b_2 ...... b_k$ according to the lexicographic order must be such that

1. $a_i = b_i \quad 1 \le i \le m - 1$ and $a_m < b_m$ for the largest possible m.
2. $b_m = a_{m+1}$.
3. $b_{j+1} = b_j + 1$ for $m \le j \le k - 1$.

In the sequence $a_1\ a_2\ .....\ a_k$, we define the maximum possible value of a_j to be $n - k + j$. Thus, the maximum possible value of a_j to be $n - k + j$. Thus, the maximum possible value of a_k is n, the maximum possible value of a_{k-1} is $n - 1$, the maximum possible value of a_{k-2} is $n - 2$, ... and the maximum possible value of a_i is $n - k + 1$. Since in $a_1\ a_2\ .....\ a_k$ the largest m for which a_m is not equal to its maximum possible value is the largest m that satisfies (1), we can determine m by examining $a_1\ a_2\ ....\ a_k$ from right to left, element by element. Once the value of m is determined, we can determine $b_m\ b_{m+1}\ ....\ b_k$ according to (2) and (3).

Example 1.8.28. *Generate the permutations of the four objects 1, 2, 3, 4.*

Solution.

$$
\begin{aligned}
& 1234 \to 1243 \to 1324 \to 1342 \to 1423 \to 1432 \to 2134 \\
& \to 2143 \to 2314 \to 2341 \to 2413 \to 2431 \to 3124 \\
& \to 3142 \to 3214 \to 3241 \to 3412 \to 3421 \to 4123 \\
& \to 4132 \to 4213 \to 4231 \to 4312 \to 4312 \to 4321
\end{aligned}
$$

Exercise 1.8

1. How many 6-digit phone numbers are there if the first digit cannot be 2 or 3 ?
2. How many three-letter words can be formed from the letters in the set $\{a, b, y\ z\}$.
3. In how many ways can be committee of three faculty members and two students can be selected from seven faculty members and eight students ?
4. How many ways can you arrange the letters in the following words ?
   (*a*) COMBINE (*b*) SUBSET
5. How many integers in {1, 2, ... 500} are divisible by 3 or 5 ?

1.9 MATHEMATICAL INDUCTION

Let $P(n)$ be a proposition or statement involving a positive integer n. Then $P(n)$ is true for all positive values of n provided

(*i*) $P(1)$ is true (1.9.1)

(*ii*) If $P(k)$ is true, then so is $P\ (k + 1)$ (1.9.2)

Condition (1.9.1) is called the *Basis step* and (1.9.2) is called *Inductive step.*

Example 1.9.1. *Using induction, show that*

$$n! \geq 2^{n-1}, \quad \text{for} \quad n = 1, 2....$$

Solution. Let $P\ (n) : n! \geq 2^{n-1}$, for $n = 1, 2,.....$

Basis step : For $n = 1$,

$$P(1) = 1! = 1 \geq 1 = 2^{1-1}.$$

Inductive step. Let $P\ (n) : n! \geq 2^{n-1}$ be true.

Then

$$
\begin{aligned}
P\ (n + 1) : (n + 1)! &= (n + 1)\ n! \\
&\geq (n + 1)\ 2^{n-1} && \text{by assumption} \\
&\geq 2.2^{n-1} && \text{since } n + 1 \geq 2 \\
&= 2^n
\end{aligned}
$$

Therefore $P(n + 1)$ is true. Hence $P(n)$ is true for all positive values of n.

Example 1.9.2. *Using induction show that $5^n - 1$ is divisible by 4, for n = 1, 2, ...*

Solution. Let $P(n) = 5^n - 1$ be divisible by 4, for $n = 1, 2,....$

Basis step. If $n = 1$, $5^n - 1 = 5 - 1 = 4$, which is divisible by 4.

Inductive step. Assume $P(n)$ is true. We prove $P(n + 1)$ is also true.

$$P(n + 1) : 5^{n+1} - 1 = 5.5^n - 1 = (5^n - 1) + 4.5^n$$

By assumption $5^n - 1$ is divisible by 4, and since 4×5^n is divisible by 4, the sum is divisible by 4. So $P(n + 1)$ is true. By induction $P(n)$ is true for all $n = 1, 2,.....$

Example 1.9.3. *Using induction, show that*

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n + 1)(2n + 1)}{6}.$$

Solution. Let $P(n) = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n + 1)(2n + 1)}{6}.$

Basis step : $P(1) = 1^2 = \frac{1.2.3}{6}$, which is true. Suppose $P(n)$ is true. Then

$$P(n + 1) = 1^2 + 2^2 + 3^2 + ... + n^2 + (n + 1)^2$$

$$= \frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2 = \frac{n(n + 1)(2n + 1) + 6(n + 1)^2}{6}$$

$$= \frac{(n + 1)[2n^2 + 7n + 6]}{6} = \frac{(n + 1)(n + 2)(n + 3)}{6}$$

$P(n + 1)$ is true. Hence the result is true for all n.

Example 1.9.4. *Let A be a finite set with n elements. Then using mathematical induction prove that cardinality of* $P(A) = 2^n$.

Solution. Let $P(n)$ be the statement that a set with n elements, has exactly 2^n subsets. For $n = 1$, there are two distinct subsets ϕ and A. So $P(1)$ is true. Assume that $P(k)$ is true. Let $P(k + 1)$ be the statement that a set with $k + 1$ elements has exactly 2^{k+1} subsets. Let A be a set with $k + 1$ elements. Let $B = A - \{x\}$. Then B has k elements. By induction B has 2^k subsets of A. To each subset $S \subset B$, we form a new set $S \cup \{x\}$. Hence we obtain another 2^k such subsets. Then A has $2^k + 2^k = 2^{k+1}$ subsets. Hence $P(k + 1)$ is true and so $P(n)$ is true for all n.

Example 1.9.5. *Using mathematical induction prove* $n^3 - n$ *is divisible by 3,* $n > 0$.

Solution. Let $P(n) : n^3 - n$ is divisible by 3.

Basis step. For $n = 1$, $1^3 - 1 = 0$ is divisible by 3. Hence the basis step is true.

Inductive step. Assume the result for $n = k$. That is, assume $k^3 - k$ is divisible by 3. For $n = k + 1$,

$$(k + 1)^3 - (k + 1) = (k + 1)[(k + 1)^2 - 1]$$

$$= (k + 1)[k^2 + 2k] = k^3 + 3k^2 + 2k = (k^3 - k) + 3(k^2 + k)$$

Since both the terms are divisible by 3, hence for $n = k + 1$ the result in true. By induction the result is true for all n.

Example 1.9.6. *Using induction prove* $n < 2^n$, $n > 0$.

Solution. Let $P(n) : n < 2^n,\ n > 0$

Basis step. For $n = 1$, $1 < 2$. Hence the basis step follows.

Inductive step. Assume the result for $n = k$.

That is, $k < 2^k$. Now $\quad k + 1 < 2^k + 1$

$$< 2^k + 2^k, \quad \text{since } 2^k > 1, \text{ for } k > 0 < 2 \times 2^k = 2^{k+1}$$

Hence by induction the result is true for all n.

Example 1.9.7. *Show that $n^4 - 4n^2$ is divisible by 3 for all $n \in N$.*

Solution. Let $P(n) : n^4 - 4n^2$ is divisible by 3.

Basis step. For $n = 1$, $1^4 - 4 \times 1^2 = -3$ is divisible by 3. Hence for $n = 1$, the result is true.

Inductive step. Assume for $n = k$ the result is true. That is assume, $k^4 - 4k^2$ is divisible by 3. Now

$$\begin{aligned}(k+1)^4 - 4(k+1)^2 &= (k+1)^2 [(k+1)^2 - 4] \\ &= (k^2 + 2k + 1)[k^2 + 2k - 3] = k^4 + 4k^3 + 2k^2 - 4k - 3 \\ &= (k^4 - 4k^2) + 4k^3 + 6k^2 - 4k - 3 = (k^4 - 4k^2) + \frac{4}{k} k^4 - \frac{4}{k} k^2 + 6k^2 - 3 \\ &= k^4 - 4k^2 + \frac{3}{k} k^4 + \frac{1}{k} [k^4 - 4k^2] + 6k^2 - 3 \\ &= (k^4 - 4k^2) + \frac{1}{k}(k^4 - 4k^2) + 3(k^3 + 2k^2 - 1)\end{aligned}$$

Since each term is divisible by 3, the result is true for $n = k + 1$. Hence the result is true for all n.

Example 1.9.8. *Show by induction $2n < 3^n$ for all $n \in N$.*

Solution. Basis step : For $n = 1$, $2 < 3^1$. Hence the result is true for $n = 1$.

Inductive step. Assume the result is true for $n = k$. That is, assume $2k < 3^k$. Now for $n = k + 1$,

$2(k + 1) = 2k + 2 < 3^k + 2 < 3^k + 3^k + 3^k$, since $3^k > 2k > 2$, for all $k \in N$.

Hence $2(k + 1) > 3.3^k = 3^{k+1}$. So the result is true for all $n \in N$.

Example 1.9.9. *Suppose that $a_{m,n}$ is defined recursively for $(m, n) \in N \times N$ by $a_{0,0} = 0$ and*

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\ a_{m,n-1} + n & \text{if } n > 0 \end{cases}$$

Show that $a_{m,n} = \dfrac{m + n(n+1)}{2}$ for all $(m, n) \in N \times N$.

Solution. We prove by generalized version of mathematical induction. This is valid when $(m, n) = (0, 0)$ we show, if the formula is true for all pairs smaller than (m, n) in the lexicographic ordering of $N \times N$, then it is true for (m, n).

Basic Step. Let $(m, n) = (0, 0)$. Then $a_{0,0} = 0$. Further when $m = n = 0$, $\dfrac{m + n(n+1)}{2}$ $= \dfrac{0 + 0 \times 1}{2} = 0$

Inductive Step. Let (m', n') is less than (m, n) such that $a_{m',n'} = \dfrac{m' + n'(n'+1)}{2}$. By recursion, if $n = 0$, then $a_{m,n} = a_{m-1,n} + 1$. Because $(m - 1, n)$ is smaller than (m, n), by induction hypothesis,

$$a_{m-1,n} = \frac{(m-1) + n(n+1)}{2}.$$

Hence $a_{m,n} = \dfrac{(m-1) + n(n+1)}{2} + 1 = \dfrac{m + n(n+1)}{2}$.

Suppose $n > 0$, so $a_{m,n} = a_{m,n-1} + n$. Since $(m, n - 1)$ is smaller than (m, n), the induction hypothesis tells that $a_{m,n-1} = \dfrac{m + (n-1)n}{2}$. So $a_{m,n} = \dfrac{m + (n-1)\,n}{2} + n = \dfrac{m + n\,(n+1)}{2}$. Hence the inductive step.

Exercise 1.9

1. Prove by induction that for all $n \geq 1$, $n^3 + 2n$ is a multiple of 3.
2. Prove that if $n \geq 1$, then $1\,(1!) + 2\,(2!) + \ldots\ldots + n(n!) = (n + 1)! - 1$.
3. Prove that every positive integer greater than or equal to 2 has a prime decomposition.
4. Using mathematical induction prove that $1^3 + 2^3 + 3^3 + \ldots\ldots.. = \dfrac{n^2(n+1)^2}{4}$.
5. Using mathematical induction prove the number $2^{2n} - 1$ is divisible by 3, for any positive integer n.

1.10 PROBABILITY THEORY

Experiment 1.10.1. An experiment is any well-defined process from which observations (data) can be obtained.

Under certain conditions, when an experiment is conducted repeatedly, it doesn't give unique results but may result in any one of the several possible outcomes. The experiment is known as *trial* and the outcomes are known as *events* or *cases*.

Example.

(*i*) *Throwing a die is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.*

(*ii*) *Tossing a fair coin is a trial and getting a head or tail is an event.*

(*iii*) *Drawing two cards from a pack of cards is a trial and getting a king and queen are events.*

Sample space 1.10.2. The set S of all possible outcomes in an experiment is called a *sample space*. Each element of a sample space is called a *sample point* or *sample*.

Example 1.10.3. (*i*) *In tossing a fair coin, there are two possible outcomes, namely head (H) and tail (T). Thus, the sample space is {H, T}.*

(*ii*) *When two coins are tossed together, the sample space is {HH, HT, TH, TT}.*

Definition 1.10.4. A sample space is said to be discrete if it has finitely many or countably infinite elements. If the elements of a sample space constitutes a continum, for *e.g.* all the points on a line, the sample space is said to be continuous.

Exhaustive events. The total number of possible outcomes in any trial is known as exhaustive events.

Example. (*i*) *In tossing a coin there are two exhaustive events, namely head and tail.*

(*ii*) *In throwing a die there are 6 exhaustive events, any one of the 6 faces 1, 2, ... 6 come in upper most.*

Favourable cases. The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event.

Example. *In throwing of two dice, the number of cases favourable to get the sum 5 is (1, 4), (4, 1), (2, 3), (3, 2).*

Mutually Exclusive events. Events are said to be *mutually exclusive* if happening of any one of them precludes the happening of all others. In other words no two of the mutually exclusive events can occur simultaneously.

Example 1.10.5. *If we toss a coin, the sample space is* $S = \{H, T\}$. *Let* A = *event of getting a head and* B = *event of getting a tail. Then* $A \cap B = \phi$. *So* A *and* B *are mutually exclusive.*

Equally Likely Events. The given events are said to be equally likely, if none of them is expected to occur in preference of the other.

Example. *If we roll an unbiased die, each outcome is equally likely to happen.*

Complementary Event. Let S be a sample space and let A be an event. Then, clearly $A \subseteq S$. Moreover $\overline{A}$, the complement of A is also a subset of S. So $\overline{A}$ is called the complementary event of A.

Independent Events. Events are said to be independent if the happening or not happening of an event doesn't depend on the occurrence of any number of the remaining events.

Example 1.10.6. *In tossing an unbiased coin, the event of getting a head in the first toss is independent of getting a head in the second, or third and subsequent throws.*

Combination of events.

Since an event is a set, we can combine events through various set operations to form new events. Thus

(*i*) $A \cup B$ is the event that occurs only when A occurs or B occurs.

(*ii*) $A \cap B$ is the event that occurs only when each one of A and B occurs.

(*iii*) $\overline{A}$ is the event that occurs only when A does not occur.

1.10.7 Mathematical or Classical Probability

If a trial results in n exhaustive, mutually exclusive and equally likely cases of which m are favourable to an event E, then the probability p of the happening of the event E is

$$\frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}.$$

Remark. (*i*) Probability p of the happening of an event is known as the probability of success and the probability of the non-happening of an event as the probability of failure.

(*ii*) If $p(E) = 1$, E is called a certain event and if $p(E) = 0$, E is called an *impossible event.*

1.10.8 Statistical or Empirical probability

If a trial is repeated a number of times under essentially homogeneous and identical conditions then the limiting value of the ratio of the number of times the event happens to the number of trials as the number of trials become indefinitely large, is called the probability of happening of the event. That is $p(E) = \underset{n \to \infty}{lt} \frac{m}{n}$.

Example 1.10.9. *A bag contains 3 red, 6 white, 7 blue balls. What is the probability that two balls drawn are white and blue ?*

Solution. Total number of balls $= 3 + 6 + 7 = 16$. Out of 16 balls, 2 can be drawn in $16\,C_2$ ways. Out of 6 white balls, 1 ball can be drawn in $6C_1$ ways and out of 7 blue balls 1 ball can be drawn in $7C_1$ ways. So total number of favourable case is $6C_1 \times 7C_1 = 6 \times 7 = 42$. The Required Probability $= \frac{42}{120} = \frac{7}{20}$.

Example 1.10.10. *One card is drawn at random from a well shuffled pack of 52 cards. Find the probability that the drawn card is*

(*i*) *a king* (*ii*) *a diamond* (*iii*) *a red card or a king.*

Solution. If S denotes the sample space, then number of exhaustive events is $n = 52$.

(*i*) If A is the event that the card drawn is a king, then the number of favourable cases is 4. So p (a king) $= \frac{4}{52}$.

(*ii*) p (a diamond) $= \frac{13}{52} = \frac{1}{4}$, where B is the event that the card drawn is a diamond and the number of favourable cases is 13.

(*iii*) Let C be the event that the card drawn is a red card or a king. We know that there are 26 red cards which include 2 kings. There are 2 more kings. Therefore number of favourable cases is $26 + 2 = 28$. So p (red card or king) $= \frac{28}{52} = \frac{7}{13}$.

Example 1.10.11. *What is the probability that a leap year selected at random will contain 53 sundays.*

Solution. A leap year contains 366 days *i.e.* 52 weeks and 2 days. Clearly there are 52 sundays in 52 weeks. For the remaining 2 days, we have the following combinations.

(*i*) Sunday and Monday
(*ii*) Monday and Tuesday
(*iii*) Tuesday and Wednesday
(*iv*) Wednesday and Thursday
(*v*) Thursday and Friday
(*vi*) Friday and Saturday
(*vii*) Saturday and Sunday.

For having 53 sundays, one of the two overdays must be a sunday. Since out of the 7 possibilities 2 favour the event that one of the two days is a Sunday.

$\therefore$ Required probability $= \frac{2}{7}$.

Example 1.10.12. *An integer is chosen at random from the first 100 natural numbers. What is the probability that the chosen integer is divisible by 4 or 6 ?*

Solution. The integers from 1 to 100 that are divisible by 4 are 4, 8, 12, ... 100. These numbers form an A.P. If this sequence contains n terms, then $100 = 4 + (n - 1) \,.\, 4$ or $n = 25$.

Again, the numbers from 1 to 100 that are divisible by 6 are 6, 12, 18, ... 96. This is also in A.P. Let it contain m terms. Then

$$96 = 6 + (m - 1)6 \quad \text{or} \quad m = 16.$$

But there are integers which are divisible by both 4 and 6. Each of such integers is divisible by 12 (*i.e.* L.C.M of 4 and 6). These numbers are 12, 24,.....96. This are also in A.P. If this contains k terms, then

$96 = 12 + (k - 1)\ 12$ or $k = 8$. Thus, if A denotes the events that the chosen integer is divisible by 4 or 6, then the number of favourable cases is $(25 + 16 - 8) = 33$. So p (a number divisible by 4 or 6) $= \frac{33}{100}$.

Example 1.10.13. *If a pair of a balanced dice is thrown, what is the probability of getting 7 ?*

Solution. Let A be the event of getting 7 if a pair of dice is thrown.

$$p(A) = \frac{\text{Favourable cases}}{\text{Exhaustive cases}} = \frac{6}{36} = 1/6$$

Note that the number of ways of getting 7 is (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3) and number of exhaustive event is $6^2 = 36$.

Various Results on Probability.

Theorem 1.10.14. If in an experiment the sample space is S, then (*i*) $p\ (A) \geq 0$, for $A \subseteq S$. That is, the probability of occurrence of an event is non-negative.

(*ii*) The probability of occurrence of an impossible event is 0 *i.e.* $p\ (\phi) = 0$.

(*iii*) $p\ (S) = 1$.

Proof. By the definition of probability we have

(*i*) $p(A) = \dfrac{\text{Favourable cases}}{\text{Exhaustive cases}} \geq 0$ (*ii*) $p\ (\varphi) = \dfrac{0}{\text{Exhaustive cases}} = 0$

(*iii*) $p\ (S) = \dfrac{\text{Favourable cases}}{\text{Exhaustive cases}} = \dfrac{n}{n} = 1$

Finite Probability Space and Probability Function. Let $S = \{A_1, A_2, ... A_n\}$ be a finite sample space. If there is a function $p : S \to R$ where R is the set of all real numbers, which associates to each element $A_i \in S$, a unique real number $p\ (A_i)$ satisfying (*i*) of Theorem 1.10.14 and Theorem 1.10.15, then p is known as a probability function on S and the set $\{p\ (A_i) : A_i \in S\}$ is known as a *probability space*.

Theorem 1.10.15. If A and B are mutually exclusive events, then

(*i*) $p\ (A \cap B) = 0$ (*ii*) $p\ (A \cup B) = p(A) + p(B)$

Proof. Since A and B are mutually exclusive, $A \cap B = \phi$. Now (*i*) $p\ (A \cap B) = p(\phi) = 0$.

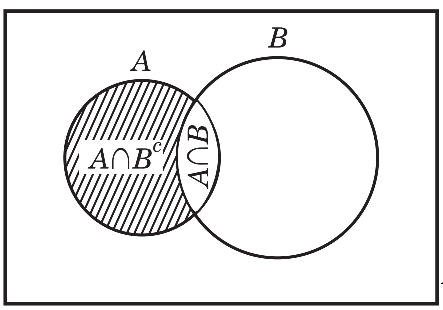


Fig. 1.8

(*ii*) Since $A \cap B = \phi$, the number of favourable cases $n\ (A \cup B) = n\ (A) + n(B)$. So

$$p(A \cup B) = \frac{n\ (A \cup B)}{n\ (S)} = \frac{n(A) + n(B)}{n\ (S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$$

$$= p(A) + p(B).$$

Theorem 1.10.16. The probability of the complementary event $\overline{A}$ of A is $p(\overline{A}) = 1 - p(A)$.

Solution. A and $\overline{A}$ are disjoint events such that $A \cap \overline{A} = S$. Then $p\ (A \cap \overline{A}) = p(A) + p(\overline{A}) = 1$, since $p(S) = 1$. Hence $p(\overline{A}) = 1 - p(A)$.

Addition Theorem 1.10.17. If A and B are two events and are not disjoint, then

$$p(A \cup B) = p(A) + p(B) - p(A \cap B).$$

Proof. $A \cup B = A \cup (\overline{A} \cap B)$. Since A and $\overline{A} \cap B$ are disjoint,

$p(A \cup B) = p(A) + p(\overline{A} \cap B)$ by Theorem 1.10.15

$= p(A) + [p\ (\overline{A} \cap B) + p(A \cap B)] - p(A \cap B)$ since $\overline{A} \cap B$ and $A \cap B$ are disjoint

$= p(A) + p(B) - p(A \cap B)$,

since $(A \cap B) \cup (\overline{A} \cap B) = B$.

Theorem 1.10.18. If $B \subset A$, then (*i*) $p\ (A \cap \overline{B}) = p(A) - p(B)$ (*ii*) $p(B) \leq p(A)$.

Proof. When $B \subset A$, B and $A \cap \overline{B}$ are mutually exclusive and their union is A.

$$p(A) = p(B \cup (A \cap \overline{B})) = p\ (B) + p(A \cap \overline{B})$$

$$\Rightarrow \quad p(A \cap \overline{B}) = p(A) - p(B).$$

We know that $p\ (A \cap \overline{B}) \geq 0 \Rightarrow p(A) - p(B) \geq 0$. Hence $p\ (A) \geq p(B)$.

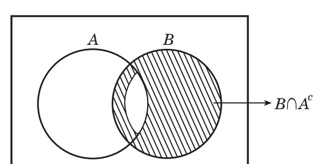


Fig. 1.9

Example 1.10.19. $p(A_1 \cup A_2) = 2/3$, $p\ (A_1 \cap A_2) = 1/6$. *Find* $p\ (\overline{A_1} \cup \overline{A_2})$ *and* $p(\overline{A_1} \cap \overline{A_2})$.

Solution. We know that $\overline{A_1 \cup A_2} = \overline{A}_1 \cap \overline{A}_2$,

$p\ (\overline{A}_1 \cup \overline{A}_2) = p\ (\overline{A_1 \cap A_2}) = 1 - p(A_1 \cap A_2) = 1 - 1/6 = 5/6$

$$p\ (\overline{A}_1 \cap \overline{A}_2) = p\ (\overline{A_1 \cup A_2}) = 1 - p(A_1 \cup A_2) = 1 - 2/3 = 1/3.$$

Example 1.10.20. *If* $p(A_1) = \frac{1}{3}$, $p(A_2) = 3/4$, $p(A_1 \cap A_2) = 1/6$, *find the probabilities of* A_1 *alone and* A_2 *alone.*

Solution. A_1 alone means $A_1 \cap \overline{A}_2$ and A_2 alone means $\overline{A}_1 \cap A_2$. Now $A_1 = (A_1 \cap A_2) \cup (A_1 \cap \overline{A}_2)$. So $p\ (A_1) = p\ (A_1 \cap A_2) + p(A_1 \cap \overline{A}_2)$ and $p(A_1 \cap \overline{A}_2) = p(A_1) - p(A_1 \cap A_2) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$ and $p(\overline{A}_1 \cap A_2) = p(A_2) - p(A_1 \cap A_2) = \frac{3}{4} - \frac{1}{6} = \frac{7}{12}$.

Example 1.10.21. *If A, B, C are independent events prove that (i) A and* $B \cup C$, *(ii) A and* $B \cap C$ *are independent.*

Solution.
$$\begin{aligned} p(A \cap (B \cup C)) &= p\ ((A \cap B) \cup (A \cap C)) \\ &= p\ (A \cap B) + p\ (A \cap C) - p(A \cap B \cap C) \\ &= p(A)\ p\ (B) + p\ (A)\ .\ p(C) - p(A)\ .\ p(B)\ .\ p(C) \\ &\qquad \text{since } A, B, C \text{ are mutually independent} \\ &= p(A)\ [p(B) + p(C) - p(B)\ .\ p(C)] \\ &= p(A)\ [p(B \cup C)] \end{aligned}$$

Hence A and $B \cup C$ are independent. Similarly $p\ (A \cap (B \cap C)) = p(A)\ .\ p(B \cap C)$.

Example 1.10.22. *If two events A and B are independent, show that* $\overline{A}$ *and* $\overline{B}$ *are also independent.*

Solution. Since A and B are independent, $p\ (A \cap B) = p(A)\ .\ p(B)$. Now

$$\begin{aligned} p\ (\overline{A} \cap \overline{B}) = p(\overline{A \cup B}) &= 1 - p(A \cup B) \\ &= 1 - (p\ (A) + p(B) - p(A \cap B)) = 1 - (p(A) + p(B) - p(A)\ .\ p(B)) \\ &= 1 - p(A) - p(B) + p(A)\ .\ p(B) = (1 - p(A)) - p(A)\ (1 - p(B)) \\ &= (1 - p(A))\ (1 - p(B)) \end{aligned}$$

Therefore $\overline{A}$ and $\overline{B}$ are independent events.

Definition 1.10.23. Conditional Probability

The conditional probability of occurrence of B when the event A has already happened is $p(B/A) = \frac{p\ (A \cap B)}{p\ (A)}$ provided $p\ (A) > 0$. The conditional probability of happening of A when B has already happened is

$$p(A/B) = \frac{p\ (A \cap B)}{p\ (B)} \text{ provided } p(B) > 0.$$

Note :

(*i*) For $p(B) > 0$, $p(A/B) \leq p(A)$. (*ii*) For $p(A) > 0$, $p(B/A) \leq p(B)$

(*iii*) The conditional probability $p(A/B)$ is not defined if $p(B) = 0$.

(*iv*) $p(B/B) = 1$.

Multiplication law of probability for Independent Events

If A and B are independent, then $p(A/B) = p(A)$ and $p(B/A) = p(B)$. Hence $p(A \cap B) = p(A)\ .\ p(B)$.

Mutually Exclusive Events and Independent Events.

Let A and B be mutually exclusive events with positive probabilities ($p(A) > 0$, $p(B) > 0$). That is, both A and B are possible events such that

$$A \cap B = \phi \Rightarrow p(A \cap B) = p(\phi) = 0.$$

...(1.10.1)

Further by compound probability theorem

$$p(A \cap B) = p(A) \,.\, p(B/A) = p(B) \,.\, p(A/B) \qquad ...(1.10.2)$$

Since $p(A) \neq 0$, $p(B) \neq 0$, from (1.10.1) (1.10.2), $p(A/B) = 0 \neq p(A)$, $p(B/A) = 0 \neq p(B)$.

Hence A and B are dependent events. Hence two possible mutually disjoint events are always dependent.

If A and B are independent events, then $p(A \cap B) = p(A) \,.\, p(B) \neq 0$. Hence A and B cannot be mutually exclusive. Hence two independent events with $p(A) > 0$, $p(B) > 0$ cannot be mutually disjoint.

Example 1.10.24. *For any three events A, B, C,* $p(A \cap \overline{B}/C) + p\ (A \cap B/C) = p\ (A/C)$

Solution. $p(A \cap \overline{B}/C) + p(A \cap B/C)$

$$= \frac{p\ (A \cap \overline{B} \cap C)}{p(C)} + \frac{p\ (A \cap B \cap C)}{p\ (C)} = \frac{p\ (A \cap \overline{B} \cap C) + p(A \cap B \cap C)}{p(C)}$$

$$= \frac{p\ ((A \cap \overline{B} \cap C) \cup (A \cap B \cap C))}{p\ (C)}$$

$$= \frac{p\ ((A \cap C) \cup (A \cap B \cap C)) \cap (\overline{B} \cup (A \cap B \cap C))}{p\ (C)}$$

Using distribution law

$$= \frac{p\ [(A \cap B \cap C) \cap (\overline{B} \cup A) \cap (\overline{B} \cup C)]}{p\ (C)}$$

Using Absorption law and distributive law.

$$= \frac{p\ ((A \cap B \cap C) \cap (\overline{B} \cup (A \cap C))}{p\ (C)} \quad \text{using distributive law}$$

$$= \frac{p((A \cap B \cap C \cap \overline{B}) \cup ((A \cap B \cap C) \cap (A \cap C))}{p\ (C)}$$

$$= \frac{p\ (A \cap C)}{p\ (C)} \text{ using absorption law} = p\ (A/C).$$

Note. $p\ (A_2 \cup A_3 \cup \ldots\ldots \cup A_n/A_1) = p(A_2/A_1) + p\ (A_3/A_1) + \ldots\ldots + p\ (A_n/A_1)$ provided $A_2, A_3, \ldots\ldots A_n$ are pairwise disjoint sets.

Example 1.10.25. *Prove* $p\ (A \cap B) \leq p(A) \leq p\ (A \cup B) \leq p(A) + p(B)$,

Solution. Since $A \cap B \subset A$, $p(A \cap B) \leq p(A)$. Also $A \subset A \cup B$, $p(A) \leq p(A \cup B)$ and $p\ (A \cup B) = p\ (A) + p(B) - p(A \cap B)$ implies $p\ (A \cup B) \leq p(A) + p(B)$.

Example 1.10.26. *A problem in statistics is given to three students A, B, C whose chance of solving it are* 1/2, 3/4, 1/4 *respectively. What is the probability that the problem will be solved if all of them try independently.*

Solution. Let *A, B, C* denote the events that the problem is solved by the students *A, B, C* respectively. Then $p(A) = 1/2$, $p(B) = 3/4$, $p(C) = 1/4$. The problem will be solved if atleast one of them solves. Thus we have to calculate the probability of occurrence of atleast one of the three events *A, B, C* or $p(A \cup B \cup C)$.

Now

$$p\ (A \cup B \cup C) = p(A) + p(B) + p(C) - p(A \cap B) - p(A \cap C) - p(B \cap C) + p(A \cap B \cap C).$$

$$= p(A) + p(B) + p(C) - p(A) \,.\, p(B) - p(A) \,.\, p(C) - p(B) \,.\, p(C) + p(A) \,.\, p(B) \,.\, p(C)$$

$$= \frac{29}{32}.$$

Example 1.10.27. *Let A and B be two events such that p(A) = 3/4, p(B) = 5/8, show that*

$p(A \cup B) \geq 3/4$ *and* $3/8 \leq p(A \cap B) \leq 5/8$.

Solution. Since $A \subseteq A \cup B, p(A) \leq p(A \cup B)$ which implies $3/4 \leq p(A \cup B)$. Now $(A \cap B) \subseteq B$ implies $p(A \cap B) \leq p(B) = 5/8$. We know

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) \leq 1$$

$$\Rightarrow \frac{3}{4} + \frac{3}{8} - 1 \leq p(A \cap B) \Rightarrow 3/8 \leq p(A \cap B) \leq 5/8.$$

Baye's Theorem 1.10.28.

Let S be a sample space. Let $A_1, A_2, ... A_k$ be disjoint subsets of S such that $A_1 \cup A_2 \cup ..... \cup A_k = S$. Let H be an event in S. That is $H \subseteq S$. Then

$$p\ (A_i/H) = \frac{p\ (A_i)\ .\ p\ (H/A_i)}{\sum_{i=1}^{n} p(A_i)\ p(H/A_i)}$$

Proof. Since $H \subseteq S$, so we write

$$H = (H \cap A_1) \cup (H \cap A_2) \cup ..... \cup (H \cap A_k)$$

where $(H \cap A_i) \cap (H \cap A_j) = \phi$ for $i \neq j$.

$$p(H) = p(H \cap A_1) + p(H \cap A_2) + ... + p(H \cap A_k)$$

$$= \sum_{i=1}^{k} p(H \cap A_i) \qquad ...(1.10.3)$$

we know $\quad p\ (H \cap A_i) = p(H)\ .\ p(A_i/H)$ and

$$p(H \cap A_i) = p\ (A_i)\ .\ p\ (H/A_i)$$

Now $\quad p\ (A_i/H) = \dfrac{p\ (A_i)\ .\ p\ (H/A_i)}{p\ (H)} \qquad ...(1.10.4)$

Substituting (1.10.3) in (1.10.4) we get

$$p(A_i/H) = \frac{p\ (A_i)\ .\ p\ (H/A_i)}{\sum_{i=1}^{k} p(A_i)\ .\ p\ (H/A_i)}$$

The probabilities $p(A_1)\ p(A_2)\ ....\ p(A_k)$ are termed priori probabilities since they exist before we gain any information from the experiment.

Example 1.10.29. *Two urn contain 4 white and 6 black balls and 4 white and 8 black. One urn is selected at random and a ball is taken out. It turns out to be white. Find the probability that it is from the first urn.*

Solution. Let A_1 and A_2 be events that the first and second urn respectively are selected. Since the urn was selected at random, $p\ (A_1) = \frac{1}{2}, p(A_2) = \frac{1}{2}$.

Let W be the event that the ball taken out is white. Then

$$W = (A_1 \cap W) \cup (A_2 \cap W)$$

with $(A_1 \cap W) \cap (H_2 \cap W) = \phi$. Now

$$p(W) = p\ (A_1 \cap W) + p\ (A_2 \cap W)$$

$$= p(A_1)\ .\ p\ (W/A_1) + p\ (A_2)\ .\ p\ (W/A_2)$$

$$= \frac{1}{2} \times \frac{4}{10} + \frac{1}{2} + \frac{4}{12} = \frac{11}{30}.$$

By Baye's theorem

$$p\,(A_1/W) = \frac{p\,(A_1)\,.\,p(W/A_1)}{p\,(W)} = \frac{\frac{1}{2}\times\frac{4}{10}}{\frac{11}{30}} = 6/11.$$

The probability that the ball is from the first urn is 6/11. Similarly $p\,(A_2/W) = 1 - 6/11 = 5/11$.

Example 1.10.30. *The contents of urns I, II, III are as follows.*

1 white	*2 black*	*3 red balls*
2 white	*1 black*	*1 red balls*
4 white	*5 black*	*3 red balls*

One urn is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from urns I, II or III.

Solution. Let A_1, A_2, A_3 denote the events that the urn I, II, III is chosen respectively and let H be the event that the two balls taken from the selected urn are white and red. Then $p\,(A_1) = p(A_2) = p(A_3) = \frac{1}{3}$.

Now
$$p\,(H/A_1) = \frac{1\,C_1\times 3\,C_1}{6C_2} = \frac{1}{5}$$

$$p\,(H/A_2) = \frac{2C_1\times 1C_1}{4C_2} = \frac{1}{3}$$

$$p\,(H/A_3) = \frac{4\,C_1\times 3C_1}{12\,C_2} = 2/11.$$

Hence
$$p\,(A_2/H) = \frac{p\,(A_2)\,.\,p\,(H/A_2)}{\sum\limits_{i=1}^{3} p\,(A_i)\,.\,p\,(H/A_i)} = \frac{\frac{1}{3}\times\frac{1}{3}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{11}} = \frac{55}{118}$$

Similarly
$$p\,(A_3/H) = \frac{\frac{1}{3}\times\frac{2}{11}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{11}} = \frac{30}{118}$$

$$p\,(A_1/H) = 1 - \frac{55}{118} - \frac{30}{118} = \frac{33}{118}.$$

Baye's Theorem for Future Events.

The probability of materialization of another event C given $p(C/A\cap H_1)$, $p\,(C/A\cap H_2)$...... is

$$p\,(C/A) = \frac{\sum\limits_{i=1}^{n} p\,(H_i)\,p\,(A/H_i)\,p\,(C/A\cap H_i)}{\sum\limits_{i=1}^{n} p\,(H_i)\,p\,(A/H_i)}$$

Example 1.10.31. *Three boxes of the same appearance have the following proportion of balls.*

I	*2 black*	*1 white*
II	*1 black*	*2 white*
III	*2 black*	*2 white*

One of the urn is selected and ball is drawn. It turns out to be white. What is the probability of drawing white ball again, if the first one is not replaced.

Solution. Let A_i denote the event of selection of i^{th} urn. Let E be the event of drawing white ball. $p(A_1) = p\ (A_2) = p(A_3) = 1/3$. Now

$$p\ (E/A_1) = \frac{1}{3},\ p(E/A_2) = 2/3,\ p(E/A_{l3}) = 2/4.$$

Let C denote the future event of drawing other white ball from the urns.

$$p\ (C/A_i \cap E) = 0,\ p(C/A_2 \cap E) = \frac{1}{2};$$

$p\ (C/A_3 \cap E) = 1/3$. Then

$$p\ (C/A) = \frac{\sum_{i=1}^{3} p(A_i)\ .\ p(E/A_i)\ p\ (C/A_i \cap E)}{\sum_{i=1}^{3} p\ (A_i)\ .\ p(E/A_i)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{3}\times 0+\frac{1}{3}\times\frac{2}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3}}{\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{1}{2}} = 1/3.$$

Example 1.10.32. *Let A and B be events with* $p(A) = \frac{1}{3}$, $p(B) = \frac{1}{4}$ *and* $p(A \cap B) = \frac{1}{5}$. *Find (i)* $p\ (A/B)$ *(ii)* $p\ (B/A)$ *(iii)* $p\ (\overline{B}/\overline{A})$

Solution.

(i)
$$p\ (A/B) = \frac{p\ (A \cap B)}{p\ (B)} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{4}{5}$$

(ii)
$$p\ (B/A) = \frac{p\ (A \cap B)}{p\ (A)} = \frac{1/5}{1/3} = 3/5$$

(iii)
$$p\ (\overline{B}/\overline{A}) = \frac{p\ (\overline{A} \cap \overline{B})}{p\ (\overline{A})} = \frac{p\ (\overline{A \cup B})}{1-p(A)} = \frac{1-p\ (A \cup B)}{1-p\ (A)}$$

$$= \frac{1-[p(A)+p(B)-p(A \cap B)]}{1-p(A)} = \frac{1-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}}{1-\frac{1}{3}} = \frac{37}{40}$$

Bernoulli's Theorem. In a single trial of an experiment if the probability of occurrence of an event be p, then the probability of occurrence of this event r times in n trials is $nC_r\, p^r\, q^{n-r}$, where $q = 1 - p$.

Example 1.10.33. *One biased coin is tossed twenty times and in each trial probability of getting head is the same and is given by 1/3. Find the probability of occurrence of*

(i) Getting exactly 18 heads

(ii) Not more than 18 heads

(iii) Atleast 4 heads

Solution. Let A be the event that we get a head in a toss of a coin. Then $p(A) = 1/3$ (given). $p(\overline{A}) = 1 - p(A) = 2/3$.

(i) p (getting exactly 18 heads in 20 tosses)

$$= 20\, C_{18}\, p^{18}\, (1-p)^2, = \frac{20 \times 19}{1 \times 2}\left(\frac{1}{3}\right)^{18} (2/3)^2$$

(ii) p (getting not more than 18 heads)

$$= 1 - p \text{ (getting either 18 or 19 or 20 heads)}$$
$$= 1 - p \text{ (getting 18 heads)} - p \text{ (getting 19 heads)}$$
$$- p \text{ (getting 20 heads)}$$
$$= 1 - \left\{740/3^{20} + 20\, C_{19}\left(\frac{1}{3}\right)^{19} (2/3) + 20\, C_{20}\, (1/3)^{20}\right\} = 1 - (801/3^{20})$$

(iii) p (getting atleast 4 heads)

$$= 1 - p \text{ (getting 0 or 1 or 2 or 3 heads)}$$
$$= 1 - \{p \text{ (getting no head)} + p \text{ (getting 1 head)}$$
$$+ p \text{ (getting 2 head)} + p \text{ (getting 3 head)}\}$$
$$= 1 - \left\{(2/3)^{20} + 20C_1\, (1/3)\, (2/3)^{19} + 20\, C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{18} + 20\, C_3\left(\frac{1}{3}\right)^3 (2/3)^{17}\right\}$$

Note. The probability of the number of success so obtained $p\,(X = x) = nC_x\, p^x\, q^{n-x}$ is called the Binomial probability distribution. Note that $q^n + nc_1\, pq^{n-1} + \ldots\ldots + nc_x\, p^x\, q^{n-x} + \ldots\ldots + p^n = (q+p)^n = 1$. If n independent trials constitute one experiment and the experiment is repeated N times, then x successes would be expected to occur is $N\, n\, C_x\, p^x\, q^{n-x}$. This is called the expected frequency of x successes in N experiments.

Note. The mean and standard deviation of Binomial distribution is np and $\sqrt{npq}$ respectively.

Example 1.10.34. *Find the binomial distribution whose mean is 6 and standard deviation is $\sqrt{2}$. Find the first two terms of the distribution.*

Solution. Mean is $np = 6$, and standard deviation $= \sqrt{npq} = \sqrt{np(1-p)}$. That is $np(1-p) = 2 \Rightarrow 6\,(1-p) = 2 \Rightarrow p = 2/3$. So $n = 9$. Now

$$f(x) = p(X = x) = nC_x\, p^x\, q^{n-x} = 9C_x\, (2/3)^x \left(\frac{1}{3}\right)^{9-x} \quad x = 0, 1, 2 \ldots\ldots 9.$$

The first term $= f(0) = 9C_0\, p^0\, q^9 = q^9 = (1/3)^9$

Second term $= f(1) = 9C_1\, p\, q^8 = 9\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8 = \frac{2}{2187}$.

Example 1.10.35. *A coin is tossed 400 times. Calculate the expected mean and standard deviation.*

Solution. Given $n = 400$, $p = 1/2$, $q = 1/2$. Mean $= np = 400/2 = 200$. Standard Deviation $= \sqrt{npq} = \frac{\sqrt{400}}{4} = 10$.

Example 1.10.36. *If 20% of bolts produced by a machine are defective. Determine the probability that out of 4 bolts choosen at random.*

(i) *1 will be defective* (ii) *less than 2 will be defective*

(iii) *atmost 2 will be defective.*

Solution.

(i) $p\,(X = 1) = nC_x\, p^x\, q^{n-x} = 4C_1\, (0.2)^1\, (0.8)^3 = 0.4096$

(ii) $p(X < 2) = p\,(X = 0) + p\,(X = 1)$

$$= 0.4096 + 4C_0\,(0.2)^{\circ}\,(0.8)^4 = 0.4096 + 0.4096 = 0.8192$$

(*iii*) $$p\,(X \le 2) = p\,(X = 0) + p\,(X = 1) + p\,(X = 2)$$

$$= 0.8192 + 4\,C_2\,(0.2)^2\,(0.8)^2\,p = 0.8192 + 0.1536 = 0.9728$$

Example 1.10.37. *Deal two packs of shuffled cards simultaneously. What is the probability that no pair of identical cards will be exposed simultaneously?*

Solution. Fix the first pack, and consider all possible rearrangements of the second pack. For each i in the range $1 \le i \le 52$ let A_i denote the set of all arrangements of the second pack which happen to have the property that the card in position i matches the card in position i of the first pack. Obviously $|A_i| = 51!$ for every i. Moreover, if $i \ne j$, then $|A_i \cap A_j| = 50!$ and so on. Let $X = \cup_i A_i$, so the probability of at least one match is $|X|/52!$. We calculate this using the inclusion-exclusion principle.

$$|X|/52! = (52!)^{-1}\left(\binom{52}{1}51! - \binom{52}{2}50! + \binom{52}{3}49! - \ldots - \binom{52}{52}0!\right)$$

$$= 1 - 1/2! + 1/3! - \ldots - 1/52! \approx 1 - \left(\sum_{i=0}^{\infty}(-1)^i/i!\right) = 1 - 1/e.$$

Thus the probability of no coincidences is (to an excellent approximation $1/e$.

Here we have use the fact that

$$e^x = 1 + x + x^2/2! + \ldots = \sum_{i=0}^{\infty} x^i/i!$$

and put $x = -1$.

Exercise 1.10

1. Let A and B be events with $p\,(A) = 3/8, p(B) = 5/8$ and $p\,(A \cup B) = 3/4$. Find (*i*) $p(A/B)$ (*ii*) $p\,(B/A)$.
2. Two fair dice are thrown. If the two numbers appearing are different, find the probability that the sum is 6.
3. An urn contains 3 red, 4 white and 5 black balls. Three balls are drawn at random. Find the probability that (*i*) all are black, and (*ii*) all are of different colours.
4. Let A and B be events with $p\,(A) = \frac{3}{8}, p(B) = \frac{1}{2}$ and $p(A \cap B) = \frac{1}{4}$. Find

   (*i*) $p\,(A \cup B)$ (*ii*) $p\,(\overline{A})$ and $p(\overline{B})$

   (*iii*) $p\,(\overline{A} \cap \overline{B})$ (*iv*) $p\,(\overline{A} \cup \overline{B})$

   (*v*) $p\,(A \cap \overline{B})$ (*vi*) $p\,(B \cap \overline{A})$
5. Suppose there are three boxes containing 2 white and 3 black balls ; 3 white and 2 black 4 white and 1 black balls respectively. There is equal probability of each box being chosen. One ball is drawn from a box at random. What ball is drawn from a box at random ? What is the probability that a ball drawn is white ?
6. A box contains five balls. Two balls are drawn and found to be white. What is the probability of all the balls being white ?

(**Hint.** *Let B be the probability that two white balls are drawn, A_1, A_2, A_3, A_4 be the probability that the box contains 2, 3, 4, or 5 white balls. Then* $p\,(A_1) = 1/4$, $p(A_2) = 1/4$, $p(A_3) = 1/4$, $p\,(A_4) = 1/4$ $p\,(B/A_1)$ *is the probability that the box contains 2 white balls and both been drawn is* $\frac{1}{4} \times \frac{2}{5} = \frac{1}{10}$

$$p(B/A_2) = \frac{1}{4} \times \frac{3}{5} = \frac{3}{20}\,;\; p\,(B/A_3) = \frac{1}{4} \times \frac{4}{5} = 1/5\,;\; p\,(B/A_4) = \frac{1}{4} \times \frac{5}{5} = 1/4.$$

7. A die is thrown 5 times. What is the probability of getting exactly 3 sixes ?

8. The probability of solving a question given to three students are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. Find the probability that the question is solved.

9. A die is rolled. If the outcome is an odd number, what is the probability that it is prime.

(**Hint.** *$S = \{1, 3, 5\}$. A be the event of getting a prime from S is {3, 5}. The required probability = 2/3.*)

10. Two numbers are selected at random from the integers 1 to 9. If the sum is even, find the probability that both numbers are odd.

Hint. *The sum of two numbers is even if either both are even or both are odd. The given numbers contain 4 even numbers and 5 odd numbers. The number of ways to choose 2 even numbers out of 4 is $4C_2 = 6$. The number of ways to choose 2 odd numbers out of 5 = $5C_2 = 10$. Number of ways to choose 2 numbers whose sum is even = 16. $\therefore$ p = 10/16 = 5/8.*

1.11. MULTISETS

Multisets are sets where an element can occur as a member more than once. For example :

$$A = \{1, 1, 1, 0, 0\}\ ;\quad B = \{a, a, a, b, c\}$$

The multisets A and B can be written as

$$A = \{3.1, 2.0\} \quad \text{and} \quad B = \{3.\ a, 1.\ b, 1.\ c\}$$

The *multiplicity* of an element in a multiset is defined to be the number of times the element appears in the multiset. The multiplicities of the elements 1 and 0 in the multiset A are 3, 2 and the multiplicities of the elements a, b, c are 3, 1, 1 respectively.

The theory of multisets is more general than the theory of sets.

The cardinality of a multiset is defined to be the cardinality of the set corresponds to, assuming that the elements in the multiset are all distinct.

Let A and B be multisets. The union of A and B denoted by $A \cup B$, is the multiset where the multiplicity of an element is the maximum of its multiplicities in A and B.

The intersection of A and B, denoted by $A \cap B$ is the multiset where the multiplicity of an element is the minimum of its multiplicities in A and B.

The *difference* of A and B, denoted by $A - B$, is the multiset where the multiplicity of an element is equal to the multiplicity of the element in A minus the multiplicity of the element in B if the difference is positive, and is equal to zero if the difference is 0 or negative.

The *sum* of A and B, denoted by $A + B$, is the multiset where the multiplicity of an element is the sum of multiplicities of the element in A and B.

Example. 1.11.1 *Let P and Q be two multisets {3.a, 2.b, 1.c} and {4.a, 3.b, 2.d} respectively. Find*

(*a*) $P \cup Q$ (*b*) $P \cap Q$ (*c*) $P - Q$ (*d*) $P + Q$.

Solution.

$$P \cup Q = \{4.a, 3.b, 1.c, 2.d\}$$
$$P \cap Q = \{3.a, 2.b\}$$
$$P - Q = \{1.c\}$$
$$P + Q = \{7.a, 5.b, 1.c, 2.d\}$$

1.12. Classified Solved Examples

Example 1.12.1 (*a*) *Define Power set and give the powersets of the following* :

(*i*) $\{a, \{b\}\}$ (*ii*) $\{1, \phi, \{\phi\}\}$

(*Visveswaraiah Tech. University, BE, March 2001*)

Solution. If S is any set, then the set of all subsets of S is called the Power set of S. The power set of S is denoted by $p(S)$.

(i) The power set of $\{a, \{b\}\}$ is

$$\{\{a\}, \{a, \{b\}\}, \{\{b\}\}, \phi\}$$

(ii) The power set of $\{1, \phi, \{\phi\}\}$ is $\{\{1\}, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}, \{1, \{\phi\}\}, \{1, \phi\}, \phi, \{1, \phi, \{\phi\}\}$.

(*b*) *Prove that "Null set is a subset of every set".*

(*Visveswaraiah Tech. University, BE, March. 2001*)

Solution. Let A, B be two sets. We say A is a subset of B if every element of A is also an element of B. So the null set containing no element is contained in any set A.

(*c*) *Let A and B be sets such that* $A \cup B \subseteq B$, $B \subseteq A$. *Find the venn diagram representation.*

Solution. Fig. 1.10

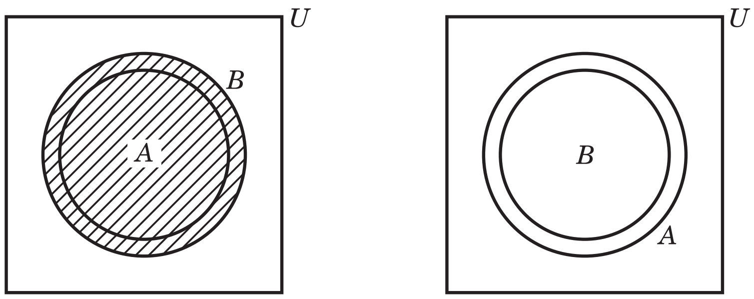
(*d*) *Prove that*

(*i*) Symmetric difference is associative.

(*ii*) $A - B = A - (A \cap B)$.

(*Visveswaraiah. Tech. University, BE, March 2001*)

Solution. Let A and B be two sets. The symmetric difference of A and B denoted by $A \Delta B$ is



$A \quad B \subseteq B$

Fig. 1.10

$$A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

To prove (*i*). Now for any sets $A, B, D,$

$$\begin{aligned}
A \Delta (B \Delta D) &= A \Delta ((B \cup D) - (B \cap D)) \\
&= A \Delta ((B \cup D) \cap (B \cap D)^C) = A \Delta (B \cup D \cap (B^C \cup D^C)) \\
&= [A \cup ((B \cup D) \cap (B^C \cup D^C))] - [A \cap ((B \cup D) \cap (B^C \cup D^C))] \\
&= [(A \cup B \cup D) \cap (A \cup B^C \cup D^C)] - [(A \cap B \cap D^C) \cup (A \cap D \cap B^C)] \\
&= [(A \cup B \cup D) \cap (A \cup B^C \cup D^C)] \cap [(A \cap B \cap D^C) \cup (A \cap D \cup B^C)]^C \\
&= (A \cup B \cup D) \cap (A \cup B^C \cup D^C) \cap (A^C \cup B^C \cap D) \cap (A^C \cup D^C \cup B)
\end{aligned}$$

...(1)

If we interchange A and D in eq. (1), then we get $(A \Delta B) \Delta D = D \Delta (A \Delta B) = D \Delta (B \Delta A)$.

$$(A \Delta B) \Delta D = (D \cup B \cup A) \cap (D \cup B^C \cup A^C) \cap (D^C \cup B^C \cup A) \cap (D^C \cup B \cup A^C)$$

which is same as (1).

Hence $\quad A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

To prove (*ii*) $\quad A - B = A - (A \cap B)$

Let $\quad x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow x \in A$ and $x \notin A \cap B$, since $x \notin B$ implies $x \notin A \cap B$.

$\Leftrightarrow x \in A - A \cap B$.

Hence, since x is arbitrary,

$$A - B = A - (A \cap B).$$

Example 1.12.2. (*a*) *Find the number of permutations of letters in the words*

(*i*) *ACCOUNTANT* (*ii*) *ENGINEERING*

(*iii*) *BOOLEAN* (*iv*) *ASSASSINATIONS.*

(*Visveswaraiah Tech. University, BE, March 2001*)

Solution.

(*i*) The letter A occur 2 times, hence there are 2 ! ways in which these A's can be rearranged among themselves. But then as this does not change the word as such. We have to divide by 2! to count the word ACCOUNTANT once. Similarly C occurs 2 times, N occur 2 times and T occur 2 times. Hence the number of ways ACCOUNTANT can be arranged is

$$\frac{10\,!}{2\,!\,2\,!\,2\,!\,2\,!}$$

(*ii*) $\frac{11\,!}{3\,!\,3\,!\,2\,!\,2\,!}$ (*iii*) $\frac{7\,!}{2\,!}$ (*iv*) $\frac{14\,!}{3\,!\,5\,!\,2\,!\,2\,!}$

Example 1.12.3. *How many different seven-person committees can be formed, each containing three female members from an available set of 20 females and four male members from an available set of 30 males*.

(*Visveswaraiah Tech. University, BE, March 2001*)

Solution. Three women's can be selected in C (20, 3) ways. Four men's can be selected in C (30, 4) ways. Therefore by multiplicative permutation law, the number of committees having 7 members are C (20, 3) C (30, 4).

(*a*) *State the pigeon hole principle and show that if any eight positive integers are chosen, two of them will have the same remainder when divided by 7.*

(*Visveswaraiah Tech. University, BE, March 2001*)

Solution. If $n + 1$ or more objects are placed in n boxes, then there is atleast one box containing two or more of the objects.

According to pigeon hole principle, for any eight positive integers, there must be two integer with same remainder, since we are dividing the integer by 7.

(*b*) Let n be an integer. If n^2 is odd, then n is odd. Suppose n is even, then $n = 2\,m$, where $m = 1, 2, \ldots\ldots$ Now $n^2 = (2\,m)^2 = 4\,m^2$, an even integer. Hence a contradiction. So n is odd.

(*c*) *Prove that for all* $n \geq 1$, $n\,! \geq 2^{n-1}$ *using Mathematical Induction*.

(*Visveswaraiah. Tech. University, BE, March 2001*)

Solution. *Basis step* : $1\,! = 1 \geq 1 = 2^{1-1}$

Inductive Step : Assume $n\,! \geq 2^{n-1}$ we prove

$(n + 1)\,! \geq 2^n$. Now $(n + 1)\,! = (n + 1)\,(n\,!) \geq (n + 1)\,.\,2^{n-1} \geq 2\,.\,2^{n-1}$, since $n + 1 \geq 2$

$= 2^n$.

Example 1.12.4 *Show that,*

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

(*Sri. Venkateswara Engg. College, Suryapet, A.P, B.Tech, March 2001*)

Solution. Proved already.

Example 1.12.5 *With the usual notations prove that*

(*i*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*Visveswaraiah Tech. University, MCA, August 2001*)

Solution.

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow x \in A \text{ or } x \in (B \cap C) \\ &\Leftrightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\ &\Leftrightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\ &\Leftrightarrow (x \in A \cup B) \text{ and } (x \in A \cup C) \quad \Leftrightarrow x \in (A \cup B) \cap (A \cup C) \end{aligned}$$

Since x is an arbitrary element,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Example 1.12.6. (*i*) *Prove by mathematical induction that*

$$1.2 + 2.3 + 3.4 + ... = \frac{n\,(n+1)\,(n+2)}{3}$$

(*ii*) $$\left[\bigcap_{i=1}^{n} A_i\right]^C = \bigcup_{i=1}^{n} A_i^C$$

(*Visveswaraiah Tech. University, M.C.A, August 2001*)

Solution.

(*i*) Basis step. $n = 1$

$$1.2 = \frac{1\,.\,2\,.\,3}{3}$$

Hence for $n = 1$, the result is true. Assume the result is true for $n = k$. Now we show the result holds for $n = k + 1$. Let $1.2 + 2.3 + ... + k\,(k+1) = \dfrac{k\,(k+1)\,(k+2)}{3}$.

Now $1.2 + 2.3 + ... + k\,(k+1) + (k+1)\,(k+2)$

$$\begin{aligned} &= \frac{k\,(k+1)\,(k+2)}{3} + (k+1)\,(k+2) \\ &= (k+1)\,(k+2)\left[\frac{k}{3} + 1\right] \\ &= \frac{(k+1)\,(k+2)\,(k+3)}{3} \end{aligned}$$

Hence the result is true for $n = k + 1$. By induction the result is true for all n.

(*ii*) Let $x \in (A_1 \cap A_2 \cap ..... \cap A_n)^C$

$$\begin{aligned} &\Leftrightarrow x \notin (A_1 \cap A_2 \cap .... \cap A_n) \\ &\Leftrightarrow x \notin A_1 \text{ or } x \notin A_2 \text{ or } ...... \; x \notin A_n \\ &\Leftrightarrow x \in A_1^C \text{ or } x \in A_2^C \text{ or } ...... \; x \in A_n^C \\ &\Leftrightarrow x \in (A_1^C \cup A_2^C \cup ...... \cup A_n^C) \end{aligned}$$

Hence $(A_1 \cap A_2 \cap .... A_n)^C = (A_1^C \cup A_2^C \cup ...... \cup A_n^C)$.

Example 1.12.7. *Prove that* $|\,A \times B\,| = |A|\;|B|$.

Solution. Let $m = |A|$ and $n = |B|$. If $n = 0$, then B and $A \times B$ are both empty and the result holds trivially. Assume $n > 0$ and let the distinct elements of B be $b_1, b_2, \ldots\ldots b_n$. For each $i = 1, 2, \ldots\ldots n$, let X_i be the set $A \times \{b_i\}$. Then X_i is a subset of $A \times B$ and $A \times B = \bigcup_{i=1}^{n} X_i$. Also for $i \neq j$, $y_i \neq y_j$ and so $A_i \cap A_j = \phi$. Further, for every i, the function $f_i : A \to X_i$ defined by $f_i(a) = (a, b_i)$ is clearly a bijection. So $|X_i| = |A| = m$ for every i. Since $\left|\bigcup_{i=1}^{n} X_i\right| = m + m + \ldots\ldots + m$ (n times) $= mn$.

In general for any finite sets $A_1, A_2, \ldots\ldots, A_n$,

$$|A_1 \times A_2 \times \ldots\ldots \times A_n| = |A_1| \times |A_2| \times \ldots\ldots \times |A_n|.$$

Example 1.12.8. *For any three sets A, B, C, prove that*

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

(*Andhra University, B.Tech. March 2001*)

Solution.

$$\begin{aligned}(x, y) \in A \times (B \cup C) &\Leftrightarrow x \in A \text{ and } (y \in B \cup C)\\ &\Leftrightarrow x \in A \text{ and } (y \in B \text{ or } y \in C)\\ &\Leftrightarrow (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\\ &\Leftrightarrow (x, y) \in A \times B \text{ or } (x, y) \in A \times C\\ &\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)\end{aligned}$$

Hence the result follows.

Example 1.12.9. *If $A \cap B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.*

(*Madras University, BE, April 1998*)

Solution. (Solved already)

Example. 1.12.10. *How many 4 digits number can be formed with the ten digits 0, 1, 2,.....,9 if*

(*i*) *Repetitions are allowed*

(*ii*) *Repetitions are not allowed*

(*iii*) *The last digit must be zero and repetitions are allowed.*

(*Bellary Eng. College, U.T, University, Karnataka, B.E., August 2001*)

Solution.

(*ii*) Since this is a problem of arranging 4 of the ten digits 0, 1, 2,...., 9, the answer is $P(10, 4) = 5040$. Among these 5040 numbers, $9 \times 8 \times 7 = 504$ of them have a leading 0. Consequently, $5040 - 504 = 4536$ of them do not have a leading 0. The first digits can be one of the nine digits 1, 2,....9, the second digits can be any of the nine remaining digits and so on.

Example 1.12.11. (*a*) *Show that $A \cup (B \cup C)^C = (A \cup B^C) \cap (A \cup C^C)$*

(*b*) *Show by induction on n that a set with n elements has precisely 2^n subsets.*

(*Bharathiar University, B.Sc. (C.T), October 2001*)

Solution.

(*a*)

$$
\begin{aligned}
& x \in A \cup (B \cup C)^C \\
& \Leftrightarrow x \in A \text{ or } x \in (B \cup C)^C \\
& \Leftrightarrow x \in A \text{ or } (x \in B^C \cap C^C) \\
& \Leftrightarrow x \in A \text{ or } (x \in B^C \text{ and } x \in C^C) \\
& \Leftrightarrow (x \in A \text{ or } x \in B^C) \text{ and } (x \in A \text{ or } x \in C^C) \\
& \Leftrightarrow x \in (A \cup B^C) \text{ and } x \in A \cup C^C \\
& \Leftrightarrow x \in (A \cup B^C) \cap (A \cup C^C)
\end{aligned}
$$

Hence the result follows.

(*b*) Proved Already.

Example 1.12.12. (*a*) (*i*) *Prove using venn diagram*

$$(A \cup B)^C = A^C \cap B^C$$

(*ii*) *If A, B, C be sets if* $A \subseteq B$, $B \cap C = \phi$, *then* $A \cap C = \phi$.

(*iii*) *Prove by induction 5 divides* $8^n - 3^n$ *for* $n \geq 1$.

(*Bharathiar University, M.Sc (SE), October 2001*).

Solution. (Proved already)

Example 1.12.13. *Find the intersection of the following sets* $\{x : x^2 - 1 = 0\}$ *and* $\{x : x^2 + 2x + 1 = 0\}$

Solution. $\{-1\}$

Example 1.12.14. *Using Mathematical induction prove* $n < 2^n$ $(n > 1)$ *(proved already).*

Example 1.12.15. *If S is any set and* $P(S)$ *is its power set and A and B belong to* $P(S)$, *prove that* $B \cap (A - B) = \phi$.

Solution. Since $A - B = A \cap B^C$, $B \cap (A - B) = B \cap (A \cap B^C) = A \cap \phi = \phi$.

(*Madras University, B.E., October 1995*).

Example 1.12.16. *List all the proper subsets of {1, 2, 3}.*

Solution. {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}.

Example 1.12.17. *State the duality principle in set theory.*

Solution. (Explained in page 15).

(*Madras University, BE April 1996*).

Example 1.12.18. *Show that for any two sets A and B,* $A - (A \cap B) = (A - B)$

Solution. (Worked out on Page 5, Example b (*i*))

(*Madras University, B.E., April 1996*).

Example 1.12.19. *Survey is taken on methods of computer travel. Each respondent is asked to check bus, train or automobile, as a major method of travelling to work. More than one answer is permitted. The results reported were as follows* :

(*i*) *30 people checked bus*
(*ii*) *35 people checked train*
(*iii*) *100 people checked automobile*
(*iv*) *15 people checked bus and train*
(*v*) *15 people checked bus and automobiles*

(*vi*) *20 people checked train and automobile*
(*vii*) *5 people checked all three methods*.

How many respondents completed their surveys ?

Solution. Let A, B, C be the persons who checked bus, train and automobiles respectively. Then by principle of inclusion and exclusion we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 30 + 35 + 100 - 15 - 15 - 20 + 5$$
$$= 120$$

The number of respondents who have completed their survey is 120.

Example 1.12.20. *If* $A \cup B = A \cup C$, *must* $B = C$ *? Explain.*

Solution. B need not be equal to C. Only when $A \cap B = A \cap C$, the result is true.

(*Madras University, BE, April 1997*)

Example 1.12.21. *How many proper non empty subsets are there for {1, 2, 3, 4}*.

Solution. There are $2^4 - 1$ proper non empty subsets for {1, 2, 3, 4}.

Example 1.12.22. *If survey of 500 television watchers produced the following information. 285 watch football games ; 195 watch hockey games ; 115 watch basket ball games ; 50 watch hockey and basket ball games ; 50 do not watch any of the three games. 45 watch football and basket ball games ; 70 watch foot ball and hockey games*.

(*i*) *How many people in the survey watch all three games* ?

(*ii*) *How many people watch exactly one of the three games* ?

(*Madras University, BE, April 1997, October 1998*)

Solution. Let A, B, C be the persons watching football, hockey, basketball games respectively.

Given : $|(A \cup B \cup C)|^c = 50$. So $|A \cup B \cup C| = 500 - 50 = 450$. By principle of inclusion and exclusion, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cup B \cup C|$

$\Rightarrow 450 = 285 + 195 + 115 - 70 - 45 - 50 + x$

$\Rightarrow x = |A \cap B \cap C| = 20$

i.e. Number of peoples watching all the three games is 20 and number of peoples watching exactly one is 325.

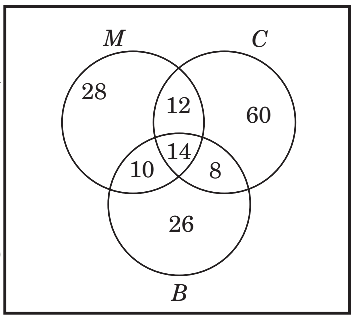


Fig. 1.11

Example 1.12.23. *If the cardinality of a set A is K, what is the cardinality of its power set P*(*A*).

Solution. Cardinality of the power set $= 2^k$.

(*Madras University, BE, October 1996*)

Example 1.12.24. *If the cardinality of the power set P*(*A*) *is 256, what is the cardinality of the set A* ?

Solution. Cardinality of the power set $P(A) = 256 = 2^8$. Then cardinality of the set $A = 8$.

Example 1.12.25. *Prove using mathematical induction*

$$2 + 5 + 8 + \ldots.. + (3n - 1) = \frac{n(3n + 1)}{2}$$

(*Madras University, BE, October 1997*)

Fig. 1.12

Example 1.12.26. **(*a*)** *If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, prove $B = C$.*

(*b*) *Among 50 students in a class, 26 passed in first semester and 21 passed in second semester examination. If 17 did not pass in either semester, how many passed in both semesters.*

(*Madras University, BE, April 1998*)

Solution. (*b*) 8 persons passed in both semester, since $|(A \cup B)^c| = 17$ implies $|A \cup B| = 33$ and hence $|A \cup B| = 20 + 21 - |A \cap B|$ implies $|A \cap B| = 8$.

Example 1.12.27. *Let $A = \{+, -\}$ and $B = \{00, 01, 10, 11\}$. List the elements of $A \times B$.*

Solution. $A \times B = \{(+ , 00), (+, 01), (+, 10), (+, 11), (-, 00), (-, 01), (-, 10), (- , 11)\}$

(*Madras University, BE, October 1998*)

Example 1.12.28. *Enumerate the elements in the following sets* :

(*a*) $\{x \in R : x^2 - 3x + 2 = 0\}$ (*b*) $\{x \in R : x^2 + 1 = 0\}$

Solution. (*a*) $x = 1, 2$ (*b*) $x = 1$

Example 1.12.29. *For any two sets A and B prove that*

$A - (A \cap B) = A - B$

Solution. (Proved in page 5)

(*Madras University, BE, April 2000*)

Example 1.12.30. (*a*) *If $A = \{x : x^2 - 1 = 0\}$, $B = \{x : x^2 - 3x + 2 = 0\}$. Find $A \cap B$ and $A - B$.*

$A = \{1, -1\}$, $B = \{1, 2\}$, $A \cap B = \{1\}$, $A - B = \{-1\}$

(*b*) *Prove by mathematical induction*

$$1 + 2 + 3 + \ldots.. + n = \frac{n\,(n+1)}{2}$$

(*Madras University, BE, April 2001*)

Solution. LHS : $P(1) = 1$, RHS : $P(1) = \frac{1 \times 2}{2} = 1$

Hence it is true for $n = 1$.

Let it be true for some integer k.

Therefore $P(k) = 1 + 2 + 3 + \ldots. + k = \frac{k\,(k+1)}{2}$

Now $P\,(k + 1) = 1 + 2 + 3 + \ldots.. + k + (k + 1) = \frac{k\,(k+1)}{2} + (k + 1)$

$$= \frac{k\,(k+1) + 2\,(k+1)}{2} = \frac{(k+1)\,[k+2]}{2}$$

Hence it is true for $k + 1$.

By induction, $1 + 2 + \ldots. + n = \frac{n(n+1)}{2}$.

Example 1.12.31. *Prove by using counting argument that*

(*i*) $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}$

(*ii*) $\binom{n}{m} = \sum_{k=0}^{r} \binom{r}{k}\binom{n-r}{m-k}$

(*B.Tech, U.P. Technical Univ., Lucknow, 2002-2003*)

Solution. (*i*) $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \dfrac{n!}{k!\,(n-k)!} + 2\,\dfrac{n!}{(k-1)!\,(n-k+1)!} + \dfrac{n!}{(k-2)!\,(n-k+2)!}$

$$= \frac{n!}{k\,(k-1)\,(k-2)!\,(n-k)!} + \frac{2n!}{(k-2)!\,(k-1)\,(n-k)!\,(n-k+1)} + \frac{n!}{(n-k+2)\,(n-k+1)\,(n-k)!\,(k-2)!}$$

$$= \frac{n!}{(k-2)!\,(n-k)!}\left\{\frac{1}{k\,(k-1)} + \frac{2}{(k-1)\,(n-k+1)} + \frac{1}{(n-k+2)\,(n-k+1)}\right\}$$

$$= \frac{n!}{(k-2)!\,(n-k)!}\left\{\frac{n^2+3n+2}{k\,(k-1)\,(n-k+1)\,(n-k+2)}\right\}$$

$$= \frac{n!\,(n+1)\,(n+2)}{(k-2)!\,(n-k)!\,(k-1)\,k\,(n-k+1)\,(n-k+2)} = \frac{(n+2)!}{k!\,(n+2-k)!}$$

$$= \binom{n+2}{k}$$

Example 1.12.32. *In a meeting of 50 scientists and poets, 35 are scientists, 30 have short hair and 25 are scientists with short hair. How many long-haired poets are there* ?

(*BE (CSE), Rajiv Gandhi Univ., Bhopal, 2001-2002*)

Solution. Let A be the set of scientists and B be the set of poets. Given $|A| + |B| = 50$ and $|A| = 35$, so $|B| = 50 - 35 = 15$. Number of peoples having short hair is 30, so number of peoples with long hair is $50 - 30 = 20$. Given 25 are scientists with short hair. So number of scientists with long hair $= 35 - 25 = 10$. Hence number of poets with long hair $= 20 - 10 = 10$.

Example 1.12.33. *Among 100 students, 32 study mathematics, 20 study physics, 45 study chemistry, 15 study mathematics and chemistry, 7 study mathematics and physics, 10 study physis and chemistry, 30 do not study any of the three subjects. Find the number of students studying exactly one of the three subjects.*

(*BE (CSE), Rajiv Gandhi. Univ., Bhopal, May-June 2002*)

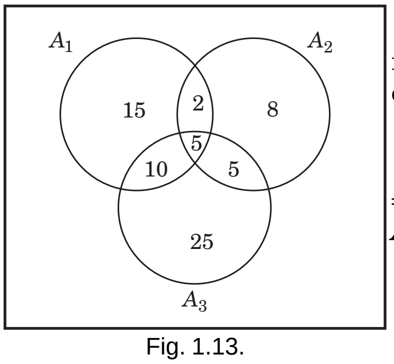


Fig. 1.13.

Solution. Let A_1 be the set of students studying Mathematics, A_2 be the set of students studying Physics, A_3 be the set of students studying Chemistry. Then, given $|A_1| = 32$, $|A_2| = 20$, $|A_3| = 45$, $|A_1 \cap A_3| = 15$, $|A_1 \cap A_2| = 7$, $|A_2 \cap A_3| = 10$ and $|(A_1 \cup A_2 \cup A_3)^c| = 30$. So $|A_1 \cup A_2 \cup A_3| = 100 - |(A_1 \cup A_2 \cup A_3)^c| = 100 - 30 = 70$. Now $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$$\Rightarrow 70 = 32 + 20 + 45 - 7 - 15 - 10 + |A_1 \cap A_2 \cap A_3|$$

$$\Rightarrow |A_1 \cap A_2 \cap A_3| = 70 - 65 = 5$$

So number of students who have taken mathematics only is $32 - (10 + 5 + 2) = 15$. Number of students who have taken physics only is $20 - (5 + 5 + 2) = 8$. Number of students who have taken chemistry only is $45 - (10 + 5 + 5) = 25$. So the number of students studying exactly one subject is $15 + 8 + 25 = 48$.

Example 1.12.34. *How many solutions are there for the equation $x_1 + x_2 + x_3 + x_4 = 20$?*

(*i*) *If all x_i must be non-negative integers.*

(*ii*) *If all x_i must be non-negative integers and x_4 is atmost 10.*

(*B. Tech., U.P. Technical University, Lucknow, 2002-2003*)

Solution. (*i*) We know the number of unordered samples of size k can be formed from a set of n elements, if repetition is allowed is $\frac{n(n+1) \ldots\ldots (n+k-1)}{k!} = (n+k-1)C_k$. That is, the number of distinct solutions to the equation $x_1 + x_2 + \ldots\ldots + x_n = k$ equals the number of unordered samples of size k (repetition allowed) from the n-element set $\{x_1, x_2, \ldots. x_n\}$ is $(n + k - 1)\, C_k$. Hence the number of solution for the equation $x_1 + x_2 + x_3 + x_4 = 20$ is $(4 + 20 - 1)C_{20} = 23C_{20}$.

(*ii*) Let $x_1 + x_2 + x_3 = 10$ and x_4 is atmost 10 making $x_1 + x_2 + x_3 + x_4 = 20$. The number of distinct solutions to the equation $x_1 + x_2 + x_3 = 10$ is $(3 + 10 - 1)C_{10} = 12C_{10} = 66$. So the number of solutions for the equation $x_1 + x_2 + x_3 + x_4 = 20$ such that all x_i must be non-negative integers and x_4 is atmost 10 is $23\ C_{20} - 66$.

Example 1.12.35. *Determine the minimum number of elements that one needs to take from the set $A = \{1, 2, 3, \ldots.. 8\}$ to be sure that two of the numbers add up to 9.*

Solution. We have four pigeon holes here namely {1, 8}, {2, 7}, {3, 6}, {4, 5}. One can observe from here that any choice of five elements of A will generate that two of the numbers add upto 9. So 5 elements are needed to take from the set A.

Example 1.12.36. *One hundred students were asked whether they had taken courses in any of the three areas namely Information Technology (IT), Bio-Technology (BT) and Management (M). The result are as follows* :

45 had taken IT, 38 had taken BT, 21 had taken management ; 18 had taken I.T and BT ; 9 had taken I.T and M ; 4 had taken BT and M and 23 had taken no courses in any of the three areas. Construct the venn diagram for the above data.

(*Bharathiar Univ., M.Sc (SE), April 2002*)

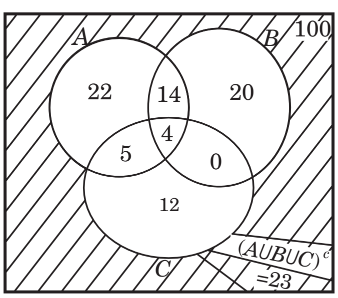


Fig. 1.14.

Solution. Out 100, 23 had taken no courses in any of the three areas. Let *A, B, C* be the set of all students who have taken IT, BT and M respectively. Then $|A \cup B \cup C|^c = 23$ and hence $|A \cup B \cup C| = 100 - 23 = 77$. Given $|A| = 45$; $|B| = 38$; $|C| = 21$; $|A \cap B| = 18$; $|B \cap C| = 4$; $|A \cap C| = 9$; $|A \cap B \cap C| = x$ (say). Then, we know

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$\Rightarrow 77 = 45 + 38 + 21 - 18 - 9 - 4 + x$$

$$\Rightarrow 77 = 73 + x$$

$$\Rightarrow x = 4.$$

The venn diagram representation is Fig. 1.14

Example 1.12.37. *A computer company must hire 25 programmers*

(*a*) *to handle systems programming jobs and 40 programmers for applications programming. Of these hired, ten will be expected to perform jobs of both types. How many programmers must be hired ?*

(*b*) *Show that the relation defined in* $N \times N$ *by* $(a, b)\, R\, (c, d)$ *iff* $a + d = b + c$ *is an equivalence relation.*

(*c*) *Let L denotes the relation "less than or equal to" and D denotes the relation 'divides'. Both L and D are defined on the set* $\{1, 2, 3, 6\}$. *Write L and D and find* $L \wedge D$. *What is the properties of L and D ? Give graphs of L and D.*

(*d*) *Using mathematical induction, prove* $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

(*Bharathiar University, B.E. (CSE), April 2003*)

Solution.

(*a*) Let S be set of systems programmers, then $|S| = 25$. Let A be set of applications programmers, then $|A| = 40$. Given $|A \cap S| = 0$. The number of programmers to be hired is $S \cup A$ and

$$|S \cup A| = |S| + |A| - |S \cap B| = 25 + 40 - 10 = 50$$

(*b*) For $(a, b) \in N \times N$, $(a, b)\, R\, (a, b)$ as $a + b = b + a$ is true. Hence R is reflexive. Let $(a, b)\, R\, (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a \Rightarrow (c, d)\, R\, (a, b)$. Hence R is symmetric. Let $(a, b)\, R\, (c, d)$, $(c, d)\, R\, (e, f) \Rightarrow a + d = b + c$ and $c + f = d + e$. Now

$$a + d + c + f = b + c + d + e \Rightarrow a + f = b + e$$

$\Rightarrow (a, b)\, R\, (e, f)$. Hence R is transitive. That is, R is an equivalence relation.

(*c*)
$$\begin{aligned} X &= \{1, 2, 3, 6\} \\ L &= \{(1, 1), (2, 2), (3, 3), (6, 6), (1, 2), (1, 3), (1, 6), (2, 3), (2, 6), (3, 6)\} \\ D &= \{(1, 1), (2, 2), (3, 3), (6, 6), (2, 6), (3, 6), (1, 6)\} \\ L \wedge D &= \{(1, 1), (2, 2), (3, 3), (6, 6), (2, 6), (3, 6), (1, 6)\} \\ &= D \end{aligned}$$

L is reflexive, non-symmetric, transitive, antisymmetric. D is reflexive, non-symmetric, transitive, antisymmetric.

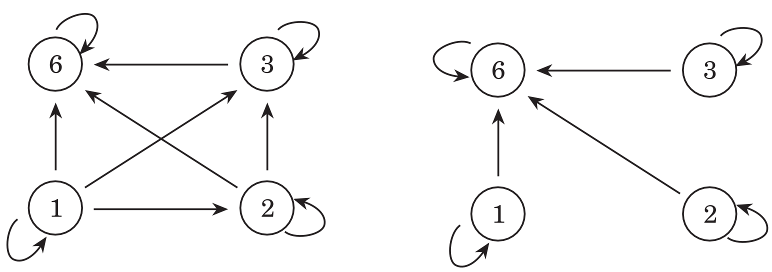


Fig. 1.15

Graph of L

(*d*) Let
$$p(n) = 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$
$$p(1) = 2 = 2^{1+1} - 2$$

Hence $P(1)$ is true. Assume for $n = k$. That is $P(k)$ is true. Consider

$$\begin{aligned} P(k+1) &= 2 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 2 + 2^{k+1} \\ &= 2 \,.\, 2^{k+1} - 2 \quad = 2^{k+2} - 2 \end{aligned}$$

Hence $P^{(k+1)}$ is true which implies $P(n)$ is true for any $n \in N$.

Example 1.12.38. **(*a*)** *Among 50 students in a class, 26 passed in first semester and 21 passed in second semester examination. If 17 did not pass in either semester, how many passed in both semesters.*

(**b**) *Show that $a^n - b^n$ is divisible by $(a - b)$ for all $n \in N$.*

(*Bharathiar Univ., M.Sc. (SE), April 2003*)

Solution.

(*a*) Let A be the set of students passed in first semester. Then $|A| = 26$. Let B be the set of students passed in second semester. Then $|B| = 21$, $|(A \cup B)^c| = 17$ which implies $|A \cup B| = 50 - 17 = 33$.

Now $|A \cup B| = |A| + |B| - |A \cap B|$

$\Rightarrow$ $33 = 26 + 21 - |A \cap B|$

$\Rightarrow$ $|A \cap B| = 26 + 21 - 33 = 14$.

(*b*) We prove the result by mathematical induction. For $n = 1$, $a - b$ is divisible by $a - b$. Hence the basis step. Let $P(k) : a^k - b^k$ is divisible by $a - b$.

Now $a^{k+1} - b^{k+1} = a^k \,.\, a - b^k \,.\, b = a^k \,.\, a - b^k \,.\, b + b^k a - b^k a = a\,(a^k - b^k) + b^k\,(a - b)$

Since $a^k - b^k$ is divisible by $a - b$, and $(a - b)$ is divisible by $(a - b)$, the sum $a\,(a^k - b^k) + b^k\,(a - b)$ is divisible by $a - b$. Hence $a^{k+1} - b^{k+1}$ is divisible by $a - b$. By induction, $a^n - b^n$ is divisible by $(a - b)$.

Example 1.12.39. *Let $S(n)$ be the statement. For $n \in Z^+$, $\sum_{i=1}^{n} i = \frac{(n + (1/2))^2}{2}$. Show that the truth of $S(k)$ implies the truth of $S(k + 1)$ for any $k \in Z^+$. Is $S(n)$ true for all $n \in Z^+$. Justify your answer.*

(*Anna University, ME (CSE), April 2003*)

Solution. Assume $S(k)$ is true. That is

$$\sum_{i=1}^{k} i = \frac{(k + (1/2))^2}{2}$$

Now
$$\sum_{i=1}^{k+1} i = 1 + 2 + \ldots.. + k + (k + 1) = \frac{(k + (1/2))^2}{2} + (k + 1)$$

$$= \frac{(k + (1/2))^2 + 2\,(k + 1)}{2} = \frac{k^2 + 3k + 9/4}{2} = \frac{(k + (3/2))^2}{2} = \frac{((k + 1) + (1/2))^2}{2}$$

Hence $S(k + 1)$ is true. It is not true for all $n \in Z^+$. For example, for $n = 1$, L.H.S. $= S(1) = 1$ and R.H.S. =

$$= \frac{[1 + (1/2)]^2}{2} = \frac{9}{4 \times 2} = \frac{9}{8}$$

Hence L.H.S. $\neq$ R.H.S.

Example 1.12.40. (*a*) *Given $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 4, 6, 8\}$, $B = \{2, 4, 5, 9\}$ and $C = \{x : x$ is a positive integer such that $x^2 \leq 16\}$.*

Compute (i) $\overline{A \cup B}$ (ii) $B \oplus C$.

(*b*) *Prove that the set of all real numbers in [0, 1] is uncountable.*

(*c*) *If f_A and f_B are characteristic functions of the sets A and B, prove that $f_{A \cap B} = f_A\, f_B$.*

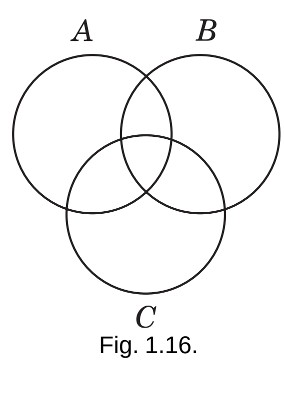


Fig. 1.16.

(*d*) *In a survey of 260 college students, the following data were obtained* :

64 had taken mathematics course

94 had taken computer science course

58 had taken business course

24 had taken mathematics and business courses

26 had taken mathematics and computer science courses

22 had taken computer science and business courses

14 had taken all the three courses

Find the number of students who have taken

(*i*) none of the courses and (*ii*) only the computer science courses

(*VTU Karnataka, BE (CSE/IT), Feb. 2003*)

Solution. (*a*)
$$U = \{1, 2, 3, \ldots\ldots, 9\}$$
$$A = \{1, 2, 4, 6, 8\}$$
$$B = \{2, 4, 5, 9\}$$
$$C = \{1, 2, 3, 4\}$$

(*i*) $\overline{A \cup B} = \{3, 7\}$

(*ii*) $B \oplus C$ is the set of elements that belong to B or C but not to both B and C. Hence
$$B \oplus C = (B - C) \cup (C - B)$$
$$= \{x : x \text{ belongs to exactly one of } B \text{ and } C\} = \{1, 3, 5, 9\}.$$

(*b*) To show that the set of real numbers in [0, 1] is uncountable set.

If possible, let the set of real numbers in [0, 1] is countable. Then all the real number in $0 \le x \le 1$ can be listed in some order, say, $x_1, x_2, \ldots\ldots$ Let the decimal representation of these real numbers be
$$x_1 = 0\,.\,a_{11}\,a_{12}\,a_{13}\,\ldots\ldots \qquad x_2 = 0\,.\,a_{21}\,a_{22}\,a_{23}\,\ldots\ldots$$
where each aij is one of the number of set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. Let C be a new real number such that $C = 0.\,b_1\,b_2\,b_3\,\ldots\ldots$, where $b_1 = 1$ if $a_{ii} = 9$ and $b_1 = 9 - a_{ii}$ if $a_{ii} = 0, 1, 2, 3, 4, 5, 6, 7, 8$ for all i. For those numbers which can be expressed in two different expansions *e.g.,* $\frac{1}{2} = 0.5000000 = 0.4999999$ we choose the expansion which ends with nines. Clearly, the number $0\,.\,b_1\,b_2\,\ldots\ldots$ is a real number between 0 and 1 that does not have trailing 0's. Then the real number C is not equal to any of $x_1, x_2, \ldots\ldots$ since it differs from the first in the first digit, the second number in the second digit, the i^{th} number in the i^{th} digit, and so on. Since there is a real number C between 0 and 1 that is not in the list which contradicts the assumption that this set is countably infinite. Hence the set of real numbers between 0 and 1 is uncountable.

(*c*) $f_{A \cap B}(x) = 1$ if $x \in A \cap B$

For $\quad x \in A$ and $x \in B$,
$$f_A(x) = 1 \text{ and } f_B(x) = 1, \quad \text{therefore} \quad f_A(x)\,f_B(x) = 1$$

Thus $\quad f_{A \cap B}(x) = f_A(x)\,f_B(x)$.

(*d*) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$
$$= 64 + 94 + 58 - 24 - 26 - 22 + 14 = 158$$

Students who had taken none of the courses is 260 – 158 = 102.

Students who had taken computer science only is 60.

Example 1.12.41 (***a***) *Prove by induction* $1^3 + 2^3 + ... + n^3 = \dfrac{n^2(n+1)^2}{4}$.

(***b***) *Find the number of ways in which a committee of 6 people can be selected from a group of 10 people if one of them is to be designed as the chair person of the committee.*

(*VTU, Karnataka, BE, Feb. 2003*)

Solution. (*a*) Basis step $n = 1$.

$$\text{L.H.S} = 1^3 = 1 \quad \text{R.H.S} = \frac{1^2\,(1+1)^2}{4} = 1, \text{ Hence for } n = 1, \text{ the result is true.}$$

Induction step. Assume

$$1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$$

$$\text{Now } 1^3 + 2^3 + ... + n^3 + (n+1)^3 = \frac{n^2\,(n+1)^2}{4} + (n+1)^3$$

$$= \frac{n^2\,(n+1)^2 + 4(n+1)^3}{4} = \frac{(n+1)^2\,[n^2 + 4\,(n+1)]}{4}$$

$$= \frac{(n+1)^2\,(n^2 + 4n + 4)}{4} = \frac{(n+1)^2\,(n+2)^2}{4}$$

Hence the result is true for all n.

(*b*) The number of ways in which a committee of 6 people can be selected from a group of 10 people is $10C_6$. Out of the grouping, one person can be selected as a chain person of the committee. That can be done in $6C_1$ ways. Hence the required number of ways is $(10C_6)\,(6C_1)$ ways.

Example 1.12.42. *If there are 14 boys and 12 girls in a class, find the number of ways of selecting one student as class representative.*

Solution. Using sum rule, there are $14 + 12 = 26$ ways of selecting one student.

Example 1.12.43. *Three persons enter into a car, where there are 5 seats. In how many ways can they take up their seats.*

Solution. The first person has a choice of 5 seats and can sit in any one of the 5 seats. The second person has a choice of 4 seats and so on. Hence, the required number of ways in which all the three persons can take the seat is $5 \times 4 \times 3 = 60$.

Example 1.12.44. *Show that in any set of eleven integers, there are two whose difference is divisible by 10.*

Solution. Let A be the given set of eleven integers, say (pigeons), B is the set $\{0, 1, 2, ... 9\}$ of pigeon holes, the possible right hand digits. By pigeonhole principle, two of integers have same right hand digit, thus their difference is divisible by 10.

Example 1.12.45. *What is coefficient of* $x^3y^2z^2$ *in* $(x + y + z)^9$?

Solution. This is same as how many ways one can choose x from three brackets, y from two bracket, z from 2 brackets in the expansion

$(x + y + z)\,(x + y + z)\,......\,(x + y + z)$ (9 times)

This can be done in $\begin{pmatrix} 9 \\ 3\ 2\ 2 \end{pmatrix} = \dfrac{9!}{3!\,2!\,2!} = 15120$ ways.

Example 1.12.46. *Find the coefficient of*

(*i*) x^{10} *in* $(1 + x^5 + x^{10} + ...)^3$ (*ii*) x^{12} *in* $(x^3 + x^4 + x^5 + ...)^3$

Solution.

(i) We know

$(1 + x^5 + x^{10} + ...)^3 = [(1 - x^5)^{-1}]^3 = (1 - x^5)^{-3} = \Sigma\, C(3 + r - 1, r)\, x^{5r}$

Since we have to find the coefficients of x^{10}, $5r = 10 \Rightarrow r = 2$. The required coefficient is $c(3 + 2 - 1, 2) = C(4, 2) = 6$.

(ii) $(x^3 + x^4 + x^5 + ...)^3 = x^9\, [1 + x + x^2 + ...]^3 = x^9\, [(1 - x)^{-1}]^3$

$= x^9\, (1 - x)^{-3} = x^9\, \Sigma\, C(3 + r - 1, r)\, x^r = \Sigma\, C(3 + r - 1, r)\, x^{9+r}$

The coefficient of x^{12} have $r + 9 = 12 \Rightarrow r = 3$, hence the required coefficient is $C(3 + 3 - 1, 3) = 10$.

Example 1.12.47.

1. *How many even numbers. of three digits can be obtained using 1, 2, 3, 4, 5 and 6, no digit being used more than once in each number?*

Solution. The unit place can be filled up by 2 or 4 or 6. This can be done in 3 ways. After filling up the unit place, we will have remaining 5 numbers. The 10^{th} place can be filled by any of the 5 numbers. This can be done in 5 ways. So 100^{th} place can be filled by any one of four numbers. Hence required number $= 3 \times 5 \times 4 = 60$.

2. *How many numbers of 6 digits which are divisible by 5 can be formed using the figures 3, 4, 5, 6, 7, 8.*

Solution. The unit place can be filled by only one way(s). The other 5 blank spaces can be filled up in $5p_5 = 120$ ways. Total number $= 5P_5 \times 1 = 120$.

3. *How many numbers between 5000 & 7000 can be formed by using the figures 1, 2, 3, 4, 5, 6, 7 and 8 if each figure is not used more than once in a number.*

Solution. To find out numbers between 5000 & 7000, we have.

Place	1000^{th}	100^{th}	10^{th}	Unit
Ways	2	7	6	5

The required answer is $2 \times 7 \times 6 \times 5 = 14 \times 30 = 420$ ways.

4. *A cricket club consists of 15 members of whom 7 are bowlers. In how many ways can a team of 11 chosen so as to include atleast 5 bowlers.*

Bowlers	*Non Bowlers*	*Total*
(7)	(8)	(15)
5	6	11
6	5	11
7	4	11

Required number of ways $= 7C_5 \times 8C_6 + 7C_6 \times 8C_5 + 7C_7 \times 8C_4 = 1050$

5. I*n how many ways can 3 persons be selected out of 15 persons so as to include always one particular person.*

Solution. First select the particular person. Then out of 14 persons, select 2 and include the particular person in all the sections required number of ways $= 14C_2$.

6. *Determine the number of positive integers n,* $1 \le n \le 2000$ *that are not divisible by 2, 3 or 5 but are divisible by 7.*

Solution. Let A denote the set of integers in $1 \le n \le 2000$ that are divisible by 2

B	||	3
C	||	5
D	||	7

To find $n\ ((\overline{A} \cap \overline{B} \cap \overline{C}) \cap D)$

$$
\begin{aligned}
n\ ((\overline{A} \cap \overline{B} \cap \overline{C}) \cap D) &= n(\overline{(A \cup B \cup C)} \cap D) \\
&= n(D) - n((A \cup B \cup C) \cap D) \\
&= n(D) - n((A \cap D) \cup (B \cap D) \cup (C \cap D)) \\
&= n(D) - [n\ (A \cap D) + n\ (B \cap D) + n\ (C \cap D) \\
&\quad - n\ (A \cap B \cap D) - n(A \cap C \cap D) - \\
&\quad n(B \cap C \cap D) + n\ (A \cap B \cap C \cap D)] \\
&= 285 - (142 + 95 + 57 - 47 - 28 - 5 + 9) = 76.
\end{aligned}
$$

7. *A computer is used for 99 hours over a period of 12 days, an integral number of hours each day. Show that on some pair of 2 consecutive days, the computer was used for atleast 17 hours.*

Solution. Among 12 days there are 6 pairs of days (pigeon holes) $\therefore\ n = 6,\ k_{+1} = 17 \Rightarrow k = 16$.

Hence among $k_{n+1} = 97$ hours.

8. *The members of a class of 27 pupils each go swimming on some of the days from Monday to Friday in a certain week. If each people goes atleast twice, show that there must be two pupils who go swimming on exactly the same days.*

Solution.

Members = pigeons

Days = pigeon holes

$n = 5$

$k_{+1} = 2 \Rightarrow k = 1$

Among $k_{n+1} = 6$ members who go swimming in exactly the same days (by generalized pigeon hole principle) $\therefore$ Among 27 pupils definetely there must be two pupils who go on swimming in the same day.

9. *How many four digit numbers begin with 4 or a 5.*

Solution. Numbers begin with 4 can be done in $1 \times 10 \times 10 \times 10 = 1000$ ways. Similarly numbers begin with 5 can be done in 1000 ways. By the addition principle, there are 1000 + 1000 = 2000 total possible outcomes.

10. *A jeweller designing a pin has decided to use five stones chosen from diamonds, rubies and emeralds. In how many ways can the stones be selected?*

Solution. We want the number of combinations of five objects out of 3 objects with repetitions allowed.

The pin might consists of one diamond, three rubies, one emerald, for instance, or five diamonds. We can represent these possibilities by representing the stones chosen by five asterisks and placing markers between the asterisks to represent the distribution among the three types of gem. For example we represent the choice of one diamond, three rubies, one emerald by

$$* \mid *** \mid *$$

and the choice of five diamonds, no rubies, no emerald by

$$***** \mid \mid$$

The number of ways to choose five items out of seven is $7C_5$ ways.

11. *Find the number of distinguishable permutations of the letters in the word ANNAMALAI. Among these permutations, in how many permutations the two N's appear next to each other.*

Solution. Number of distinguishable permutations = $\frac{9!}{4!\ 2!}$. Considering two N's as a single letter, the required permutations is $\frac{8!}{4!}$.

12. *The total number of bookings in 31 days in a month in a railway reservation counter is 7613. Show that on any one of these 31 days, there are atleast 246 bookings. Find also the minimum total number of bookings in 31 days to guarantee atleast 246 bookings on any one of these 31 days.*

Solution. If the maximum number of bookings for each day is 245, then maximum number of bookings for 31 days $= 245 \times 31 = 7595$, where it is given as 7613. Hence there will be atleast 246 bookings on any of 31 days.

Minimum total number of bookings to satisfy the given condition is $7595 + 1 = 7596$.

13. *How many 6-letter words not necessarily meaningful can be formed from the letters of CARACAS.*

Solution. Total number of letters = 7, 2C's, 3 A's, 1 R's and 1S. For 6 letter words, we have four options "1C, 3A, 1R, 1S", "2C, 2A, 1R, 1S", "2C, 3A, O R, 1S", "2C, 3A, 1R, OS".

$$\text{Total number } = \frac{6!}{1!\ 3!\ 1!\ 1!} + \frac{6!}{2!\ 2!\ 1!\ 1!} + \frac{6!}{2!\ 3!\ 0!\ 1!} + \frac{6!}{2!\ 3!\ 1!\ 0!} = 420$$

14. *Find the total number of three digit even integers that can be formed from the digits 1, 2, 3, 4, 5, 6, 7 if no digit is repeated in any of these integers.*

Solution. The last digit can be filled up by 2, 4, or 6. Hence the required number of even integers $= 6 \times 5 \times 3 = 90$.

15. *A die is thrown twice. What is the probability of getting the sum 9, given that the number turned in one of the dice is 6?*

Solution.

$$S = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

$$\therefore \qquad n(s) = 11$$

$$E = \{(3, 6), (6, 3)\}$$

$$n(E) = 2.$$

Hence probability $= \frac{2}{11}$.

16. *A coin is tossed two times. If you get atleast one head, you win Rs. 100, otherwise you lose Rs. 200. What is your expectation?*

Solution. p (atleast one head) $= \frac{3}{4}$

p (no head) $= \frac{1}{4}$

Expectation $= \frac{3}{4}(100) + \frac{1}{4}(-200) = 25$.

17. *A problem in discrete structure is given independently to 4 students A, B, C, D whose chances of solving it are 0.3, 0.5, 0.7 and 0.9 respectively. What is the probability that the problem is solved atleast by one of them? Also find the probability that the problem is solved by atmost one of them.*

Solution. P (at least one to solve)

$= 1 - P$ (no body solves) $= 1 - (0.7)(0.5)(0.3)(0.1) = 0.9895.$

P (at most one to solve) $= P$ (no body solves) $+ P$ (only one to solve)

$= (0.7)\,(0.5)\,(0.3)\,(0.1) + (0.3)\,(0.5)\,(0.3)\,(0.1) + (0.7)\,(0.5)\,(0.3)\,(0.1) + (0.7)\,(0.5)\,(0.7)\,(0.1)$

$+ (0.7)\,(0.5)\,(0.3)\,(0.9) = 0.1445$

18. *In a cell phone manufacturing company 50%, 30%, 20% of the phones are produced in the units A, B and C respectively. It is estimated that 90%, 80%, and 70% of the phones produced in units A, B and C respectively are accepted after inspection. A cell phone produced in this company is drawn at random and found to be in acceptable condition. Using Bayes theorem find the probability that the cell phone drawn was produced in unit C.*

Solution.

$$P(C \mid_{\text{acceptable}}) = \frac{P(\text{acceptable} \mid_{C}\ P(C)}{[P(\text{acceptable} \mid_{A})\ P(A) + P(\text{acceptable} \mid_{B})\ P(B) + P(\text{acceptable} \mid_{C})\ P(C)]}$$

$$= \frac{(0.7)(0.2)}{(0.9)(0.5) + (0.8)(0.3) + (0.7)(0.2)} = \frac{14}{83} = 0.1687.$$

1.13. Classified Unsolved Problems.

1. Consider the subsets $A = \{1, 7, 8\}$, $B = \{1, 6, 9, 10\}$, $C = \{1, 9, 10\}$**,** where $U = \{1, 2, 3, ... 10\}$. List the non-empty minsets generated by A, B, C. Do they form a partition of U.

(*Madras University, BE, October 2000*)

2. (*a*) Define Union of sets, disjoint sets. Let A and B be two arbitrary sets. Show that $P\,(A \cap B) = P(A) + P(B)$.

(*b*) Let A, B, C be subsets of universal set U. Given that $A \cap B = A \cap C$ and $\overline{A} \cap B = \overline{A} \cap C$. Is it necessary that $B = C$?

3. (*i*) Let $A = \{1, 3, 5\}$, $B = \{1, 2, 3\}$, $C = \{2, 3, 4\}$. Find $A \times (B \cap C)$.

(*BE, Maharashtra*).

(*ii*) Using Venn diagram, verify

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(*iii*) State the principle of mathematical induction.

(*iv*) Using mathematical induction prove that $n\,! \geq 2^n$ for $n \geq 4$

[*Bharathiar University, M.Sc (S.E) April 2002*]

4. (*i*) Of the 32 people who save paper or bottles (both) for recycling, 30 save paper and 14 save bottles. Find the number of people who save only paper.

(*ii*) A debating team consists of r boys and s girls. Find the number n of ways they can sit in a row if (*i*) the boys and girls are each to sit together, (*ii*) just the girls are to sit together.

(*Anna University, MCA, Dec. 2001*)

5. The members of a class of 27 pupils each go swimming on some of the days from Monday to Friday in a certain week. If each pupil goes atleast twice, show that there must be two pupils who go swimming on exactly the same days. (*Anna Univ., MCA, Dec. 2001*)

6. Prove by induction $\sum\limits_{i=1}^{n} i\,(i-1) = \dfrac{n}{3}\,(n^2 - 1)$ (*Anna Univ., MCA, Dec. 2001*)

7. Using Mathematical induction show that

$$\sum_{t=1}^{n} t^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{t=1}^{n} t\right]^2$$

(*Anna Univ., ME (CSE), Dec. 2001*)

8. Let A, B, C be subsets of the universal set $\overline{U}$.

(i) Given that $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, is it necessary that $B = C$? Justify your answer.

(ii) Define infinite sets and countability. What is the cardinality of the following sets

(i) $Z = \{\ldots\ -4, -3, -2, -1, 0, 1, 2, \ldots\}$

(ii) $N \times N$, N is the set of natural numbers.

9. (a) Let $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, c, f, g\}$, $B = \{a, e\}$, $C = \{b, h\}$. Compute

(i) $\overline{A \cup B}$ (ii) $\overline{A \cap B}$

(iii) $B \oplus B$ (iv) $\overline{U}$

(v) $\overline{B} \cup \overline{C}$ (vi) $A - B$.

(b) Let A, B, C be finite sets. Show that

(i) $|A \cup B| = |A| + |B| - |A \cap B|$

(ii) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.

(c) Using characteristic functions show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

(*V.T.U. Karnataka, BE (CSE), July/Aug. 2002*)

10. Prove by mathematical induction that

$$1^2 + 3^2 + 5^2 + \ldots\ldots.. + (2n-1)^2 = \frac{n\,(2n+1)\,(2n-1)}{3}$$

(*VTU, Karnataka, BE (CSE), July/August 2002*)

11. (i) How many bit strings of length 10 begin and end with a 1 ?

(ii) Using mathematical induction, prove that $2^n < n!$ for every positive integer n with $n \geq 4$.

(iii) A committee of k people is to be chosen from a set of 7 women and 4 men. How many ways are there to form the committee if

1. the committee has 5 people, 3 women and 2 men
2. the committee can be any positive size but must have equal numbers of women and men
3. the committee has 4 people and atleast two are men
4. the committee has 4 people, two of each sex and Mr. and Mrs. Singh cannot be on the committee. (*Anna Univ., MCA, Nov./Dec. 2002*)

(**Hint :** (i) $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 = 2^8$

(ii) Proved already

(iii) 1. $7C_3 \times 4C_2$ 2. $7C_1 \times 4C_1 + 7C_2 \times 4C_2 + 7C_3 \times 4C_3 + 7C_4 \times 4C_4$

3. $4C_2 \times 7C_2 + 4C_1 \times 7C_3 + 4C_o \times 7C_4$ 4. $4\ C_2 \times 5C_2$

12. (a) Using induction principle prove that $4n < (n^2 - 7)$ for all $n \geq 6$.

(**Hint :** Assume $n^2 - 7 > 4n, n \geq 6$, then $(n+1)^2 - 7 = (n^2 - 7) + 2n + 1 > 4n + 2n + 1 > 4n + 4$, since $n \geq 6$. Hence the result is true for $n + 1$ and hence the result is true for all n.)

(b) State the principle of Inclusion and Exclusion. Determine the number of positive integers n, where $1 \leq n \leq 100$ and n is not divisible by 2, 3 or 5.

(**Hint :** Find number of integers that are divisible by 2, 3 or 5. Then 100 – that number, gives the required result).

(c) Determine the number of positive integers n, $1 \leq n \leq 2000$ that are not divisible by 2, 3, or 5 but are divisible by 7. (*Anna Univ., ME, Dec. 2002*)

13. Prove the following by mathematical induction

(a) $$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots. + \frac{1}{(2n-1)\,(2n+1)} = \frac{n}{2n+1}$$

(b) Explain the pigeon hole principle with the help of suitable example.

(*B.E, IV Sem IT, Rajiv Gandhi, Tech. Univ., Bhopal, June 2002*)

14. If A and B are two non-empty sets then prove that

(i) $B - A = B \cap A$

(ii) $B \Delta A = (B - A) \cup (A - B)$

(*M.C.A., I Sem., Rajiv Gandhi Tech. Univ. Dec. 2002*)

15. (i) State the principle of inclusion and exclusion.

(ii) What do you understand by multisets ?

(*B.E (III Sem. CSE), Rajiv. Gandhi. Tech. Univ., 2001-2002*)

16. Show that

$$1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$$

by induction (*BE (CSE), Rajiv. Gandhi. Tech. Univ., 2001-2002*)

17. Assume A, B, C are arbitrary sets. You don't have to prove (i) anything or provide counter examples but only state if the following statements are true or false.

(i) $\{a, \phi\} \in \{a, \{a, \phi\}\}$

(ii) If $A \in B$ and $B \subseteq C$, then $A \in C$

(iii) If $A \in B$ and $B \subseteq C$, then $A \subseteq C$

(iv) If $A \subseteq B$ and $B \in C$, then $A \in C$

(v) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$

(vi) Prove the following or provide a counter example

$$A \cup B \subseteq A \cap B \Rightarrow A = B.$$

(vii) Fibonacci numbers F_n are defined by

$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let the sequence of numbers G_n be defined by

$G_1 = 1, G_2 = 3, G_n = G_{n-1} + G_{n-2} + 1$ for all $n \geq 3$. Prove, using induction, that $G_n = 2F_n - 1$ for all $n \geq 1$.

($viii$) Use Induction to show that

$$2 + 4 + 6 + ... + 2n = n^2 + n.$$

(ix) For all integers, $n \geq 2$, prove that

$$\sum_{i=1}^{n-1} i\,(i+1) = \frac{(n\,(n-1)\,(n+1))}{3}$$

(x) In a teaching room there are 20 seats, which are formed by 5 rows, each 4 tables wide. A certain class using that room has 9 students.

In how many ways can these 9 students be seated in that teaching room ?

(*B.Tech, V sem. U.P. Technical. Univ., Lucknow, 2002-2003*)

18. Show that

$1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n-1} n^2 = (-1)^{n-2} \dfrac{n\,(n+1)}{2}$ by mathematical induction.

(*BE (CSE), Rajiv Gandhi, Univ., May-June 2002*)

19. (a) Prove that $2^n < n!$ for all $n \geq 4$ and $n \in Z^+$.

(b) How many positive integers ≤ 100 are relatively prime to 100 ?

(*Anna Univ., M.E., April 2003*)

20. State the principle of inclusion and Exclusion. Determine the number of positive integers n, $1 \leq n \leq 2000$ that are not divisible by 2, 3 or 5 but are divisible by 7.

(*Anna Univ, M.E., April 2003*)

21. If $P = \{a, c, e\}$, $Q = \{100, 101, 102\}$, $R = \{m, c, e, 101\}$, compute

$$((Q \cup P - P \cap R) \times R$$

where $\cup$, —, $\cap$, X are set-theoretic binary operations. *(IGNOU, MCA II yr, June 2003)*

22.(*a*) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with 6 members if it must have more women than men.

(*b*) A computer is used for 99 hours over a period of 12 days, an integral number of hours each day. Show that on some pair of 2 consecutive days, the computer was used for atleast 17 hours.

(*c*) Using mathematical induction prove that

$$1^2 + 2^2 + \ldots + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}, \text{ where } n \text{ is a +ve integer.}$$

(*d*) A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian. 14 have taken courses both French and Russian. If 2092 students have taken atleast one of Spanish, French, Russian, how many students have taken a course in all three languages.

(*e*) How many ways are there to form a three letter sequence using the letters *a, b, c, d, e, f*

(*i*) with repetition of letters allowed ?

(*ii*) without repetition and containing the letter *e* ?

(Anna Univ., MCA, April/May 2003)

23. Define multinomial theorem.

The multinomial theorem states that for all real numbers $a_1, a_2, \ldots\ldots, a_k, n \in N$, we have

$$(a_1 + a_2 + \ldots.. + a_k)^n = \sum_{n_1 + n_2 + \ldots + n_k = n} \binom{n}{n_1 \ldots n_k} a_1^{n_1}\ a_2^{n_2} \ldots.\ a_k^{n_k}$$

Here $\binom{n}{n_1\, n_2 \ldots.\, n_k} = \frac{n!}{n_1!\, n_2! \ldots.\, n_k!}$, is called the multinomial coefficient

Given real numbers $a_1, a_2, \ldots.\, a_k$, $(a_1 + a_2 + \ldots. + a_k)^n = (a_1 + a_2 + \ldots.. + a_k)\,(a_1 + a_2 + \ldots. + a_k)\ldots(a_1 + a_2 + \ldots\, a_k)$. After performing this product but before collecting like terms, a typical term in this product has the from $a_1^{n_1}\, a_2^{n_2} \ldots.\, a_k^{n_k}$. The coefficient of $a_1^{n_1}\, a_2^{n_2} \ldots.\, a_k^{n_k}$ after collecting like terms is equal to a_1, n_2 factors equal to a_2, so on, as we multiply the n copies of $(a_1 + a_2 + \ldots.\, a_k)$. This is precisely the multinomial coefficient $\binom{n}{n_1\, n_2 \ldots.\, n_k}$.

24. Show that if any 11 numbers are chosen from the set {1, 2, ..., 20}, then one of them will be a multiple of another.

Hint : Create 10 or fewer pigeon holes in a such a way that each number chosen can be assigned to only one pigeonhole and when x and y are assigned to the same pigeonhole, either x/y or y/x. There are 8 prime numbers between 1 and 20 but knowing that x and y are multiples of the same prime will not guarantee that either x/y or y/x.

We try again. There are 10 odd numbers between 1 and 20. Every positive integer $n = 2^k\, m$, where m is odd, $k \geq 0$. If 11 numbers are chosen from {1, 2, ..., 20}, then two of them must have the same odd part, follows by pigeonhole principle.

Let n_1 and n_2 be chosen numbers with same odd part m. That is, $n_1 = 2^{k_1}\, m$; $n_2 = 2^{k_2}\, m$ for same k_1, k_2 with $k_1 \geq k_2$. If $k_1 \geq k_2$, then n_1 is a multiple of n_2 ; Otherwise n_2 is a multiple of n_1.