# 1

# Introduction to Vector Analysis

**(Addition of vectors — Subtraction of vectors — Multiplication of a vector by a scalar — Multiplication of vectors : Scalar and vector products — Cartesian system of co-ordinates — Triple product : Scalar and vector triple products — Curvilinear co-ordinates : cylindrical and spherical systems — Transformation relations between Cartesian and cylindrical systems ; transformation relations between cartesian and spherical systems. Typical examples and Exercises).**

A fairly good knowledge of vectors is an essential pre-requisite to the study of theory of Electromagnetic fields. Therefore, before introducing the concept of a field, it is necessary to introduce the vector concept in order to enable the readers to understand and appreciate the meaning of a field and to follow the subsequent chapters with great ease. Those who are already conversant with vectors will certainly find this chapter a very useful review and introductory to the notations followed in this book.

Any physical quantity may be represented either as a scalar or as a vector. By definition, a scalar is a quantity which is wholly characterized by its magnitude, and a vector is defined as a quantity fully characterized by magnitude and direction.

Mass, volume, density, time, temperature, charge, etc., are all examples of scalar quantities. Velocity, force, acceleration, momentum, temperature gradient, etc. are parameters which can be represented by means of vectors. To be more specific, vector quantities are those which can be portrayed on a map or model by a "directed line segment", the length of the line representing the magnitude (to a suitable scale), the orientation depicting the direction and an arrow indicating the sense. For example, in Fig. 1.1, the line *OA* represents a velocity vector $v$, the length of *OA* denoting the numerical value of velocity (say, in metres per second to an appropriate scale) ; the angle of inclination $\alpha$ with a reference axis signifies the orientation of the vector with the reference ; the arrow suggests that the velocity of motion is directed away from the origin *O* along *OA*.

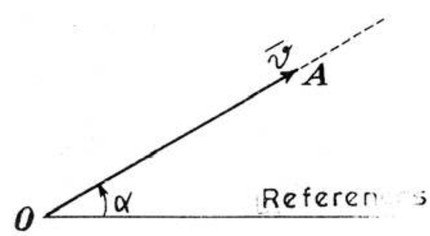



Fig. 1.1. A velocity vector *A*.

Basic rules and properties of vectors are briefly given below :

## 1.1. ADDITION OF VECTORS

The sum of two or more vectors is called the vector sum which is the resultant of the vectors added.

Add vector **A** to **B** or **B** to **A**. The resultant will be the same. See Fig. 1.2. This is evident from parallelogram law.

$$\mathbf{B} + \mathbf{A} = \mathbf{A} + \mathbf{B} \qquad \text{...(1.1)}$$


Fig. 1.2. Commutative law.

It involves no distinction for the order in which one vector is added to the other. This is referred to as "Commutative law" of addition of vectors.

Fig. 1.3 illustrates what is known as "Associative law".

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \qquad ...(1.2)$$

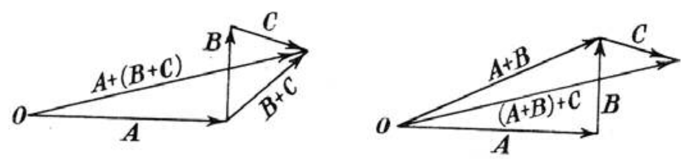



Fig. 1.3. Associative law.

It makes no difference whether the sum of **B** and **C** is added to **A**, or **C** is added to the sum of **A** and **B**. The above law is applicable to the addition of any number of vectors.

## 1.2. SUBTRACTION OF VECTORS

**A – B** (vector *A* minus vector *B*) implies that the vector *B* is subtracted from *A*, or that the vector *B* is reversed and added to the vector *A*. In effect, the subtraction is a process of reversing the vector to be subtracted and following the rule for addition. See Fig. 1.4.

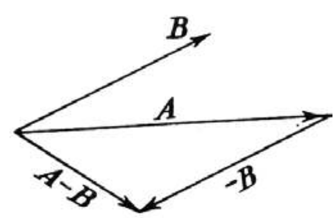



Fig. 1.4. Subtraction of vectors.

## 1.3. MULTIPLICATION OF A VECTOR BY A SCALAR

Multiplication is also a process involving the addition of vectors in a different way. If $k$ is a scalar multiplier and **A** is a vector quantity, then $k\mathbf{A}$ is a vector whose orientation is the same as the vector *A*, but the length and the magnitude is $k$ times that of *A*. For example, if $k = 2$, $k\mathbf{A}$ is a vector, twice the length of the vector *A* and in the same direction. See Fig. 1.5.

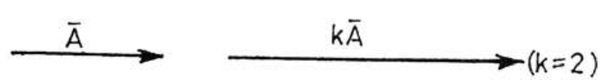



Fig. 1.5. Multiplication of a vector by a scalar.

Similarly, from 'Distributive law', we have

$$k\,(\mathbf{A} + \mathbf{B} + \mathbf{C}) = k\mathbf{A} + k\mathbf{B} + k\mathbf{C} \qquad ...(1.3)$$

## 1.4. MULTIPLICATION OF VECTORS

Multiplication of vectors is performed in two ways : one gives a scalar result and the other yields a vector result.

Scalar product of two vectors is defined as the scalar quantity whose magnitude is given by the product of the magnitudes of the vectors multiplied by the cosine of the angle between them. The scalar product of two vectors $A$ and $B$ is denoted through a DOT. It may also be referred to as a DOT PRODUCT.

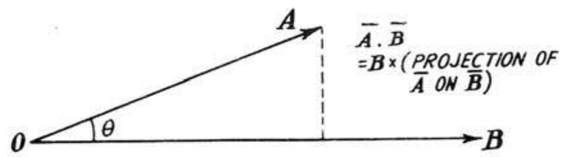



Fig. 1.6. Scalar product of vectors $A$ and $B$.

Symbol $\quad \mathbf{A} \cdot \mathbf{B} = AB \cos \theta$

The scalar product is the same as the product of $B$ and projection of $A$ on $B$ ; or the product of $A$ and projection of $B$ on $A$. See Fig. 1.6.

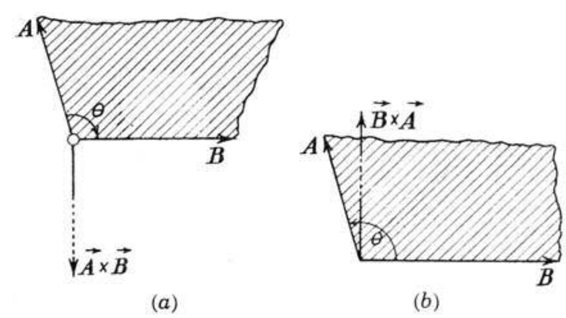
Scalar or dot product obeys "Commutative law", as the order in which multiplication is performed is immaterial, yielding the same result.

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \qquad \text{...(1.4)}$$

As distributive law holds good for addition of vectors, we can also write that

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \qquad \text{...(1.5)}$$

Vector product of two vectors $A$ and $B$ is represented in the form $\mathbf{A} \times \mathbf{B}$ (read as $A$ cross $B$). The product is a vector with magnitude equal to the product of the magnitudes of $A$ and $B$ and sine of the angle between them and is directed perpendicularly to the plane containing $A$ and $B$ and has the sense as given by the "Right hand rule". See Fig. 1.7. It is clearly seen that although the magnitudes of the cross products $A \times B$ and $B \times A$ are the same, the two vector results are of opposite sign, indicating that the vectors are oppositely directed. This clearly demonstrates that "Commutative law" does not apply to vector products.




(*a*) (*b*)

Fig. 1.7. (*a*) Cross product $\mathbf{A} \times \mathbf{B}$.
(*b*) Cross product $\mathbf{B} \times \mathbf{A}$.

$$\mathbf{A} \times \mathbf{B} = -\, \mathbf{B} \times \mathbf{A} \qquad \text{...(1.6)}$$

The distributive law holds as in the case of scalar product. However, the order of operation should be preserved.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad \text{...(1.7)}$$

**Note.** It is readily seen that the magnitude of the vector product $\mathbf{A} \times \mathbf{B}$, in Fig. 1.7, is the same as the area of the parallelogram with sides $A$ and $B$.

## 1.5. CARTESIAN SYSTEM OF CO-ORDINATES

The position and orientation of a space vector can be clearly defined and identified by choosing an appropriate system of co-ordinate axes $X$, $Y$, $Z$. It is customary to make use of the right-handed system of axes ; *viz.,* A screw with right-handed thread will advance in the positive $Z$-direction when turned through 90° from the positive $X$-axis towards the $Y$-axis, the three axes $X$, $Y$, $Z$ being mutually perpendicular. See Fig. 1.8.

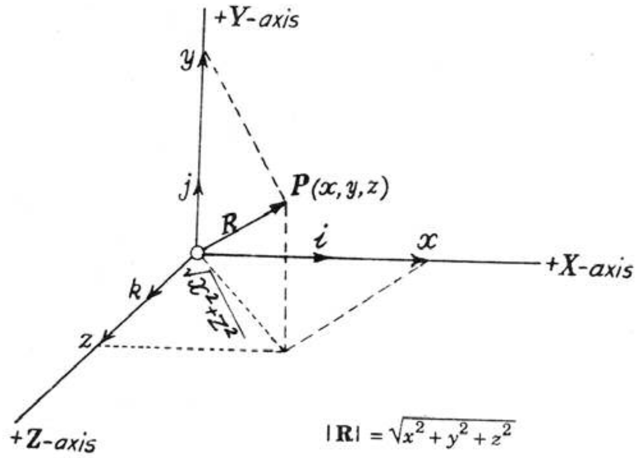



Fig. 1.8. Cartesian system of co-ordinates ; $i, j, k$ are unit vectors in positive $X$, $Y$, $Z$-directions, respectively.

Consider any point $P$ in space at a distance $R$ from the origin $O$ ; $\mathbf{R}$ is a vector directed from the origin to the point $P$ $(x, y, z)$. If $i, j, k$ are unit vectors in the positive $X$, $Y$, $Z$-directions respectively,

$$\mathbf{R} = xi + yi + zk \qquad ...(1.8)$$

In general, if $\mathbf{A}$ is any vector whose components along $X$, $Y$ and $Z$-axes are $A_x$, $A_y$ and $A_z$, respectively, then

$$\mathbf{A} = A_x i + A_y j + A_z k \qquad ...(1.9)$$

The unit vectors in $X$, $Y$ and $Z$ directions may be written as $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_z$ respectively, with the result that the equation (1.9) may be reproduced as follows :

$$\mathbf{A} = A_x \mathbf{u}_x + A_y \mathbf{u}_y + A_z \mathbf{u}_z \qquad ...(1.9A)$$

From the definition of a cross product, it follows that for a right-handed system explained above, the following relations apply :

$$\left.\begin{array}{lll} i \times j = k & or & \mathbf{u}_x \times \mathbf{u}_y = \mathbf{u}_z \\ j \times k = i & or & \mathbf{u}_y \times \mathbf{u}_z = \mathbf{u}_x \\ k \times i = j & or & \mathbf{u}_z \times \mathbf{u}_x = \mathbf{u}_y \end{array}\right\} \qquad ...(1.10)$$

As the cross products are non-commutative, it follows that

$$i \times j = -j \times i = k$$

Similarly $\quad j \times k = -k \times j = i$

and $\quad k \times i = -i \times k = j$

Further, $\quad i \times i = j \times j = k \times k = 0 \qquad ...(1.11)$

as the sine of zero angle is zero.

The dot products between the unit vectors are

$$\left.\begin{array}{l} i \cdot i = j \cdot j = k \cdot k = 1 \\ i \cdot j = j \cdot k = k \cdot i = 0 \end{array}\right\} \qquad ...(1.12)$$

as any two unit vectors in the orthogonal system are prependicular to each other. If, as in equation (1.9), we have another vector $B$ given as

$$B = B_x i + B_y j + B_z k$$

or $\quad B = B_x \mathbf{u}_x + B_y \mathbf{u}_y + B_z \mathbf{u}_z \qquad ...(1.13)$

the dot product of the vectors $A$ and $B$ is given by

$$\begin{aligned} \mathbf{A} \cdot B = AB \cos (\mathbf{A}, \mathbf{B}) &= (A_x i + A_y j + A_z k) \cdot (B_x i + B_y j + B_z k) \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned} \qquad ...(1.14)$$

From the above relation, the angle between the vectors $A$ and $B$ can be evaluated right from the definition of the product. Accordingly,

$$\cos(\mathbf{A}, \mathbf{B}) = \frac{A_x B_x + A_y B_y + A_z B_z}{AB} \qquad ...(1.15)$$

It may be noted that $\frac{A_x}{A}, \frac{B_x}{B}$, etc., are direction cosines of the line segments $A$ and $B$, etc.

The cross-product of **A** and **B** is given by

$$\mathbf{A} \times \mathbf{B} = (A_x i + A_y j + A_z k) \times (B_x i + B_y j + B_z k)$$
$$= (A_y B_z - A_z B_y) i + (A_z B_x - A_x B_z) j + (A_x B_y - A_y B_x) k \qquad ...(1.16A)$$

which may also be written in the form of a determinant as follows :

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad ...(1.16B)$$

## 1.6. TRIPLE PRODUCTS

We have seen that the dot product of two vectors is a scalar and the cross product of two vectors is another vector. The vector product can be dotted or crossed with a third vector giving rise to a triple product.

### 1.6.1. Scalar triple product

Examine the triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$**.** What does this mean ? Referring to Fig. 1.9, let us assume that the vectors $B$ and $C$ are in the $X$–$Y$ plane. The magnitude of the cross product $\mathbf{B} \times \mathbf{C}$ is clearly equal to the area of the parallelogram $OBEC$ which forms the base of a parallelopiped. The dot product of **A** with the vector product $\mathbf{B} \times \mathbf{C}$ would be the product of the $Z$-component of **A** (*viz.*, component of **A** normal to the base) and the area of the base in the $X$-$Y$ plane. The triple product would, therefore, represent the volume of the parallelopiped generated by the vectors $A$, $B$ and $C$ depicted in Fig. 1.9.

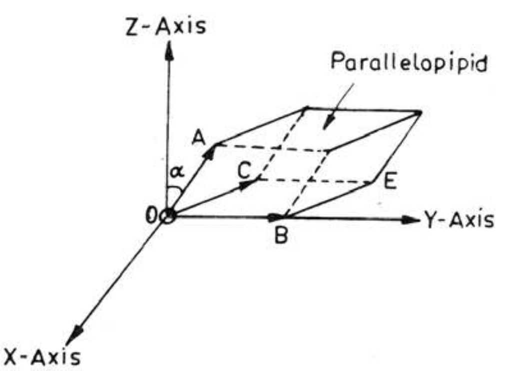



Fig. 1.9. Scalar triple product.

In matrix form, $$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \qquad ...(1.17)$$

### 1.6.2. Vector triple product

The vector product of vector $A$ with the cross product of vectors $B$ and $C$ would be another vector.

For example let **D** be the resultant vector, viz.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{D} \qquad ...(1.18)$$

As cross products are non-commutative, it may be noticed that

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) \qquad ...(1.19)$$

The vector triple product may be simplified by the $BAC - CAB$ rule as follows

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B}) \qquad ...(1.20)$$

The proof for the above mathematical identity is left to the readers as an exercise.

(**Hint.** By writing out both sides in component form, the identity can be proved.)

**Example 1.1.** *Given the two vectors* $A = 3i + 5j$ *and* $B = 4i + 2j$, *find the vector results* **A + B, A – B, B – A,** *and sketch the results.*

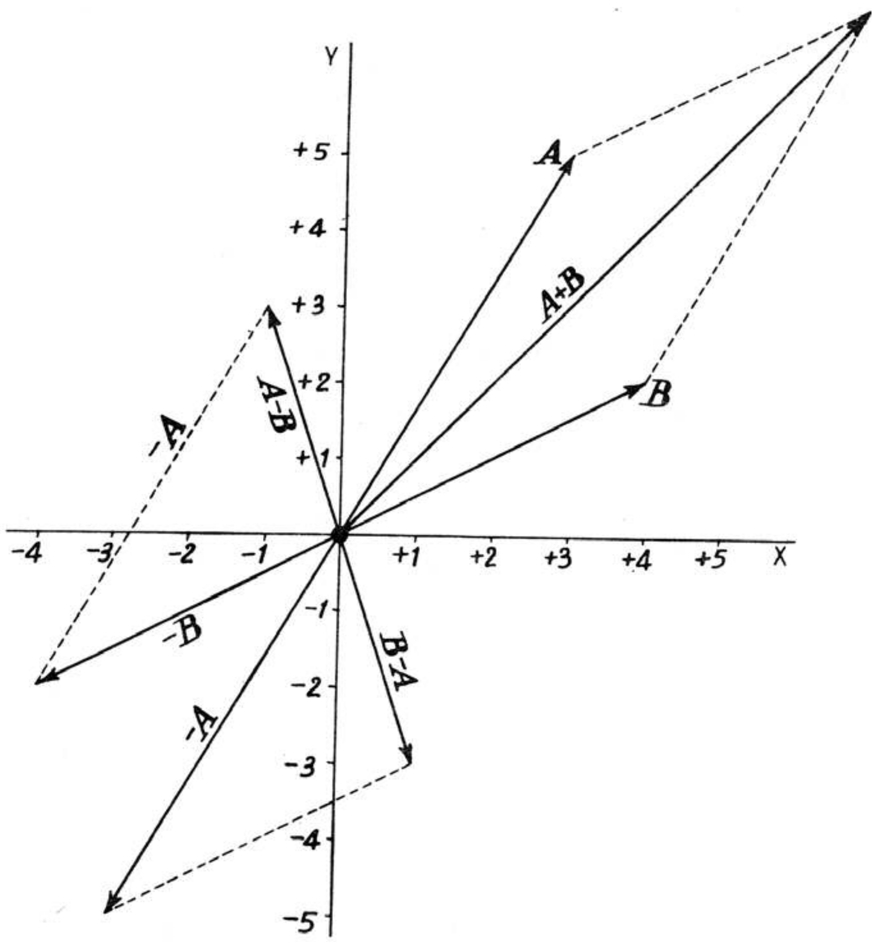



Fig. 1.10. For Example 1.1.

**Solution.** The problem deals with only two dimensions, *viz.*, in $X$–$Y$ plane.

The vectors $A = 3i + 5j$ ...(*i*)

and $B = 4i + 2j$ ...(*ii*)

are shown in Fig. 1.10.

The vector $A$ has a component of + 3 units along $X$-axis and + 5 units along $Y$-axis. Similarly, vector $B$ has components + 4 and + 2 respectively along $X$ and $Y$ axes.

Adding (*i*) and (*ii*), $A + B = 7i + 7j$ ...(*iii*)

Subtracting (*ii*) from (*i*), $A - B = -i + 3j$

Similarly, $B - A = i - 3j$

**Example 1.2.** *Given the two vector combinations*

$$\mathbf{A} + \mathbf{B} = 2i + 3j - 3k$$

*and*

$$\mathbf{A} - \mathbf{B} = 4i + j + k,$$

(*a*) *find* $A$ and $B$ *in vector form ;*

(*b*) *and also the dot products of* **A + B** *and* **A – B**. *Also evaluate* **A · B.**

**Solution.** $\mathbf{A} + \mathbf{B} = 2i + 3j - 3k$ ...(*i*)

$\mathbf{A} - \mathbf{B} = 4i + j + k$ ...(*ii*)

Adding (*i*) and (*ii*), $2\mathbf{A} = 6i + 4j - 2k$ ;

(*a*) whence $\mathbf{A} = 3i + 2j - k.$

Similarly, subtracting (*ii*) from (*i*),

$$\mathbf{B} = -i + j - 2k$$

(*b*) $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = 2i \cdot 4i + 3j \cdot j + (-3k \cdot k)$

$= 8 \quad + 3 \quad - 3 = 8.$

$\mathbf{A} \cdot \mathbf{B} = -3 + 2 + 2 = 1.$

**Example 1.3.** *Show that the vectors*

$$\mathbf{A} = 4i + 2j - k$$

*and* $\mathbf{B} = 2i - 2j + 4k$

*are perpendicular to each other.*

**Solution.** Two finite vectors $A$ and $B$ are perpendicular to each other if their dot product is zero.

In this example, let us check up if the dot product $\mathbf{A} \cdot \mathbf{B}$ is zero.

The scalar product of the vectors $A$ and $B$ is equal to

$$4i \cdot 2i - 2j \cdot 2j - k \cdot 4k = 8 - 4 - 4 = 0$$

It follows that each of them being non-zero, the vectors are mutually perpendicular.

**Example 1.4.** *Prove that the vectors*

$$\mathbf{A} = 3i + j - 2k, \ \ \mathbf{B} = 2i - 4j + k$$

*and* $\mathbf{C} = 5i - 3j - k$

*form the sides of a right-triangle.*

**Solution.** The three vectors can form the sides of a triangle if two of them added give the third one. Further, if any two of them are perpendicular to each other, the triangle is a right-angled one.

Examine the three vector relations :

$\mathbf{A} = 3i + j - 2k$ ...(*i*)

$\mathbf{B} = 2i - 4j + k$ ...(*ii*)

$\mathbf{C} = 5i - 3j - k$ ...(*iii*)

Adding (*i*) and (*ii*) gives (*iii*)

Furthermore, $\mathbf{A} \cdot \mathbf{B} = 3 \times 2 - 1 \times 4 - 2 \times 1 = 0$, proving that the vectors $A$ and $B$ are perpendicular to each other. Hence the result.

**Example 1.5.** *By squaring both sides of the vector relation* $\mathbf{A} = \mathbf{B} - \mathbf{C}$ *and interpreting the result geometrically, prove the "law of cosines".*

**Solution.** We know that

$$\mathbf{A} \cdot \mathbf{A} = A^2$$

Right hand side $= (\mathbf{B} - \mathbf{C}) \cdot (\mathbf{B} - \mathbf{C})$

$A^2 = \mathbf{B} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{C} - \mathbf{C} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{C}$

$= B^2 + C^2 - 2\, \boldsymbol{B} \cdot \boldsymbol{C}$

But $\mathbf{B} \cdot \mathbf{C} = BC \cos (\mathbf{B}, \mathbf{C}) = BC \cos \alpha$

Thus $A^2 = B^2 + C^2 - 2\, BC \cos (\mathbf{B}, \mathbf{C})$ ...(1.21)

(law of cosines)

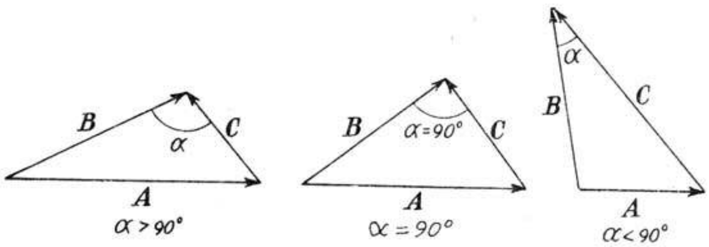



Fig. 1.11. For Example 1.5.

**Example 1.6.** *Show that* $\mathbf{a} = i \cos \alpha + j \sin \alpha$ *and* $\mathbf{b} = i \cos \beta - j \sin \beta$ *are unit vectors in the X–Y plane, inclined at an angle* $\alpha$ and $-\beta$ *respectively with the X-axis. By appropriate product relation between the vectors, derive the expansion formula for* $\cos(\alpha + \beta)$.

**Solution.** $\mathbf{a} = i \cos \alpha + j \sin \alpha$ is a vector whose components along $X$ and $Y$ axes are $\cos \alpha$ and $\sin \alpha$ respectively, and the magnitude of the vector is $\sqrt{(\cos \alpha)^2 + (\sin \alpha)^2} = 1$ ; $\mathbf{a}$ is thus a unit vector.

$$\mathbf{b} = i \cos \beta - j \sin \beta$$

Here the angle $(-\beta)$ is negative. However, the length of the vector is $\sqrt{(\cos \beta)^2 + (\sin \beta)^2} = 1$. Hence $\mathbf{b}$ is also a unit vector. See Fig. 1.12 showing the two unit vectors $a$ and $b$. The angle between the two unit vectors is $\alpha + \beta$.

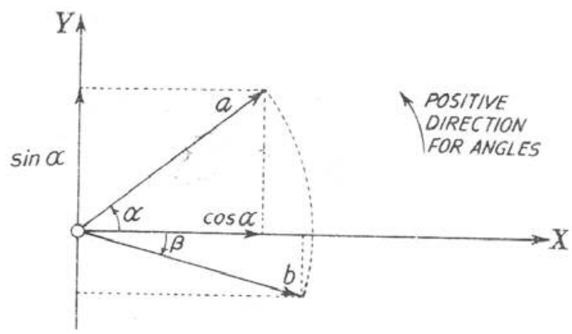



Fig. 1.12. For Example 1.6.

The dot product of the vectors is 1 cos (**a, b**)

Thus,
$$\cos(\alpha + \beta) = \mathbf{a} \cdot \mathbf{b}$$
$$= (i \cos \alpha + j \sin \alpha) \cdot (i \cos \beta - j \sin \beta),$$
$$= \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Similarly, replacement of $+\beta$ by $-\beta$ yields

$$\cos(\alpha - \beta) = (\cos \alpha \cos \beta + \sin \alpha \sin \beta) \quad \text{...(1.22)}$$

(Cosine formula)

**Example 1.7.** *Find the product* $\mathbf{A} \times \mathbf{B}$, *given that*

$$\mathbf{A} = -4\mathbf{u}_x + 3\mathbf{u}_y - 4\mathbf{u}_z$$

*and*
$$\mathbf{B} = 2\mathbf{u}_x - \mathbf{u}_y + 2\mathbf{u}_z.$$

**Solution.** In matrix form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ -4 & +3 & -4 \\ 2 & -1 & +2 \end{vmatrix}$$
$$= (6 - 4)\,\mathbf{u}_x + (-8 + 8)\,\mathbf{u}_y + (4 - 6)\,\mathbf{u}_z$$
$$= 2\,\mathbf{u}_x - 2\,\mathbf{u}_z$$

**Example 1.8.** *Using the definition of "cross product", find the angle between the vectors*

$$2\,\mathbf{u}_x - 2\mathbf{u}_y + \mathbf{u}_z \qquad \textit{and}$$
$$2\mathbf{u}_x - \mathbf{u}_y - 2\mathbf{u}_z.$$

**Solution.** The angle between two vectors **A** and **B** may be determined from the relation

$$\sin(\mathbf{A}, \mathbf{B}) = \frac{|\mathbf{A} \times \mathbf{B}|}{|\mathbf{A}|\ |\mathbf{B}|}$$

Let the vectors given be **A** and **B** respectively.

In matrix form,
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ 2 & -2 & 1 \\ 2 & -1 & -2 \end{vmatrix}$$

$$= (4 + 1)\,\mathbf{u}_x + (2 + 4)\,\mathbf{u}_y + (-2 + 4)\,\mathbf{u}_z$$
$$= 5\mathbf{u}_x + 6\mathbf{u}_y + 2\mathbf{u}_z$$
$$|\mathbf{A} \times \mathbf{B}| = \sqrt{5^2 + 6^2 + 2^2} = \sqrt{65}$$
$$|\mathbf{A}| = \sqrt{2^2 + 2^2 + 1^2} = 3$$
$$|\mathbf{B}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$
$$\sin(\mathbf{A}, \mathbf{B}) = \frac{\sqrt{65}}{(3)\,(3)} = 0.896$$

Angle between the vectors $A$ and $B = \sin^{-1}(0.896) = \mathbf{63.6°}$.

## 1.7. CURVILINEAR SYSTEM OF CO-ORDINATES

In the foregoing articles and examples, only cartesian (or rectangular) system was used. Quite often, in examples with symmetrical geometrical configurations and fields, curvilinear co-ordinates find useful applications especially in cases of cylindrical and spherical configurations.

Consider the equation

$$f(x, y, z) = p \qquad ...(1.23)$$

where $p$ is a constant. This equation represents a family of surfaces in space. For example, $z = p$ depicts the surface parallel to the $X$–$Y$ plane, in cartesian system of co-ordinates.

In the general curvilinear system of co-ordinates, we consider any point $p$ as the point of intersection of three mutually perpendicular surfaces. Mathematically, we may think of three describing equations such as

$$\left.\begin{aligned} f_1(x, y, z) &= p_1 \\ f_2(x, y, z) &= p_2 \\ f_3(x, y, z) &= p_3 \end{aligned}\right\} \qquad ...(1.24)$$

which may be chosen such that the three families of surfaces are mutually perpendicular (orthogonal). The point $P$ is completely defined if the values of $p_1, p_2$ and $p_3$ corresponding to the three surfaces are specified. The variables $p_1, p_2, p_3$ are then the "curvilinear co-ordinates" of the point $P$.

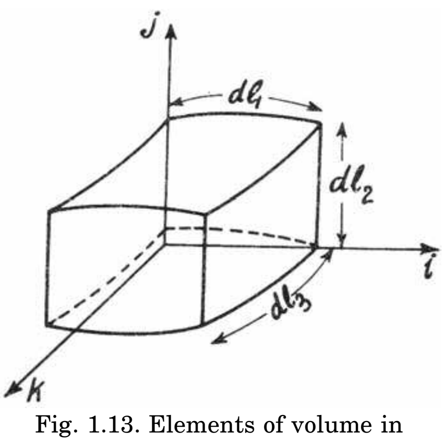



Fig. 1.13. Elements of volume in curvilinear co-ordinates.

Consider an infinitesimal volume shown in Fig. 1.13. The elemental length $dl_1$ is perpendicular to the surface defined by the co-ordinate $p_1$. The length $dl_1$ represents the infinitesimal separation between the surfaces $p_1$ and $p_1 + dp_1$ ; this elemental length may be expressed in the form

$$dl_1 = h_1 dp_1 \quad ...(1.25)$$

where $h_1 = f(p_1, p_2, p_3)$

In the same manner, the elemental lengths $dl_2$ and $dl_3$ may also be written as follows :

$$dl_2 = h_2 dp_2 \quad ...(1.26)$$

$$dl_3 = h_3 dp_3 \quad ...(1.27)$$

The above three are the most general equations. In particular, in the case of cartesian system,

$$h_1 = h_2 = h_3 = 1$$

$$\left.\begin{aligned} dl_1 &= dx = dp_1 \\ dl_2 &= dy = dp_2 \\ dl_3 &= dz = dp_3 \end{aligned}\right\} \quad ...(1.28)$$

It may be recalled that in the cartesian system, $i, j, k$ are unit vectors parallel to $X, Y, Z$ axes (in the positive direction) respectively, viz., $i = \mathbf{u}_x, j = \mathbf{u}_y$ and $k = \mathbf{u}_z$ ; but in the general curvilinear system the unit vectors $i, j, k$ are defined as the vectors of unit length normal respectively to the surfaces characterized by the co-ordinates $p_1, p_2, p_3$ and oriented in the positive direction, *viz.,* towards the increasing values of the respective co-ordinates. As in the cartesian system, the cross product relations

$$\left.\begin{aligned} i \times j &= k \\ j \times k &= i \\ k \times i &= j \end{aligned}\right\} \quad ...(1.28\ a)$$

are satisfied.

In general, the volume element can be written as

$$dv = dl_1\, dl_2\, dl_3 = h_1 h_2 h_3\, (dp_1 dp_2 dp_3) \quad ...(1.29)$$

As the cartesian system has already been introduced, the next two articles are devoted to the introduction to cylindrical and spherical systems.

### 1.7.1. Cylindrical system of co-ordinates

In this system of co-ordinates, any point in space is considered as the point of intersection of the following surfaces :

a circular cylinder (radius $r$ constant)
a vertical plane ( $\phi$ constant)
another plane with $z$ constant. (This plane is parallel to $XY$ plane at a distance $z$ from it.)

The cylindrical system is fully illustrated in Fig. 1.14. The co-ordinates are $r$, $\phi$ and $z$. The unit vectors $i, j, k$ at any point $P$ satisfy the cross product relation $i \times j = k$. It may be noted that $i$ is the unit vector in the direction of increasing $r$ ; $j$ is the unit vector in the direction of increasing $\phi$ and $k$ is the same as in cartesian system, being the unit vector in the direction of increasing $z$. In the general curvilinear system, the unit vectors do not maintain the same direction in space as the point $P$ moves about, but they remain mutually orthogonal, in compliance with the right–handed system.

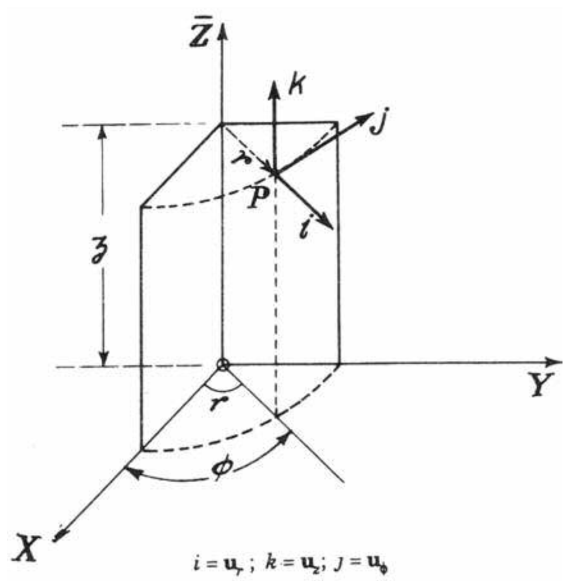



Fig. 1.14. Cylindrical system with co-ordinates, $r$, $\phi$, $z$.

A differential volume element in the cylindrical system of co-ordinates is depicted in Fig. 1.15.

In this case,

$$dl_1 = h_1 dp_1 = dr \; ; \; \mathbf{dl}_1 = (dr)\, i$$
$$dl_2 = h_2 dp_2 = rd\phi \; ; \; dl_2 = (rd\phi)\, j \qquad \text{...(1.30)}$$
$$dl_3 = h_3 dp_3 = dz \; ; \; dl_3 = (dz)\, k$$

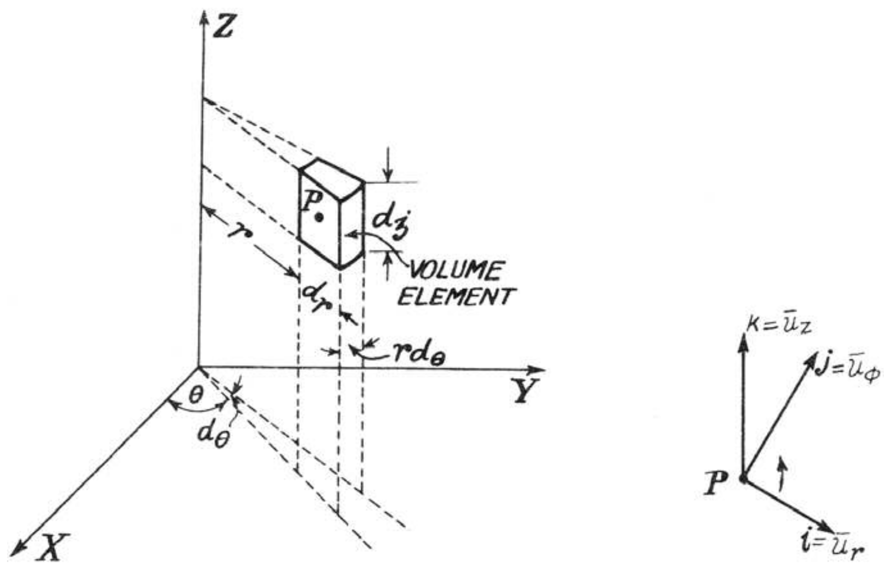



Fig. 1.15. Volume element in cylindrical system.

It can be seen readily that

$$h_1 = 1,\; p_1 = r$$
$$h_2 = r,\; p_2 = \phi \qquad \text{...(1.30 } a\text{)}$$
$$h_3 = 1,\; p_3 = z$$

Any elemental length $dl$ can be written in the vector form as follows :

$$\mathbf{dl} = (dl_1)\, i + (dl_2) j + (dl_3)\, k$$
$$= (dr)\, i + (rd\phi) j + (dz)\, k \qquad \text{...(1.31)}$$
$$dl = \sqrt{(dr)^2 + (rd\phi)^2 + (dz)^2} \qquad \text{...(1.32)}$$

The volume element is

$$dv = (dl_1)\,(dl_2)\,(dl_3)$$
$$= (dx)\,(dy)\,(dz) \qquad \text{...(1.33)}$$

(in cartesian system).

The volume element in the cylindrical system is given by

$$dv = (dr)\,(rd\phi)\,(dz)$$
$$= r\, dr\, d\phi\, dz \qquad \text{...(1.34)}$$

**1.7.2. Spherical system of co-ordinates**

In this system, we consider any point as the point of intersection of the following surfaces :

a sphere (radius $R$ constant)

a cone ($\theta$ constant, the angle between $R$ and $Z$-axis)

a plane ($\phi$ constant).

Fig. 1.16 shows the point $P$ as a point of intersection of the spherical surface, conical surface and plane surface defined by the co-ordinates $R$, $\theta$ and $\phi$. The unit vectors $i, j, k$ at any point are the

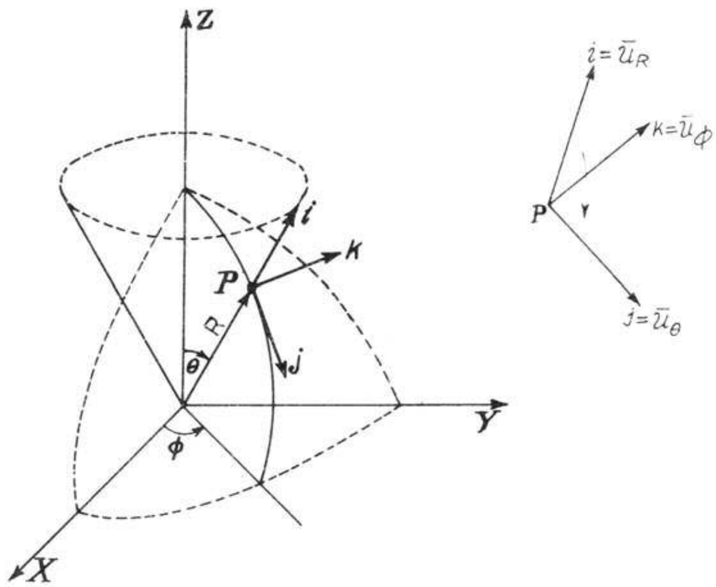



Fig. 1.16. Spherical system with co-ordinates ($R$, θ, φ).

vectors $\mathbf{u}_R$, in the direction of increasing $R$, $\mathbf{u}_\theta$ in the direction of increasing θ and $\mathbf{u}_\phi$ in the direction of increasing φ respectively. The unit vectors satisfy the cross product relation

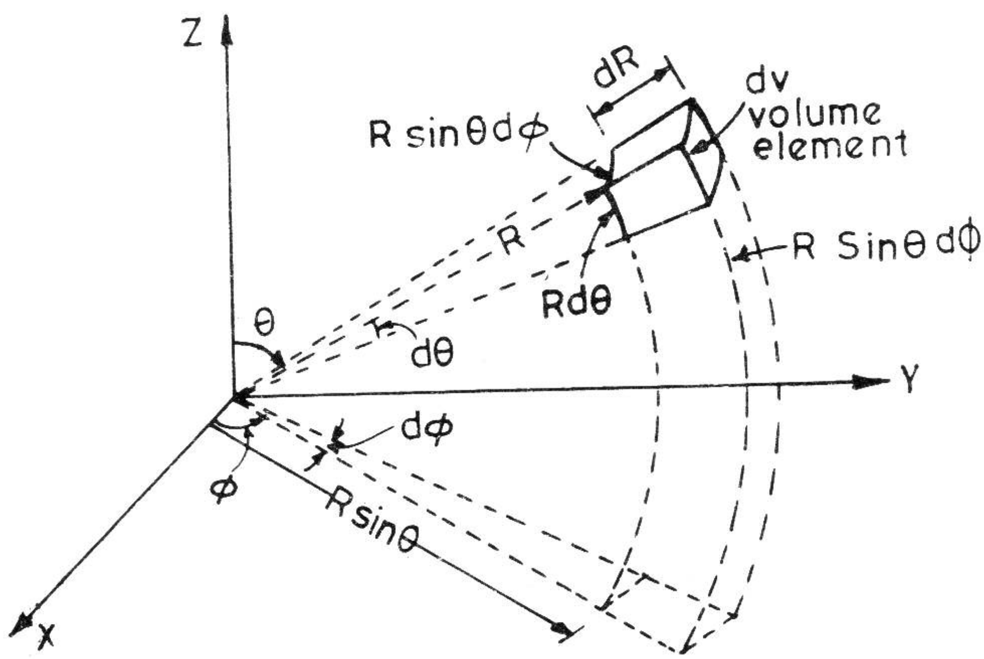
$$i \times j = k$$

or

$$\mathbf{u}_R \times \mathbf{u}_\theta = \mathbf{u}_\phi$$

It should be noted that the angle φ is the same in the cylindrical and spherical systems.

A differential volume element in the spherical system is shown in Fig. 1.17.




$$dv = (dR)\,(Rd\theta)\,(R \sin\theta\, d\phi)$$

Fig. 1.17. Volume element in spherical system of co-ordinates.

In the spherical system,

$$dl_1 = h_1\, dp_1 = (dR)\ ;\ dl_1 = dR$$
$$dl_2 = h_2\, dp_2 = (R d\theta)\ ;\ dl_2 = R d\theta \qquad ...(1.35)$$
$$dl_3 = h_3\, dp_3 = (R \sin\theta\, d\phi)\ ;\ dl_3 = R \sin\theta\, d\phi$$

Here

$$h_1 = 1\ ;\ p_1 = R$$
$$h_2 = R\ ;\ p_2 = \theta \qquad ...(1.36)$$
$$h_3 = R \sin\theta\ ;\ p_3 = \phi$$

An elemental length in the spherical system can be written in the form

$$\mathbf{dl} = (dl_1)\, i + (dl_2)\, j + (dl_3)\, k$$
$$= (dR)\, i + (R d\theta)\, j + (R \sin\theta\, d\phi)\, k. \qquad ...(1.37)$$
$$dl = \sqrt{(dR)^2 + (R\, d\theta)^2 + (R \sin\theta\, d\phi)^2} \qquad ...(1.38)$$

The volume element may be expressed as

$$dv = (dl_1)\,(dl_2)\,(dl_3)$$
$$= (dR)\,(R\, d\theta)\,(R \sin\theta d\phi)$$
$$= R^2 \sin\theta\, dR\, d\theta\, d\phi \qquad ...(1.39)$$

Table 1.1 given below summarizes the correspondence of curvilinear system with cartesian, cylindrical and spherical systems of co-ordinates :

**Table 1.1**

| *Curvilinear* | *Cartesian (cylindrical)* | *Cylindrical* | *Spherical* |
|---|---|---|---|
| $p_1$ | $x$ | $r$ | $R$ |
| $p_2$ | $y$ | $\phi$ | $\theta$ |
| $p_3$ | $z$ | $z$ | $\phi$ |
| $h_1$ | 1 | 1 | 1 |
| $h_2$ | 1 | $r$ | $R$ |
| $h_3$ | 1 | 1 | $R \sin\theta$ |

## 1.8. TRANSFORMATION BETWEEN CARTESIAN AND CYLINDRICAL SYSTEMS

It may sometimes be necessary to transform a vector from one system to another system of co-ordinates. In this article, we shall derive the relations required for transformation of a vector relation or quantity between cartesian and cylindrical forms.

Let a vector $F$ in the cartesian (rectangular) system be

$$\mathbf{F} = F_x i + F_y j + F_z k$$
$$= F_x \mathbf{u}_x + F_y \mathbf{u}_y + F_z \mathbf{u}_z \qquad ...(1.40)$$

where $F_x, F_y, F_z$ are functions of $x, y, z$.

Our objective is to transform the vector from the above form to the cylindrical system of co-ordinates viz.,

$$\mathbf{F} = F_r \mathbf{u}_r + F_\phi\, \mathbf{u}_\phi + F_z \mathbf{u}_z \qquad ...(1.41)$$

where $F_r, F_\phi, F_z$ are functions of $r, \phi, z$.

Referring to Fig. 1.18, it can be readily seen that the vector ($r$) directed from $A$ to $P$ can be resolved into components $x$ and $y$. The co-ordinate transformation relations obtain as follows :

$$\left.\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan\phi &= y/x \\ z &= z \end{aligned}\right\} \qquad \text{...(1.42)}$$

The co-ordinates in cylindrical form may be transformed into cartesian form by the following relations.

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned}\right\} \qquad \text{...(1.43)}$$

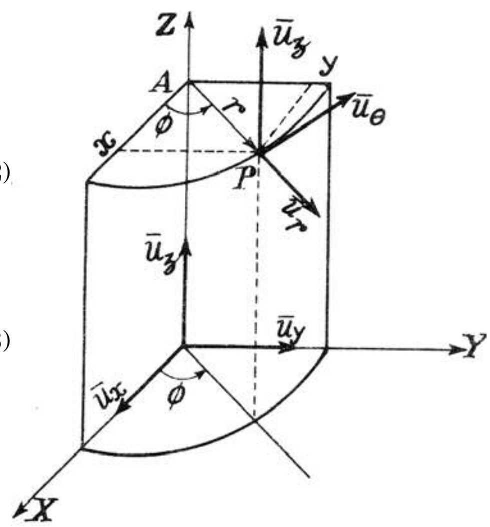



Fig. 1.18. Transformation between cartesian and cylindrical systems.

The variable $z$ and the unit vector $k$ or $\mathbf{u}_z$ are the same in cartesian and cylindrical forms. However, the unit vectors $i$ and $j$ should be clearly distinguished in the two cases. In order to avoid confusion, we shall denote the vectors in cartesian system as $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_z$ and in cylindrical system as $\mathbf{u}_r$, $\mathbf{u}_\phi$, $\mathbf{u}_z$. Now the equations (1.40) and (1.41) may be rewritten as

$$\mathbf{F} = F_x\,\mathbf{u}_x + F_y\mathbf{u}_y + F_z\,\mathbf{u}_z \qquad \text{...(1.43}a\text{)}$$

$$\mathbf{F} = F_r\mathbf{u}_r + F_\phi\,\mathbf{u}_\phi + F_z\mathbf{u}_z \qquad \text{...(1.43}b\text{)}$$

As $F_r$ is a component of $F$ in the radial direction, this component can be evaluated from the dot product $\mathbf{F}\cdot\mathbf{u}_r$. Similarly, $F_\phi$ is a component of $F$ in the direction of increasing $\phi$ and is given by the dot product of $\mathbf{F}$ with $\mathbf{u}_\phi$.

Thus,
$$F_r = \mathbf{F}\cdot\mathbf{u}_r$$
$$F_\phi = \mathbf{F}\cdot\mathbf{u}_\phi$$

Substituting for $F$ from equation (1.43 $a$),

$$\left.\begin{aligned} F_r &= (F_x\mathbf{u}_x + F_y\mathbf{u}_y + F_z\mathbf{u}_z)\cdot\mathbf{u}_r \\ &= F_x\mathbf{u}_x\cdot\mathbf{u}_r + F_y\mathbf{u}_y\cdot\mathbf{u}_r \\ &\quad (\text{as } \mathbf{u}_z\cdot\mathbf{u}_r = 0) \end{aligned}\right\} \qquad \text{...(1.44)}$$

Similarly,
$$\left.\begin{aligned} F_\phi &= F_x\,\mathbf{u}_x\cdot\mathbf{u}_\phi + F_y\mathbf{u}_y\cdot\mathbf{u}_\phi \\ &\quad (\text{as } \mathbf{u}_z\cdot\mathbf{u}_\phi = 0) \end{aligned}\right\} \qquad \text{...(1.45)}$$

It is clearly seen from Fig. 1.18 that

$$\left.\begin{aligned} \text{u}\mathbf{u}_x\cdot\mathbf{u}_r &= \cos\phi \\ \mathbf{u}_y\cdot\mathbf{u}_r &= \sin\phi \\ \mathbf{u}_x\cdot\mathbf{u}_\phi &= -\sin\phi \\ \mathbf{u}_y\cdot\mathbf{u}_\phi &= \cos\phi \end{aligned}\right\} \qquad \text{...(1.46)}$$

Substituting these relations into Eqs. (1.44) and (1.45), we obtain

$$F_r = F_x\cos\phi + F_y\sin\phi$$
$$F_\phi = -F_x\sin\phi + F_y\cos\phi$$

The complete vector expression in cylindrical form can be written in terms of cartesian (rectangular) components as follows :

$$\mathbf{F} = (F_x\cos\phi + F_y\sin\phi)\,\mathbf{u}_r + (-F_x\sin\phi + F_y\cos\phi)\,\mathbf{u}_\phi + F_z\mathbf{u}_z \qquad \text{...(1.47)}$$

**Transformation from cylindrical to cartesian (rectangular) form :**

If **F** is in cylindrical form, $F_x$ and $F_y$ may be expressed in terms of the original **F** in cylindrical form as follows :

$$F_x = \mathbf{F} \cdot \mathbf{u}_x$$
$$F_y = \mathbf{F} \cdot \mathbf{u}_y$$

Thus,
$$F_x = (F_r\mathbf{u}_r + F_\phi\mathbf{u}_\phi + F_z\mathbf{u}_z) \cdot \mathbf{u}_x$$
$$= F_r \cos\phi - F_\phi \sin\phi = F_r \frac{x}{\sqrt{x^2+y^2}} - F_\phi \frac{y}{\sqrt{x^2+y^2}}$$

Similarly,
$$F_y = F_r \frac{y}{\sqrt{x^2+y^2}} + F_\phi \frac{x}{\sqrt{x^2+y^2}}$$
$$F_z = F_z \qquad \text{(same as in cylindrical system)}$$
$$\mathbf{F} = \left(F_r \frac{x}{\sqrt{x^2+y^2}} - F_\phi \frac{y}{\sqrt{x^2+y^2}}\right)\mathbf{u}_x + \left(F_r \frac{y}{\sqrt{x^2+y^2}} + F_\phi \frac{x}{\sqrt{x^2+y^2}}\right)\mathbf{u}_y + F_z\mathbf{u}_z \qquad ...(1.48)$$

The transformation relations between cartesian and cylindrical systems are summarized in Table 1.2.

**Table 1.2**

| *Cartesian to cylindrical* | *Cylindrical to cartesian* |
|---|---|
| $F_r = F_x \cos\phi + F_y \sin\phi$ | $F_x = F_r \frac{x}{\sqrt{x^2+y^2}} - F_\phi \frac{y}{\sqrt{x^2+y^2}}$ |
| $F_\phi = -F_x \sin\phi + F_y \cos\phi$ | $F_y = F_r \frac{y}{\sqrt{x^2+y^2}} + F_\phi \frac{x}{\sqrt{x^2+y^2}}$ |
| | $F_z = F_z$ |

## 1.9. TRANSFORMATION BETWEEN CARTESIAN AND SPHERICAL SYSTEMS

A vector in cartesian system can be transformed into spherical system in exactly the same manner as in the preceding article.

It is seen from Fig. 1.19 that

$$\left.\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}\right\} \qquad ...(1.49)$$

Given the cartesian co-ordinates, the corresponding spherical co-ordinates can be derived as follows :

$$R^2 = x^2 + y^2 + z^2$$
$$\cos\theta = z/R = z/(x^2+y^2+z^2)^{1/2} \qquad ...(1.50)$$
$$\tan\phi = y/x$$

**Transformation of vector F from cartesian to spherical systems :**

We shall now transform the vector $F$ in (1.43 $a$) into spherical system of co-ordinates, *viz.*,

$$\mathbf{F} = F_R\mathbf{u}_R + F_\theta\mathbf{u}_\theta + F_\phi\mathbf{u}_\phi \qquad ...(1.51)$$

Now, by definition,
$$F_R = \mathbf{F} \cdot \mathbf{u}_R \qquad ...(1.52\ a)$$
$$F_\theta = \mathbf{F} \cdot \mathbf{u}_\theta \qquad ...(1.52\ b)$$
$$F_\phi = \mathbf{F} \cdot \mathbf{u}_\phi \qquad ...(1.52\ c)$$

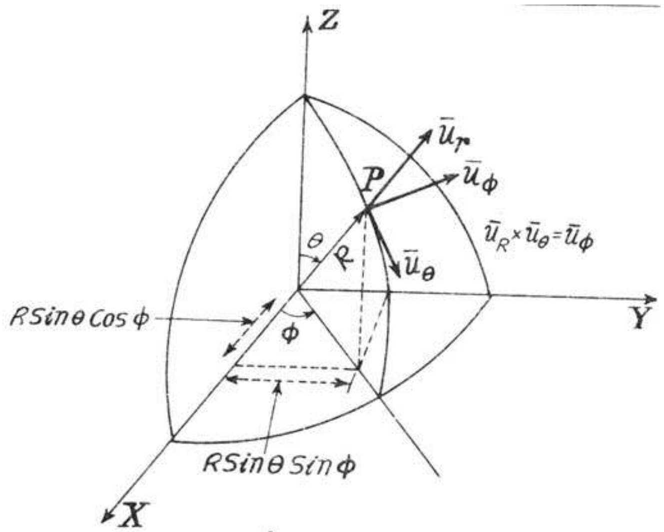



Fig. 1.19. Transformation between cartesian and spherical systems (Also for Example 1.10).

Consequently $\quad F_R = F_x \mathbf{u}_x \cdot \mathbf{u}_R + F_y \mathbf{u}_y \cdot \mathbf{u}_R + F_z \mathbf{u}_z \cdot \mathbf{u}_R$

The dot product relations between the unit vectors are

$$\mathbf{u}_x \cdot \mathbf{u}_R = \sin\theta\cos\phi$$
$$\mathbf{u}_y \cdot \mathbf{u}_R = \sin\theta\sin\phi$$
$$\mathbf{u}_z \cdot \mathbf{u}_R = \cos\theta$$

Hence $F_R = F_x \sin\theta\cos\phi + F_y \sin\theta\sin\phi + F_z \cos\theta$ ...(1.53)

Similarly, the following relations may be established

$$F_\theta = F_x \cos\theta\cos\phi + F_y \cos\theta\sin\phi - F_z \sin\theta \quad \text{...(1.54)}$$
$$F_\phi = -F_x \sin\phi + F_y \cos\phi \quad \text{...(1.55)}$$

With a good background for establishing transformation relations from one system to another, the readers are advised to develop the relations for the components of $F$ in cartesian form, viz., for $F_x$, $F_y$ and $F_z$ from the vector expressions in spherical form. This is left as an exercise to them.

The relations for mutual transformation between cartesian (rectangular) and spherical systems are summarized in Table 1.3.

**Table 1.3**

| *Cartesian to spherical* | *Spherical to cartesian* |
|---|---|
| $F_R = F_x \sin\theta\cos\phi + F_y \sin\theta\sin\phi + F_z\cos\theta$ | $F_x = \dfrac{F_R\, x}{\sqrt{x^2+y^2+z^2}} + \dfrac{F_\theta\, xz}{\sqrt{(x^2+y^2)\,(x^2+y^2+z^2)}} - \dfrac{F_\phi\, y}{\sqrt{(x^2+y^2)}}$ |
| $F_\theta = F_x \cos\theta\cos\phi + F_y\cos\theta\sin\phi - F_z\sin\theta$ | $F_y = \dfrac{F_R\, y}{\sqrt{x^2+y^2+z^2}} + \dfrac{F_\theta\, yz}{\sqrt{(x^2+y^2)\,(x^2+y^2+z^2)}} + \dfrac{F_\phi\, x}{\sqrt{x^2+y^2}}$ |
| $F_\phi = -F_x\sin\phi + F_y\cos\phi$ | $F_z = \dfrac{F_R\, z}{\sqrt{x^2+y^2+z^2}} - \dfrac{F_\theta\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$ |

**Example 1.9.** *Give the cylindrical co-ordinates of the point whose cartesian co-ordinates are $x = 3$ ; $y = 4$ ; $z = 5$ units and show the same on a sketch.*

**Solution.** $x = 3$ ; $y = 4$ and $z = 5$ units.

Using the equation (1.42), the cylindrical co-ordinates are found as follows :

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5 \text{ units.}$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(4/3) = 53.1°.$$

$$z = 5 \quad \text{units (same as in cartesian system)}$$

The point $P$ and its co-ordinates in both systems (cartesian and cylindrical) are clearly depicted in Fig. 1.20.

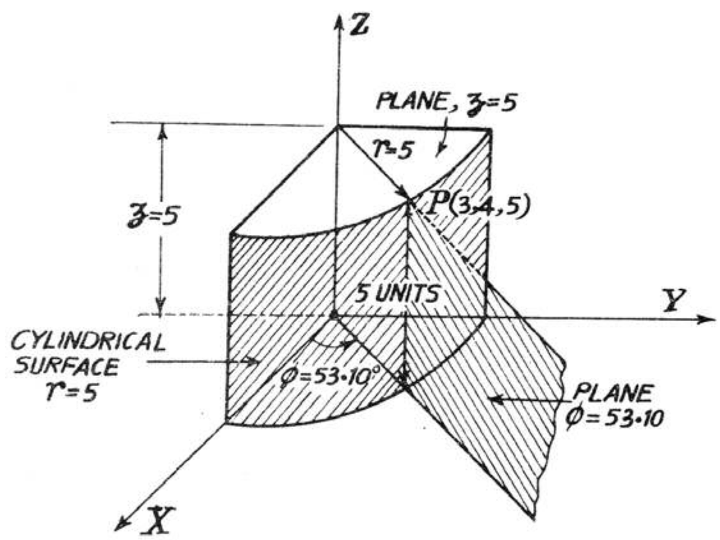



Fig. 1.20. For Example 1.9.

**Example 1.10.** *Give the spherical co-ordinates of the point whose cartesian co-ordinates are $x = 3$ ; $y = 4$ ; $z = 5$ units and show the same on a sketch.*

**Solution.** $x = 3$ ; $y = 4$ and $z = 5$ units.

Referring to equation (1.50),

$$R^2 = x^2 + y^2 + z^2 = 3^2 + 4^2 + 5^2$$

whence $R = 5\sqrt{2} = 7.07$ units.

$$\cos\theta = z/R = 5/(5\sqrt{2}) = 1/\sqrt{2}$$

$$\theta = 45°.$$

$$\tan\phi = y/x = 4/3$$

$$\phi = 53.1°$$

The point $P$ is displayed on Fig. 1.19.

**Example 1.11.** *A circle in the X-Y plane is centred at the origin and has a radius of 2 units. Determine the unit vector* $\mathbf{u}_p$ *(a) in cartesian and (b) polar systems, drawn tangentially to the circle at the point whose co-ordinates are $x = \sqrt{3}$ ; $y = 1$ and $z = 0$ and is in the general direction of increasing $y$.*

**Solution.** This is a two-dimensional problem. ($z = 0$)

The circle is centred at the origin and is in the $X$, $Y$ plane, as shown in Fig. 1.21. The point $P$ has the co-ordinates

$$x = \sqrt{3} \text{ and } y = 1.$$

Hence $\quad r = \sqrt{x^2 + y^2} = \sqrt{(3 + 1)} = 2$ units. (as given) (*a*) To determine the unit vector $\mathbf{u}_p$ at $P$ in cartesian co-ordinates :

The radius vector $r$ makes an angle $\phi$ with $X$-axis, *viz.*,

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(1/\sqrt{3}) = 30°.$$

Referring to the Fig. 1.21, it is seen that the unit vector $\mathbf{u}_p$ is oriented at an angle of 120° with $+X$ axis and 60° with $-X$ axis.

Hence $$\mathbf{u}_p = (-\cos 60°)\,\mathbf{u}_x + (\sin 60°)\,\mathbf{u}_y$$
$$= -\frac{1}{2}\mathbf{u}_x + \sqrt{3}/2\,\mathbf{u}_y$$

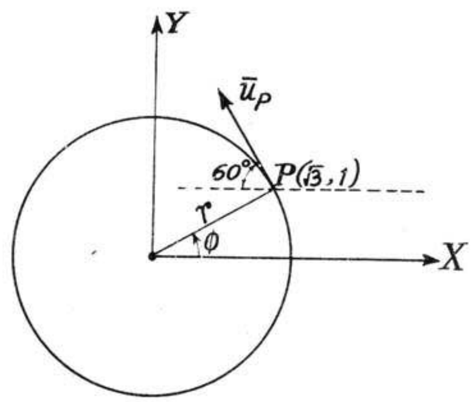


Fig. 1.21. For Example 1.11.

(*b*) To determine the unit vector $\mathbf{u}_p$ at the point $P$ in polar co-ordinates :

Referring to Table 1.2,

$$F_r = F_x \cos\phi + F_y \sin\phi$$
$$F_\phi = -F_x \sin\phi + F_y \cos\phi$$

In part (*a*) $$F_x = -1/2$$
$$F_y = \sqrt{3}/2 \text{ and } \phi = 30°$$

Now, in polar co-ordinates,

$$\mathbf{u}_p = F_r\mathbf{u}_r + F_\phi\mathbf{u}_\phi$$

where $$F_r = -\frac{1}{2}\times\sqrt{3}/2 + \sqrt{3}/2\times\frac{1}{2} = 0$$
$$F_\phi = +\frac{1}{2}\times\frac{1}{2} + \sqrt{3}/2\times\sqrt{3}/2 = 1$$

Thus, in polar co-ordinates,

$$\mathbf{u}_p = \mathbf{u}_\phi = 1\ \angle 120°$$

as $\phi$ = angle that the unit vector $\mathbf{u}_p$ makes with the $X$-axis in the positive direction = 120° · $\mathbf{u}_p$ is in the direction of increasing $\phi$ and has no component in the radial direction, since it is tangential to the circle at the point $P$.

**Example 1.12.** *Locate on a sketch and specify the cylindrical co-ordinates of the points whose co-ordinates are*

(*a*) $x = 1$ ; $y = 1.5$ *and* $z = 2.5$ *units ;*

(*b*) $x = -1$ ; $y = 2$ *and* $z = -3$ ;

(*c*) $x = 1$ ; $y = -3$ *and* $z = 2$.

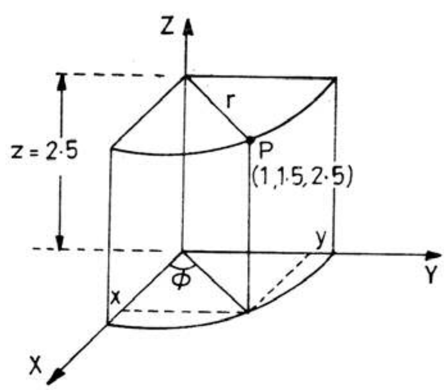


Fig. 1.22

Solution. (*a*) $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1.5^2} = 1.803$ units

$\tan\phi = 1.5$ ; whence $\phi = 56.3°$

$z = 2.5$ units.

The co-ordinates are $r = 1.803$ ; $\phi = 56.3°$ ; $z = 2.5$ units. See Fig. 1.22. The point is shown as $P$ in the diagram.

(*b*) The point is shown as $Q$ in Fig. 1.23. If a perpendicular is drawn from the point $Q$ to the $X$–$Y$ plane, it meets the $XY$ plane at $Q'$.

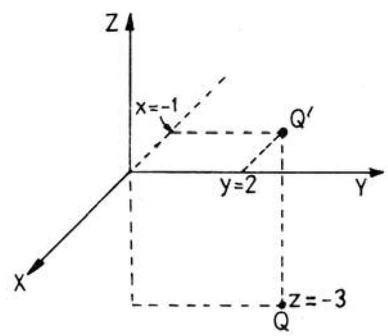



Fig. 1.23. For Example 1.12 (*a*)

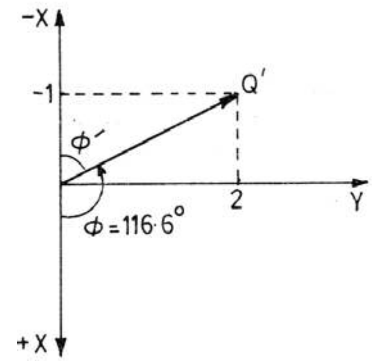



Fig. 1.24. For Example 1.12 (*b*)

For clarity, the point $Q'$ is depicted in the $X$–$Y$ plane redrawn on the plane of this page, to indicate the angle $\phi$ measured from $X$ axis in the positive direction. See Fig. 1.24.

It is seen that the supplementary angle $\phi'$ is given by the relation

$$\tan\phi' = 2 \text{ ; whence } \phi' = 63.43°$$
$$\phi = 180 - 63.43 = 116.57°, \text{ (say) } 116.6°$$
$$r = \sqrt{(-1)^2 + 2^2} = \sqrt{5} = 2.236 \text{ units.}$$

The co-ordinates are $r = 2.236$ units ; $\phi = 116.6°$ and $z = -3$ units.

(*c*) Fig. 1.25 shows the projection of the point $S$ (1, – 3, 2) by drawing a perpendicular to the $XY$ plane. $S'$ is the point projected from $S$ and is located on $XY$ plane at a distance $r$ from the origin.

$$r = \sqrt{1^2 + (-3)^2} = \sqrt{10} = 3.16 \text{ units.}$$

$\phi'$ = the angle made by $OS'$ with $-Y$ axis = 18.43°. Hence, the angle $\phi$ between the $+X$ axis and the vector $OS'$ is – 71.57° which is the same as the angle 360 – 71.57 = 288.43° made by $OS'$ in the positive direction with $+X$ axis.

The co-ordinates are $r = 3.16$ ; $\phi = -71.57°$ or 288.43° and $z = 2$ units.

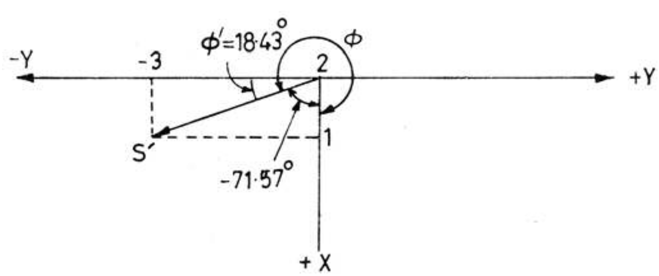



Fig. 1.25. For Example 1.12 (*c*).

**Example 1.13.** *A vector field is given by the expression*

*(a)* $\mathbf{F} = (1/r)\,\mathbf{u}_r$ *(in cylindrical co-ordinates) ;*

*(b)* $\mathbf{F} = (1/R)\,\mathbf{u}_R$ *(in spherical co-ordinates).*

*Determine* **F** *in each case in cartesian form, at a point with x = 1, y = 1 and z = 1 unit.*

**Solution.** (*a*) $\mathbf{F} = \frac{1}{r}\,\mathbf{u}_r$ (in cylindrical form)

$= F_r\,\mathbf{u}_r + F_\phi\,\mathbf{u}_\phi + F_z\,\mathbf{u}_z$ (in general)

From the data, it is clear that $F_\phi = F_z = 0$.

In other words, $F_r = 1/r$, meaning that the field varies only with the distance $r$ from the origin.

At the point $(x = 1, y = 1, z = 1)$

$$r = \sqrt{(1^2 + 1^2)} = \sqrt{2}\ ;$$
$$F = \frac{1}{\sqrt{2}}$$

From the Table, 1.2,

$$F_x = F_r \frac{x}{\sqrt{x^2 + y^2}} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2} \qquad \text{(as } F_\phi = 0)$$
$$F_y = F_r \frac{y}{\sqrt{x^2 + y^2}} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}$$

Hence, the field at (1, 1, 1) is given in cartesian system as

$$\mathbf{F} = \frac{1}{2}(\mathbf{u}_x + \mathbf{u}_y)$$

The vector field has no component in $Z$-direction, as it varies only with the distance of the point (at which field is required) from $Z$-axis and lies in a plane parallel to $X$–$Y$ plane.

(*b*) $\mathbf{F} = (1/R)\, \mathbf{u}_R$ (in spherical system)

In general (in spherical system),

$$\mathbf{F} = F_R \,/\, \mathbf{u}_R + F_\theta\, \mathbf{u}_\theta + F_\phi\, \mathbf{u}_\phi.$$

In this example, the field varies only as a function of the radial distance of the point from the origin, the point assumed to lie on a sphere with radius $R$. The components $F_\theta$ and $F_\phi$ are, therefore, non-existent.

$$F_R = 1/R$$

At the point in question, *viz.*,

$$x = 1, y = 1, z = 1,$$
$$R = \sqrt{(1 + 1 + 1)} = \sqrt{3}$$
$$F_R = 1/\sqrt{3}$$

From Table 1.3,

$$F_x = F_R\left(\frac{x}{R}\right) = \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{1}{3} \qquad \text{(as } F_\theta = F_\phi = 0)$$
$$F_y = F_R \frac{y}{R} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$$
$$F_z = F_R\left(\frac{z}{R}\right) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$$

Hence, at (1, 1, 1), $\mathbf{F} = \frac{1}{3}(\mathbf{u}_x + \mathbf{u}_y + \mathbf{u}_z)$.

**Example 1.14.** *Express the unit vectors* $\mathbf{u}_R$, $\mathbf{u}_\theta$, $\mathbf{u}_\phi$ *in the spherical system in terms of the unit vectors* $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_z$ *of the cartesian system.*

**Solution.** Refer to Fig. 1.19 and to the Article 1.9 on Transformation between cartesian and spherical systems. It is seen that

$$\mathbf{u}_R \cdot \mathbf{u}_x = \sin\theta \cos\phi$$
$$\mathbf{u}_R \cdot \mathbf{u}_y = \sin\theta \sin\phi$$

$$\mathbf{u}_R \cdot \mathbf{u}_z = \cos\theta$$

Consequently, $\mathbf{u}_R = (\sin\theta\cos\phi)\,\mathbf{u}_x + (\sin\theta\sin\phi)\,\mathbf{u}_y + (\cos\theta)\,\mathbf{u}_z$

Transforming the multipliers into cartesian co-ordinates, we obtain

$$\mathbf{u}_R = \frac{x}{R}\mathbf{u}_x + \frac{y}{R}\mathbf{u}_y + \frac{z}{R}\mathbf{u}_z$$

Similarly, prove the following relations :

$$\mathbf{u}_\theta = (\cos\theta\cos\phi)\,\mathbf{u}_x + (\cos\theta\sin\phi)\,\mathbf{u}_y + (-\sin\theta)\,\mathbf{u}_z$$

$$= \frac{1}{Rr}(zx\,\mathbf{u}_x + yz\,\mathbf{u}_y - r^2\,\mathbf{u}_z)$$

where $r^2 = x^2 + y^2$

and $R^2 = x^2 + y^2 + z^2$

$$\mathbf{u}_\phi = \frac{1}{\sqrt{x^2+y^2}}\left\{x\,\overline{\mathbf{u}}_y - y\,\overline{\mathbf{u}}_x\right\}$$

**Example 1.15.** *Given $\mathbf{A} = 3\mathbf{u}_x + 4\mathbf{u}_y$ and $\mathbf{B} = 12\mathbf{u}_y - 5\mathbf{u}_z$, find the angle between them by using (a) the dot product; (b) the cross product.*

**Solution.** (*a*) Dot product

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta$$

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|}$$

$$|\mathbf{A}| = \sqrt{3^2+4^2} = 5$$

$$|\mathbf{B}| = \sqrt{12^2+5^2} = 13$$

$$\mathbf{A}\cdot\mathbf{B} = (3\mathbf{u}_x + 4\mathbf{u}_y)\cdot(12\mathbf{u}_y - 5\mathbf{u}_z)$$

$$= 3\,(0) + (4)\,(12) - 5\,(0) = 48$$

$$\cos\theta = \frac{48}{(5)\,(13)} = \frac{48}{65} \quad \text{whence} \quad \theta = 42.4°$$

(*b*) Cross Product

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ 3 & 4 & 0 \\ 0 & 12 & -5 \end{vmatrix} = -20\,\mathbf{u}_x + 15\mathbf{u}_y + 36\mathbf{u}_z$$

$$|\mathbf{A}|\,|\mathbf{B}| = 65$$

$$|\mathbf{A}\times\mathbf{B}| = \sqrt{20^2+15^2+36^2} = 43.83$$

$$\sin\theta = \frac{|\mathbf{A}\times\mathbf{B}|}{|\mathbf{A}|\,|\mathbf{B}|} = \frac{43.83}{65} = 0.674$$

whence $\theta = 42.4°$ (as above)

**Example 1.16.** *Given the three vectors*

$$\boldsymbol{A} = \boldsymbol{2u_x} + \boldsymbol{3\,u_y} \quad ; \quad \boldsymbol{B} = \boldsymbol{2u_x} + \boldsymbol{u_z} \quad ; \quad \boldsymbol{C} = \boldsymbol{3u_y} + \boldsymbol{2\,u_z}$$

*evaluate the triple products $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$ and $(\boldsymbol{A})\cdot(\boldsymbol{B}\times\boldsymbol{C})$. Compare the results.*

**Solution.**

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 3\mathbf{u}_x - 2\mathbf{u}_y - 6\mathbf{u}_z$$

$$\mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = (3\mathbf{u}_x - 2\mathbf{u}_y - 6\mathbf{u}_z)\cdot(3\mathbf{u}_y + 2\mathbf{u}_z)$$

$$= -(2)\,(3) - (6)\,(2) = -18 \qquad \ldots(i)$$

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ 2 & 0 & 1 \\ 0 & 3 & 2 \end{vmatrix} = -3\mathbf{u}_x - 4\mathbf{u}_y - 6\mathbf{u}_z$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = (2\mathbf{u}_x + 3\mathbf{u}_y) \cdot (-3\mathbf{u}_x - 4\mathbf{u}_y + 6\mathbf{u}_z)$$
$$= -(2)(3) - (3)(4) = -18 \qquad ...(ii)$$

It is seen from (*i*) and (*ii*) that

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$$

The scalar triple products in the same cyclic order are equal.

**Example 1.17.** *A vector **F** is directed from the point $P_1$ (1, – 2, 3) to the point $P_2$ (2, – 5, 4). Determine the unit vector in the direction of **F**.*

**Solution.** The vector **F** is given by the directed line segment $\mathbf{P}_1\,\mathbf{P}_2$.

$$\mathbf{F} = \mathbf{P}_1\,\mathbf{P}_2 = (2\mathbf{u}_x - 5\,\mathbf{u}_y + 4\,\mathbf{u}_z) - (\mathbf{u}_x - 2\,\mathbf{u}_y + 3\,\mathbf{u}_z)$$
$$= \mathbf{u}_x - 3\,\mathbf{u}_y + \mathbf{u}_z$$
$$|\,\mathbf{F}\,| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$$

Hence the unit vector $\mathbf{u}_F$ is given by the ratio of the vector **F** to its magnitude.

Unit vector
$$\mathbf{u}_F = \frac{\mathbf{u}_x - 3\,\mathbf{u}_y + \mathbf{u}_z}{\sqrt{11}}$$
$$= (1/\sqrt{11})\,\mathbf{u}_x - (3/\sqrt{11})\,\mathbf{u}_y + (1/\sqrt{11})\,\mathbf{u}_z$$

All important terms frequently encountered in vector analysis as applied to FIELD THEORY are dealt with adequately in the later chapters. The three significant terms of our interest are : "Gradient", "Divergence" and "Curl". While the first is introduced with reference to 'potential gradient' in Chapter 3, enough light is thrown on "Divergence" and "Curl" of vector field quantities in Chapter 5, with typical examples incorporated therein.

The students are advised to solve the drill problems and questions given in the Exercises under each Chapter.

## EXERCISES

1. If **A** and **B** are the sides of a parallelogram, and **C** and **D** are the diagonals, and θ is the angle between *A* and *B*, show that
$$C^2 + D^2 = 2\,(A^2 + B^2) \text{ and}$$
$$C^2 - D^2 = 4\,AB\cos\theta$$
2. Given two vectors $\mathbf{a} = 5i + 3j$ and $\mathbf{b} = 3i - 4j$, evaluate **a** + **b**, **a – b**, **b – a** and sketch the results. Also find (*i*) $\mathbf{a} \cdot \mathbf{b}$ and (*ii*) $\mathbf{a} \times \mathbf{b}$.
3. Four vectors *A*, *B*, *C*, *D* are given as :
$$\mathbf{A} = 2i + 3j + 4k \qquad \mathbf{B} = i + 2j + 2k$$
$$\mathbf{C} = 3i + j + 2k \qquad \mathbf{D} = i - j - 2k$$
all drawn from the origin, and the vector tips *A*, *B* and *C*, *D* are joined. Check if the lines *AB* and *CD* are (*i*) perpendicular ; (*ii*) parallel.
Also determine the ratio of the lengths *AB* and *CD*.
4. Prove that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \times \mathbf{b}) \times \mathbf{a}$.
5. Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.
Interpret the result geometrically.

**6.** Given $\mathbf{a} = i \cos \alpha - j \sin \alpha$ and $\mathbf{b} = i \cos \beta + j \sin \beta$

derive the formula for expansion of $\sin (\alpha + \beta)$ using the above vectors. Illustrate by a suitable figure.

**7.** Show that the two vectors, $\mathbf{A} = 6i + j - 5k$ and $\mathbf{B} = 5 (i - j + k)$ are perpendicular to each other.

**8.** Given the two vectors $\mathbf{C} = 4i + 3j + 5k$ and $\mathbf{D} = i - 2j + 2k$, determine the angle between them.

**9.** A field vector is given by the expression

$$\mathbf{A} = (x/R)\, \mathbf{u}_x + (y/R)\, \mathbf{u}_y + (z/R)\, \mathbf{u}_z \ ; \quad \text{where, } R = \sqrt{(x^2 + y^2 + z^2)}$$

Transform the vector into cylindrical system of co-ordinates.

**10.** A vector in cylindrical system of co-ordinates is given as $(1/r)\, \mathbf{u}_r$. Rewrite the same in cartesian system of co-ordinates.

**11.** A vector field $F$ is given by the relation

$\mathbf{F} = (1/R)\, \mathbf{u}_R$ (in spherical system of co-ordinates).

Transform this vector into cartesian (rectangular) system.

**12.** Transform the vector in Problem 10 (in cylindrical form) into spherical system of co-ordinates.

**13.** Transform the vector in Problem 11 (in spherical form) into cylindrical system of co-ordinates.

**14.** Locate on a sketch and specify the co-ordinates in spherical system of the points whose rectangular (cartesian) co-ordinates are given as :

($a$) $x = 2 ; y = 1 ; z = 3$ ($b$) $x = -2 ; y = -1 ; z = -3$ ($c$) $x = 1 ; y = -3 ; z = 2$.

**15.** Find the angle $\theta$ between the face diagonals $A$ and $B$ of a cube as shown in Fig. 1.26.

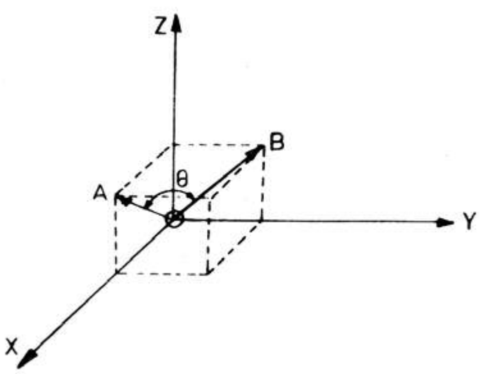



Fig. 1.26. For Example 15.

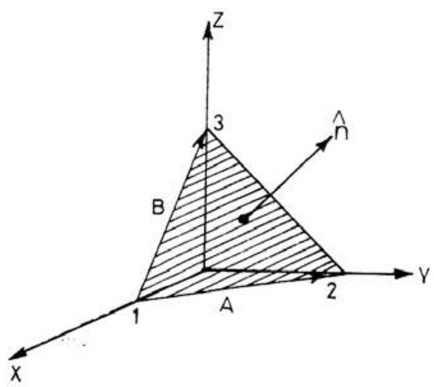



Fig. 1.27. For Example 18.

**16.** Prove the identity : $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$

**17.** Prove or disprove that the cross product is associative.

*viz.,* $$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \overset{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

**18.** $\hat{n}$ is a unit outward normal to the plane shown in Fig. 1.27. Find the unit vector in cartesian form.