1

Measurement Systems and Their Characteristics

1.1. Introduction

The development of science and technology is very much dependent upon a parallel development of measuremenet techniques. Knowledge largely depends on measurement, and the technology of measurement, called instrumentation, serves not only science but all branches of engineeering, medicine and almost every spheres of human life. Measuring instruments are used in monitoring and control of processes and operations. Most specialized instruments are used in experimental scientific and engineering work.

Measurement methods may be classified into direct and indirect methods. In direct methods of measurement the quantity to be measured is compared directly against a standard of same knind of quantity. The magnitude of the quantity being measured is expressed in terms of a chosen unit for the standard and a numerical multiplier. A length can be measured in tems of metre and a numerical constant. Thus, a 10 metre length meas a length ten times greater than a metre.

Direct methods of measurement are though simple, it is not always possible, feasible and practicable to use them. The involvement of a man in these methods makes them inaccurate and less sensitive. For these reasons, engineering applications use measuremenet systems which are indirect methods of measurement. A currentis measured by an ammeter which gives a deflection of a pointer on a scale corresponding to the current. Thus, the current is not compared with a standard current, rather it is converted into a force which causes the pointer to deflect.

An electrical signal is a versatile quantity because of the fact that it can be easily amplified; attenuated, measured, rectified, modified, modulated, transmitted, and controlled. This fact created the interest to use electrical methods to measure non-electrical quantities. For this purpose a device known as transducer is used to convert the non-electrical quantities into electrical quantities. Then the quantities are indirectly manipulated with speed and versatility found in electrical measurement systems. Further, higher speed and versatility found in electronic instruments make them more popular. The electronic instruments, now-a-days, are computing, manipulating, and processing information in much the same way as the mind. for these reasons, the importance of studying electronic instruments is increased.

Increase in availability and types of computer facilities, and decrease in the cast of various modulus required for digital systems are accelrating the development of digital instrumentation for the measurement. The digital form of measurement is also used to display the measured quantity in readable numbers instead of a deflection of a pointer on a scale which completely eliminates a number of human errors.

1.2. Preliminaries of Measurements

Though today we have very sophisticated measurement systems, we can not think of a measurement without error. The error can be reduced by selecting a proper method of measurement and by taking some necessary precautions at the time of measurement. Recording the measured data also plays an important role.

Choice of Measurement Method. It is an important work to select a suitable method of measurement before starting it. At the time of selection, the following points should be kept in mind :

1. Apparatus available
2. Accuracy desired
3. Time required
4. Difficulties in measurement
5. Necessary conditions of measurement.

A method must be selected that makes use of available apparatus to obtain the desired result without sacrifying the desired accuracy. During the time of selection it should be kept in mind that the difficulties faced during measurement canbe easily overcome. At the same time necessary conditions must be fulfilled. It is not wise to choose a method giving higher accuracy than desired one at the cost of time and money. The method should be as simple as possible and consistent with requirements of the task. It is always beneficial to study carefully different apparatus before starting the actual measurement. Before starting measurement with a particular method, it is advisable to look whether there is a better and simpler method. Wide experiences in the field of measurement are very helpful in selecting a method.

After selection of the method and the apparatus, it is important that they be intelligently used. For this each piece of apparatus and its method of operation should be thoroughly understood. The line diagram of electrical circuits should be drawn in the beginning. It saves time and minimizes the possibilities of wrong connections of apparatus. Before energising it is necessary to check measuring instruments, other apparatus and electrical circuits. Taking necessary precautions gives better result.

Record Preparation. Record preparation of any experiment is not less important. Therefore, the data necessary for preparation of record must be written carefully. These data are recorded mostly in a bound note-book. Sometimes data may be written on loose sheets and after arranging them properly they can be permanently bound together. The habit of memorizing the data or writing them in short is not good because there is every possibility of forgetting some of the data which may cause great inconvenience. Without the line diagram of electrical circuits and specifications of all apparatus, a report can never be said to be complete. Report should be such that the experiment can be repeated with the same method and apparatus at any time. Thus error in the result due to unusual functioning of the instrument or due to any other reason can be removed. Any unusual behaviour of apparatus should be noted on the data sheet, and, if recorded data are rejected or discarded, the reasons for the action should be recorded. In short, the report must be such that any other person can get every information about the experiment just by going through the record or even can repeat the experiment at any time.

Precautions in Measurement. Certain precautions are essential which must be taken to ensure the safe and efficient use of instruments and also to get better result. There are some precautions that should be taken in general regardless of the instruments and the type of measurement undertaken. In making electrical connections, it should be seen that contact surfaces are clean, nuts are firmly tightened, wires and cables have sufficient cross-section for the expected current, and insulation is appropriate for the voltage in use. Sliding contacts should be cleaned occasionally. For the measurement of a large alternating current or voltage, a low range instrument with an instrument transformer should be preferred to a large range instrument. Before energising a circuit all components should be checked to ensure the proper connections, and appropriate range of apparatus. Protective resistors should always be inserted where necessary. At the time of opening a circuit the first break should be made at the terminal nearest the power source. The reversed procedure should be adopted at the time of making the connection, *i.e.* the connection at the power terminal should be made at the last. The operator should be careful where there is a chance of electric shocks.

Other precautions are applicable directly to instruments rather than to the general circuit. The position of the range switch of a multi range instrument should be checked before closing the circuit. If the initial current is much higher than the steady-state current, the current coils of instruments should be protected against the initial high current by a short-circuiting switch. When delicate instruments, such as micro-ammeter or pivoted galvanometers, are moved, they should be protected against mechanical damage by shorting the terminals to provide heavy overdamping. Where a coil clamp is provided it should always be set when the instrument is moved. Handling of instruments should be careful giving special attentionto laboratory standards.Pivoted instruments should never be placed where they may expose to vibration.

1.3. Characteristics of Instruments

We have discussed that the selection of an instrument which is most suitable for a proposed measurement is very important. To make intelligent choice, there must be some quantitative bases for comparing one instrument with the possible alternatives. The performance characteristics of instruments are generally divided into *static characteristics* and *dynamic characteristics*. The static characteristics involve the measurement of constant or only quite slowly varying quantities. Under these conditions a set of performance criteria are defined. These criteria give a meaningful description of the quality of measurement withoiut involving dynamic descriptions. Many other quantities vary rapidly wih time. Under these conditions, the dynamic relations between the instrument input and output must be examined by the use of some mathematical description, generally differential equations. These performance criteria are called the dynamic characteristics. The static characteristics are discussed in this section, while the dynamic characteristics will be discussed in succeeding sections.

State Characteristics. The static charactersitics are concerned with the measurement of quantities that are constant or vary only quite slowly. To get the constant output the instruments are calibrated by comparison with some standards of known accuracy. This process, in one form or another, is called static *calibration* and all the static performance characteristics are obtained by this. We shall, therefore, devote sometimes to clear the concept of this term.

Static Calibration. There may be one or more inputs influencing the output(s) of an instrument. Static calibration refers generally to a situation where all inputs except one are kept at some constant values. The concerned input is changed over some range of constant values, causing the output(s) to vary over some range of constant values. This input-output relation comprises a static calibration valid under the conditions that all the other inputs are constant. The procedure may be repeated, in turn for each input, to develop a family of static input-output relations. The overall instrument static behaviour may be obtained by some suitable form of superposition of individual effects or in some cases by variation of several inputs simultaneously. In practice there may be many modifying and/or interfering inputs each of which might have quite small effects and which would be impractical to control. Thus, situations stated above is ideal one and can only be approached, but never reached, in practice.

It is important to exercise considerable care in choosing the standard instrument when calibrating the response of an instrument to its desired input by comparing with the standard one. It is not possible to calibrate the instrument with an accuracy greater than that of the standard. As a rule it can be followed that the standard instrument should be at least about 10 times as accurate as the instrument being calibrated. The standard may be the primary standard or the secondary standard (which itself has been calibrated against a primary standard). Thus, the accuracy of measurement can ultimately be traced to the relevant primary standard. This ability to trace the accuracy of measurement to the primary standard is called *traceability*.

Now, we discuss about general static characteristics which are of general interest of every instrument. Some of them are : (*i*) Accuracy and precision (*ii*) Error (*iii*) Sensitivity (*iv*) Drift (*v*) Resolution and threshold (*vi*) Hysteresis and dead zone (*vii*) Linearity (*viii*) Repeatability.

Accuracy and Precision. Accuracy plays an important role in the measurement of any quantity. So, it is necessary to discuss about it. The measurement of a quantity is based on some international fundamental standards. These fundamental standards are perfectly accurate, while others are derived from these. These derived standards are not perfectly accurate in spite of all precautions. In general, measurement of any quantity is done by comparing with derived standards which themselves are not perfectly accurate. So, the error in measurement is not only due to error in methods but also due to standards (derived) not being perfectly accurate. Thus measurement with 100% accuracy is not possible with any method. So, a measurement without error is impossible. Now, it is the duty of the person performing the experiment to keep the error within limit.

The final value of any quantity obtained by any method of measurements is being influenced by so many factors. For example, the value of resistance measured is influenced by temperature, current density along the wire, tension in the wire and other factors. Sometimes it is possible that due to some factors the error may be positive while due to some other factor it may be negative. This may result in accurate measurements as positive and negative errors would cancel each other. An average value of large number of readings, may give result free from error. But these are not always possible. In general error in a method or in an instrument remains constant at any time and so the average value may not be accurate.

When an ammeter with an error of ± 1% indicates exactly 10A, the true value of current is somewhere between 9.9A and 10.1A. Thus, the measurement accuracy is 1% which defines the closeness of the measured value to the true value.

The word 'precision' is often used in place of accuracy as if they are interchangeable. But this is not true. However, they are related. So, they should be distinguished. Accuracy of measurement is defined as the deviation of the measured value from true value. On the other hand, precision of measurement is defined as the maximum deviation of different readings from true value. Thus it is a measure of consistency in measurement. An example will clarify the point. A thermometer does not give 100% accurate value of temperature, but, whenever the same temperature is measured, it gives the same reading. The thermometer is said to be a precision thermometer though it is not perfectly accurate.

Let us consider an ammeter with digital display and reads up to three decimal points. Say, it reads 8.135 A (see Fig. 1.1). If current increases or decreases by 1 mA, the reading becomes 8.136A or 8.134A. Thus, the current is measured with the precision of 1 mA. When the current is between 8.135A an 8.136A, the displayed value would always be 8.135A.

Fig. 1.1. Digital ammeter display.

Significant Figures. The number of significant figures gives an indication of the precision of the measurement. Significant figures in which the result is expressed give information regarding the magnitude and the measurement precision of the measured quantity. The more significant figures indicate the greater precision of measurement.

A significant figures may be any one of the digits 1, 2, ..., 7, 8, 9. Zero is a significant figure except when used to fix the decimal point or to fill the places of unknown or discarded digits. Thus the significant figures in number 0.0052 (the length of a rod in Km) are 5 and 2, while in 2056 these are 2, 0, 5 and 6. For a number 390 000 (the population of a city), the zeros may or may not be the members of the significant figures. To avoid the uncertainty caused by zeros to the left of the decimal point, one should write this as 3.9×10^5 if two significant figures are intended, 3.90×10^5 if three, 3.900×10^5 if four and so forth. The significant figures are dependent on the unit in which the precision of measurement is intended. To clear the statement consider an example. If a capacitor is specified as having a capacitance of 25 μF, its significant figures are two. Here it indicates that the precision of measurement is of the range of 1 μF. If the value of the capacitor is given as 25.0 μF, then the capacitance is closer to 25.0 μF, than it is to 24.9 μF or 25.1 μF. Thus, the sigificant figure is three. In this case, the precision of measurement is upto one tenth of μF.

Errors in Measurement. By now it has become clear that a quantity can never be measured with perfect accuracy in practice. So it is necessary to know the limit of maximum possible error in any measurement. Measurements without this have no meaning. It has no meaning in saying that the resistance of a resistor is 100 ohm. But, if it is said that the resistance of a resistor is 100 ± 2.5 ohm, then it means that the resistor can be used wherever a resistance of value varying from 97.5 ohm to 102.5 ohm can be tolerated. Some definitions are being given below.

Absolute-error. 'Absolute-error' is also called as 'maximum possible error'. Error in measurement,

$$\delta R = A_m - A \qquad \text{...(1.3-1)}$$

where A_m = measured value

A = accurate value

Absolute error ($\in_0$) is the limit of error in measurement. In other words δR must never be higher than $\in_0$. So,

$$|\in_0| = \text{max. } |A_m - A| \qquad \text{...(1.3-2)}$$

Relative error. Absolute-error does not give any information about accuracy. For example, –1 volt error in measurement of 1100 volt is negligible, but –1 volt error in measurement of 10 volt is never acceptable. Relative-error is the ratio of absolute error with the accurate value. So, relative error,

$$\in_r = \frac{\in_0}{A} \qquad ...(1.3\text{-}3)$$

If $\in_0$ is negligibly small as compared to A_m, then Eq. (1.3-3) can be written as

$$\in_r = \frac{\in_0}{A_m} \qquad ...(1.3\text{-}4)$$

Generally, relative error is given in per cent of measured value,

i.e. per cent error $= 100 \in_r$ ...(1.3-5)

Correction. Correction is negative of error. So correction,

$$\delta C = -\delta R \qquad ...(1.3\text{-}6)$$

Also $A = A_m + \delta C$ ...(1.3-7)

So, addition of correctionin in the measured value gives accurate value.

Detail study about the different types of errors and their statistical analysis are given in Chapter 2.

Static Sensitivity. In general, sensitivity is defined as the ratio of the incremental output to the incremental input. When an input-output calibration curve is a straight line as that of Fig. 1.2 (*a*), the static sensitivity of the instrument is the slope of the calibration curve. If the calibration curve is not a straight line, which is normally the case, the sensitivity is not constants, it will vary with the input as shown in Fig. 1.2 (*b*). For a meaningful definition of sensitivity the output quantity must be taken as the actual physical output observed, not the meaning attached to the scale numbers. For an example, the actual physical output of a voltmeter is the angular deflection of the pointed and the unit of sensitivity, therefore, will be radian/volt.

Drift. We discussed about the sensitivity of an instrument to the desired input. The instrument is also sensitive to interfering and/or modyfing inputs. Due to these undesired inputs the output of the instrument drift from the accurate value. So, it may be of interest to know the sensitivity of the instrument to these undesired inputs. This will help in allowing correction of the readings. As an example, consider a dynamometer type instrument. Temperature can cause a change in the resistance of the coils that will result in a change in output reading even though the voltage has not changed. In this sense the temperature is an interfering input. Also, the temperature can change the constant of the controlling spring, in turn giving a change in the voltage sensitivity. In this sense, the temperature is a modifying input. The first effect is often called a *zero drift* while the second effect is called a *sensitivity drift or scale-factor drift.*

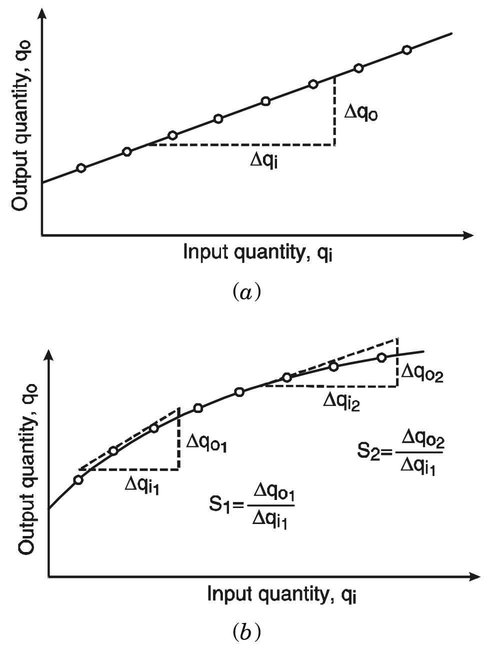


Fig. 1.2. (*a*) Linear response (*b*) non linear response.

Resolution and Threshold. If the input to an instrument is increased slowly from some arbitrary non-zero value, it will be observed that the output of the instrument does not change at all until there is a certain minimum increment in the input. This minimum increment in input is called resolution of the instrument. Thus, the resolution is defined as *the minimum input increment that gives definite numerical change in the output.*

If the input to an instrument is increased very slowly from zero value there will be some minimum value of input below which no output can be detected. *This minimum value of input is defined as the threshold* of the instrument. Thus, the resolution defines the minimum measurable input change while the threshold defines the minimum measurable input. Both resolution and threshold may be given either in absolute value or as a percentage of full scale reading.

Hysteresis and Dead Zone. If the voltage to a moving iron voltmeter, for an example, is slowly and smoothly varies from zero to full scale value and then back to zero, the input-output cue may appear as in Fig. 1.3 (*a*). The different loading and unloading curves are due to the magnetic hysteresis of the iron. This is hysteresis effect. For an instsrument with central zero the response will be as in Fig. 1.3(*b*).

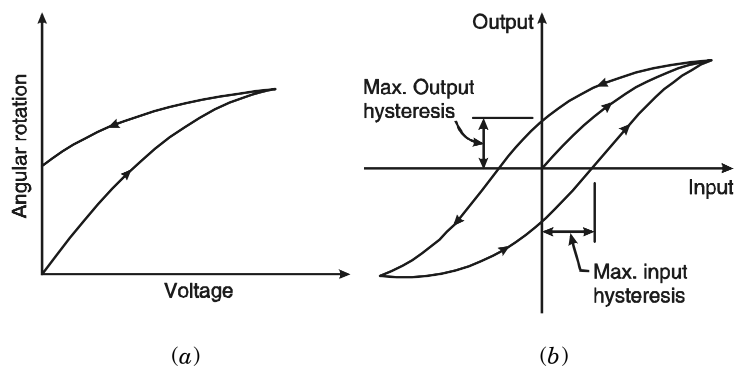


Fig. 1.3 Hysteresis loop.

The terms dead zone, dead band, and dead space are sometimes used in place of the hysteresis. However, they may be defined as the total range of possible values of input for a given output. They may, thus, be numerically equal to twice the hysteresis defined in Fig. 1.3 (*b*).

Linearity. Though an instrument with non-linear calibration curve may be highly acurate, there are many applications were linear behaviour of the instrument is most desirable. In case of linear behaviour of the instrumenet its sensitivity remains constant and it is convenient to find the measured value of the quantity from the scale reading just by multiplyhing it with a constant. When the behaviour of the instrument is non linear, the sensitivity varies with the quantity and for obtaining the measured value of the quantity, we need a calibration curve or equation giving relation between the input and output of the instrument. Because of above advantages of the linear behaviour, we prefer to sacrify some accuracy and fit an approximate linear calibration curve for a non-linear calibration curve.

The linearity is defined as the maximum deviation of any calibration point from a reference straight line. The reference straight linemay be the least squares fit of the calibration points. The least squares fittings of the straight line means that the sum of squares of the vertical deviation of the data points from the fitted line is minimum. The linearity is generally expressed as a percentage of full scale reading.

Repeatability. Repeatability is defined as the closeness of a number of measured values of the same quantity under the same conditions (such as the same observer, the same method, the same apparatus, and the same environment). This is affected by internal noise and drift. The repeatibility is expressed in percentage of the true value.

1.4. Dynamic Characteristics

In contrast to the static characteristics, the dynamic characteristics are concerned with the measurement of quantities that vary with time. To study the characeristics of such quantities the first step is to have a mathematicl model of the measurement quantities or the measurement systems. Generally a dynamic system is represented by a differential equation. Other forms of mathematical models of a dynamic system may be the transfer function model and the state space model. There are two methods of analysis of a dynamic response : *time domain analysis* and *frequency domain analysis*. We consider the measurement systems which are approximated as a linear time invariant system. Most of the measurement system can be represented by a first order or second order model.

Mathematical Model. A measurement system, in general, can be represented by an-nth order differential equation.

$$\frac{d^n q_o(t)}{dt^n} + a_{n-1}\frac{d^{n-1} q_o(t)}{dt^{n-1}} + \ldots + a_1 \frac{dq_o(t)}{dt} + a_o q_o(t)$$
$$= b_m \frac{d^m q_i(t)}{dt^m} + b_{m-1}\frac{d^{m-1} q_i(t)}{dt^{m-1}} + \ldots + b_1 \frac{dq_i(t)}{dt} + b_0 q_i(t) \quad \ldots(1.4\text{-}1)$$

where $q_o(t)$ = output quantity

$q_i(t)$ = input quantity

t = time

a's and b's = constant co-efficient depending on the parameters of the system

$n > m$

$\frac{d^r q(t)}{dt^r}$ = rth derivative of $q(t)$ w.r.t.time

Transfer function. In case the initial conditions in Eq. (1.4-1) are zero, the simple way of solving the differential Eq. (1.4-1) is to utilize the Laplace transform technique. Taking Laplace transform on both sides of Eq. (1.4-1), it can be written as

$$s^n Q_o(s) + a_{n-1}s^{n-1}Q_o(s) + \ldots + a_1 s Q_o(s) + a_0 Q_o(s)$$
$$= b_m s^m Q_i(s) + \ldots + b_1 s Q_i(s) + b_0 Q_i(s)$$

or
$$\frac{Q_o(s)}{Q_i(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + b_1 s + a_0} \qquad \ldots(1.4\text{-}2)$$

where $Q_o(s)$ = Laplace transform of the output quantity $q_o(t)$

$Q_i(s)$ = Laplace transform of the input quantity $q_i(t)$.

Define,
$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + b_1 s + a_0} \qquad \ldots(1.4\text{-}3)$$

$G(s)$ is called the *transfer function* of the system. The block diagram representation of the system is shown in Fig. 1.4.

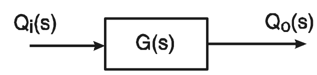


Fig. 1.4 Transfer function representation.

To get the solution of the differential equation we can write,

$$Qo(s) = G(s)\,Q_i(s) \qquad \ldots(1.4\text{-}4)$$

The time response $q_o(t)$ can be obtained by taking inverse Laplace transform of the right hand side of Eq. (1.4-4). The time repose $q_o(t)$ has two components. One component decays with time to zero and is independent of the type of input. It only depends on the dynamics of the system. This component is called the *transient response* of the system. Second component is the response which exists after the transient response is died out. This component is called the *steady state response* of the system.

The modern technique to solve an nth order differential equation, even with non-zero initial conditions, is the *state space technique*.

State space representation. For many applications, it becomes more convenient to express the system behaviour in terms of n first order differential equations. The matrix form of these n first order differential equations is known as the state equation. A simple procedure to convert the nth order transfer function $G(s)$ given by Eq. (1.4-3), hence the differential equation given by Eq. (1.4-1), to a state equation (not unique) is given as follows :

The block diagram of Fig. 1.4 can be broken into two blocks as in Fig. 1.5.

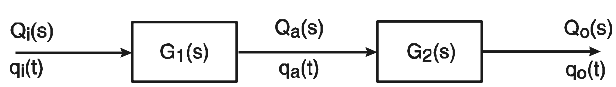


Fig. 1.5. Block diagram.

where $$G_1(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \ldots\ldots + a_1 s + a_0}$$

$$G_2(s) = b_m s^m + b_{m-1}s^{m-1} + \ldots\ldots + b_1 s + b_0$$

$q_o(t)$ = defined as auxiliary output.

Then, the dynamics of the auxiliary output is given by

$$\frac{d^n q_a(t)}{dt^n} + a_{n-1}\frac{d^{n-1} q_a(t)}{dt^{n-1}} + \ldots\ldots + a_1 \frac{dq_o(t)}{dt} + a_0 q_a(t) = q_i(t) \quad \ldots(1.4\text{-}5)$$

Now, we obtain a set of n first order differential equation of Eq. (1.4-5). Let

$$\begin{aligned}
x_1(t) &= q_a(t) \\
\dot{x}_1(t) &= x_2(t) = \dot{q}_a(t) \\
\dot{x}_2(t) &= x_3(t) = \ddot{q}_a(t) \\
&\vdots \\
x_{n-1}(t) &= x_n(t) = \overset{(n-1)}{q_a}(t)
\end{aligned}$$

Hence $\dot{x}_n(t) = -a_0 x_1(t) - a_1 x_2(t) \ldots\ldots - a_{n-1}x_n(t) + q_i(t) = q_a^n(t)$

The variables $x_1, x_2 \ldots\ldots x_n$ are called state variables. Dot (.) indicates derivatives w.r.t. time, $q_a^n(t)$ indicating nth derivative of $q_a(t)$ w.r.t. time.

In matrix form :

$$\dot{\mathrm{x}}(t) = A\dot{\mathrm{x}}(t) + Bq_i(t) \quad \ldots(1.4\text{-}6)$$

$$q_a(t) = C_a \mathrm{x}(t) \quad \ldots(1.4\text{-}7)$$

where $$A = \phi = 2\tan^{-1}\left(\frac{\sqrt{1-\delta^2}}{-\delta}\right)$$

$$\mathrm{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$C_a = [1, 0\ \ 0......0] \qquad ...(1.4\text{-}8)$$

The actual output is given by

$$q_0(t) = b_0 q_a(t) + b_1 \dot{q}_a(t) + ... + b_m q_a^m(t)$$

$$= b_0 x_1(t) + b_1 x_2(t) + ... + b_m x_{m+1}(t)$$

In matrix form :

$$q_0(t) = C\mathrm{x}(t)$$

where $C = [b_0\ b_1......b_m\ 0......0]$ ...(1.4-9)

In most cases constants b_1, b_2, ... b_m are zero and $b_0 = 1$ and the actual output is same as the auxiliary output. The output matrix $C = C_a$. Eq. (1.4-6) and Eq. (1.4-9) together are called state space model of the dynamic system given by Eq. (1.4-1).

1.5. Time Domain Analysis

In time domain analysis, the time is taken as independent variable and the time response (variation of output with time) is evaluated. A measurement system can be actuated by any kind of input functions. But, for analysis of the system we take only known inputs, called standard inputs. The response of a dynamic system generally consists of two components : (*a*) the transient response and (*b*) steady state response. If $q_o(t)$ is a time response, then in general,

$$q_o(t) = q_t(t) + q_{ss}(t)$$

where $q_t(t)$ and $q_{ss}(t)$ denote the transient and steady state parts of the response respectively.

We shall analyse the time responses of first order and second order systems. Order of a system is the highest order of derivatives of the differential equation representing the system. Thus a zero order system, in reality is represented by an algebraic equation and have static characteristics only. An example of a first order system is a *RL* circuit shown in Fig. 1.6. A step input of *E* volt is applied to the circuit. The loop equation is

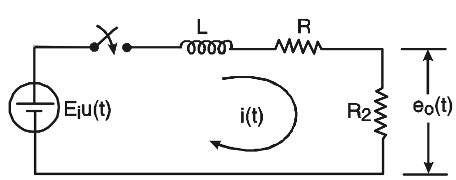


Fig. 1.6. *RL* circuit.

$$L\frac{di(t)}{dt} + (R_1 + R_2)\,i(t) = E_i u(t) \qquad ...(1.5\text{-}1)$$

Also, $$e_0(t) = R_2 i(t) \qquad ...(1.5\text{-}2)$$

Replacing $i(t)$, we have

$$\frac{L}{R_2}\frac{de_0(t)}{dt} + \frac{R_1 + R_2}{R_2} e_0(t) = E_i u(t) \qquad ...(1.5\text{-}3)$$

If $$a_1 \triangleq \frac{L}{R_2} \quad \text{and} \quad a_0 \triangleq \frac{R_1 + R_2}{R_2}$$

Then, $$a_1 \frac{de_0(t)}{dt} + a_0 e_0(t) = E_i u(t) \qquad ...(1.5\text{-}4)$$

which is a first order differential equations.

A series *RLC* circuit, shown in Fig. 1.7, can represent a second order system. A unit step input of E_i volt is applied to the circuit.

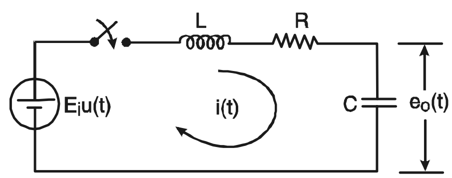


Fig. 1.7. *RLC* circuit.

From the loop equation,

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i\,dt = E_i u(t) \qquad ...(1.5\text{-}5)$$

and $$e_0(t) = \frac{1}{C}\int i\,dt$$

Hence $$i(t) = C\frac{de_0(t)}{dt}$$

Eq. (1.5-5) can be written as,

$$LC\frac{d^2 e_0(t)}{dt^2} + RC\frac{de_0(t)}{dt} + e_0(t) = E_i u(t)$$

or $$\frac{d^2 e_0(t)}{dt^2} + \frac{R}{L}\frac{de_0(t)}{dt} + \frac{1}{LC} e_0(t) = \frac{E_i}{LC} u(t) \qquad ...(1.5\text{-}6)$$

If $$a_1 = \frac{R}{L},\ a_0 = \frac{1}{LC} \text{ and } \frac{E_i}{LC} = b_0,$$

Then $$\frac{d^2 e_0(t)}{dt^2} + a_1\frac{de_0(t)}{dt} + a_0 e_0(t) = b_0\, u(t) \qquad ...(1.5\text{-}7)$$

represents a second order system.

Standard inputs. In the time domain analysis, the following test signals are often used :

1. Step input 2. Ramp input 3. Impulse input.

Step input. In this case there is an instantaneous change in the input variable. Its mathematical representation is

$$q_i(t) = K \qquad t > 0$$
$$q_i(t) = 0 \qquad t < 0 \qquad \text{...(1.5-8)}$$

Where K is a constant. Alternatively,

$$q_i(t) = K\,u(t) \qquad \text{...(1.5-9)}$$

Where $u(t)$ is the unit step function. The function $q_i(t)$ is not defined at initial time ($t = 0$). Its Laplace transform $Q_i(s) = \dfrac{K}{s}$. The step function is shown in Fig. 1.8(a).

Ramp (or velocity) input. Mathematically a ramp function is represented by

$$q_i(t) = K{\cdot}t \qquad t > 0$$
$$q_i(t) = 0 \qquad t < 0 \qquad \text{...(1.5-10)}$$

Alternatively,

$$q_i(t) = Kt\,u(t) \qquad \text{...(1.5-11)}$$

The Laplace transform is,

$$Q_i(s) = \frac{R}{s^2} \qquad \text{...(1.5-12)}$$

This function is also called velocity function. It is shown in Fig. 1.8(b).

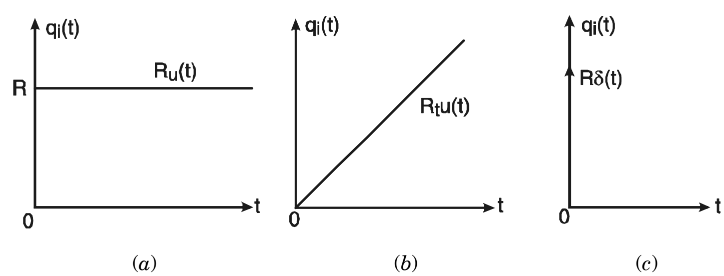


Fig. 1.8(a) Step input (b) ramp input (c) impulse input.

*Impulse function.*It is shown in Fig. 1.7(c) and is represented mathematically by

$$q_i(t) = K\delta(t) \qquad \text{...(1.5-13)}$$

Where $\delta(t)$ is the unit impulse function and its Laplace transform is unity. Hence,

$$Q_i(s) = K \qquad ...(1.5\text{-}14)$$

Response of a first order system. We shall study, now, the response of a first order system when standard input signals are applied. A first order system is generally representd by a first order differential equations,

$$a_1 \frac{dq_0(t)}{dt} + a_0 q_0(t) = b_0 q_i(t) \qquad ...(1.5\text{-}15)$$

Taking Laplace transform,

$$a_1 s\, Q_o(s) - a_1 q_o(0) + a_0 Q_o(s) = b_0 Q_i(s)$$

Where $q_o(0)$ is the initial value of the output. Let time constant and static gain be,

$$T = \frac{a_1}{a_0} \qquad m = \frac{b_0}{a_0}$$

Without loss of generality the gain m may be taken as unity. Hence;

$$Q_o(s) = \frac{Q_i(s)}{1 + Ts} + \frac{Tq_o(0)}{1 + Ts} \qquad ...(1.5\text{-}16)$$

Under the zero initial condition, *i.e.* $q_o(0) = 0$, the Eq. (1.5-16) reduces to

$$\frac{Q_o(s)}{Q_i(s)} = G(s) = \frac{1}{1 + Ts} \qquad ...(1.5\text{-}17)$$

where $G(s)$ is the transfer function of the first order system.

Response to step input. From Eq. (1.5-17)

$$Q_o(s) = \frac{Q_i(s)}{1 + Ts}$$

Since $$Q_i(s) = \frac{K}{s}$$

$$Q_o(s) = \frac{K}{s(1 + Ts)} = K\left(\frac{1}{s} - \frac{T}{1 + Ts}\right)$$

Taking inverse Laplace transform,

$$q_o(t) = K\left(1 - e^{-\frac{t}{T}}\right) \qquad ...(1.5\text{-}18)$$

This response is plotted in Fig. 1.9. From Eq. (1.5-18), the magnitude of the output for $t = T$ is

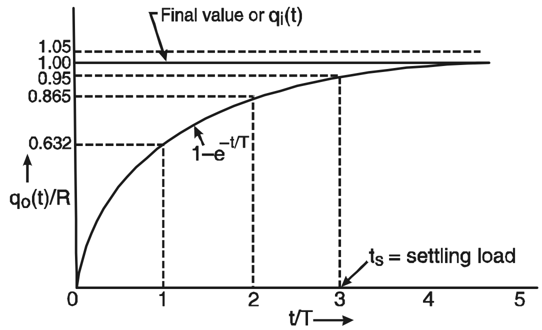


Fig. 1.9. Step response of first order system.

$$q_o(t) = 0.632\, K$$

Thus the time constant of the system is defined as the time to reach the response to 63.2% of its final value. The steady state value of the ouptut is given by

$$q_{ss} = \lim_{t\to\infty} q_0(t) = \lim_{t\to\infty} K\left(1 - e^{-\frac{t}{T}}\right) = K$$

This shows that the steady state error is zero. The dynamic error is given by

$$e_d = K - K\left(1 - e^{-\frac{t}{T}}\right) = Ke^{-\frac{t}{T}}$$

and the per unit error

$$\frac{e_d}{K} = \mathrm{e}^{-\frac{t}{T}} \qquad ...(1.5\text{-}19)$$

The plot of the error with time is shown in Fig. 1.10.

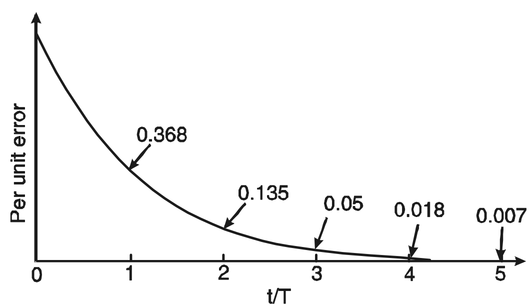


Fig. 1.10. Error in step response.

In case initial condition is not zero the response canbe obtained from

$$q_o(t) = K\left(1 - e^{-\frac{t}{T}}\right) + q_o(0)\, e^{-\frac{t}{T}} \qquad \text{...(1.5-20)}$$

Response to ramp input. From Eq. (1.5-12) the Laplace transform of a ramp function, $r(t) = Kt\,u(t)$, is

$$Q_i(s) = \frac{K}{s^2}$$

So, from Eq. (1.5-17),

$$Q_o(s) = \frac{K}{s^2(1+Ts)} = \frac{K}{s^2} - \frac{KT}{s} + \frac{KT^2}{1+Ts}$$

Taking the inverse Laplace transform,

$$q_o(t) = K\left\{t - T\left(1 - e^{-\frac{t}{T}}\right)\right\} \qquad \text{...(1.5-21)}$$

The response is plotted in Fig. 1.11. The dynamic error is as given below,

$$e_d = \text{input} - \text{output}$$

$$= Kt - K\left\{t - T\left(1 - e^{-\frac{t}{T}}\right)\right\} \qquad \text{...(1.5-22)}$$

or

$$\frac{e_d}{K} = T\left(1 - e^{-\frac{t}{T}}\right) \qquad \text{...(1.5-23)}$$

The steady state error

$$e_{ss} = \lim_{t\to\infty} T\left(1 - e^{-\frac{t}{T}}\right) = T \qquad \text{...(1.5-24)}$$

Thus, the steady state error is not zero, and the response will never track the input. It is equal to time constant in magnitude.

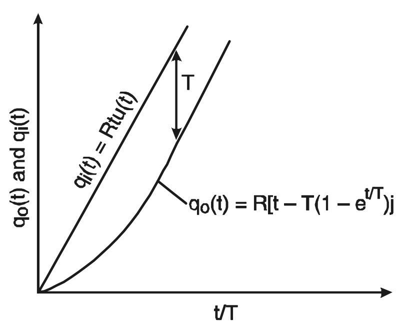


Fig. 1.11. Response with ramp input of a first order system.

Response to impulse input. Let the impulse function be of strength K. Hence, its Laplace transform, Eq. (1.5-14), is

$$Q_i(s) = K$$

From Eq. (1.5-17),

$$Q_o(s) = \frac{K}{1+Ts}$$

Taking inverse Laplace transform,

$$q_o(t) = \frac{K}{T} e^{-\frac{t}{T}} \qquad \text{...(1.5-25)}$$

Response of second-order system. Now, the responses of a second order system for standard signals are analysed. The second order system, in general, is represented by a differential equation.

$$a_2 \frac{d^2 q_0(t)}{dt^2} + a_1 \frac{dq_0(t)}{dt} + a_0 q_0(t) = b_0 q_i(t) \qquad \text{...(1.5-26)}$$

The transfer function representation is

$$\frac{Q_0(s)}{Q_i(s)} = \frac{m\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \qquad \text{...(1.5-27)}$$

where $m = \frac{b_0}{a_0}$ = static gain (or sensitivity)

$\omega_n = \sqrt{\frac{a_0}{a_2}}$ = undamped natural frequency (rad/s)

$\delta = \frac{a_1}{2\sqrt{a_0 a_2}}$ = damping ratio

The transient response of the system represented by Eq. (1.5-27) is determined by the roots of the characteristic equation,

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0 \qquad \text{...(1.5-28)}$$

The roots of Eq. (1.5-28) are

$$s_1,\ s_2 = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1} \qquad \text{...(1.5-29)}$$

There are four cases depending on the value of δ.

Case I. When $\delta > 1$. (overdamped case). The roots are real and unequal and are given by Eq. (1.5-29). In this case there is no oscillation and the response approaches the final value in finite time.

Case II. When $\delta = 1$ (critically damped case). The roots are real and equal. The response has no oscillation and it approaches the final value asymmtotically.

Case III. When $\delta = 0$ (undamped case). The roots are imaginary and equal in magnitude. The response is oscillatory and will never reach to the final value.

Case IV. When $\delta < 1$ (underdamped case). Here the oscillations decays with time. The oscillation is dependent on the value of δ $(0 < \delta < 1)$.

Response to unit step input. The second order system is represented (for $m = 1$) by the following equation,

$$\frac{Q_o(s)}{Q_i(s)} = G(s) = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \qquad \text{...(1.5-30)}$$

In case m is not unity, the response is just to be multiplied by m. For the unit step input,

$$Q_o(s) = \frac{\omega_n^2}{s\left(s^2 + 2\delta\omega_n s + \omega_n^2\right)} \qquad \text{...(1.5-31)}$$

For the step input of amplitude K, the unit step response is mulitplied by K. The time response $q_0(t)$ is obtained by taking inverse Laplace transform of Eq. (1.5-31). The response is given by the following equation,

$$q_0(t) = 1 + \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta}} \sin\left(\omega_n \sqrt{1-\delta^2}\, t + \phi\right) \qquad \text{...(1.5-32)}$$

where $\phi = \tan^{-1}\left(\frac{\sqrt{1-\delta^2}}{\delta}\right)$

The variation of the output response of the system for various values of damping δ against the normalized time $\omega_n t$ is plotted in Fig. 1.12. It is clear that the response becomes more oscillatory as δ decreases in value ($0 < \delta < 1$). When $\delta \geq 1$ there is no oscillation in the response ; the output never exceeds the reference input.

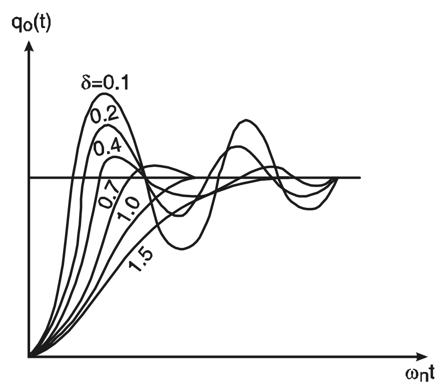


Fig. 1.12. Step responses of second order system.

The response is usually characterised by some specifications. Now, we shall defined those specifications, see Fig. 1.13.

1. *Overshoot.* The overshoot is defined as the maximum deviation of the output from the input during the transient state. It is often represented as a percentage of the final value, that is,

$$\text{Per cent overshoot} = \frac{\text{Maximum overshoot}}{\text{Final desired value}} \times 100 \qquad \text{...(1.5-33)}$$

This is also recognised as a measure of the relative stability of the system. More overshoot means poor stability.

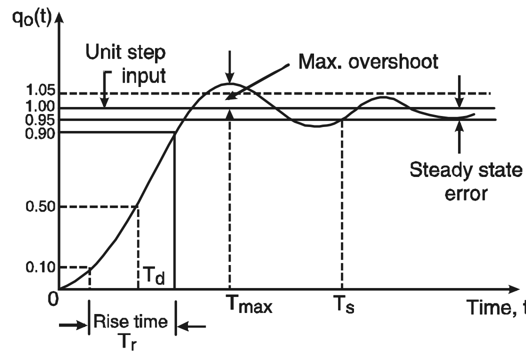


Fig. 1.13. System specification.

2. *Time delay.* The time delay (or delay time) T_d is defined normally as the time required for the response to reach 50% of the final desired value.

3. *Rise time.* The rise time T_r is equal to the time required for the response to rise from 10% to 90% of the desired final value.

4. *Settling time.* The settling time T_s is the time required for the response to reach first time within 5% of the final value. It is often taken as 3 times the time constant of the system.

We have already seen the dependence of the response on the damping ratio and natural frequency of the system.

Now, we shall derive a relation between the overshoot and the damping ratio of the system. Differentiate the time response $q_0(t)$, Eq. (1.5-32), with time and equate it to zero to obtain the time corresponding to maximum overshoot of the response from the final value. Hence, from Eq. (1.5-32),

$$\frac{dq_0(t)}{dt} = -\frac{\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \sin(\omega t + \phi) + \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \omega_n \sqrt{1-\delta^2} \cos(\omega t + \phi) \quad \text{...(1.5-34)}$$

$$= 0 \qquad (t \geq 0)$$

Take $\omega = \omega_n \sqrt{1-\delta^2}$

$$\phi = \tan^{-1} \frac{\sqrt{1-\delta^2}}{-\delta}$$

$$= \sin^{-1}\sqrt{1-\delta^2} = \cos^{-1}(-\delta) \quad \text{...(1.5-35)}$$

From Eqs. (1.5-34) and (1.5-35),

$$\frac{\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \sin \omega_n \sqrt{1-\delta^2}\, t = 0 \qquad (t \geq 0)$$

which gives two solutions :

$$t = \infty$$

and $$\omega_n \sqrt{1 - \delta^2}\ t = n\pi \qquad [n = 0, 1, 2, ...]$$

Since the first peak of the output response occurs at $n = 1$, hence

$$t_{max} = \frac{\pi}{\omega_n \sqrt{1 - \delta^2}} \qquad ...(1.5\text{-}36)$$

For this value of time, the maximum value of response obtained from Eq. (1.5-32) is

$$q_0\left(t_{max}\right) = 1 + e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}$$

Hence, maximum overshoot

$$= q_0\left(t_{max}\right) - 1 = e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}$$

and the percentage overshoot $= 100\ e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}$ ...(1.5-37)

It is clear from Eq. (1.5-37) that the overshoot of step response of a second order system depends onthe system damping ratio. Fig. 1.14 shows the relation between percentage overshoot and damping ratio.

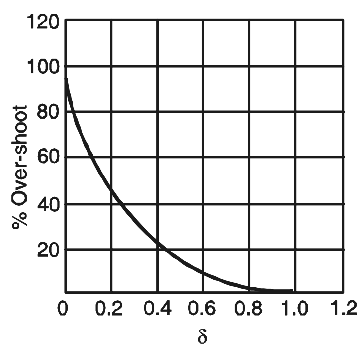


Fig. 1.14. Over shoot Vs. damping ratio.

From Eq. (1.5-32), the steady state error,

$$e_{ss} = \lim_{t \to \infty} \left\{q_0(t) - 1\right\} = 0 \qquad ...(1.5\text{-}38)$$

Response to unit ramp input. For a unit ramp input, $q_i(t) = tu(t)$, applied to the second order system, Eq. (1.5-30), the output response becomes.

$$q_0(t) = L^{-1}\left[\frac{\omega_n^2}{s^2 + \left(s^2 + 2\delta\omega_n s + \omega_n^2\right)}\right]$$

or $$q_0(t) = t - \frac{2\delta}{\omega_n} + \frac{e^{-\delta\omega_n t}}{\omega_n\sqrt{1-\delta^2}} \sin\left(\omega_n\sqrt{1-\delta^2}\ t - \phi\right) \qquad ...(1.5\text{-}39)$$

where $$\phi = 2\tan^{-1}\left(\frac{\sqrt{1-\delta^2}}{-\delta}\right)$$

This shows that transient response is similar to that for step input. But the response does not agree at steady state.

The steady state error,

$$e_{ss} = \lim_{t\to\infty} \{q_o(t) - q_i(t)\} = -\frac{2\delta}{\omega_n} \qquad \text{...(1.5-40)}$$

The response against the normalized time $w_n t$ is plotted in Fig. 1.15.

Response to impulse input. For a unit impulse input $q(t) = \delta(t)$, the response becomes

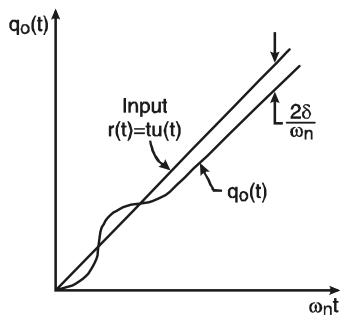


Fig. 1.15. Response to ramp input.

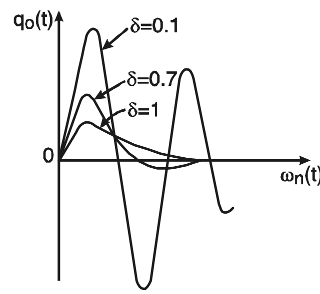


Fig. 1.16. Response to impulse input.

$$q_0(t) = L^{-1}\left[\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right]$$

or $$q_0(t) = \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\,\omega_n \sin\left\{\omega_n\sqrt{1-\delta^2}\;t\right\} \qquad \text{...(1.5-41)}$$

The impulse response against normalized time $\omega_n t$ is plotted in Fig. 1.16.

1.6. Frequency Domain Analysis

In Section 1.5, we have considered the output response of a system in time domain. The frequency domain analysis is popular because of the amount of computation involved in obtaining the time response. The frequency response is often obtained by means of graphical method.

In frequency domain analysis, a sinusoidal signal with varying frequency is used as the test signal. For linear systems the output response is also sinusoidal. To obtain the frequency response, the operator, in the transfer function is replaced by $j\omega$; ω (= 2π frequency) being the angular frequency. This is often called sinuosidal transfer function. If the amplitude of the sinusoidal signal is unity, *i.e.* $q_i(t) = \sin \omega t$, the output is given by

$$Q_o(j\omega) = G(j\omega) \quad \text{...(1.6-1)}$$

Thus, the response is a complex quantity and is represented by its magnitude and phase angle, *i.e.*

$$M = |Q_o(j\omega)| = |G(j\omega)| \quad \text{...(1.6-2)}$$

$$\phi = \underline{/Q_o(j\omega)} = \underline{/G(j\omega)}$$

The frequency response is therefore, represented generally by its magnitude and phase angle plots against frequency ($0 \le \omega < \infty$).

Logarithmic plotting of frequency response. Logarithmic plots of both magnitude and frequency are known as **Bode Plots.** The two Bode plots of a frequency response are :

(*a*) Plot of the magnitude in decibel (db) *versus* $\log_{10} \omega$.

(*b*) Plot of the phase angle *versus* $\log_{10} \omega$.

The advantage of Bode plots is that the product factors in the expression of $G(jw)$ become additive terms. Also, it is very easy to plot the curves withoiut knowing their actual values at a number of frequencies. The transfer function of measurement systems may be written as

$$G(s) = \frac{K(1 + T_1 s)(1 + T_2 s)}{s(1 + T_3 s)\ (1 + T_4 s)\ (1 + bs + as^2)} \quad \text{...(1.6-3)}$$

It is clear from above expression that $G(s)$ contains, in general, the following factors :

1. Constant gain, K.

2. Pole at the origin which represents an integrator. There may be multiple poles at the origin.

3. Zero at the origin which represents a differentiator. It is not given in Eq. (1.6-3).

4. Real poles and zeros.

5. Complex conjugate poles and zeros in pair. Complex zeros are not given in Eq. (1.6-3).

The magnitude of the response of Eq. (1.6-3) in db is given by

$$\begin{aligned} 20\ \log_{10}\ |G(j\omega)| &= 20\ \log_{10}\ K + 20\ \log_{10}\ |1 + j\omega\ T_1| \\ &\quad + 20\ \log_{10}\ |1 + j\omega T_2|\ - 20\ \log_{10}\ |j\omega| \\ &\quad - 20\ \log_{10}\ |1 + j\omega T_2|\ - 20\ \log_{10}\ |1 + j\omega T_4| \\ &\quad - 20\ \log_{10}\ \left|1 + jb\omega - a\omega^2\right| \end{aligned}$$

Similarly, the phase angle may be written as,

$$\begin{aligned} \arg G(j\omega) &= \arg K + \arg (1 + j\omega\, T_1) + \arg (1 + j\omega T_2) \\ &\quad - \arg(90°) - \arg (1 + j\omega T_3) \\ &\quad - \arg (1 + j\omega\, T_4) - \arg (1 + jb\omega - a\omega^2) \end{aligned}$$

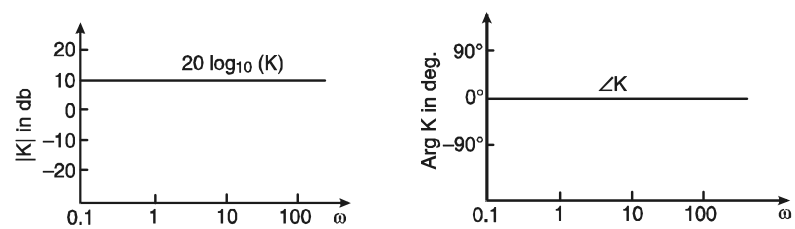
Thus, it is possible to plot curves for individual terms separately and to get the complete plot, it is just to add them together. We shall, therefore, first discuss the method of plotting individual terms.

Constant gain *K*

$$|K|_{db} = 20 \log_{10} |K| = \text{constant}$$

$$\angle K = 0° \quad \text{or} \quad 180° \qquad \text{...(1.6-4)}$$

The magnitude in *db* and phase angle plots against ω on $\log_{10}$ scale are shown in Fig. 1.17.



(*a*) (*b*)

Fig. 1.17. Bode plots of gain *K*.

Poles or zones at origin $(jw)^{\pm n}$. In general poles at origin are represented by $(j\omega)^{-n}$ and zeros by $(j\omega)^{n}$, where n is the mutliplicity of poles or zeros.

$$M = \left|(j\omega)^{\pm n}\right|_{db} = 20\ \log_{10} \left|(j\omega)^{\pm n}\right|$$

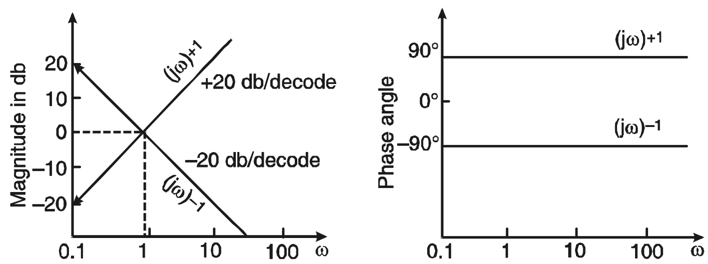
$$= \pm 20\ n\ \log_{10}\ \omega \qquad \text{...(1.6-5)}$$

The slope of the curve, which is a straight line when ω is taken on $\log_{10}$ scale, is given by

$$\text{Slope} = \frac{d\ (\pm 20n\ \log_{10}\ \omega)}{d\ (\log_{10}\ \omega)} = \pm 20n\ \ \text{db/decade} \qquad \text{...(1.6-6)}$$

The phase angle $\phi = \angle(j\omega)^{\pm n} = \pm \frac{\pi n}{2}$ ...(1.6-7)

The curves ($n = 1$) are plotted in Fig. 1.18.



(*a*) (*b*)

Fig. 1.18. Bode plots of poles and zeros at origin.

Real zero, $(1 + j\omega T)$. The magnitude in db,

$$M = |1 + j\omega T|_{db} = 20 \log_{10} (\sqrt{1 + \omega^2 T^2}) \quad \text{...(1.6-8)}$$

and the phase angle

$$\phi = \underline{/(1 + j\omega T)} = \tan^{-1}(\omega T) \quad \text{...(1.6-9)}$$

To get asymptotic plots, let, for small frequencies, $\omega T \ll 1$. Then

$$|1 + j\omega T|_{db} = 20 \log_{10}(1) = 0 \quad \text{...(1.6-10)}$$

At high frequencies $\omega T \gg 1$,

$$\begin{aligned} |1 + j\omega T|_{db} &= 20 \log_{10} \omega T \\ &= 20 \log_{10} \omega + 20 \log_{10} T \quad \text{...(1.6-12)} \end{aligned}$$

This is a equation of a straight line with a slope of 20 db/decade. For the intersection point of low and high frequencies asymptotic lines, equate Eq. (1.6-11) to zero. The frequency obtained in this way is called the corner frequency ω_c. Hence

$$20 \log_{10} \omega_c T = 0$$

or

$$\omega_c = \frac{1}{T} \quad \text{...(1.6-12)}$$

From Eq. (1.6-9) the phase angle for $\omega T \ll 1$ is zero while that for $\omega T \gg 1$ is 90°. At corner frequency $\phi = 45°$.

Real pole, $(1 + j\omega T)^{-1}$

The magnitude in decibel,

$$M = |(1 + j\omega T)^{-1}|_{db}$$

$$= -20 \log_{10} \sqrt{1 + \omega^2 T^2} \qquad \text{...(1.6-13)}$$

and the phase angle, $\phi = \underline{/(1 + j\omega T)^{-1}} = -\tan^{-1} \omega T \qquad \text{...(1.6-14)}$

Similar to the real zero, the decibel magnitude, Eq. (1.6-13), is taken as zero for low values of ω (*i.e.* $\omega T \le 1$). But for high frequencies the slope of the asymptotic line is – 20 db/decade. Again the corner frequency

$$\omega_c = \frac{1}{T}$$

From Eq. (1.6-14),

$\phi = 0°$ for $\omega T \ll 1$

$\phi = -90°$ for $\omega T \gg 1$

$\phi = -45°$ for $\omega T = \omega_e$.

The magnitude and phase angles of a real zero and a real pole are plotted in Fig. 1.19.

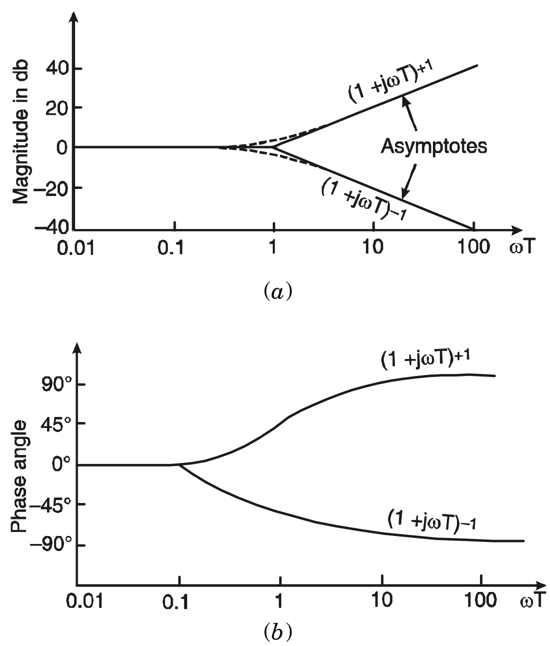


Fig. 1.19. Bode plots of real poles and zeros.

Complex pair of poles, $(1 + jb\omega - a\omega^2)^{-1}$

The decibel magnitude,

$$M = \left|(1 + jb\omega - a\omega^2)^{-1}\right|_{db}$$

$$= -20\log_{10}\sqrt{\left[\left(1-a\omega^2\right)^2+(b\omega)^2\right]} \quad \text{...(1.6-15)}$$

and the phase angle, $\phi = \underline{/\left(1 + jb\omega - a\omega^2\right)^{-1}}$

$$= -\tan^{-1}\frac{b\omega}{\left(1-a\omega^2\right)} \quad \text{...(1.6-16)}$$

For very low frequencies $\omega\sqrt{a} \ll 1$, both M and ϕ are zero. For every high frequencies $\omega\sqrt{a} \gg 1$,

$$M = -20\log_{10}(a\omega^2)$$

$$= -40\log_{10}\left(\sqrt{a}\,\omega\right) \quad \text{...(1.6-17)}$$

and $$\phi = -\tan^{-1}\left(-\frac{b}{a\omega}\right) = -180° \quad \text{...(1.6-18)}$$

At frequenc $\omega = \dfrac{1}{\sqrt{a}}$

$$M = 20\log_{10}\left(\frac{\sqrt{a}}{b}\right)$$

$$\phi = -90°$$

For maximum value of decibel magnitude :

$$\frac{dM}{d\omega} = 0$$

Hence $$\omega = \omega_r = \frac{1}{\sqrt{a}}\sqrt{1-\frac{b^2}{2a}} \quad \text{...(1.6-19)}$$

This frequency is known as resonance frequency ω_r. Substituting $\omega = \omega_r$ in Eq. (1.6-15), the maximum decibel magnitude is given by

$$M_{max} = 20\log_{10}\frac{2a}{b\sqrt{4a-b^2}} \quad \text{...(1.6-20)}$$

The decibel magnitude and phase angle plots against ω (on $\log_{10}$ scale) are shown in Fig. 1.20. The maximum value depends on b and a.

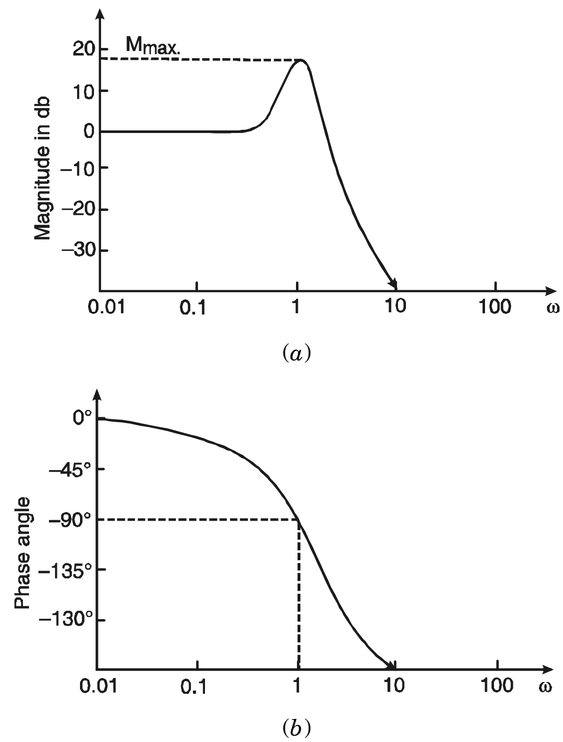


Fig. 1.20. Bode plots of complex poles.

Frequency response of first order instruments. The sinusoidal transfer function of a first order instrument may be obtained, for $s = j\omega$, from Eq. (1.5-17). Hence the output is given by

$$Q_0(j\omega) = \frac{1}{1 + j\omega T} \qquad \text{...(1.6-21)}$$

The decibel magnitude and phase angle can be obtained from Eqs. (1.6-13) and (1.6-14), *i.e.*

$$M = |Q_o(j\omega)|_{db} = -20 \log_{10}\left(\sqrt{1 + \omega^2 T^2}\right) \qquad \text{...(1.6-22)}$$

$$\phi = \underline{/Q_o(j\omega)} = -\tan^{-1} \omega T \qquad \text{...(1.6-23)}$$

The asymptotic plots of magnitude and phase angle against ω (on a $\log_{10}$ scale) are shown in Fig. 1.19.

A first order instrumenet approaches perfection if $Q_o(j\omega)$ approaches $1 \angle 0°$, *i.e.* the decibel magnitude and phase angle are zero. This means that a first order instrument tends to be a zero order instrument. This is possible when the product ωT is sufficiently small. For a particular value of T there will be some input frequency ω below that the measurement is accurate. Alternatively, if an input of high frequency is to be measured, the instrument time constant T must be sufficiently small.

Frequeny response of second order system. The sinusoidal transfer function may be obtained from Eq. (1.5-27) for $s = j\omega$. So, the output is given by

$$Q_o(j\omega) = \frac{1}{1 + j\dfrac{2\delta\omega}{\omega_n} - \left(\dfrac{\omega}{\omega_n}\right)^2} \quad \text{...(1.6-24)}$$

Hence, by substituting $b = \dfrac{2\delta}{\omega_n}$ and $a = \dfrac{1}{\omega_n^2}$ in Eqs. (1.6-15) and (1.6-16), the decibel magnitude and phase angle are given by

$$M = \left|Q_o(j\omega)\right|_{db} = -20\ \log_{10}\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{2\delta\omega}{\omega_n}\right)^2} \quad \text{...(1.6-25)}$$

$$\phi = \underline{/Q_o(j\omega)} = -\tan^{-1}\frac{2\delta\omega/\omega_n}{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]} \quad \text{...(1.6-26)}$$

The resonance frequency and the maximum magnitude are given, from Eq. (1.6-19) and (1.6-20), by

$$\omega_r = \omega_n\sqrt{1 - 2\delta^2} \quad \text{...(1.6-27)}$$

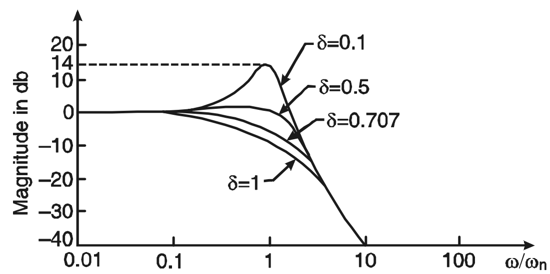
$$M_{max} = 20\ \log_{10}\frac{1}{2\delta\sqrt{1 - \delta^2}}$$

$$= -20\ \log_{10}(2) - 20\ \log_{10}(\delta) - 10\ \log_{10}\left(1 - \delta^2\right) \quad \text{...(1.6-28)}$$

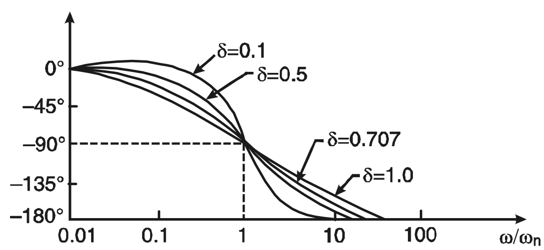
For real value of ω_r, the value of $\delta \leq 0.707$. So, the peak in the magnitude will occur only when $\delta < 0.707$. The maximum magnitude and resonance frequencies for various values of δ ($0 \leq 0.707$) are given in Table 1.1. The Bode plots for various d are shown in fig. 1.21.

Table 1.1. M_{max} **and** ω_r/ω_n **for various d**

δ	M_{max}	ω_r/ω_n
0	∞	1
0.1	22.922	0.99
0.5	1.584	0.707
0.707	0	0



(*a*)



(*b*)

Fig. 1.21. Bode plots (or frequency response) of second order system.

Bandwidth. The bandwidth is defined as the frequency at which the magnitude $|Q_o(j\omega)|$ has dropped to 70.7% of its zero frequency magnitude, or 3 db down from zero frequency level. Generally, the bandwidth indicates the noise filtering characteristics of the instrument. It also gives a measure of the transient response properties. A large bandwidth usually indicates that signals of high frequency (noises usually are of high frequency) will be passed on to the outputs. Thus, the transient response may have a faster rise time with a larger overshoot. Conversely, for small bandwidth only low frequency signals are passed (noises are filtered being of high frequencies) accompanied by slow and sluggish time response.

1.7. Illustrative Examples

Example 1. *Give the number of significant figures in each of the following* :

(*a*) 341 (*b*) 0.57 (*c*) 25.27

(*d*) 0.000 05 (*e*) 5.10×10^5 (*f*) 20 000.

Solution. (*a*) Significant figures are three (3, 4, 1) as it is closer to 341 than to 340 or 342.

(*b*) Significant figures are *two* (5, 7).

(*c*) Significant figures are *four* (2, 5, 2, 7).

(*d*) Significant figures are *five* (0, 0, 0, 0, 5) as it is closer to 0.000 05 than to 0.000 04 or 0.000 06. Significant figures will be *one* (5) if it can be written as 0.5×10^{-4}.

(*e*) Significant figures are *three* (5, 1, 0) as it is closer to 5.10×10^5 than to 5.09×10^5 or 5.11×10^5.

(*f*) If value is closer to 20 000 than to 19 999 or 20 001, the significant figures are *five* (2, 0, 0, 0). If it is closer to 20×10^3 than to 19×10^3 or 21×10^3, the significant figures are two (2, 0).

Example 2. *Two resistors R_1 and R_2 are connected in series. $R_1 = 28.5\ \Omega$ and $R_2 = 35.62\ \Omega$ with an uncertainty of one unit in the last digit of each number. Calculate the total series resistance.*

Solution. The two resistors are connected in series, hence the total series resistance is equal to the sum of resistance of R_1 and R_2.

For addition, round off* the more accurate numbers to one more decimal digit than is contained in the least accurate number. Hence the total series resistance.

$$R_s = R_1 + R_2 = 28.5 + 35.62 = 64.12\ \Omega$$

Now, round off the result to the same decimal places as the least accurate number. Then

$$R_s = \mathbf{64.1\ \Omega.}$$

Example 3. *Find the total resistance if the resistors R_1 and R_2 in Example 2 are connected in parallel.*

Solution. The total resistance

*For rounding off a number to n significant figures, discard all digits to the right of the nth places. If the first discarded digits is less than one-half a unit in nth place, leave the nth digit unchanged. If the first discarded digit is greater than one-half, increase the nth digit by 1. In case the first discarded digit is exactly one-half, leave the nth digit unchanged if it is an even digit and add 1 to it if is odd.

$$R_p = \frac{R_1 R_2}{R_1 + R_2} = \frac{X}{Y}(\text{say})$$

From example –2, $Y = 64.1\ \Omega$

To multiply R_1 and R_2 first round off R_2 to two decimal points

$$X = R_1 R_2 = 28.5 \times 35.62 = 1015.17$$

Now, round off this value to the same number of decimal places as the least accurate numbers (*i.e.* R_1). So, $X = 1015.2$.

Therefore, $$R_p = \frac{1015.2}{64.1} = 15.83...$$

After rounding off,

$$R_p = \mathbf{15.8\ \Omega.}$$

Example 4. *One inductor with reactance of 125.135 Ω is connected in series with a capacitor of reactance of 98.92 Ω. Calculate the total series reactance.*

Solution. The total reactance is given by

$$X_i = X_i - X_c$$

So, to get total reactance the capacitive reactance is subtracted from the inductive reactance. For subtraction round off the more accurate number to the same number of decimal places as the less accurate number. Hence

$$X_i = 125.14 - 98.92 = \mathbf{26.22\ \Omega.}$$

Example 5. *A voltmeter is tested by comparing it with a voltmeter for which the static correction given in the correction curve for 100 V is 0.05 V. The two voltmeters read 102 V and 100 V respectively. Find the absolute and relative error in the voltmeter under test. What is the static correction to be made.*

Solution. The true voltage

$$V_{true} = \text{reading of second voltmeter + correction}$$
$$= 100 + 0.05 = 100.05 \text{ V}$$

Hence the error $\delta R = V_m - V_{true}$

$$= 102 - 100.05 = 1.95 \text{ V}$$

Absolute error $\epsilon_0 = |\delta R| = 1.95$ V

Relative error $\epsilon_r = \frac{\epsilon_0}{V_{true}} = \frac{1.95}{100.05}$

$$= \mathbf{0.0195 = 1.95\%}$$

Static correction $\delta c = -\delta R = -\mathbf{1.95\ V}$

Example 6. *In a permanent magnet moving coil ammeter the pointer moves through an angle of 35° when current to be measured* *is changed by 50 mA. The ammeter is spring controlled. Find the sensitivity of the instrument.*

Solution. The relation between the deflection and current is linear. Hence, sensitivity is given by

$$S_i = \frac{\Delta q_0(t)}{\Delta q_i(t)} = \frac{35}{50} = 0.7°/\text{mA}.$$

Example 7. *A 0—50 V voltmeter has 100 scale division that can be read to $\frac{1}{2}$ division. Determine the resolution of the meter in volt. What will be its value in percentage of full scale ?*

Solution. 1 scale division

$$= \frac{50}{100} = 0.5 \text{ V}$$

$$\text{Resolution} = \frac{1}{2} \text{ division} = \frac{1}{2} \times 0.5 = \mathbf{0.25\ V}$$

$$\% \text{ Resolution} = \frac{\frac{1}{2}}{100} \times 100 = \mathbf{0.5\%}$$

Example 8. *An instrument is represented by a first order transfer function*

$$G(s) = \frac{1}{1 + Ts}$$

If response to a unit step input reaches 63.2% of its final value in 3.5 second, find the time constant T of the instrument.

Solution. At $t = 3.5$ second, $q_o(t) = 0.632$

From Eq. (1.5-18) for $K = 1$, we know that

$$q_0(t) = 0.632 \qquad (t = T)$$

Hence $\quad T = \mathbf{3.5\ second.}$

Example 9. *Find steady state error in a first order instrument when excited by a unit ramp input, if the dynamic error is 1.264 for t = time constant T of the instrument.*

Solution. From Eq. (1.5-23), the dynamic error

$$e_d = T\left(1 - e^{-\frac{t}{T}}\right)$$

where T is the time constant of the instrument and K in Eq. (1.5-23) is unity in this case.

$$\therefore \qquad 1.264 = T\left(1 - e^{-\frac{T}{T}}\right)$$

or $\qquad 1.264 = 0.632\,T$

$$T = \frac{1.264}{0.632} = 2 \text{ second}$$

Now, from Eq. (1.5-24), the steady state error is

$$e_{ss} = T = \mathbf{2}.$$

Example 10. *An instrument is represented by a transfer function*

$$G(s) = \frac{1}{s^2 + s + 1}$$

Find the percentage overshoot if the instrument is excited by a unit step input. Find, also, the steady state error in unit ramp response.

Solution. From the given transfer function,

$$\omega_n = 1 \qquad \text{and} \qquad \delta = \frac{1}{2\omega_n} = 0.5$$

From Eq. (1.5-36),

$$\text{Percentage overshoot} = 100\, e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}}$$

$$= 100\, e^{-\frac{0.5x}{\sqrt{0.75}}} = \mathbf{13.5\%}$$

From Eq. (1.5-39), the steady state error,

$$e_{ss} = -\frac{2\delta}{\omega_n} = -\frac{2 \times 0.5}{1} = -\mathbf{1}.$$

Example 11. *Find the state space representation of a second order instrument represented by a differential equation,*

$$\frac{d^2 q_0(t)}{dt^2} + 2\frac{dq_0(t)}{dt} + 3\, q_o(t) = q_i(t).$$

Solution.

Let $\quad q_o(t) = x_1(t)$

$$\dot{x}_1(t) = \frac{dq_o(t)}{dt} = x_2$$

$$\dot{x}_2(t) = \frac{d^2 q_o(t)}{dt^2} = -3q_o(t) - 2\frac{dq_o(t)}{dt} + q_i(t)$$

$$= -3x_1(t) - 2x_2(t) + q_i(t)$$

In matrix form (or state space form)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} q_i(t)$$

$$q_o(t) = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Example 12. *Overshoot in a step response of a second order instrument is 25%. Find the peak (maximum) magnitude (in db) of its requency response.*

Solution. Percentage overshoot $= 100\, e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}}$

$$\therefore \qquad 0.25 = e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}}$$

or $$-\frac{\pi\delta}{\sqrt{1-\delta^2}} = -1.382$$

or $$\delta^2 = 0.162$$

or $$\delta = 0.402$$

The peak magnitude in db is given from Eq. (1.6-28) as

$$M_{max} = 20 \log_{10} \frac{1}{2\delta\sqrt{1-\delta^2}}$$

$$= 20 \log_{10} \frac{1}{2 \times 0.402\sqrt{1-0.162}}$$

$$= 20 \log_{10} 1.36 = \mathbf{2.67\ db.}$$

OBJECTIVE QUESTIONS

1. Select the correct statement from the four statements given below.

(*a*) Static characteristics are concerned with the measurement of consant or slowly varying quantities.

(*b*) In case of measurement of slowly varying quantities, the dynamic relation between the instrument input and output is considered.

(*c*) Differential equation of the instrument is considered while analyzing static.

(*d*) None of these.

2. What are the significant figures in 5.10×10^5.

(*a*) Two (5, 1) (*b*) Three (5, 1, 0) (*c*) Two (1, 0) (*d*) One (1).

3. Two resistors $R_1 = 28.5$ ohm and $R_2 = 35.62$ are connected in series. With an uncertainty of one unit in the last digit of each number. Find the total series resistance.

(*a*) 64.1 (*b*) 64.12 (*c*) 64.0 (*d*) 64

4. A voltmeter is tested by comparing it with a voltmeter for which the static correction is 0.05 V. The two voltmeters read 102 V and 100 V respectively. The static correction in the voltmeter under test is

(*a*) 1.95 v (*b*) 2.0 V (*c*) – 1.95 V (*d*) – 2.0 V

5. In a permanent magnet moving coil ammeter the pointer moves through an angle of 25° when the current to be measured is changed by 50 mA. The instrument is spring controlled. The sensitivity of the meter is given by

(*a*) 0.7 degree/mA (*b*) 0.5 mA/degree
(*c*) 0.7 mA/degree (*d*) 0.5 degree/mA.

6. A 0-50 V voltmeter has 100 scale divisions that canbe read to half of a division. The resolution of the meter in percent is

(*a*) 0.5% (*b*) 1.0% (*c*) 0.25% (*d*) 2%

7. An instrument's transfer function is

$$G(s) = \frac{1}{1 + Ts}$$

If the response to a unit step input reaches 63.2% of its final value in 3.5 second, then the time consant T is given by

(*a*) 0.632 second (*b*) 3.5 second
(*c*) 5.54 second (*d*) none of these

8. An instrument's transfer function is

$$G(s) = \frac{1}{s^2 + s + 1}$$

Then the damping ratio is given by

(*a*) 0.707 (*b*) 0.5 (*c*) 0.635 (*d*) 0.25

9. An instrument's transfer function is

$$G(s) = \frac{1}{s^2 + s + 1}$$

Then the natural frequency is given by

(*a*) 0.707 (*b*) 0.5 (*c*) 1.0 (*d*) 0.25

10. An instrument's transfer function is

$$G(s) = \frac{1}{s^2 + s + 1}$$

Then the percentage overshoot is given by

(*a*) 11.707% (*b*) 12.55% (*c*) 14.635% (*d*) 13.5%

11. An instrument's transfer function is

$$G(s) = \frac{1}{s^2 + s + 1}$$

Then the steady state error in unit ramp response is givenby

(*a*) 0.707 (*b*) 1.0 (*c*) –1.0 (*d*) 1.25

Answers

1. (*a*)	2. (*b*)	3. (*a*)	4. (*c*)
5. (*d*)	6. (*a*)	7. (*b*)	8. (*b*)
9. (*c*)	10. (*d*)	11. (*c*)	

REVIEW QUESTIONS

1. What are the points to be considered while choosing a measurement method ? What are important points to be considered while preparing the measurement records ? Mention general precautions to be taken in measurement.

2. What do you understand by static characteristics of an instrument ? Write short notes on the following terms related to static characteristics.

(*a*) Static calibration (*b*) Accuracy and precision

(*c*) Significant figures (*d*) Linearity and repeatedity

3. What differences are there between :

(*a*) Static sensitivity and drift (*b*) Resolution and threshold

(*c*) Hysteresis and dead zone

4. What are different types of methods for dynamic modelling of instruments ? Briefly discuss each of them.

5. What are different Standard inputs used in time domain analysis of an instrument ? Discuss them.

6. Obtain time response of a first order instrument when unit step input is applied. Is there any overshoot in the response ? How do you define the time constant ? Derive expression for steady state and dynamic errors.

7. Derive expression for time response of a second order instrument for unit step input.

8. Define the following terms with reference to step response of a second order instrument :

(*a*) Overshoot (*b*) Settling time (*c*) Time dalay (*d*) Rise time.

9. What is the standard input used for frequency response analysis ? How do you obtain the frequency response ? Why is logarithmic plotting of frequency response preferred ?

10. Discuss Bode plots method of frequency response.

11. Derive expressions for resonance frequency and maximum magnitude in frequency response of a second order instrument.

EXERCISES

1. Mention the number of significant figures in each of the following :

(*i*) 3.32. (*ii*) 0.125 (*iii*) 241 (*iv*) 0.0025

(*v*) 2.10×10^6 (*vi*) 200 000.

2. Two capacitors $C_1 = 25.1\ \mu F$ and $C_2 = 60.15\ \mu F$ are connected in parallel. Calculate the total parallel capacitance. Give the number of significant figures.

3. When a 100 V is measured by a voltmeter it indicates 99.5 V. Find the relative error and static correction for the voltmenter.

4. A potentiometer has 100 turns, find the resolution in volt and in percentage of full range when 70 volt is applied to the potentiometer.

5. An instrument's transfer function is given by

$$G(s) = \frac{4}{s^2 + 3s + 4}$$

Find the damping ratio δ and natural frequency ω_n of the instrument. Also calculate the percentage overshoot in the step response.

6. Obtain the state space equations for the second order instrument of problem 5.

7. For the instrument of problem 5. Calculate the resonance frequency and maximum magnitude of the frequency response.