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Vector Analysis

In the study of electromagnetic field theory a great saving in complexity of notation may be accomplished by the use of notation of vector analysis. Vector analysis is divided into two general categories-vector algebra and vector calculus. *Vector algebra* concerns algebraic operations on vectors, such as addition, subtraction, and manipulation whereas *vector calculus* concerns differential and integral operations involving vector functions. In providing this valuable shorthand, vector analysis also brings to the forefront the physical ideas involved in equations.

The present chapter is intended to give a brief but self-contained exposition of basic vector analysis. In order to facilitate our numerical calculations, this chapter will also introduce certain coordinate systems particularly orthogonal coordinate systems in which the three surfaces used to define the coordinate system are orthogonal to each other. We will concentrate on the most important and commonly used orthogonal coordinate systems-rectangular (cartesian), cylindrical, spherical.

Following are the reasons for beginning our study of electromagnetic fields with the analysis of vectors :

1. The electromagnetic field quantities are conveniently described in terms of vectors.

2. The fundamental laws governing these basic vectors field quantities can be precisely stated using the concepts of vector calculus.

3. To have a qualitative understanding of these laws and their implications.

4. The study of vector analysis also has numerous other applications in mechanics, fluid flow, heat flow, etc.

1.1 SCALARS AND VECTORS

In the study of elementary physics several kinds of quantities have been encountered. These quantities can be arranged in two groups, scalars and vectors. The term *scalar* refers to a quantity whose value may be represented by a (positive and negative) real number and is defined as

A scalar is a quantity that is completely characterized by its magnitude and algebraic sign.

Examples of some physical quantities that are scalars are mass, time, work, potential, current, etc. A simple extension of the idea of scalar is a scalar field, *i.e.*, a function of position that is completely specified by its magnitude at all points in space. Since scalars are real number hence they are represented as A, B, C, a, b, c, etc.

On the other hand, some physical quantities, for their specification, need magnitude as well as direction. Such quantities are called *vectors*. That is,

A vector is a quantity that is completely characterized by its magnitude and direction.

As examples of vectors we cite position from a fixed origin, velocity, acceleration, force, etc. The generalization to a vector field gives a function of position that is completely specified by its magnitude and direction at all points in space.

In order to explain the above terms, let us consider that a particle is displaced from position P to position Q, then the displacement, **PQ**, has a magnitude given by the length of PQ and also a direction given by the direction of PQ, that is, from P to Q. This is shown in Fig. 1.1. It should be noted that the displacement is a vector quantity while the length is a scalar. Some other examples include electric field and a potential gradient.

A vector is represented on a diagram by an arrow line whose length is proportional to the magnitude and the arrow indicating the direction of the vector. There are two types of vectors, namely, *localized* and *free vectors*. For the complete specification of a localized vectors, the point at which the vector acts should also be specified whereas there is no such restriction for free vectors. Force is an example of localized vector and couple that of a free vector.

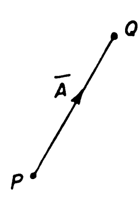


Fig. 1.1.

To distinguish vectors from scalars, bold-face remain type letters *e.g.* **A, B, a** may be used for vectors. The magnitude of a vector **A** is denoted by $|\mathbf{A}|$ or by A. The other suitable and usually used notations are

$$A\ ; \vec{A}, \mathbf{A}, \underline{A}$$

In electromagnetics, we usually come across scalar and vector fields. A field (scalar or vector) may be defined mathematically as some function of the vector connecting an arbitrary origin to a general point in space. In other words, *if at each point in a region any physical function has some value, then the region is called a field.* Usually it is possible to assign some physical effect responsible for the field, such as the force on a compass needle in the earth's magnetic field, or the movement of smoke particle in the field defined by the vector velocity of air in some region of space.

In general, there are two types of fields, namely, scalar and vector. If the value of the physical function at each point is a scalar quantity, then the field is a scalar field. Examples are :

(*a*) the temperature of atmosphere,

(*b*) the temperature throughout the bowl of soup,

(*c*) the height of the surface of the earth above sea level,

(*d*) the density at any point in the earth or in a similar non-homogeneous body

This is so because a scalar quantity has some specific value which may vary with position and time at a point in the given region.

When the value of the function at any point is a vector quantity, the field is called a vector field. Examples are :

(*a*) the wind velocity of the atmosphere.

(*b*) the force of gravity on a mass in space.

(*c*) the force on a charged body placed in an electric field,

(*d*) magnetic field of earth,

(*e*) the voltage gradient in a cable, and

(*f*) the temperature gradient in a soldering-iron tip.

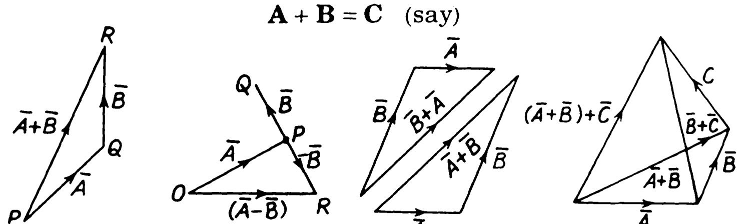
1.2. VECTOR ALGEBRA

With the definitions of vectors and vector fields, let us formulate the rules of vector arithmetic, vector algebra, and of vector calculus.

1.2A. Addition of Vectors

If a vector **A** is represented by a line $\overrightarrow{PQ}$, and a vector **B** by the line $\overrightarrow{QR}$, then the sum of the vectors **A** and **B**, represented by the line $\overrightarrow{PR}$ shown in Fig. 1.2*a*, is given by

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \quad \text{(say)}$$



(*a*) (*b*) (*c*) (*d*)

Fig. 1.2.

If **A** be a vector and m a real positive number, then $m\mathbf{A}$ is defined to be a vector having the same direction as **A** and magnitudes m times that of **A**. However, if m be a negative, equal to $(-n)$ say, then $-n\mathbf{A}$ is a vector in a direction opposite to **A** and magnitude n times that of **A**. In particular, $(-\mathbf{A})$ is a vector equal and opposite to **A**.

The vector difference $(\mathbf{A}-\mathbf{B})$ of two vectors **A** and **B** can be expressed as the sum $\mathbf{A}+(-\mathbf{B})$. The vector $(\mathbf{A}-\mathbf{B})$ is shown as vector $\overrightarrow{OR}$ in Fig. 1.2*b*.

From Fig. 1.2*c*, it is apparent that the *vector addition obeys the commutative law i.e.*

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}.$$

In case, there are more than two vectors, say, **A, B** and **C**, they obey the Associative law *i.e.*

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$

The same is evident from Fig. 1.2*d*.

Unit Vector. A vector **A** may also be expressed as $\mathbf{A}=A\,\mathbf{a}$, where A is the magnitude of **A** and **a** is a vector of unit magnitude and direction that of **A**. The vector **a** is known as an unit vector. Two different unit vectors differ only in direction. A unit vector is indicated by **a, u, 1**. Sometimes they are also expressed with suffices (such as $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$) which shows the direction of unit vector. Thus, vector $\mathbf{a}_x$ is a unit vector along the positive x-direction ; the unit vector $\mathbf{a}_y$ along the positive y-direction ; and the unit vector $\mathbf{a}_z$ along the positive z-direction in the cartesian coordinate system.

Null Vector. A vector of zero magnitude is called a null-vector or zero vector. We denote it by **O**, evidently

$$\mathbf{A}+\mathbf{O}=\mathbf{A} \quad \text{and} \quad \mathbf{A}-\mathbf{A}=\mathbf{O}$$

Furthermore, it is easier to prove that the multiplication of sum of two vectors **A** and **B** by a scalar m, equals the sum of vectors $m\mathbf{A}$ and $m\mathbf{B}$, *i.e.*

$$m(\mathbf{A}+\mathbf{B})=m\mathbf{A}+m\mathbf{B}.$$

Two vectors are said to be equal if they are equal in magnitude, having the same direction and same unit.

Example 1.1. *Show by vector methods that the line joining a vertex of a parallelogram with the middle point of an opposite side, cuts a diagonal at a point of trisection.*

Solution. Let $ABCD$ be the parallelogram, E the middle point of CD, and F the point of intersection of AE and diagonal BD. Let

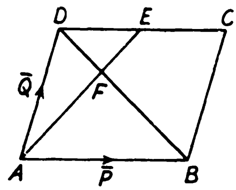


Fig. 1.3

$$\vec{AB} = \vec{P}, \vec{AD} = \vec{Q}$$

By vector addition,

$$\vec{AD} + \vec{DB} = \vec{AB}$$

$$\therefore \quad \vec{DB} = \vec{AB} - \vec{AD} = \vec{P} - \vec{Q}$$

Let DF be m times DB, then

$$\vec{DF} = m\vec{DB}$$

From ΔADF, the vector addition gives

$$\vec{AD} + \vec{DF} = \vec{AF}$$

$$\therefore \quad \vec{AF} = \vec{Q} + m\vec{DB}$$

$$= \vec{Q} + m\,(\vec{P} - \vec{Q}) \qquad ...(i)$$

Since E is the middle point of the side CD, hence

$$\vec{DE} = \tfrac{1}{2}\vec{DC} = \tfrac{1}{2}\vec{AB} = \tfrac{1}{2}\vec{P}$$

From ΔADE, the vector addition gives

$$\vec{AD} + \vec{DE} = \vec{AE} \quad \therefore \quad \vec{AE} = \vec{Q} + \tfrac{1}{2}\vec{P}$$

Assuming AF to be a submultiple of AE, so that

$$\vec{AF} = n\vec{AE} = n\,(\vec{Q} + \tfrac{1}{2}\vec{P}) \qquad ...(ii)$$

Comparing equations (*i*) and (*ii*),

$$\vec{Q} + m(\vec{P} - \vec{Q}) = n(\vec{Q} + \tfrac{1}{2}\vec{P})$$

i.e.

$$\vec{Q}\,[1 - m - n] + \vec{P}\left[m - \frac{n}{2}\right] = 0$$

Since $\vec{P}$ and $\vec{Q}$ have different directions, the above equation will be satisfied only if

$$1 - m - n = 0 \text{ and } m - \frac{n}{2} = 0$$

Therefore, $m = \frac{1}{3}$ and $n = \frac{2}{3}$.

Hence $DF = \frac{1}{3}DB$, and F becomes a point of tri-section.

1.2B. Resolution of a Vector

If the sum of two vectors **A** and **B** equals a third vector **C**, then **A** and **B** are said to be the components of **C**. In other words, vector **C** may be resolved into its component vectors, **A** and **B**. Further, it may be evident that a given vector can be resolved into components in a variety of ways.

If two directions are specified by unit vectors **a** and **b**, then any vector **C** in their plane can be resolved uniquely into components parallel to **a** and **b** ; and thus Fig. 1.4*a* gives

$$C = \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$
$$= |\overrightarrow{OA}|\ \mathbf{a} + |\overrightarrow{AC}|\ \mathbf{b}$$
$$= m\ \mathbf{a} + n\ \mathbf{b} \qquad ...(1)$$

where m and n are suitable scalars. In general if **C** is any vector in space, we can choose three vectors **a, b** and **c**. The vectors **a, b, c** are sometimes called the *base vectors.*

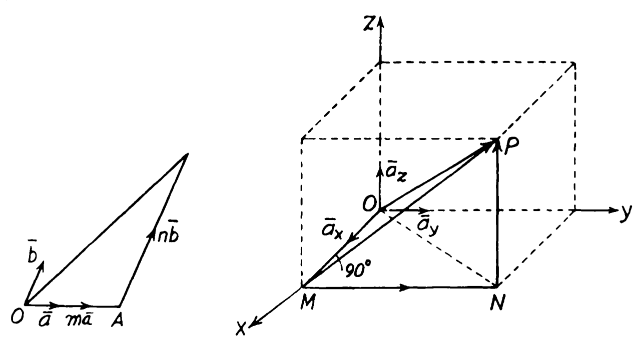
For convenience, the base vectors are often chosen as three mutually perpendicular unit vectors in the directions of x, y and z of cartesian co-ordinate system. These may be denoted by $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ respectively. From Fig. 1.4*b*, by vector resolution

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MN} + \overrightarrow{NP} \qquad ...(2)$$

where $|\overrightarrow{OM}| = A_x$ = magnitude of component of $\overrightarrow{OP}$ along the x-direction.

$|\overrightarrow{MN}| = A_y$ = magnitude of $\overrightarrow{OP}$ along the y-direction

$|\overrightarrow{NP}| = A_z$ = magnitude of $\overrightarrow{OP}$ along the z-direction.



(*a*) (*b*)

Fig. 1.4

Hence, substituting $\overrightarrow{OM} = A_x\ \mathbf{a}_x$, $\overrightarrow{MN} = A_y\ \mathbf{a}_y$ and $\overrightarrow{NP} = A_z\ \mathbf{a}_z$ in the above equation, we have

$$\overrightarrow{OP} = A_x\ \mathbf{a}_x + A_y\ \mathbf{a}_y + A_z\ \mathbf{a}_z \qquad ...(3)$$

In particular, the *position vector* $\mathbf{R}_p$ of a point $P(A_x, A_y, A_z)$ is given by

$$\mathbf{R}_p = x\ \mathbf{a}_x + y\ \mathbf{a}_y + z\ \mathbf{a}_z \qquad ...(4)$$

In terms of the base vectors, the sum of two vectors or the product of a vector by a scalar can be written as :

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z) + (B_x\,\mathbf{a}_x + B_y\,\mathbf{a}_y + B_z\,\mathbf{a}_z) \\ &= (A_x + B_x)\,\mathbf{a}_x + (A_y + B_y)\,\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z \qquad \text{...(5)} \end{aligned}$$

$$m\mathbf{A} = mA_x\,\mathbf{a}_x + mA_y\,\mathbf{a}_y + mA_z\,\mathbf{a}_z \qquad \text{...(6)}$$

The magnitude of a vector $\mathbf{A}\,(=\overrightarrow{OP})$ is determined in terms of A_x, A_y and A_z as follows :

$$\begin{aligned} |\mathbf{A}| &= OP = \sqrt{(ON)^2 + (NP)^2} \\ &= \sqrt{(OM)^2 + (MN)^2 + (NP)^2} \\ &= \sqrt{A_x^2 + A_y^2 + A_z{}^2} \qquad \text{...(7)} \end{aligned}$$

If α, β and γ be the angles between the vector $\overrightarrow{OP}$ and the positive directions of x, y, z respectively, then cosine of those angles are called the *direction cosines.* From Fig. 1.4*b*, it can be written that

$$\left.\begin{aligned} \cos\alpha &= \frac{A_x}{|\mathbf{A}|} = \frac{A_x}{A} \\ \cos\beta &= \frac{A_y}{|\mathbf{A}|} = \frac{A_y}{A} \\ \cos\gamma &= \frac{A_z}{|\mathbf{A}|} = \frac{A_z}{A} \end{aligned}\right\} \qquad \text{...(8)}$$

Therefore,

$$\left.\begin{aligned} A_x &= A\cos\alpha \\ A_p &= A\cos\beta \\ A_z &= A\cos\gamma \end{aligned}\right\} \qquad \text{...(9)}$$

and

$$\begin{aligned} \mathbf{A} &= A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\mathbf{a}_z \\ &= A\,[\cos\alpha\,\mathbf{a}_x + \cos\beta\,\mathbf{a}_y + \cos\gamma\,\mathbf{a}_z)] \qquad \text{...(10)} \end{aligned}$$

The unit vector corresponding to vector **A** is determined by the relation

$$\frac{\mathbf{A}}{A} = \cos\alpha\,\mathbf{a}_x + \cos\beta\,\mathbf{a}_y + \cos\gamma\,\mathbf{a}_z = \mathbf{a} \qquad \text{...(11)}$$

From this, it follows that the magnitude of **a** is

$$|\mathbf{a}| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1 \qquad \text{...(12)}$$

Further, any three numbers proportional to the direction cosines of a line having the same proportionality constant are called *direction ratios* of the line. Let a, b, c be the direction ratios of the line and cos α, cos β, cos γ its direction cosines ; then

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c}$$

$$= \frac{\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}} \qquad ...(13)$$

Therefore

$$\left.\begin{aligned} \cos\alpha &= \frac{a}{\sqrt{a^2+b^2+c^2}} \\ \cos\beta &= \frac{b}{\sqrt{a^2+b^2+c^2}} \\ \text{and} \quad \cos\gamma &= \frac{c}{\sqrt{a^2+b^2+c^2}} \end{aligned}\right\} \qquad ....(14)$$

1.2C. The Scalar or Dot Product

The scalar or dot product of two vectors **A** and **B** (written as **A** · **B**, and read as "**A** dot **B**") is defined as the product of the magnitude of **A**, the magnitude of **B** and the cosine of the smaller angle between them,

$$\mathbf{A}\cdot\mathbf{B} \;\Delta\; AB\cos\theta \qquad ...(15)$$

The dot or scalar product gives a scalar quantity and it obeys the commutative law *i.e.*

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \qquad ...(16)$$

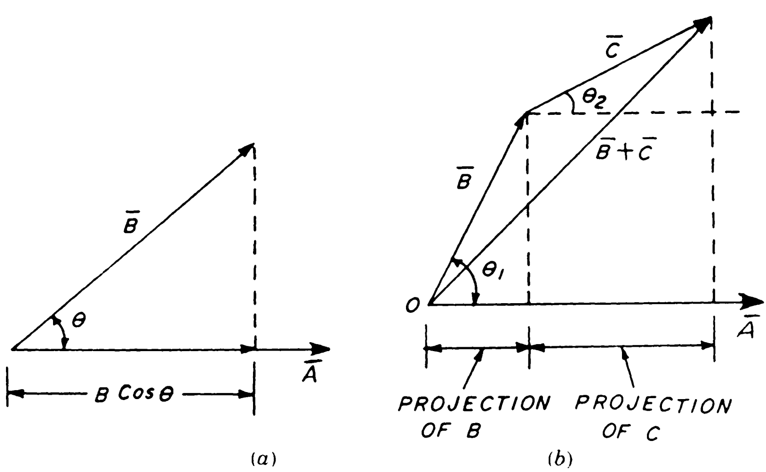


Fig. 1.5

For this, Fig. 1.5*a*, suggests to express the left side as

$$\mathbf{A}\cdot\mathbf{B} = A\,(B\cos\theta) = A \text{ (Projection of } \mathbf{B} \text{ on } \mathbf{A})$$
$$= B \text{ (Projection of } \mathbf{A} \text{ on } \mathbf{B}) = \mathbf{B}\cdot\mathbf{A}.$$

A physical example of the scalar product is $\mathbf{F} \cdot \mathbf{L}$, the work-done by a constant force $\mathbf{F}$ when its point of application undergoes a displacement $\mathbf{L}$. In case, the force varies along the path, integration is necessary to find the total work and the result becomes

$$\text{Work done } = \int \mathbf{F} \cdot d\mathbf{L}$$

similarly, the total magnetic flux ψ_m crossing a surface of area S is given by $\mathbf{B} \cdot \mathbf{S}$. If the magnetic flux density, $\mathbf{B}$, is perpendicular and uniform over the surface ;

$$\phi_m \ \psi_m = \mathbf{B} \cdot \mathbf{S} = BS \cos \theta = BS \cos 0° = BS$$

where $\mathbf{S}$ is the surface area of magnitude S and direction normal to the area at every point. However, if the flux-density $\mathbf{B}$ is not constant and uniform over the surface, the total flux ψ_m is given by the integral

$$\phi_m \ \psi_m = \int \mathbf{B} \cdot d\mathbf{S}$$

Distributive law. The dot product of a vector $\mathbf{A}$ with the vector sum of $\mathbf{B}$ and $\mathbf{C}$ equals to the sum of dot product of $\mathbf{A}$ with $\mathbf{B}$ and dot product of $\mathbf{A}$ with $\mathbf{C}$, that is,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \qquad ...(17)$$

This law is known as *distributive law*. In order to show the validity of Eq. 17, let us refer to Fig. 1.5*b* from which it follows that

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot [\text{Projection of } (\mathbf{B} + \mathbf{C}) \text{ on } \mathbf{A}] \\ &= \mathbf{A} \cdot [\text{Projection of } \mathbf{B} \text{ on } \mathbf{A} + \text{Projection of } \mathbf{C} \text{ on } \mathbf{A}] \\ &= \mathbf{A} \cdot [\text{Projection of } \mathbf{B} \text{ on } \mathbf{A}] + \mathbf{A} \times [\text{Projection of } \mathbf{C} \text{ on } \mathbf{A}] \\ &= AB \cos \theta_1 + AC \cos \theta_2 \\ &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \end{aligned}$$

which indicates that the dot product obeys distributive law.

Furthermore, unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ are perpendicular to each other hence

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \ \mathbf{a}_x = 0 \qquad ...(18)$$

and

$$\mathbf{a}_x \cdot \mathbf{a}_x = 1 \cdot 1 \cdot \cos 0° = 1 \qquad ...(19)$$

$$\mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \qquad ...(20)$$

Also a vector dotted with itself yields the magnitude squared,

or

$$\begin{aligned} \mathbf{A} \cdot \mathbf{A} &= AA \cos 0 = A^2 \\ &= |\mathbf{A}|^2 = A_x A_x + A_y A_y + A_z A_z \\ &= A_x^2 + A_y^2 + A_z^2 \end{aligned}$$

Next, we wish to find a general expression for $\mathbf{A} \cdot \mathbf{B}$, for this let

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

and

$$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$$

Then $\mathbf{A} \cdot \mathbf{B}$ is given by

$$\mathbf{A} \cdot \mathbf{B} = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z \cdot \mathbf{a}_z) \cdot (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \qquad ...(21)$$

Using Eqs. 18–20 in Eq. 21, yields

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z \qquad ...(22)$$

Probably the most important application of the dot product is that of finding the component of a vector in the given direction. If **a** be a unit vector in any direction, then

$$\mathbf{A} \cdot \mathbf{a} = |\mathbf{A}|\ |\mathbf{a}| \cos\theta_{Aa} = |\mathbf{A}| \cos\theta_{Aa} \qquad ...(23)$$

is the component of **A** in the direction of the unit vector **a**. Thus $\mathbf{A} \cdot \mathbf{a}_x$ is the component of **A** in the direction of $\mathbf{A}_x$, or

$$\mathbf{A} \cdot \mathbf{a}_x = A_x = A \cos\alpha \qquad ...(24)$$

Hence, the problem of finding the component of a vector in any desired direction becomes the problem of finding a unit vector in that direction.

Example 1.2. *Write in component form the expression for the vector which extends from*

(*a*) *The origin to C(– 1, – 5, 3)*

(*b*) *C(– 1, – 5, 3) to D (3, – 5, 2)*

Solution. Let there be two general points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the position vectors $\mathbf{r}_P, \mathbf{r}_Q$ may be written in terms of the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$. Thus

$$\mathbf{r}_P = x_1\,\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z = \overrightarrow{OP} \qquad ...(i)$$

and

$$\mathbf{r}_Q = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_y = \overrightarrow{OQ} \qquad ...(ii)$$

Then, the vector which extends from point P to Q will be given by the vector relation (Fig. 1.6)

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \qquad ...(iii)$$

Introducing (*i*) and (*ii*) in (*iii*), we have

$$\overrightarrow{PQ} = (x_2 - x_1)\,\mathbf{a}_x + (y_2 - y_1)\,\mathbf{a}_y + (z_2 - z_1)\,\mathbf{a}_z$$

Hence the vector which extends from point P to Q will be given by :

$$\overrightarrow{PQ} = (x_2 - x_1)\,\mathbf{a}_x + (y_2 - y_1)\,\mathbf{a}_y + (z_2 - z_1)\,\mathbf{a}_z$$

(*a*) Point $P \equiv$ origin (0, 0, 0) $\equiv O$

and $Q \equiv$ Point C (– 1, – 5, 3)

Then, $\overrightarrow{OC} = -1\,\mathbf{a}_x - 5\,\mathbf{a}_y + 3\,\mathbf{a}_z$

(*b*) $\overrightarrow{CD} = (3+1)\,\mathbf{a}_x + (-5+5)\,\mathbf{a}_y + (2-3)\,\mathbf{a}_z = 4\,\mathbf{a}_x - \mathbf{a}_z.$

Example 1.3. *Find the magnitude of the vector*

(*a*) $\mathbf{A} = 3\,\mathbf{a}_x + 4\,\mathbf{a}_y - 5\,\mathbf{a}_z$

(*b*) $\mathbf{B} = 2\cos\alpha\,\mathbf{a}_x + 2\sin\alpha\,\mathbf{a}_y + 7\,\mathbf{a}_z.$

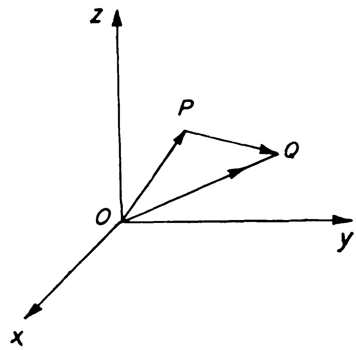


Fig. 1.6

Solution. If **A** has components A_x, A_y, A_z along the x, y, z axes respectively, then

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\mathbf{a}_y + A_z\,\mathbf{a}_z$$

and the magnitude of A is given by

$$\mathbf{A}\cdot\mathbf{A} = A_x^2 + A_y^2 + A_z^2 = A^2$$

Hence $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

(*a*) $A_x = 3, A_y = 4, A_z = -5$

$A = \sqrt{9+16+25} = 5\sqrt{2} = 5 \times 1.414 = 7.07$

(*b*) $B_x = 2\cos\alpha, B_y = 2\sin\alpha, B_z = 7$

Hence, $B = \sqrt{4\cos^2\alpha + 4\sin^2\alpha + 49}$

$= \sqrt{4+49} = \sqrt{53} = 7.28.$

Example 1.4. *Determine the component form of the unit vector*

(*a*) *which is directed from (3, 1, – 2) towards (4, – 1, 0)*

(*b*) *which lies in the direction of the vector* $6\,\mathbf{a}_x - 3\,\mathbf{a}_y + 6\,\mathbf{a}_z$.

Solution. (*a*) The vector which is directed from P (3, 1, – 2) towards Q (4, – 1, 0) is given by

$$\overrightarrow{PQ} = (4-3)\,\mathbf{a}_x + (-1-1)\,\mathbf{a}_y + (0+2)\,\mathbf{a}_z$$

$$= \mathbf{a}_x - 2\,\mathbf{a}_y + 2\,\mathbf{a}_z = |\overrightarrow{PQ}|\ \mathbf{a}_{PQ}$$

where $|\overrightarrow{PQ}|$ is the magnitude of the vector $\overrightarrow{PQ}$ and $\overrightarrow{\mathbf{a}_{PQ}}$ is the unit vector along the direction from P to Q.

Hence $\overrightarrow{\mathbf{a}_{PQ}} = \dfrac{1}{|\overrightarrow{PQ}|}\,[\overrightarrow{PQ}]$

$$= \frac{1}{\sqrt{1+4+4}}\,[\mathbf{a}_x - 2\,\mathbf{a}_y + 2\,\mathbf{a}_z]$$

$$= \frac{1}{3}\,\mathbf{a}_x - \frac{2}{3}\,\mathbf{a}_y + \frac{2}{3}\,\mathbf{a}_z$$

(b) Let the given vector be **A** then

$$\mathbf{A} = 6\,\mathbf{a}_x - 3\,\mathbf{a}_y + 6\,\mathbf{a}_z = |\mathbf{A}|\,\mathbf{a}_A$$

where $\mathbf{a}_A$ is the unit vector which lies in the direction of **A**, therefore,

$$\mathbf{a}_A = \frac{1}{|\mathbf{A}|}\,[\mathbf{A}]$$

$$= \frac{1}{\sqrt{36 + 9 + 36}}\,[6\,\mathbf{a}_x - 3\,\mathbf{a}_y + 6\,\mathbf{a}_z]$$

$$= \frac{2}{3}\,\mathbf{a}_x - \frac{1}{3}\,\mathbf{a}_y + \frac{2}{3}\,\mathbf{a}_z.$$

Example 1.5. *Use the dot product to find the angle between the two vectors.*

(a) $3\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z$ *and* $-\mathbf{a}_x + 2\mathbf{a}_y + 7\mathbf{a}_z$

(b) $2\,\mathbf{a}_x$ *and* $-\mathbf{a}_x + 2\,\mathbf{a}_y + 7\mathbf{a}_z$.

Solution. From the definition of dot product,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$

$$= A_xB_x + A_yB_y + A_zB_z$$

$$= 3 \times (-1) + (-2) \times 2 + 1 \times 7$$

or $$\cos\theta = \frac{1}{\sqrt{9 + 4 + 1}} \cdot \frac{t}{\sqrt{1 + 4 + 49}}\,[-3 - 4 + 7] = 0$$

$$\theta = 90°$$

Hence the two vectors are at right angles.

(b) Given $\mathbf{A} = 2\,\mathbf{a}_x$,

$$A_x = 2$$

$$A_y = A_z = 0$$

and $\mathbf{B} = -a_x + 2\,\mathbf{a}_y + 7\mathbf{a}_z$

$$B_x = -1$$

$$B_y = 2,$$

$$B_z = 7$$

$$A = |\mathbf{A}| = 2$$

and $B = \sqrt{1 + 4 + 49} = \sqrt{54}$

Therefore from the definition of dot product of two vectors, we have,

$$AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

or $$AB\cos\theta = -1 \times 2 = -2$$

$$\cos\theta = \frac{-2}{2 \times \sqrt{54}} = -\frac{1}{\sqrt{54}} = -\frac{1}{7.34}$$

$$\theta = 90 + \cos^{-1}\left(\frac{1}{7.34}\right) = 97.8^\circ.$$

Example 1.6. *Show that the cosines of the angle ψ between the vectors* **A** *and* **B** *is given by the sum of the products of their direction cosines.*

Solution. Let $\cos \alpha_1$, $\cos \beta_1$, $\cos \gamma_1$ be the direction cosines of **A** and $\cos \alpha_2$, $\cos \beta_2$ $\cos \gamma_2$ be the direction cosines of **B**, then

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

$$= \sqrt{A_x^2 + A_y^2 + A_z^2}\left[\frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\,\mathbf{a}_x + \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\,\mathbf{a}_y + \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\,\mathbf{a}_z\right]$$

or $$\mathbf{A} = A\,[\cos \alpha_1\,\mathbf{a}_x + \cos \beta_1\mathbf{a}_y + \cos \gamma_1\,\mathbf{a}_z]$$

where $$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$A_x = A \cos \alpha_1$$

$$A_y = A \cos \beta_1$$

and $$A_z = A \cos \gamma_1$$

similarly $$\mathbf{B} = B\,[\cos \alpha_1\,\mathbf{a}_x + \cos \beta_2\mathbf{a}_y + \cos \gamma_2\,\mathbf{a}_z]$$

$\therefore$ $$\mathbf{A}\cdot\mathbf{B} = AB \cos \psi$$

$$= AB\,[\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2]$$

where ψ is the angle between vectors **A** and **B**

$\therefore$ $$\cos \psi = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2$$

or $$\psi = \cos^{-1}\,[\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2]$$

Example 1.7. *Show that*

$$\mathbf{a} = \cos \alpha\,\mathbf{a}_x + \sin \alpha\,\mathbf{a}_y$$

and $$\mathbf{b} = \cos \beta\,\mathbf{a}_x - \sin \beta\,\mathbf{a}_y$$

are unit vectors along the x-y plane, inclined at angles α and – β respectively, with the x-axis. By appropriate product relation between the vectors, derive the expansion formula for cos α (α + β) and cos (α – β).

Solution. Let $\mathbf{a}_x$ and $\mathbf{a}_y$ be the unit vectors along the positive x and positive y-directions ; then for vector $\vec{OA}$, $\vec{OC}$ will represent the vector $|\vec{OA}| \cos \alpha\,\vec{\mathbf{a}}_\alpha$ and $\vec{CA}$ the vector $|\vec{OA}| \sin \alpha\,\mathbf{a}_y$, such that

$$\vec{OA} = |\vec{OA}| \cos \alpha\,\mathbf{a}_x + |\vec{OA}| \sin \alpha\,\mathbf{a}_y \qquad ...(i)$$

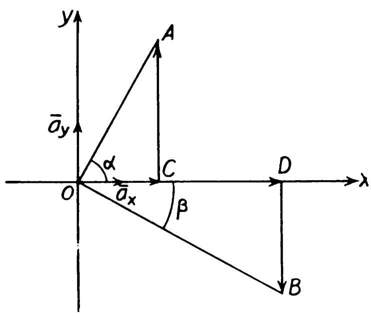


Fig. 1.7

similarly $\overrightarrow{OD}$ will represent the vector $|\overrightarrow{OB}| \cos\beta \, \mathbf{a}_x$ and $\overrightarrow{DB}$ the vector $-\ |\overrightarrow{OB}| \sin\beta \, \mathbf{a}_y$. This vector $\overrightarrow{DB}$ will be drawn by assuming an equal but opposite unit vector $(-\mathbf{a}_y)$. Thus,

$$\overrightarrow{OB} = |\overrightarrow{OB}| \cos\beta\mathbf{a}_x - |\overrightarrow{OB}| \sin\beta\mathbf{a}_y \quad ...(ii)$$

Equations (*i*) and (*ii*) are similar to the given equations for **a** and **b** respectively.

$$\text{Magnitude of } \mathbf{a} = |\overrightarrow{OA}| = \sqrt{(OC)^2 + (CA)^2}$$

$$= \sqrt{(\cos\alpha)^2 + (\sin\alpha)^2} = 1 \quad ...(iii)$$

$$\text{and magnitude of} \quad \mathbf{b} = |\overrightarrow{OB}|$$

$$= \sqrt{(\cos\beta)^2 + (-\sin\beta)^2} = 1 \quad ...(iv)$$

Thus, **a** and **b** are unit vectors.

It also appears, from Fig. 1.7 that angle between **a** and **b** is $(\alpha + \beta)$ and this will be calculated with the help of the dot product of two vectors **a** and **b**.

$$\mathbf{a} \cdot \mathbf{b} = ab \cos(\alpha + \beta) = 1.1 \cos(\alpha + \beta)$$

$$= \cos(\alpha + \beta) \quad ...(v)$$

But,

$$\mathbf{a} \cdot \mathbf{b} = (\cos\alpha\mathbf{a}_x + \sin\alpha\mathbf{a}_y) \cdot (\cos\alpha\mathbf{a}_x - \sin\beta\mathbf{a}_y)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad ...(vi)$$

Therefore, equations (*v*) and (*vi*) give

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Replacing $+\beta$ by $-\beta$, we obtain

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Example 1.8. *By squaring both sides of the equation* $\mathbf{A} = \mathbf{B} - \mathbf{C}$ *and interpreting the result geometrically, prove the "law of cosines".*

Solution. Given :

$$\mathbf{A} = \mathbf{B} - \mathbf{C} \quad ...(i)$$

or

$$\mathbf{A} + \mathbf{C} = \mathbf{B}$$

This shows that **A, B** and **C** form the sides of a triangle shown in Fig. 1.8. Squaring both sides of Eq. (*i*), yields,

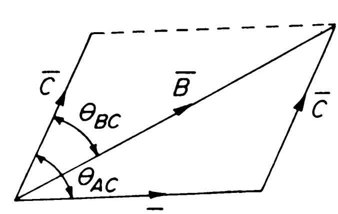


Fig. 1.8

$$\mathbf{A}\cdot\mathbf{A} = (\mathbf{B}-\mathbf{C})\cdot(\mathbf{B}-\mathbf{C})$$
$$A^2 = \mathbf{B}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{C} - \mathbf{B}\cdot\mathbf{C} + \mathbf{C}\cdot\mathbf{C}$$
$$= B^2 - 2\mathbf{B}\cdot\mathbf{C} + C^2$$
$$A^2 = B^2 + C^2 - 2BC\cos\theta_{BC}$$

which is the "law of cosines".

1.2D. Cross Product or Vector Product

The vector product (of **A** and **B**) written with a cross between the two vectors as **A** × **B** and read "**A** cross **B**", is a vector normal to the plane containing **A** and **B** and having magnitude equal to the product of the magnitudes of **A**, **B** and the sine of the angle θ, where θ is the angle measured in the anticlockwise direction from first vector (**A**) to second (**B**) one. Its sense is that **A**, **B** and **A** × **B** from a right-handed system or in the direction of advance of a right handed screw as **A** is turned into **B.** This direction is illustrated in Fig. 1.9*a*. In Fig. 1.9*b*, **A** and **B** are shown to be in the same plane ; then :

S = **A** × **B** will be in the direction shown.

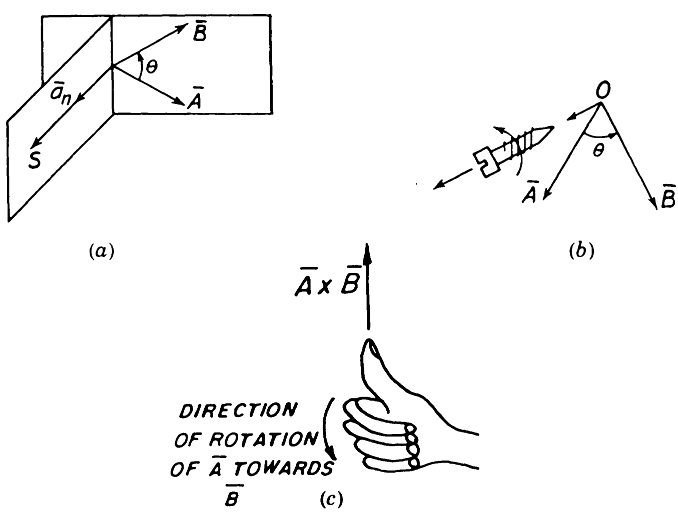


Fig. 1.9

In order to make it more clear, we may follow the right hand thumb rule. According to this, if the figures of the right hand indicates the sense of rotation from first vector towards the other (from **A** towards **B**), that is, in anticlockwise direction then the thumb will point the direction for the product **A** × **B** [Fig. 1.9*c*].

Thus, from the definition,

$$\mathbf{A}\times\mathbf{B} \;\Delta\; AB\sin\theta\,\mathbf{a}_n \qquad \text{...(25)}$$

where $\mathbf{a}_n$ is the unit vector and '*n*' stands for the normal, that is, $\mathbf{a}_n$ will be perpendicular to the plane containing **A** and **B** [Fig. 1.9*c*].

Following the above definitions, it follows that

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \qquad ...(26)$$

that is, the direction of $\mathbf{B} \times \mathbf{A}$ will be opposite to that of $\mathbf{A} \times \mathbf{B}$, hence the commutative law of multiplication does not hold good for cross-product of vectors. But the cross-product also obeys the distributive law like dot product, that is,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad ...(27)$$

If **A** and **B** are two vectors with the same direction, then evidently $\mathbf{A} \times \mathbf{B}$ is a vector of zero magnitude as the angle θ between **A** and **B** is zero. These two vectors are then said to be parallel vectors. This is also true when **A** and **B** have opposite directions.

A case of special importance is

$$\mathbf{A} \times \mathbf{A} = 0 \qquad ...(28)$$

In particular, $\mathbf{a}_x \times \mathbf{a}_x = \mathbf{a}_y \times \mathbf{a}_y = \mathbf{a}_z \times \mathbf{a}_z = 0$ ...(29)

It may also be seen that $\mathbf{a}_x \times \mathbf{a}_y$ is a vector of unit magnitude perpendicular to $\mathbf{a}_x$ and $\mathbf{a}_y$ and therefore it is a unit vector $\mathbf{a}_z$ along the positive *z*-direction.

Then, $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$ ...(30)

Also, $\mathbf{a}_y \times \mathbf{a}_x = -\mathbf{a}_z$. ...(31)

Similarly, $\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x, \quad \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$

and $\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y, \quad \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$ ...(31)

It should be noted that we have a positive sign when $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ are in the same cyclic order, and a negative sign, otherwise.

When **A** and **B** are given in terms of base (unit) vectors, *viz,*

$$\mathbf{A} = A_x\, \mathbf{a}_x + A_y \cdot \mathbf{a}_y + A_z \cdot \mathbf{a}_z$$

and $$\mathbf{B} = B_x\, \mathbf{a}_x + B_y\, \mathbf{a}_y + B_z\, \mathbf{a}_z,$$

then, $$\mathbf{A} \times \mathbf{B} = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \times (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z)$$
$$= A_xB_x\, \mathbf{a}_x \times \mathbf{a}_x + A_xB_y\mathbf{a}_x \times \mathbf{a}_y$$
$$+ A_xB_z\mathbf{a}_x \times \mathbf{a}_z + \text{two sets of similar terms.}$$

or $$\mathbf{A} \times \mathbf{B} = 0 + |(A_xB_y\mathbf{a}_z - A_xB_z\mathbf{a}_y) + \text{similar terms}$$
$$= (A_yB_z - A_zB_y)\, \mathbf{a}_x + (A_zB_x - A_xB_z)\, \mathbf{a}_y + (A_xB_y - A_yB_x)\, \mathbf{a}_z$$

This can also be written in the form of a determinant, in a more easily remembered form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad ...(33)$$

The cross product may be used to replace the right-hand rule familiar to all electrical and communication engineers. Consider the force on a straight conductor of length **L**, where the direction assigned to **L** corresponds to the direction of the steady current I and a uniform magnetic field of flux-density **B** is present. Using vector notation, the force experienced by the conductor may be computed form,

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}$$

The idea of cross-product if incorporated in a co-ordinate system, leads to have a right handed coordinate system wherein if x is rotated in an anticlockwise direction towards y, the z-axis must be indicated by the direction of thumb. Fig. 1.10, shows the various possibilities to have a right-handed cartesian coordinate system.

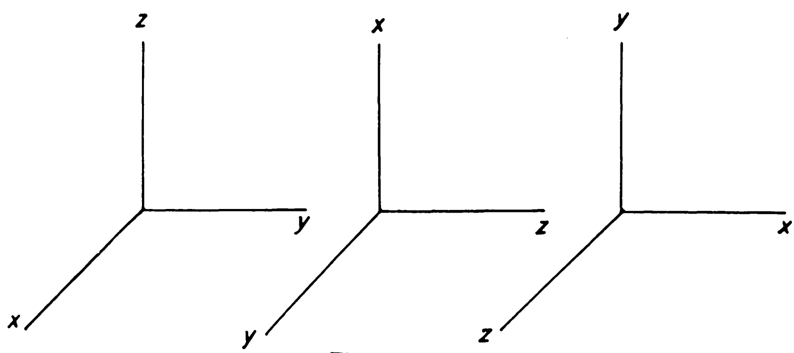


Fig. 1.10

Example 1.9. *Given* $\mathbf{A} = 2\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$, $\mathbf{B} = 6\mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$, *find* $\mathbf{A} \times \mathbf{B}$ *and the unit vector perpendicular to both* **A** *and* **B**.

Solution. We have

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 2 & -1 \\ 6 & -3 & 2 \end{vmatrix}$$

$$= \mathbf{a}_x (4 - 3) - \mathbf{a}_y (4 + 6) + \mathbf{a}_z (-6 - 12)$$

$$= \mathbf{a}_x - 10\mathbf{a}_y - 18\mathbf{a}_z$$

The unit vector perpendicular to **A** and **B** is given by :

$$\frac{\mathbf{A} \times \mathbf{B}}{AB \sin\theta} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

$$= \frac{\mathbf{a}_x - 10\mathbf{a}_y - 18\mathbf{a}_z}{\sqrt{1+100+324}}$$

$$= \frac{1}{5\sqrt{17}}\mathbf{a}_x - \frac{2}{\sqrt{17}}\mathbf{a}_y - \frac{18}{5\sqrt{17}}\mathbf{a}_z.$$

Example 1.10. *As shown in Fig. 1.11, a rigid body rotates with an angular velocity* ω *about an axis through O. Find the linear velocity* **V** *of a point P of the body.*

Solution. Let OA be the axis of rotation ; then w is directed towards OA. Let the radius vector $\overrightarrow{OP}$ be **r**. Then $OP = r$ and $NP = r \sin\theta$, where N is the foot of the perpendicular from P on OA and θ is the angle between ω and **r**.

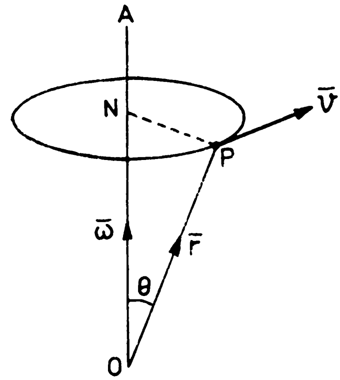


Fig. 1.11

Due to the rotation, P moves perpendicular to the plane OPN with a velocity

$$\vec{\omega} \cdot \overrightarrow{NP}, \text{ i.e. } wr \sin\theta$$

Hence, $\mathbf{v} = \omega \times \mathbf{r}$

1.2E. Product of Three Vectors

Let there be three vectors **A, B** and **C**. Now, there are three ways in which the three vectors may be multiplied.

(*a*) **A (B · C)** : This signifies the product of a vector **A**, with the scalar product of vectors, **B** and **C.**

(*b*) **Scalar Triple Product : A · (B × C)**

Let the vectors **A, B** and **C** occupy the position as indicated in Fig. 1.12, with the three edges of a parallelopiped. Let, θ be the angle between **B** and **C**, and ϕ the angle between **A** and $\mathbf{a}_n$, unit vector normal to the plane containing **B** and **C** (*i.e.*, the unit vector along the directly of the vector **S** = **B** × **C**).

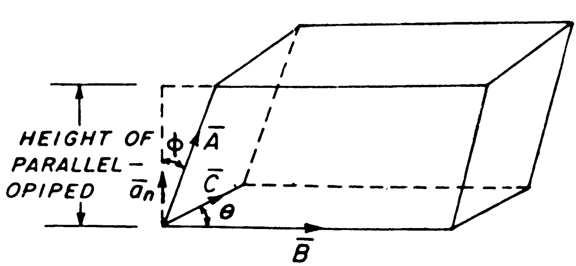


Fig. 1.12

The scalar triple product $\{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})\}$

$$= \mathbf{A} \cdot \{BC \sin \theta \, \mathbf{a}_n\} = (\mathbf{A} \cdot \mathbf{a}_n) \, BC \sin \theta$$
$$= (A \cos \phi) \, BC \sin \theta$$
$$= \text{(Height of the parallelopiped) (Area of the base of it)}$$
$$= \text{Volume of the parallelopiped} \quad ...(34)$$

This shows that the scalar triple product $[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})]$ represents the volume of the parallelopiped formed by the vectors **A, B** and **C**. This geometrical representation also shows that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \quad ...(35)$$

that is, the positions of the dot and the cross in a scalar triple product are interchangeable.

or $$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \quad ...(36)$$

(*c*) **Vector Triple Product.** $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

If **A, B, C** are any three vectors, then multiplication indicated by $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ results in a vector triple product. The vector $\{\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\}$ will be perpendicular to the vector $\mathbf{S} = \mathbf{B} \times \mathbf{C}$ and vector **A**, whereas vector $\mathbf{S} = \mathbf{B} \times \mathbf{C}$ is perpendicular to the plane containing **B** and **C** (Fig. 1.13). Hence, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ will lie in the plane of vectors **B** and **C**. Therefore, it is now possible to resolve this product into components parallel to **B** and **C** and thus, it will be expressed as :

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = m\mathbf{B} + n\mathbf{C} \quad ...(37)$$

where m and n are suitable numbers.

Multiply both sides of Eq. 37 scalarly by vector **A**, then

$$\mathbf{A} \cdot \{\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\} = m(\mathbf{A} \cdot \mathbf{B}) + n \, (\mathbf{A} \cdot \mathbf{C}) \quad ...(38)$$

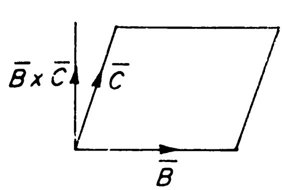


Fig. 1.13

The left hand side is a scalar triple product of **A, A** and $(\mathbf{B} \times \mathbf{C})$. Since $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is also perpendicular to **A**, therefore, the left hand side of Eq. 38 is zero, so that,

$$m(\mathbf{A} \cdot \mathbf{B}) + n \, (\mathbf{A} \cdot \mathbf{C}) = 0$$

or $$\frac{m}{\mathbf{A} \cdot \mathbf{C}} = \frac{n}{-(\mathbf{A} \cdot \mathbf{B})} = p \quad \text{(say)} \quad ...(39)$$

Introducing these values of m and n in 37, we have :

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = p(\mathbf{A} \cdot \mathbf{C}) \, \mathbf{B} - p \, (\mathbf{A} \cdot \mathbf{B}) \, \mathbf{C} \quad ...(40)$$

Both sides in Eq. 40 are equally balanced in **A, B, C.** Hence p must be some numerical constant independent of **A, B, C.** To find the magnitude of p, we shall take the special case, when

$$\mathbf{A} = \mathbf{B} = \mathbf{a}_x$$

and $\mathbf{C} = \mathbf{a}_y$, then

$$\mathbf{a}_x \times (\mathbf{a}_x \times \mathbf{a}_y) = p\,(\mathbf{a}_x \cdot \mathbf{a}_y)\,\mathbf{a}_x - p\,(\mathbf{a}_x \cdot \mathbf{a}_x)\,\mathbf{a}_y$$

or $\mathbf{a}_x \times \mathbf{a}_z = \mathbf{O} - p\mathbf{a}_y$

or $-\mathbf{a}_y = -p\mathbf{a}_y$

or $p = 1$

Substituting, $p = 1$, in Eq. 40, we have the relation,

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}} \qquad ...(41)$$

Furthermore, Eq. 41 may be written as

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) \\ &= -[(\mathbf{C} \cdot \mathbf{B})\,\mathbf{A} - (\mathbf{C} \cdot \mathbf{A})\mathbf{B}] \\ &= (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\,\mathbf{A}\end{aligned}$$

Thus, $$\boxed{(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\,\mathbf{A}} \quad ...(42)$$

Example 1.11. *Prove that*

(*a*) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \begin{vmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{C} \\ \mathbf{A} \cdot \mathbf{D} & \mathbf{B} \cdot \mathbf{D} \end{vmatrix}$

(*b*) $(\mathbf{A} \times \mathbf{B}) \cdot [(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})] = [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})]^2$

Solution. (*a*) Let $\mathbf{P} = \mathbf{C} \times \mathbf{D}$

then $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{P} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{P})$

since dot and cross are interchangeable in scalar triple product

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} \times \mathbf{P}) &= \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] \\ &= \mathbf{A} \cdot [\mathbf{C}\,(\mathbf{B} \cdot \mathbf{D}) - \mathbf{D}\,(\mathbf{B} \cdot \mathbf{C})] \\ &= (\mathbf{A} \cdot \mathbf{C})\,(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})\,(\mathbf{B} \cdot \mathbf{C})] \\ &= \begin{vmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{C} \\ \mathbf{A} \cdot \mathbf{D} & \mathbf{B} \cdot \mathbf{D} \end{vmatrix}\end{aligned}$$

(*b*) Putting $\mathbf{B} \times \mathbf{C} = \mathbf{P}$

$$\begin{aligned}(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) &= \mathbf{P} \times (\mathbf{C} \times \mathbf{A}) \\ &= (\mathbf{P} \cdot \mathbf{A})\,\mathbf{C} - (\mathbf{P} \cdot \mathbf{C})\,\mathbf{A} \\ &\quad [(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}]\,\mathbf{C} - [(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{C}]\mathbf{A} = m\mathbf{C}.\end{aligned}$$

where $m = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}$

and $(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{C} = 0$

Hence $(\mathbf{A} \times \mathbf{B}) \cdot [(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})]$

$$\begin{aligned}&= (\mathbf{A} \times \mathbf{B}) \cdot m\mathbf{C} \\ &= m[\mathbf{A} \times \mathbf{B}] \cdot \mathbf{C} = m[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})]\end{aligned}$$

as dot and cross are interchangeable, substituting the value of m, we have

$$(\mathbf{A} \times \mathbf{B}) \cdot [(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})] = [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})]^2$$

Example 1.12. *An unknown vector* **v** *satisfies the equation*

$$\mathbf{v}\,(\mathbf{a}\cdot\mathbf{v}) + \mathbf{a}\times(\mathbf{a}\times\mathbf{v}) = \mathbf{b}$$

where **a** *and* **b** *are known vectors and angle between* **a** *and* **b** *is acute. Find* (**a** · **v**) *and also solve for* **v**.

Solution. Given that **v** satisfies the equation

$$\mathbf{v}(\mathbf{a}\cdot\mathbf{v}) + \mathbf{a}\times(\mathbf{a}\times\mathbf{v}) = \mathbf{b} \qquad ...(i)$$

Multiplying both sides scalarly by **a**, we have,

$$(\mathbf{a}\cdot\mathbf{v})\,(\mathbf{a}\cdot\mathbf{v}) + \mathbf{a}\cdot[\mathbf{a}\times(\mathbf{a}\times\mathbf{v})] = \mathbf{a}\cdot\mathbf{b} \qquad ...(ii)$$

and making use of the identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C} \qquad ...(iii)$$

We have, $\mathbf{a}\times(\mathbf{a}\times\mathbf{v}) = (\mathbf{a}\cdot\mathbf{v})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{v}$

$$= (\mathbf{a}\cdot\mathbf{v})\mathbf{a} - a^2\mathbf{v} \qquad ...(iv)$$

From (*ii*) and (*iv*)

$$(\mathbf{a}\cdot\mathbf{v})^2 + \mathbf{a}\cdot[(\mathbf{a}\cdot\mathbf{v})\mathbf{a} - a^2\mathbf{v}] = \mathbf{a}\cdot\mathbf{b}$$

$$(\mathbf{a}\cdot\mathbf{v})^2 + a^2(\mathbf{a}\cdot\mathbf{v}) - a^2\,(\mathbf{a}\cdot\mathbf{v}) = \mathbf{a}\cdot\mathbf{b}$$

$$(\mathbf{a}\cdot\mathbf{v})^2 = \mathbf{a}\cdot\mathbf{b}$$

Hence, $$\mathbf{a}\cdot\mathbf{v} = \sqrt{\mathbf{a}\cdot\mathbf{b}} \qquad ...(v)$$

Rewriting the equation (*i*)

$$\mathbf{v}\,(\mathbf{a}\cdot\mathbf{v}) + \mathbf{a}\times(\mathbf{a}\times\mathbf{v}) = \mathbf{b}$$

Then, $$\mathbf{v}\,(\mathbf{a}\cdot\mathbf{v}) = \mathbf{b} - \mathbf{a}\times(\mathbf{a}\times\mathbf{v})$$

$$= \mathbf{b} - (\mathbf{a}\cdot\mathbf{v})\mathbf{a} + a^2\mathbf{v}, \text{ from } (iv)$$

$\therefore$ $$\mathbf{v}\,(\mathbf{a}\cdot\mathbf{v}) + (\mathbf{a}\cdot\mathbf{v})\,\mathbf{a} - a^2\mathbf{v} = \mathbf{b} \qquad ...(vi)$$

Substituting the value of (**a** · **v**) from (*v*) in (*vi*) yields

$$\mathbf{v}(\sqrt{\mathbf{a}\cdot\mathbf{b}}) + \mathbf{a}\,(\sqrt{\mathbf{a}\cdot\mathbf{b}}) - \mathbf{v}\mathbf{a}^2 = \mathbf{b}$$

or $$\mathbf{v}\,[\sqrt{\mathbf{a}\cdot\mathbf{b}} - \mathbf{a}^2] = \mathbf{b} - \mathbf{a}\,(\sqrt{\mathbf{a}\cdot\mathbf{b}})$$

Hence, $$\mathbf{v} = \frac{\mathbf{b} - \mathbf{a}\,(\sqrt{\mathbf{a}\cdot\mathbf{b}})}{[\sqrt{\mathbf{a}\cdot\mathbf{b}} - \mathbf{a}^2]} \qquad ...(vii)$$

1.2F. Vector Representation of a Surface

Fig. 1.14 shows a plane surface of arbitrary shape. This surface may be represented by a vector **S** whose length (magnitude) corresponds to the magnitude of the surface area and direction normal (outward) to the plane of the surface. This vector **S** can also be conveniently expressed

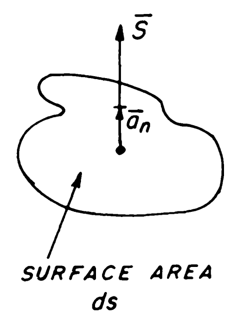


Fig. 1.14

in terms of a unit normal vector $\mathbf{a}_n$ perpendicular to the surface that is,

$$\mathbf{S} = S\mathbf{a}_n \qquad ...(43)$$

The direction of **S** should follow the right hand screw rule.

1.3. VECTOR CALCULUS

1.3A. Differentiation of a Vector

We shall often be concerned with the rates of change of scalar and vector quantities, with both time and space co-ordinates, and thus with the time and space derivatives.

Let t be a scalar variable and $\mathbf{A}(t)$ be a vector function of t (*i.e.* **A** has a definite magnitude and direction for every value of t). In a time δt, vector **A**, as in Fig. 1.5*a* may change by $\delta\mathbf{A}$ both in magnitude and direction. Then, the differential coefficient of **A** with respect to t is defined as a limit by the equation

$$\frac{d\mathbf{A}}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{A}(t + \delta t) - \mathbf{A}(t)}{\delta t} \qquad ...(44)$$

The differential coefficient $d\mathbf{A}/dt$ is itself a vector. In order to obtain its geometrical representation, let $\vec{OP}$ be the vector $\mathbf{A}(t)$ and $\vec{OQ}$. The vector $\mathbf{A}(t + \delta t)$, *i.e.* vector $\mathbf{A}(t)$ is drawn for different values of t from the same point O. The other end of the vector will trace out a space curve as depicted in Fig. 1.15.

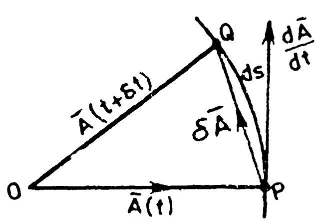


Fig. 1.15

Referring to Fig. 1.15, we have

$$\mathbf{A}(t + \delta t) - \mathbf{A}(t) = \vec{OQ} - \vec{OP} = \vec{PQ} = \delta\mathbf{A} \quad \text{(say)}$$

Hence
$$\frac{d\mathbf{A}}{dt} = \lim_{\delta t \to 0} \frac{\delta\mathbf{A}}{\delta t} \qquad ...(45)$$

The direction of $d\mathbf{A}/dt$ is the limiting direction $\delta\mathbf{A}/\delta t$ or that of $\delta\mathbf{A}$. But as Q tends to P, PQ tends to the tangent line at P. Thus, the direction of $d\mathbf{A}/dt$ is along the tangent to the space curve traced out by the tip of the vector $\mathbf{A}(t)$. If the length of the arc of this curve be S from a fixed point on it upto P, then the magnitude of $d\mathbf{A}/dt$ is given by

$$\left|\frac{d\mathbf{A}}{dt}\right| = \lim_{\delta t \to 0} \frac{|\delta\mathbf{A}|}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{|\delta\mathbf{A}|}{\delta s} \cdot \frac{\delta s}{\delta t} = \frac{ds}{dt} \qquad ...(46)$$

Since the ratio $\frac{|\delta A|}{\delta s} = \frac{\text{Chord } PQ}{\text{arc } PQ} \to 1$ as δt approach zero.

Let $\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z$, at any time t, and after time δt, it becomes $\mathbf{A}(t + \delta t)$ such that $\mathbf{A}(t + \delta t) = (A_x + \delta A_x)\,\mathbf{a}_x + (A_y + \delta A_y)\,\mathbf{a}_y + (A_z + \delta A_z)\,\mathbf{a}_z$ then from,

$$\frac{d\mathbf{A}}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{A}(t + \delta t) - \mathbf{A}(t)}{\delta t}$$

it follows that,

$$\frac{d\mathbf{A}}{dt} = \lim_{\delta t \to 0} [\mathbf{a}_x (A_x + \delta A_x) + \mathbf{a}_y (A_y + \delta A_y) + \mathbf{a}_z (A_z + \delta A_z) - (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z)]/\delta t$$

$$= \lim_{\delta t \to 0} \frac{\mathbf{a}_x \cdot \delta A_x + \mathbf{a}_y \cdot \delta A_y + \mathbf{a}_z \cdot \delta A_z}{\delta t}$$

$$\boxed{\frac{d\mathbf{A}}{dt} = \left(\frac{dA_x}{dt}\right)\mathbf{a}_x + \left(\frac{dA_y}{dt}\right)\mathbf{a}_y + \left(\frac{dA_z}{dt}\right)\mathbf{a}_z} \qquad \ldots(47)$$

The time derivative of a vector is then equal to the vector sum of the time-derivatives of its components.

If A is a function of several scalar variables, say ; x, y, z and t, then in a similar manner, the partial differential coefficients $\frac{\partial \mathbf{A}}{\partial x}, \frac{\partial \mathbf{A}}{\partial y}$ etc. can also be defined. If $\mathbf{A}$ and $\mathbf{B}$ are vector functions and m, a scalar function of t, then

$$\frac{d(m\mathbf{A})}{dt} = \frac{dm}{dt}\mathbf{A} + m\frac{d\mathbf{A}}{dt};$$

$$\frac{d(\mathbf{A} \cdot \mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} \qquad \ldots(48)$$

and

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}$$

The proof of the last formula is given below. The others can be proved in a similar way.

Let $\mathbf{A}(t + \delta t) = \mathbf{A}(t) + \delta\mathbf{A} = \mathbf{A} + \delta\mathbf{A}$, for brevity and $\mathbf{B}(t + \delta t) = \mathbf{B}(t) + \delta\mathbf{B} = \mathbf{B} + \delta\mathbf{B}$, then

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \lim_{\delta t \to 0} \frac{(\mathbf{A} + \delta\mathbf{A}) \times (\mathbf{B} + \delta\mathbf{B}) - \mathbf{A} \times \mathbf{B}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{[\mathbf{A} \times \mathbf{B} + \mathbf{A} \times \delta\mathbf{B} + \delta\mathbf{A} \times \mathbf{B} + \delta\mathbf{A} \times \delta\mathbf{B} - \mathbf{A} \times \mathbf{B}]}{\delta t}$$

$$= \lim_{\delta t \to 0} \left(\mathbf{A} \times \frac{\delta \mathbf{B}}{\delta t} + \frac{\delta \mathbf{A}}{\delta t} \times \mathbf{B} + \frac{\delta \mathbf{A}}{\delta t} \times \delta \mathbf{B} \right)$$

$$= \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt} \qquad \text{(neglecting 3rd term)}$$

Example 1.13. *If* $\mathbf{A} = t^3 \, \mathbf{a}_x + (3t^2 - 2t) \, \mathbf{a}_y + \left(2t - \frac{1}{t}\right) \mathbf{a}_z$ *find,* $\frac{d\mathbf{A}}{dt}$ *and the magnitude of* $\frac{d\mathbf{A}}{dt}$ *when* $t = 1$.

Solution. Given

$$\mathbf{A} = t^3 \, \mathbf{a}_x + (3t^2 - 2t) \, \mathbf{a}_y + \left(2t - \frac{1}{t}\right) \mathbf{a}_z$$

On differentiation, we get

$$\frac{d\mathbf{A}}{dt} = 3t^2 \, \mathbf{a}_x + (6t - 2) \, \mathbf{a}_y + \left(2 + \frac{1}{t^2}\right) \mathbf{a}_z$$

When $t = 1$

$$\frac{d\mathbf{A}}{dt} = 3\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z$$

and $$\left| \frac{d\mathbf{A}}{dt} \right| = \sqrt{9 + 16 + 9} = \sqrt{34} \cong 5.91.$$

Example 1.14. *If* **a** *be a variable unit vector, show that (i)* $\frac{d\mathbf{a}}{dt}$ *is a vector normal to* **a**, *(ii)* $\frac{d\mathbf{a}}{d\theta}$ *is a unit vector normal to* **a**, θ *being the angle through which* **a** *turns.*

Solution. Since **a** is a unit vector, then $\mathbf{a} \cdot \mathbf{a} = 1$. Differentiating this, we have :

$$\frac{d\mathbf{a}}{dt} \cdot \mathbf{a} + \mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0$$

$$\therefore \qquad 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0$$

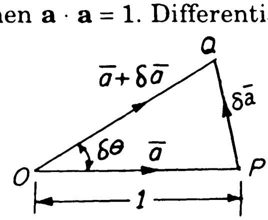


Fig. 1.16

This shows that **a** and $\frac{d\mathbf{a}}{dt}$ are at right angles.

Let $\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OQ} = \mathbf{a} + \delta\mathbf{a}$ be two neighbouring values of the given vector **a** making an angle $\delta\theta$ with each other. Then $\overrightarrow{PQ} = \delta\mathbf{a}$,

and $$\frac{d\mathbf{a}}{d\theta} = \lim_{\delta\theta \to 0} \frac{\delta\mathbf{a}}{\delta\theta}$$

Then, it may be seen that $\frac{d\mathbf{a}}{d\theta}$ is normal to **a**, since $\delta\mathbf{a}$ is normal to **a** in the limiting position. Moreover,

$$\left|\frac{d\mathbf{a}}{d\theta}\right| = \lim_{\delta\theta \to 0} \left|\frac{\delta\mathbf{a}}{\delta\theta}\right| = \lim_{\delta\theta \to 0} \frac{(PO)\cdot\delta\theta}{\delta\theta}$$
$$= OP = 1$$

Hence $\frac{d\mathbf{a}}{d\theta}$ is a unit vector normal to **a**.

1.3B. Scalar and Vector Fields

One of the most important concepts in the subject of electromagnetic fields is that of a field. There are two types of fields commonly encountered in our study : scalar fields and vector fields. The temperature distribution in a room is the most common example of a *scalar field*. As shown in Fig. 1.17*a*, the contours of constant temperature T_1, T_2, T_3 assume that there is a smooth and uniform variation in the contour change. This temperature distribution is called a field because the temperature will have values at various times and position in the room (region). This is denoted by $T(x, y, z, t)$.

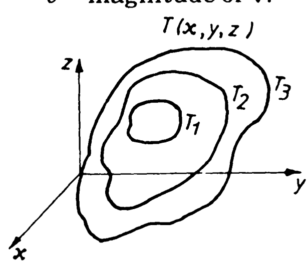
The basic difference between a scalar field and a *vector field* is that the quantity of interest is a vector field is assumed to have a directional property as well as a magnitude at points in the region. A common example of a vector field is the flow of a fluid in a constricted pipe (Fig. 1.17*b*).

If R be any region in space and at each point of this region a vector $\mathbf{v} = \mathbf{v}(x, y, z)$ is given, then R is said to be a vector field. Each vector **v** of the field is regarded as a localized vector attached to the corresponding point (x, y, z). Here, **v** can be expressed in terms of its components.

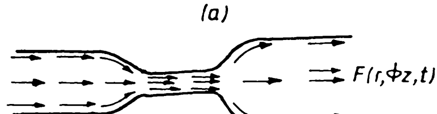
$$\mathbf{v} = v_x\,\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z$$

or $$\mathbf{v} = v\mathbf{a}_r,$$

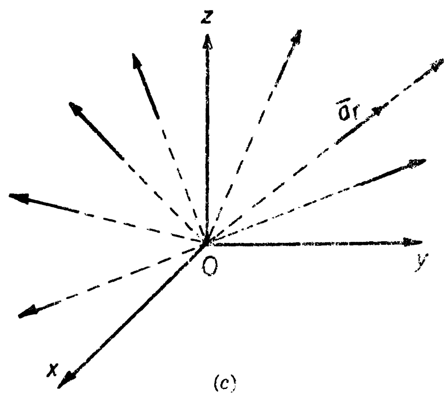
where $$v = \text{magnitude of } \mathbf{v}.$$



(a)



(b)



(c)

Fig. 1.17

$\mathbf{a}_r$ = the unit vector along the line joining O to $P(x, y, z)$ in positive r-direction v_x, v_y and v_z are the components of v along the three axes respectively, all being functions of x, y and z (Fig. 1.17c).

Similarly, if at each point of a region R in space, a scalar $\phi = \phi(x, y, z)$ be given, then ϕ defines a scalar field in R. An example is the temperature at every point of a mass of heated liquid or electrostatic potential at any point due to a point charge (may be assumed to be situated at the origin O).

A simple example of a vector field is the field defined by the radius vector $\mathbf{r} = x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z$. A physical example of a vector field is afforded by the velocities of the particles of a fluid under flow or electric field due to a point charge (may be assumed to be situated at the origin O).

1.3C. Gradient of a Scalar Field

We shall be interested in one particular function of the space derivatives of a scalar quantity—*the gradient*—and in two particular functions of the space derivatives of a vector quantity—*the divergence* and the *curl*. Again, many other functions could be defined, but those studied here are unique as they are useful to describe certain physical quantities.

Let $\phi(x, y, z)$ be a function defining a scalar field at a point (x, y, z) in cartesian co-ordinate system. If the co-ordinates x, y, z are increased by dx, dy and dz respectively, then

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz \qquad ...(49)$$

In equation 49, the scalar quantity has been assumed to be continuous and differentiable function of the co-ordinates. We now wish to know the change $d\phi$ over the distance $d\mathbf{r}$.

Now $d\phi$ is the scalar product of two vectors **A** and **B** as follows :

$$\mathbf{A} = \mathbf{a}_x \frac{\partial \phi}{\partial x} + \mathbf{a}_y \frac{\partial \phi}{\partial y} + \mathbf{a}_z \frac{\partial \phi}{\partial z} \qquad ...(50)$$

$$\mathbf{B} = \mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, d_z \qquad ...(51)$$

The vector **A**, whose components are the rates of change of ϕ with distance along the co-ordinate axes, is called the *gradient of the scalar quantity* ϕ. The gradient is commonly abbreviated as 'grad' and the operation on the scalar ϕ defined by the term gradient is indicated by the symbol, $\overline{\nabla}$ called 'del' or 'nabla'.

Thus, $$\mathbf{A} = \text{grad}\ \phi\ + \overline{\nabla}\, \phi \qquad ...(52)$$

For the general 3-dimensional case, the operator 'del' is defined as

$$\overline{\nabla} = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \right) \qquad ...(53)$$

The partial differentiations indicated are to be carried out on whatever scalar quantity stands to the right of the $\overline{\nabla}$ symbol.

Rewriting equation 49,

$$d\phi = \left(\mathbf{a}_x \frac{\partial \phi}{\partial x} + \mathbf{a}_y \frac{\partial \phi}{\partial y} + \mathbf{a}_z \frac{\partial \phi}{\partial z} \right) \cdot (\mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz)$$

$$= \nabla \phi \cdot d\mathbf{r} \qquad ...(54)$$

where $$d\mathbf{r} = \mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz$$

The equation $\phi\,(x, y, z) =$ constant, represents a certain surface and as we change the value of the constant, we obtain a family of surfaces known as equipotential surfaces (contours). In Fig. 1.18 curves S_1 corresponding to $\phi = C_1$ and S_2 corresponding to $\phi = C_2$ represent two such equipotential surfaces. Let point P with position vector **r** lies on $\phi = C_1$ and Q with position vector $\mathbf{r} + d\mathbf{r}$ lies on $\phi = C_2$ as shown in Fig. 1.18. Then

$$PQ = d\mathbf{r}$$

If d_n denotes the distance along the normal from the point P to surface S_2, we may write

$$dn = \mathbf{a}_n \cdot d\mathbf{r}$$

where $\mathbf{a}_n$ is the unit vector normal to the surface S_1 at P.

We have

$$d\phi = \frac{\partial \phi}{\partial n}\, dn = \frac{\partial \phi}{\partial n}\, (\mathbf{a}_n \cdot d\mathbf{r}) \qquad ...(55)$$

$$= \overline{\nabla}\, \phi \cdot d\mathbf{r} \quad \text{[from Eq. 54]} \qquad ...(56)$$

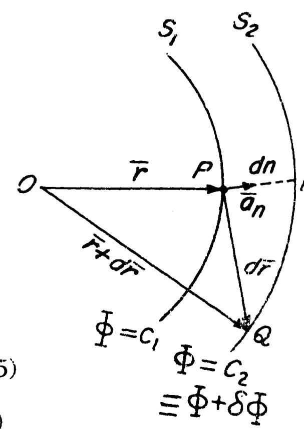


Fig. 1.18

and, in particular, if $d\mathbf{r}$ lies in the surface S_1, we have

$$d\phi = (\overline{\nabla}\phi) \cdot d\mathbf{r} = 0 \qquad ...(57)$$

showing that the vector $\overline{\nabla}\phi$ is normal to the surface ϕ = constant. Since the vector $d\mathbf{r}$ is arbitrary, we have from Eq. (56).

$$\overline{\nabla}\phi = \left(\frac{\partial\phi}{\partial n}\right)\mathbf{a}_n \qquad ...(58)$$

Hence, $\overline{\nabla}\phi$ is a vector whose magnitude is equal to the maximum rate of change of ϕ with respect to the space variables and direction normal to the surface ϕ = constant (*i.e.* $\mathbf{a}_n$).

Furthermore, if δl denotes the length PQ and $\mathbf{a}_m$, the unit vector in direction PQ, then the directional derivative $\partial\phi/\partial l$, is defined by the relation

$$\frac{\partial\phi}{\partial t} = \lim_{\delta l \to 0} \frac{\partial\phi}{\partial l} \qquad ...(59)$$

where ∂l is the difference between values of ϕ at Q and P.

Since $\qquad \delta l = |\delta\mathbf{r}|,$

and $\qquad \partial l = \dfrac{\partial n}{\cos(\angle NPQ)}$

then $\qquad \delta l = \dfrac{\delta n}{\mathbf{a}_m \cdot \mathbf{a}_n} \qquad ...(60)$

Substitution of the value of δl from Eq. (60) in Eq. (59), yields :

$$\frac{\partial\phi}{\partial l} = \lim_{\delta l \to 0} \left[(\mathbf{a}_m \cdot \mathbf{a}_n)\frac{\delta\phi}{\delta n}\right]$$

$$= \mathbf{a}_m \cdot \mathbf{a}_n \frac{\partial\phi}{\partial n} \qquad ...(61)$$

Introducing (58) in (61), we have

$$\boxed{\frac{\partial\phi}{\partial l} = \mathbf{a}_m \cdot \nabla\phi} \qquad ...(62)$$

According to Eq. (62), the directional derivative $\partial\phi/\partial l$ is the component of $\overline{\nabla}\phi$ along the unit vector $\mathbf{a}_m$. It also follows that $|\overline{\nabla}\phi|$ gives the maximum rate of change of ϕ.

Moreover, following Eq. (54), the magnitude of $d\phi$ can be written as

$$d\phi = \overline{\nabla}\phi \cdot d\mathbf{r} = |\overline{\nabla}\phi|\ |d\mathbf{r}| \cos\theta \qquad ...(63)$$

where θ is the angle between the vectors $\nabla\phi$ and $d\mathbf{r}$.

The direction of $d\mathbf{r}$ is chosen such that $d\phi$ is a maximum. The obvious solution will be with $\cos\theta = 1$, or $\theta = 0°$ that is the direction of $\nabla\phi$.

In terms of its components, the magnitude of $\overline{\nabla}\phi$ is

$$|\overline{\nabla}\phi| = \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\theta \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]^{1/2} \quad ...(64)$$

To summarize, the gradient of a scalar function is a vector with the following properties :

1. Its components at any points are the rates of change of the function along the directions of the coordinates axes at that point.

2. Its magnitude at the point is the maximum rate of change of the function with distance.

3. Its direction is that of the maximum rate of change of the function, and it points towards larger values of the function.

The gradient is thus a vector point-function derived from a scalar point-function.

Example 1.15. *Find a unit vector normal to the surface*

$$x^2y + 2xz = 4$$

at the point (2, – 2, 3).

Solution. A vector normal to a surface is given by the gradient of that function. Thus grad

$$\overline{\nabla}\ \phi = \mathbf{a}_x \frac{\partial \phi}{\partial x} + \mathbf{a}_y + \frac{\partial \phi}{\partial y} + \mathbf{a}_z \frac{\partial \phi}{\partial z}$$

$$= \mathbf{a}_x \frac{\partial}{\partial x}(x^2y + 2xz) + \mathbf{a}_y \frac{\partial}{\partial y}(x^2y + 2xz) + a_z \frac{\partial}{\partial z}(x^2y + 2xz)$$

$$= (2\,xy + 2\,z)\ \mathbf{a}_x + x^2\mathbf{a}_y + 2x\mathbf{a}_z$$

Thus, the vector normal to the surface ($x^2y + 2xz = 4$) at point (2, – 2, 3) is $\overline{\nabla}\phi$ given by

$$\overline{\nabla}\phi = -2\mathbf{a}_x + 4\mathbf{a}_y + 4\mathbf{a}_z$$

and the magnitude of $\overline{\nabla}\phi$ is

$$|\overline{\nabla}\phi| = \sqrt{4 + 16 + 16} = 6$$

Hence a unit vector normal to the surface

$$= \frac{\overline{\nabla}\phi}{|\overline{\nabla}\phi|} = \frac{1}{6}(-2\mathbf{a}_x + 4\mathbf{a}_y + 4\mathbf{a}_z)$$

$$= -\frac{1}{3}\mathbf{a}_x + \frac{2}{3}\mathbf{a}_y + \frac{2}{3}\mathbf{a}_z.$$

Example 1.16. *Find the directional derivative of $x^2 + y^2 + 4xyz$ at (1, – 2, 2) in the direction 2 $\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_x$.*

Solution. The gradient of the function ($x^2 + y^2 + 4xyz$) is

$$\overline{\nabla}\phi = \mathbf{a}_x\,(2x + 4yz) + \mathbf{a}_y\,(2y + 4xz) + \mathbf{a}_z\,(4xy)$$

and at point (1, – 2, 2), $\overline{\nabla}\phi$ becomes

$$\overline{\nabla}\phi = -14\,\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$$

The directional derivative along $(2\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z)$

$$= \mathbf{a}_m \cdot \overline{\nabla}\phi$$

$$= \frac{1\,(2\,\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z)}{\sqrt{2^2 + 2^2 + 1}} \cdot (1 - 14\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z)$$

$$= \frac{1}{3}\,(-28 - 8 - 8) = -14\tfrac{2}{3}.$$

Example 1.17. *Show that the gradient of the product of two scalar functions m and n can be expressed by :*

$$\overline{\nabla}\,(mn) = m\overline{\nabla}n + n\overline{\nabla}\,m$$

Solution. $$\overline{\nabla}(mn) = \mathbf{a}_x \frac{\partial}{\partial x}(mn) + \mathbf{a}_y \frac{\partial}{\partial y}(mn) + \mathbf{a}_z \frac{\partial}{\partial z}(mn)$$

$$= \mathbf{a}_z\left(m\frac{\partial n}{\partial x} + n\frac{\partial m}{\partial x}\right) + \mathbf{a}_y\left(m\frac{\partial n}{\partial x} + n\frac{\partial m}{\partial y}\right) + \mathbf{a}_z\left(m\frac{\partial n}{\partial z} + n\frac{\partial m}{\partial z}\right)$$

$$= m\left(\mathbf{a}_x\frac{\partial n}{\partial x} + \mathbf{a}_y\frac{\partial n}{\partial y} + \mathbf{a}_z\frac{\partial n}{\partial z}\right) + n\left(\mathbf{a}_x\frac{\partial m}{\partial x} + \mathbf{a}_y\frac{\partial m}{\partial y} + \mathbf{a}_z\frac{\partial m}{\partial z}\right)$$

$$= m\,\overline{\nabla}\,n + n\overline{\nabla}m.$$

1.3D. Divergence and Curl of Vector Field

Let $\mathbf{A}\,(x, y, z)$ defines a vector field. The operation of the vector operator $\overline{\nabla}$ on $\mathbf{A}$ gives two different results corresponding to the scalar and vector multiplication of two vectors.

Let us consider, first the scalar multiplication of $\overline{\nabla}$ and $\mathbf{A}$. The product being a scalar quantity is known as the divergence of the vector field $\mathbf{A}$, and is denoted by div. $\mathbf{A}$, $(\overline{\nabla}\mathbf{A})$. Thus if,

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y \cdot \mathbf{a}_y + A_z\mathbf{a}_z.$$

$$\overline{\nabla}\cdot\mathbf{A} = \left(\mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z}\right)\cdot(A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)$$

$$= \mathbf{a}_x\left(\mathbf{a}_x\frac{\partial A_x}{\partial x} + \mathbf{a}_y\frac{\partial A_y}{\partial x} + \mathbf{a}_z\frac{\partial A_z}{\partial z}\right) + \mathbf{a}_y\left(\mathbf{a}_x\frac{\partial A_x}{\partial y} + \mathbf{a}_y\frac{\partial A_y}{\partial y} + \mathbf{a}_z\frac{\partial A_x}{\partial y}\right) + \mathbf{a}_z\left(\frac{\partial A_x}{\partial z} + \mathbf{a}_y\frac{\partial A_y}{\partial z} + \mathbf{a}_z\frac{\partial A_z}{\partial z}\right)$$

since $\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1$

and $\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = 0$

Thus $$\overline{\nabla} \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Next, consider the product

$$\overline{\nabla} \times \mathbf{A} = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \times (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z)$$

$$= \mathbf{a}_x \times \left[\frac{\partial A_x}{\partial x}\,\mathbf{a}_x + \frac{\partial A_y}{\partial x}\,\mathbf{a}_y + \frac{\partial A_z}{\partial_z}\,\mathbf{a}_z\right] + \text{other two similar terms.}$$

$$= \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial_z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_y}{\partial y}\right)$$

or $$\overline{\nabla} \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \text{curl } \mathbf{A}$$

The vector $\overline{\nabla} \times \mathbf{A}$ is known as the curl of vector field **A**, and is denoted by curl **A** since cross product also shows the rotation hence *if curl* **A** *= 0, the field* **A** *is called irrotational.* The divergence and curl of a vector field have been further considered in sections 1.4 and 1.5.

Example 1.18. *Find the divergence and curl of the vector*

$$\mathbf{A} = xyz\,\mathbf{a}_x + 3x^2y\mathbf{a}_y + (xz^2 - y^2z)\mathbf{a}_z.$$

Solution. From definition.

$$\text{div. } \mathbf{A} = \overline{\nabla} \cdot \mathbf{A} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + (3xz - y^2)$$

$$= 3x^2 - y^2 + z\,(y + 2x)$$

$$\text{Curl } \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$\text{Curl } \mathbf{A} = \mathbf{a}_x\left[\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right]$$

$$+ \mathbf{a}_y\left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial x}(xz^2 - y^2z)\right] + a_z\left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right]$$

$$= \mathbf{A}_x\,(-2yz - 0) + \mathbf{a}_y\,(xy - z^2) + \mathbf{a}_z\,(6xy - xz)$$

$$= -2yz\,\mathbf{a}_x + (xy - z^2)\mathbf{a}_y + (6xy - xz)\,\mathbf{a}_z$$

Example 1.19. *Prove that :*

(*a*) $\overline{\nabla} \cdot (\phi \mathbf{A}) = (\nabla\phi) \cdot \mathbf{A} + \phi(\overline{\nabla} \cdot \mathbf{A})$

(*b*) $\overline{\nabla}\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \overline{\nabla})\mathbf{B} + (\mathbf{B} \cdot \overline{\nabla})\mathbf{A} + \mathbf{A} \times (\overline{\nabla} \times \mathbf{B}) + \mathbf{B} \times \overline{\nabla} \times \mathbf{A}$

(*c*) $\overline{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \overline{\nabla})\,\mathbf{A} - \mathbf{B}\,(\overline{\nabla} \cdot \mathbf{A} - (\mathbf{A} \cdot \overline{\nabla})\mathbf{B} + \mathbf{A}\,(\overline{\nabla} \cdot \mathbf{B})$

Solution. (*a*) $\overline{\nabla} \cdot (\phi\mathbf{A}) = \left(\mathbf{a}_x \dfrac{\partial}{\partial x} + \mathbf{a}_y \dfrac{\partial}{\partial y} + \mathbf{a}_z \dfrac{\partial}{\partial z}\right) \cdot (\phi\mathbf{A})$

$$= \left(\sum \mathbf{a}_x \frac{\partial}{\partial x}\right) \cdot (\phi\mathbf{A})$$

$$= \sum \mathbf{a}_x \cdot \frac{\partial \phi}{\partial x}\mathbf{A} + \sum \mathbf{a}_x \cdot \phi\frac{\partial \mathbf{A}}{\partial x}$$

$$= \left(\sum \mathbf{a}_x \frac{\partial \phi}{\partial x}\right)\mathbf{A} + \phi\left(\sum \mathbf{a}_x \frac{\partial}{\partial x} \cdot \mathbf{A}\right)$$

$$= (\overline{\nabla}\phi) \cdot \mathbf{A} + \phi(\overline{\nabla} \cdot \mathbf{A})$$

(*b*) Since

$$\mathbf{A} \times (\overline{\nabla} \times \mathbf{B}) = \mathbf{A} \times \left(\sum \mathbf{a}_x \frac{\partial}{\partial x} \times \mathbf{B}\right)$$

$$= \sum \mathbf{A} \times \left(\mathbf{a}_x \times \frac{\partial \mathbf{B}}{\partial x}\right)$$

$$= \sum \left(\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{a}_x - \sum \left(\mathbf{A} \cdot \mathbf{a}_x\right)\frac{\partial \mathbf{B}}{\partial x}$$

$$= \sum \left(\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{a}_x - \left(\mathbf{A} \cdot \sum \mathbf{a}_x - \frac{\partial}{\partial x}\right)\mathbf{B}$$

$$\mathbf{A} \times (\overline{\nabla} \times \mathbf{B}) = \sum \left(\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{a}_x - (\mathbf{A} \cdot \nabla)\,\mathbf{B}$$

so that, $\mathbf{A} \times (\overline{\nabla} \times \mathbf{B}) + (\mathbf{A} \cdot \overline{\nabla})\mathbf{B} = \sum \left(\mathbf{A} \cdot \dfrac{\partial \mathbf{B}}{\partial x}\right)\mathbf{a}_x$

Similarly, $\mathbf{B} \times (\overline{\nabla} \times \mathbf{A}) + (\mathbf{B} \cdot \overline{\nabla})\mathbf{A} = \sum \left(\mathbf{B} \cdot \dfrac{\partial \mathbf{A}}{\partial x}\right)\mathbf{a}_x$

Adding up, we get

$$\mathbf{A} \times (\overline{\nabla} \times \mathbf{B}) + \mathbf{B}(\overline{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\,\mathbf{B} + (\mathbf{B} \cdot \overline{\nabla})\mathbf{A}$$

$$= \sum \mathbf{a}_x \left(\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x} + \mathbf{B} \cdot \frac{\partial \mathbf{A}}{\partial x}\right)$$

$$= \sum \mathbf{a}_x \frac{\partial}{\partial x}(\mathbf{A} \cdot \mathbf{B}) = \overline{\nabla}(\mathbf{A} \cdot \mathbf{B})$$

(*c*) Putting $\overline{\nabla} = \overline{\nabla}_A = \overline{\nabla}_B$, we see that

$$\overline{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \nabla_A \times (\mathbf{A} \times \mathbf{B}) + \overline{\nabla}_B\,(\mathbf{A} \times \mathbf{B})$$

$$= (\overline{\nabla}_A \cdot \mathbf{B})\,A - (\overline{\nabla}_A \cdot \mathbf{A})\mathbf{B} + (\overline{\nabla}_B \cdot B)\,\mathbf{A} - (\overline{\nabla}_B \cdot \mathbf{A})\mathbf{B}$$

$$= (\mathbf{B} \cdot \overline{\nabla}_A)\mathbf{A} - (\nabla_A \cdot \mathbf{A})\mathbf{B} + \mathbf{A}\,(\nabla_B \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla_B)\mathbf{B}$$

Replacing $\overline{\nabla}_A$ and $\overline{\nabla}_B$ by $\overline{\nabla}$ (since $\overline{\nabla} \cdot \mathbf{A} = \overline{\nabla}_A \cdot \mathbf{A} + \overline{\nabla}_B \cdot \mathbf{A} = \overline{\nabla}_A \cdot \mathbf{A} + 0$), we have

$$\overline{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \overline{\nabla})\mathbf{A} - \mathbf{B}\,(\overline{\nabla} \cdot \mathbf{A}) + \mathbf{A}\,(\overline{\nabla} \cdot \mathbf{B}) - (\mathbf{A} \cdot \overline{\nabla})\,\mathbf{B}$$

1.3E. Repeated Operations by $\overline{\nabla}$

Let $\phi\,(x, y, z)$ be a function defining a scalar field, then the gradient of this field, namely $\overline{\nabla}\phi$ defines a vector field. The divergence of this vector field is

$$\text{div (grad } \phi) = \text{div } (\overline{\nabla}\phi) = \overline{\nabla} \cdot \overline{\nabla}\phi$$

$$= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\mathbf{a}_x \frac{\partial \phi}{\theta x} + \mathbf{x}_y \frac{\partial \phi}{\partial z} + \mathbf{q}_z \frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

The operator $(\overline{\nabla} \cdot \overline{\nabla})$ is often written as ∇^2 and is known as the "*Laplacian*" operator. This operator is a scalar operator.

The curl of the vector field $\overline{\nabla}\phi$ is curl (grad ϕ) = $\overline{\nabla} \times (\overline{\nabla}\phi)$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{vmatrix}$$

This shows that the field $\overline{\nabla}\phi$ is irrotational. It follows therefore, that, if **A** is a vector field expressed by, $\mathbf{A} = \overline{\nabla}\phi$, then curl $A = 0$ and this shows that an irrotational vector field **A** can be expressed as the gradient of scalar field ϕ.

Example 1.20. *Prove that,*

$$\textit{curl (curl } \mathbf{A}) = \overline{\nabla} \times (\overline{\nabla} \times \mathbf{A})$$

$$= \textit{grad (div } \mathbf{A}) - \nabla^2 \mathbf{A}$$

Solution. curl (curl **A**) = $\overline{\nabla} \times (\overline{\nabla} \times \mathbf{A})$

Using the relation,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

$$\overline{\nabla} \times (\overline{\nabla} \times \mathbf{A}) = (\overline{\nabla} \cdot \mathbf{A})\overline{\nabla} - (\overline{\nabla} \cdot \nabla)\mathbf{A}$$

$$= \text{grad (div } \mathbf{A}) - \nabla^2 \mathbf{A}$$

Example 1.21. *Determine the constant a so that the vector* $\mathrm{v} = (x + 3y)\,\mathbf{a}_x + (y - 2z)\,\mathbf{a}_y + (x + az)\,\mathbf{a}_z$ *is a solenoidal.*

Solution. Since

$$\mathbf{V} = (x + 3y)\,\mathbf{a}_x - (y - 2z)\,\mathbf{a}_y + (x + az)\,\mathbf{a}_z$$

is a solenoidal, therefore, the divergence of this vector **V** is zero.

Then the divergence of **V**, is given by

$$\overline{\nabla} \cdot \mathbf{V} = \overline{\nabla} \cdot [(x + 3y)\, \mathbf{a}_x + (y - 2z)\, \mathbf{a}_y + (x + az)\, \mathbf{a}_z] = 0$$

or
$$\left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \times [(x + 3y)\, \mathbf{a}_x + (y - 2z)\, \mathbf{a}_y + (x + \mathbf{a}_z) = 0$$

or
$$\overline{\nabla} \cdot \mathbf{V} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az) = 0$$

or $1 + 1 + a = 0$

or $a = -2.$

Example 1.22. *A vector* **V** *is called irrotational if* $\overline{\nabla} \times \mathbf{V} = 0$, *find constants* a, b, c *so that,*

$$\mathbf{V} = (x + 2y + az)\, \mathbf{a}_x + (bx - 3y - z)\, \mathbf{a}_y + (4x + cy + 2z)\, \mathbf{a}_z$$

is irrotational.

Solution. Vector

$$\mathbf{V} = (x + 2y + az)\, \mathbf{a}_x + (b_x - 3y - z)\, \mathbf{a}_y + (4x + cy + 2z)\, \mathbf{a}_z$$

Taking curl of **V**, we have,

$$\overline{\nabla} \times \mathbf{V} = \overline{\nabla} \times [(x + 2y + az)\, \mathbf{a}_x + (bx - 3y - z)\, \mathbf{a}_y + (4x + cy + 2z)\, \mathbf{a}_z)]$$

$$= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \times [(x + 2y + az)\, \mathbf{a}_x + (bx - 3y - z)\, \mathbf{a}_y + (4x + cy + 2z)\, \mathbf{a}_2)]$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= \mathbf{a}_x \left[\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right] + \mathbf{a}_y \left[\frac{\partial}{\partial x}(4x + cy + 2z) + \frac{\partial}{\partial z}(x + 2y + az)\right] + \mathbf{a}_z \left[\frac{\partial}{\partial x}(bx - 3y + z) - \frac{\partial}{\partial y}(x + 2y + az)\right]$$

$$= (c - 1)\, \mathbf{a}_x - (4 - a)\, \mathbf{a}_y + (b - 2)\, \mathbf{a}_z$$

Since vector **V** is irrotational,

$$\overline{\nabla} \times \mathbf{V} = 0$$

Therefore, $c - 1 = 0 \Rightarrow c = 1$

$4 - a = 0 \Rightarrow a = 4$

$b - 2 = 0 \Rightarrow b = 2.$

1.3F. Integration of a Vector

If $\mathbf{A}(t)$ and $\mathbf{Q}(t)$ be two vector functions of a scalar variable t, connected by the relation

$$\frac{d\mathbf{A}(t)}{dt} = \mathbf{a}(t)$$

then $A(t)$ is said to be the integral of $\mathbf{a}(t)$ and is written as

$$\int \mathbf{a}(t)\, dt = \mathbf{A}(t) + \mathbf{C}$$

This integral is an indefinite integral and $\mathbf{C}$ is an arbitrary constant.

(*a*) **Line Integrals.** Let c be the given curve in the xy-plane and let $f(x, y)$ be a function of x and y. Divide the curve c into n-parts of lengths $\delta l_1, \delta l_2, \ldots \delta l_n$, and let (x_r, y_r) be any general point on the rth part δl_r of the curve (Fig. 1.19 *a*).

Then the limit of the sum

$$f(x_1, y_1)\,\delta l + f(x_2, y_2)\,\delta l_2 + \ldots + f(x_n, y_n)\,\delta l_n \qquad \ldots(65)$$

When $n \to \infty$, and the length of each part $\delta l_r \to 0$, is called the *line integral* of $f(x, y)$ along the curve C. It is denoted by

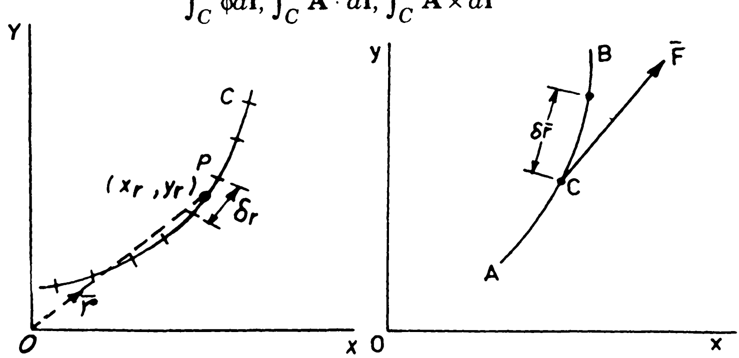
$$\int_C f(x, y)\, dl$$

On similar lines, the line integral of a function $f(x, y, z)$ along a space curve C can also be defined.

The concept of the line integral can also be extended to vector integration by taking the vector elements $\delta\mathbf{l}_1, \delta\mathbf{l}_2, \ldots \delta\mathbf{l}_n$ [Eq. 65] and then performing the vector addition.

Three types of line integrals may thus be defined :

$$\int_C \phi d\mathbf{l}, \int_C \mathbf{A}\cdot d\mathbf{l}, \int_C \mathbf{A}\times d\mathbf{l}$$



(*a*) (*b*)

Fig. 1.19

The middle integrals is a scalar while the other two are vectors. Since $d\mathbf{l} = d\mathbf{r}$, where $\mathbf{r}$ is the position vector of a point on the curve C, therefore, the three line integrals may also be written as

$$\int_C \phi d\mathbf{r}, \int_C \mathbf{A} d\mathbf{r}, \int_C \mathbf{A} \times d\mathbf{r}$$

An example of the line integral is affected by the work-done by a force $\mathbf{F}$ acting on a particle which moves in a curve C. The work-done during the displacement $\delta\mathbf{r}$ is $\mathbf{F} \cdot \delta\mathbf{r}$. So, the total work-done by the force is

$$\lim_{\delta_r \to 0} \sum \mathbf{F} \cdot \delta\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

If the force $\mathbf{F}$ arises from a potential ϕ, then

$$\mathbf{F} = \text{grad } \phi$$

$$= \mathbf{a}_x \frac{\partial\phi}{\partial x} + \mathbf{a}_y \frac{\partial\phi}{\partial y} + \mathbf{a}_z \frac{\partial\phi}{\partial_z}$$

Also $$d\mathbf{r} = \mathbf{a}_z \, dx + \mathbf{a}_y dy + \mathbf{a}_z d_z$$

So $$\mathbf{F} \cdot d\mathbf{r} = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi$$

Therefore, the work-done in moving the particle from A to B along the curve C

$$= \int_c d\phi = \int_A^B d\phi = \Big[\phi\Big]_A^B$$

We see that in this case, the work done depends only on the initial and final points, A and B, and not on the path followed in reaching B from A. Such a field of force is called a *conservative* field.

Example 1.23. *Find the work done when a force* $\mathbf{F} = (x^2 - y^2 + x)\mathbf{a}_x - (2xy + y)\mathbf{a}_y$ *moves a particle in the xy-plane from (0, 0) to (1, 1) along the parabola* $y^2 = x$. *Is the work done different when the path is the straight line* $y = x$ *? Refer to Fig. 1.19c.*

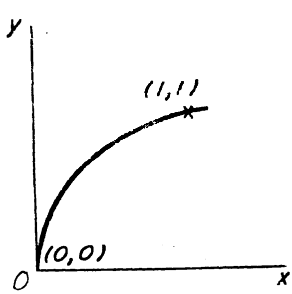


Fig. 1.19 (c)

Solution. In xy-plane, the position vector $\mathbf{r}$ is

$$\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y$$

and this gives

$$d\mathbf{r} = \mathbf{a}_x \cdot dx + \mathbf{a}_y dy$$

The workdone $$= \int_c \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_C [(x^2 - y^2 + x)\,\mathbf{a}_x - (2xy + y)\,\mathbf{a}_y] \cdot [\mathbf{a}_x dx + \mathbf{a}_y dy]$$

$$= \int_C [(x^2 - y^2 + x)\,dx - (2xy + y)\,dy]$$

Since $x = y^2$

$\therefore$ $dx = 2ydy$

Work done $= \int_C [(y^4 - y^2 + y^2)\,2ydy - (2y^3 + y)\,dy]$

$$= \int_0^1 (2y^5 - 2y^3 - y)\,dy$$

$$= \left[2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} - \frac{1}{2} = -\frac{2}{3}$$

In case the particle moves from (0, 0) to (1, 1) along $y = x$, the workdone is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [x^2 - y^2 + x\,dx - (2xy + dy))]$$

$$= \int_0^1 [xdx - (2x^2 + x)\,dx] = -\frac{2}{3}$$

The function ϕ may be determined by the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$. That is,

$$\phi = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_C [x^2 - y^2 + x]\,dx - (2xy + y)\,dy]$$

$$= \left[\frac{x^3}{3} - y^2x + \frac{x^2}{2} - xy^2 - \frac{y^2}{2}\right]$$

$$= \frac{x^3}{3} - 2xy^2 + \frac{x^2}{2} - \frac{y^2}{2}$$

Further, for $y = x$; we notice that, the gradient of a function ϕ is given by

$$\phi = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2xy^2 - \frac{1}{2}y^2$$

may be related to $\mathbf{F}$ by $\mathbf{F} = \text{grad}\,\phi$. Since the integral for both the paths give the same result, therefore, the field is conservative and the work-done does not depend on the path followed.

Example 1.24. *If* $\mathbf{F} = 2y\mathbf{a}_x - z\mathbf{a}_y + x\mathbf{a}_z$, *evaluate* $\int_C \mathbf{F} \times d\mathbf{r}$ *along the curve* $x = \cos t$, $y = \sin t$, $z = 2 \cot t$, *from* $t = 0$ *to* $t = \pi/2$.

Solution.

$$\mathbf{F} \times d\mathbf{r} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$= -(zdz - xdy)\,\mathbf{a}_x + (xdx - 2ydz)\,\mathbf{a}_y + (2ydy + zdx)\,\mathbf{a}_z$$

$$= (4 \cot t \sin t - \cos^2 t)\, dt\mathbf{a}_x + (4 \sin^2 t - \cot t \sin t)\, dt\, \mathbf{a}_y$$
$$+ (2 \sin t \cot t - 2 \cot t \sin t)\, dt\, \mathbf{a}_z,$$

On c

$$\therefore \quad \int_c \mathbf{F} \times d\mathbf{r} = \int_0^{\pi/2} [(4 \cot t \sin t - \cos^2 t)\,\mathbf{a}_x$$
$$+ (4 \sin^2 t - \cot t \sin t\,)\,\mathbf{a}_y]\, dt$$

$$= (4 \cdot \tfrac{1}{2} - \tfrac{1}{2} \cdot \tfrac{1}{2}\pi)\,\mathbf{a}_x + (4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi - \tfrac{1}{2})\,\mathbf{a}_y$$

$$= \left(2 - \frac{\pi}{4}\right)\mathbf{a}_x + \left(\pi - \frac{1}{2}\right)\mathbf{a}_y$$

(*b*) **Surface and Volume Integrals.** Let S be a surface in space and let $f(x, y, z)$ be a function defined at every point of this surface. Divide the surface S into n parts δS_1, δS_2, ... δS_n and let (x_r, y_r, z_r) be a point on the rth part, *i.e.*, δS_r. The then limit of the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta S_r$$

as $n \to \infty$ and the area of each part $\delta S_r \to 0$ defines

$$\iint_S f(x, y, z)\, ds,$$

the surface integral of $f(x, y, z)$ over S.

The surface element $\delta \mathbf{s}_r$ can also be regarded as a vector having a magnitude equal to the surface area and direction along an outward normal to the surface element. Again, three types of surface integrals may be defined :

$$\iint_S \phi\, d\mathbf{s},\ \iint_S \mathbf{A} \cdot d\mathbf{s},\ \iint \mathbf{A} \times d\mathbf{s}$$

Denoting by $\mathbf{a}_n$ the unit vector normal to the surface, the above integrals can also we written as

$$\iint_S \phi\,(\mathbf{a}_n\, ds), \iint_S \mathbf{A}\cdot\mathbf{a}_n\, ds), \iint_S \mathbf{A}\times(\mathbf{a}_n\, ds)$$

The middle integral measures the flux of **A** across the surface. If **A** = density × fluid velocity

$$\mathbf{A} = \rho\mathbf{v} \text{ then}$$

$$\mathbf{A}\cdot d\mathbf{s} = (\mathbf{A}\cdot\mathbf{a}_n)\, ds$$

$$= (\rho\mathbf{v}\cdot\mathbf{a}_n)\, ds = \rho v ds \cos\theta$$

Therefore, $\iint_S \mathbf{A}\cdot d\mathbf{s} = \iint_S \rho v ds \cos\theta$

In particular if we imagine a fluid flowing through a uniform circular cylinder, then $\theta = 0$, and the above integral will represent the mass of the fluid flowing out of the surface S Fig. 1.19(d). This formula is also used to calculate the electric or magnetic (ϕ or ψ_m) flux in electromagnetic system.

The volume integrals $\iiint_V \phi dV$ and $\iiint_v \mathbf{A} dV$

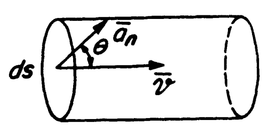


Fig. 1.19 (d)

are defined by dividing the volume V in n elementary volumes ; $\delta V_1, \delta V_2, \ldots \delta V_n$ and by assuming any point (x_r, y_r, z_r) inside the rth subdivision δV_r such that as n tends to infinity, the dimensions of each subdivision tends to zero, and

$$\lim_{n\to\infty}\left[\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r\right] \equiv \iiint_V f(x, y, z)\, dV.$$

The first of these is a scalar and the second, a vector.

1.4. FLUX AND DIVERGENCE—THE DIVERGENCE THEOREM

It is often necessary to calculate the flux of a vector through a surface. The flux $d\phi$ of a vector **A** through an infinitesimal surface $d\mathbf{s}$ is given by

$$d\phi = \mathbf{A}\cdot d\mathbf{s} \quad \ldots(66)$$

where the vector $d\mathbf{s}$ representing the element of area is normal to its surface. The flux $d\phi$ is component of the vector normal to the surface multiplied by the area of the surface. For a finite surface, the total flux is then obtained by integrating $\mathbf{A}\cdot d\mathbf{s}$ over the entire surface :

$$\phi = \int_S \mathbf{A} \cdot d\mathbf{s} \qquad \text{...(67)}$$

For a closed surface bounding a finite volume, the vector $d\mathbf{s}$ is taken to point outward.

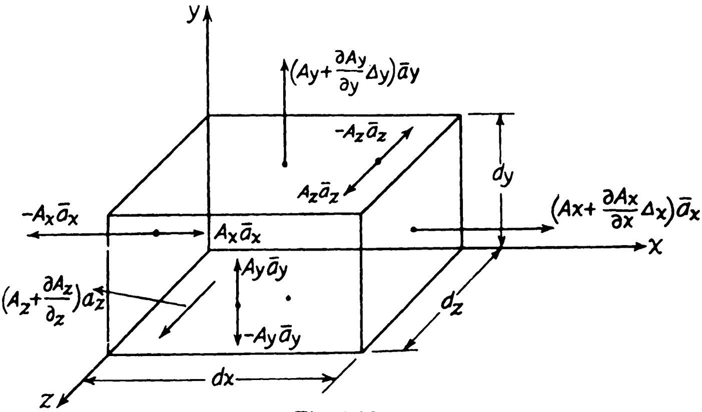


Fig. 1.20.

As an example of flux, let us consider fluid flow. Here, we can define a vector $\rho\mathbf{V}$, ρ being the fluid density and $\mathbf{V}$ the fluid velocity at a point. The flux through any closed surface is the net rate at which fluid (mass) leaves the volume bounded by the surface.

The outward flux of a vector through a closed surface can be calculated either by using Eq. 67 or as follows. Let us consider an infinitesimal volume $dx\,dy\,dz$ and a vector $\mathbf{A}$ (as in Fig. 1.20) whose components A_x, A_y and A_z are functions of the coordinate x, y and z. If the volume considered is of an infinitesimal size then the first order variations of the vector $\mathbf{A}$ will be significant.

Let A_x be the average value of $\mathbf{A}$ on the left face in the x-direction. Since A_x is directed towards the left face (inward), hence the outward normal component of $\mathbf{A}$ at the left side face is $-A_x$. If the vector field changes between left and right faces, the normal component of $\mathbf{A}$ at the right face can by Taylor's theorem, be represented by an infinite series

$$A_x + \frac{\partial A_x}{\partial x}\Delta x + \frac{\partial^2 A_x}{\partial x^2}\frac{(\Delta x)^2}{2\,!} + \ldots\ldots\ldots$$

where Δx is very small, the square and higher order terms may be neglected so that at the right side face, we have the normal component of $\mathbf{A}$

$$A_x + \frac{\partial A_x}{dx}\Delta x$$

In a like manner, the normal component of **A** at the bottom face $(-A_y)$ and the top face is

$$A_y + \frac{\partial A_y}{\partial y}\Delta y$$

Similarly, at the back face, it is $-A_z$ and at the front face is

$$A_z + \frac{\partial A_z}{\partial z}\Delta z$$

Now the outward flux of **A** over the left side face

$$= d\,\phi_L = -A_x \Delta y \Delta z$$

and over the right side face is

$$d\phi_R = \left(A_x + \frac{\partial A_x}{\partial x}\Delta x\right)\Delta y \cdot \Delta z \qquad ...(68)$$

Then the net outward flux through the two faces is

$$d\phi_L + d\phi_R = \frac{\partial A_x}{\partial x}\Delta x \cdot \Delta y \cdot \Delta z = \frac{\partial A_x}{\partial x}\Delta v \qquad ...(69)$$

where Δv is the volume of infinitesimal element.

If we calculate the net flux through the other pairs of faces in the same manner, we find the total outward flux for the element of volume Δv to be

$$d\phi_{total} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta v \qquad ...(70)$$

For finite volume v, the total outward flux is then given by integral

$$\phi_{total} = \int_v \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)dv \qquad ...(71)$$

At any given point in the volume, the quantity

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

is thus the outgoing flux per unit volume. We call this, the divergence of the vector **A** at the point. The divergence of **A** is abbreviated as div **A** and defined as

$$\text{div}\,\mathbf{A} = \lim_{\Delta v \to 0} \frac{d\phi_{total}}{\Delta v}$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad ...(72)$$

According to the rules for the scalar product, we can write the divergence **A** as

$$\boxed{\text{div. } \mathbf{A} = \overline{\nabla} \cdot \mathbf{A}} \qquad ...(73)$$

where the operator $\overline{\nabla}$ has already been discussed.

The operator $\overline{\nabla}$ has physical meaning, not by itself, but only when it operates on a function appearing to the right of it.

In Eq. 71, the total outward flux is also equal to the surface of the outward normal component of **A**, thus

$$\int_S \mathbf{A} \cdot d\mathbf{s} = \int_v \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dv \qquad ...(74)$$

Then
$$\boxed{\int_S \mathbf{A} \cdot d\mathbf{s} = \int_v (\overline{\nabla} \cdot \mathbf{A})\, dv} \qquad ...(75)$$

This is the *divergence theorem relating the surface integral to volume integral. It is obvious from Eq. (75) that the left hand side involves only the values of* **A** *on the surface* **S**, *whereas the right-hand side involves the values of* **A** *throughout the volume v enclosed by* **S.** This important theorem will be used frequently.

If the volume v is allowed to shrink sufficiently such that $\overline{\nabla} \cdot \mathbf{A}$ does not vary appreciably over it, then

$$\int_S \mathbf{A} \cdot d\mathbf{s} = (\overline{\nabla} \cdot \mathbf{A})\, v \qquad ...(76)$$

and divergence can therefore be defined as

$$\boxed{\overline{\nabla} \cdot \mathbf{A} = \lim_{v \to 0} \frac{1}{v} \int_s \mathbf{A} \cdot ds} \qquad ...(77)$$

As we have seen, the divergence is the outward flux per unit volume as the volume approaches zero. Further, *if the divergence of a vector field is zero, the field is termed as solenoidal.*

1.5. LINE INTEGRAL AND CURL OF A VECTOR FIELD

The line integral $\int_a^b \mathbf{A} \cdot d\mathbf{l}$ evaluated from a point 'a' to the point 'b' for the work done W by a force **F** acting along some specified path is given by

$$W = \int \mathbf{F} \cdot d\mathbf{l} \qquad ...(78)$$

where both **F** and $d\mathbf{l}$ must of course be known functions of the coordinates.

For illustration, let us calculate the workdone by a force **F**, which is in the y-direction and has a magnitude proportional to y, when it moves around the circular path from a to b as shown in Fig. 1.21. Since $F = Kr \sin\theta$, $dl = r\,d\theta$, and the scalar product introduces a factor $\cos\theta$. (K is a proportionality constant between F and y)

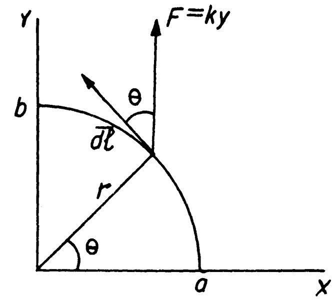


Fig. 1.21. The force **F** proportional to y, and its point of application moves from a to b. The work done is given by the line integral of $\mathbf{F} \cdot dl$ over the curve shown.

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{l}$$

$$= \int_0^{\pi/2} (Kr \sin\theta)\,(rd\,\theta)\cos\theta$$

$$= \frac{Kr^2}{2} \qquad \text{....(79)}$$

A vector field in which the line integral of $\mathbf{A} \cdot d\mathbf{l}$ *around any closed path is zero is said to be conservative.* Thus, for a conservative field,

$$\int \mathbf{A} \cdot d\mathbf{l} = 0 \qquad \text{...(80)}$$

The circle on the integral sign indicates that the path is closed.

For an infinitesimal element of path $d\mathbf{l}$ in xy-plane, and from the definition of the scalar product,

$$\mathbf{A} \cdot d\mathbf{l} = A_x\,\Delta x + A_y \Delta y \qquad \text{...(81)}$$

Thus, for any closed path in the xy-plane,

$$\int \mathbf{A} \cdot d\mathbf{I} = \int A_x\,dx + \int A_y\,dy \qquad \text{...(82)}$$

Let us now refer to Fig. 1.22 to evaluate $\int \mathbf{A} \cdot \mathbf{dl}$. Let vector **A** has a component A_x (average) along the path 12, and A_y along the path

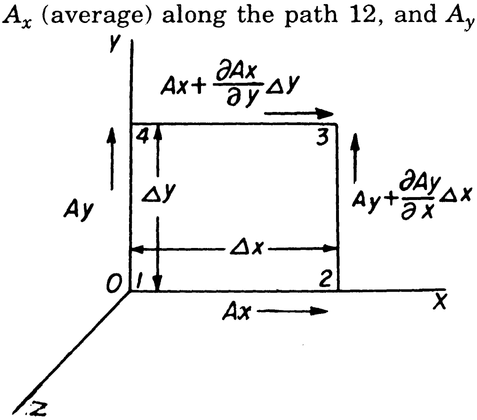


Fig. 1.22.

14. If the field is not uniform, its value at edges 23 and 43 may be expressed to a first approximation by

$$A_y + \frac{\partial A_y}{\partial x} \Delta x$$

and
$$A_x + \frac{\partial A_x}{\partial y} \Delta y$$

Performing the integration of $\int \mathbf{A} \cdot \mathbf{dl}$ in anti-clockwise direction, we have

$$\int \mathbf{A} \cdot \mathbf{dl} = A_x \Delta x + \left(A_y + \frac{\partial A_y}{\partial x} \Delta x\right) \Delta y - \left(A_x + \frac{\partial A_x}{\partial y} \Delta y\right) \Delta x - A_y \Delta y \quad ...(83)$$

Collecting the coefficients of Δ_x and Δ_y in (83) and comparing these with the corresponding terms in (82), we then obtain

$$\int A_x \, dx = -\frac{\partial A_x}{\partial y} \Delta y \, \Delta x$$

and
$$\int A_y \, dy = \frac{\partial A_y}{\partial x} \Delta x \, \Delta y$$

Further,
$$\int \mathbf{A} \cdot \mathbf{dl} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \Delta x \, \Delta y \quad ...(84)$$

If we let
$$g_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \quad ...(85)$$

then
$$\mathbf{A} \cdot \mathbf{dl} = g_3 ds \quad ...(86)$$

where $ds = \Delta x \, \Delta y$ is the area enclosed on the xy-plane by the infinitesimal path. It may be emphasized that Eq. (86) is correct only if the line integral is evaluated in the direction such that a right hand screw advances in the positive z-direction. The above requirement is met by performing the integration in an anticlockwise direction.

When the small path is arbitrarily oriented in space, then $\int \mathbf{A} \cdot \mathbf{dl}$ is the sum of the projections of $d\mathbf{s}$ on the co-ordinate planes, multiplied by functions similar to g_z in Eq. (86) :

$$\int \mathbf{A} \cdot \mathbf{dl} = ds \cos(\angle \mathbf{a}_n, x) g_x + ds \cos(\mathbf{a}_n, y) g_y + ds \cos(\mathbf{a}_n, z) g_z \quad ...(87)$$

$$= ds_x g_x + ds_y g_y + ds_z g_z \quad ...(88)$$

where, for example, $\cos(\angle a_n, x)$ is the cosine of the angle between the vector $d\mathbf{s}$, which is normal to the surface and the x-axis, and where

$$g_x = \frac{\partial A}{\partial y} - \frac{\partial A_y}{\partial z} \qquad ...(89)$$

$$g_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \qquad ...(90)$$

$$g_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y} \qquad ...(91)$$

From Eq. (88) it appears that the right hand side is just the scalar product of $d\mathbf{s}$ with a vector **C**, such that $C_x = g_x$, $C_y = g_y$ and $C_z = g_z$. The vector **C** is called the curl of **A** :

$$\text{Curl } \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial x} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \qquad ...(92)$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \overline{\nabla} \times \mathbf{A} \qquad ...(93)$$

Rewriting Eq. (88) for a small path of integration, we have

$$\boxed{\int \mathbf{A} \cdot d\mathbf{l} = (\overline{\nabla} \times \mathbf{A}) \cdot d\mathbf{s}} \qquad ...(94)$$

Here again, care must be taken about the direction in which the line integral is evaluated. This direction of integration must be related to the direction in which the right side of Eq. (94) is positive by the right-hand screw rule. The above equation provides us with a definition of the curl that is same for all the co-ordinate systems :

$$\boxed{(\overline{\nabla} \times \mathbf{A})_n = \lim_{\Delta s \to 0} \frac{\int \mathbf{A} \cdot d\mathbf{l}}{\Delta S}} \qquad ...(95)$$

This equation shows that the *component of the curl of a vector normal to a surface ΔS is equal to the line-integral of the vector around the boundary of the surface divided by the area of the surface for the limiting case in which the surface area approaches zero.*

As an example, let us calculate the curl of the gravitational force. We shall first calculate the components in a formal manner in rectangular co-ordinates and then calculate these components with the help of the above definition of the curl.

For the first approach, we choose a co-ordinate system in which, the z-axis is perpendicular to the earth's surface and pointing upward, and in which the x-axes and y-axes are parallel to the earth's surface. Then, the gravitational force **F** on a mass m has the following components :

$$F_x = F_y = 0,$$

$$F_z = -mg \qquad ...(96)$$

Then
$$\bar{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -mg \end{vmatrix} = 0 \qquad ...(97)$$

If we proceed in the second manner, we choose small rectangles around which $\int \mathbf{A} \cdot \mathbf{dl} = \int \mathbf{F} \cdot \mathbf{dl}$ is to be evaluated. To obtain the z-component of the curl, we choose a path in the xy-plane. Since $\mathbf{F}$ is perpendicular to $\mathbf{dl}$ on all parts of the path, $\mathbf{F} \cdot \mathbf{dl} = 0$ around this path, and so $(\bar{\nabla} \times \mathbf{F})_z = 0$. To find the y-component of the curl, we choose a path lying in the xz-plane. On the two parts of the path parallel to the z-axis there are contributions to the integral, but they are equal and opposite, thus $(\bar{\nabla} \times \mathbf{F})_y = 0$. By similar arguments, $(\nabla \times \mathbf{F})_x$ is also zero.

We could, of course, have said from the beginning that the curl of the gravitational force field is zero, for we know that such a force field is conservative. Further, Eq. (93) shows that the curl of a vector that is not a function of x, y, or z is zero.

As a second illustration, let us consider the velocity of fluid in a fluid stream in which the velocity $\mathbf{v}$ is proportional to the distance from the bottom of the stream. We choose the z-axis parallel to the direction of flow, and the x-axis perpendicular to the stream bottom, as in Fig. 1.24. Then

$$v_x = 0, v_y = 0, v_z = cx \qquad ...(98)$$

Now, the value of the curl may be evaluated from Eq. (98). For $(\bar{\nabla} \times \mathbf{V})_x$, we choose a path parallel to the yz-plane and in the line integral $\int \mathbf{V} \cdot \mathbf{dl}$, around such a path it is seen that the contributions are equal and opposite on the parts parallel to the z-axis, hence

$$(\bar{\nabla} \times \mathbf{v})_x = 0$$

Likewise, $$(\bar{\nabla} \times \mathbf{v})_z = 0.$$

Now, in order to calculate the y component we choose a path parallel to the xz-plane and evaluate the line integral $\int \mathbf{A} \cdot \mathbf{dl}$ around it in the sense that would advance a right-hand screw in the positive y-direction. On the parts of the path parallel to the x-axis, $\mathbf{v} \cdot \mathbf{dl} = 0$, since $\mathbf{v}$ and $\mathbf{dl}$ are perpendicular. On the bottom part of the path, at a distance x from the yz-plane,

$$\int \mathbf{V} \cdot dl = cx\, \Delta z \qquad ...(99)$$

whereas at $(x + \Delta x)$

$$\int \mathbf{V} \cdot \mathbf{dl} = -c\,(x + \Delta x)\, \Delta z \qquad ...(100)$$

Therefore, for the whole path,

$$\int \mathbf{V} \cdot \mathbf{dl} = -c \Delta x\, \Delta z \qquad ...(101)$$

and the y-component of the curl is

$$(\bar{\nabla} \times \mathbf{v})_y = \lim_{S \to 0} \frac{\int \mathbf{v} \cdot \mathbf{dl}}{S} = -\frac{c\,\Delta x\,\Delta z}{\Delta z\,\Delta z} = -c \qquad \text{...(102)}$$

Calculating $(\bar{\nabla} \times \mathbf{v})$ in a formal manner,

$$\bar{\nabla} \times \mathbf{v} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & cx \end{vmatrix} = -c\,\mathbf{a_y} \qquad \text{...(103)}$$

which is the same result as above.

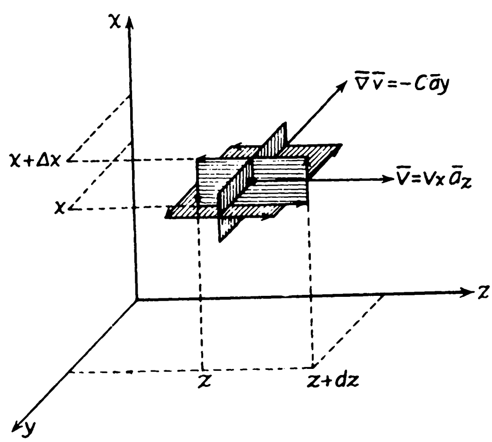


Fig. 1.23

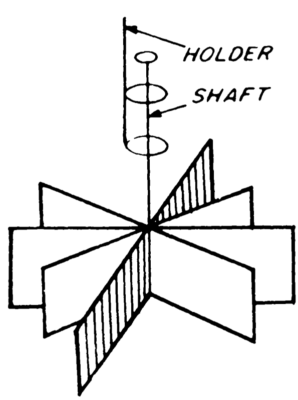


Fig. 1.24. Paddle wheel for measuring curl.

A physical interpretation of the curl of **V** can be obtained with the aid of the curl-meter, or paddle-wheel device of Fig. 1.24. If this device is inserted with its shaft vertical to the trough (Fig. 1.25 *a*)

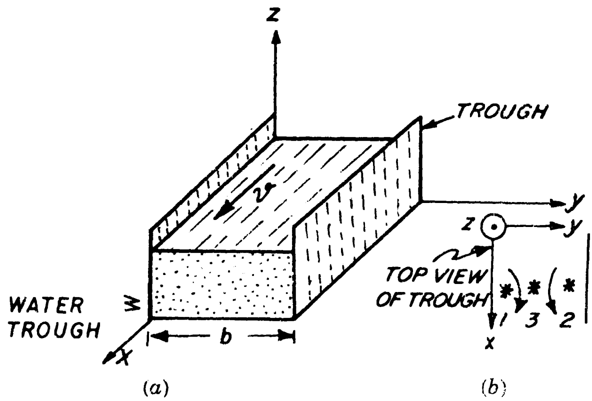


Fig. 1.25

with the assumed sinusoidal variation for the velocity of the water, it spins clockwise when it is on the left of the centre of the trough (position '1' in Fig. 1.25 *b* and counterclockwise when it is on the right of the centre of the trough (position '2' in Fig. 1.25 *b*, corresponding to negative and positive values of curl. At the centre of the trough (position '3' in Fig. 1.25 *b*), the curl meter does not rotate ; since the forces on the paddles are balanced. This corresponds to the curl of **v** being zero. The rate of rotation of the paddle wheel shaft is proportional to the curl of **V** at the point where it is inserted. Thus, it rotates fastest near the edges of the trough. At any point the rate of rotation is also maximum with the shaft vertical (rather than inclined to the vertical), indicating that $\overline{\nabla} \times \mathbf{v}$ is in the z-direction. It is assumed that the paddle wheel is small enough to avoid the effects due to the flow appreciably and to indicate closely the conditions at a point.

If the curl meter with vertical shaft is inserted in water with uniform velocity, it will not rotate and consequently curl **v** equals zero.

1.6. STOKE'S THEOREM

For the small path of integral, we have

$$\int \mathbf{A} \cdot d\mathbf{l} = (\overline{\nabla} \times \mathbf{A}) \cdot d\mathbf{s} \qquad ...(104)$$

Eq. (104) is true only for a path so small that $\overline{\nabla} \times \mathbf{A}$ can be considered constant over the surface $d\mathbf{s}$ bounded by the path. In case, the path is large enough such that this condition is not met, the equation will be extended to arbitrary paths by dividing the surface into infinitesimal areas $d\mathbf{s}_1, d\mathbf{s}_2$, etc. as shown in Fig. 1.26, so that for any one of these small areas, we have

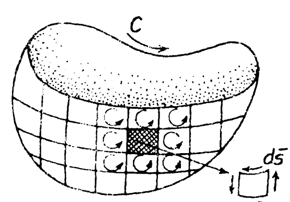


Fig. 1.26. Proof of stoke's theorem for a general vector field.

$$\int \mathbf{A} \cdot d\mathbf{l}_i = (\overline{\nabla} \times \mathbf{A}) \cdot d\mathbf{s}_i \qquad ...(105)$$

We add the left hand sides of these equations for all the $d\mathbf{s}_i$ and then we add all the right hand sides. The sum of the left hand sides is the line integral around the external boundary, since there are always two equal and opposite contributions to the sum along every common side between adjacent $d\mathbf{s}_i$. The sum of the right sides is merely the integral of $(\overline{\nabla} \times \mathbf{A}) \cdot d\mathbf{s}$ over the finite surface. Thus, for an arbitrary path

$$\int \mathbf{A} \cdot d\mathbf{l} = \int_S (\overline{\nabla} \times \mathbf{A}) \cdot d\mathbf{s} \qquad ...(106)$$

This is *stoke's theorem. It relates a line integral to a surface integral over any surface of which the line integral path is a boundary.*

Now, the conditions for a vector-field to become conservative can be obtained through the use of stoke's theorem, which suggests that the line integral of $\mathbf{A} \cdot \mathbf{dl}$ around an arbitrary closed paths is zero provided $\overline{\nabla} \times \mathbf{A} = 0$ everywhere. This condition makes use of the substitution

$$\mathbf{A} = \overline{\nabla} f \qquad ...(107)$$

for then,

$$A_x = \frac{\partial f}{\partial x}, \quad A_y = \frac{\partial f}{\partial x} \text{ and } A_z = \frac{\partial f}{\partial z} \qquad ...(108)$$

and

$$(\overline{\nabla} \times \mathbf{A})_y = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial y}$$

$$= \frac{\partial^2 f}{\partial y\, \partial z} - \frac{\partial^2 f}{\partial z \cdot \partial y} = 0 \qquad ...(109)$$

and so on for the other components of the curl. Then $\overline{\nabla} \times \mathbf{A} = 0$, and the function $\mathbf{A}$ is conservative.

The above discussion shows that the vector field $\mathbf{A}$ is conservative if $\mathbf{A}$ can be expressed as the gradient of a scalar point function f, since the curl of a gradient is always zero,

$$\overline{\nabla} \times (\overline{\nabla} f) = 0 \qquad ...(110)$$

We shall now show that the divergence of the curl is always zero, from stoke's theorem (106)

$$\int \mathbf{A} \cdot d\mathbf{l} = \int_S (\overline{\nabla} \times \mathbf{A})\, d\mathbf{s} \qquad ...(111)$$

in which $\mathbf{A}$ is any vector point-function. Remembering that stoke's theorem holds for any surface bounded by the path of integration of the line integral, we consider two different surfaces bounded by the same path. These two surface enclose a region of space, and since they have a common boundary.

$$\int_{S_1} (\overline{\nabla} \times \mathbf{A})\, d\mathbf{s}_1 = \int_{S_2} (\overline{\nabla} \times \mathbf{A}) \cdot d\mathbf{s}_2 \qquad ...(112)$$

Here $d\mathbf{s}_1$ and $d\mathbf{s}_2$ represent vectors in directions determined by the sense in which the line integral is evaluated. If we reverse the direction of the vector $d\mathbf{s}_2$ and call it $d\mathbf{s}_2' = -d\mathbf{s}_2$, where both $d\mathbf{s}_1$ and $d\mathbf{s}_2$ point either inward or outward from the volume enclosed by the two surfaces, then

$$\int_{S_1} (\overline{\nabla} \times \mathbf{A})\, d\mathbf{s}_1 + \int_{S_2'} (\overline{\nabla} \times \mathbf{A})\, d\mathbf{s}_2' = 0 \qquad ...(113)$$

This expression gives the total flux of $(\overline{\nabla} \times \mathbf{A})$, either inward or outward, for the volume enclosed by the two surfaces. Then, from the divergence theorem

$$\int_V \overline{\nabla} \cdot (\overline{\nabla} \times \mathbf{A})\, dV = 0 \qquad ...(114)$$

Since this expression must be true for any two surfaces bounded by any arbitrary closed path in the field, it follows that, everywhere,

$$\overline{\nabla} \cdot (\overline{\nabla} \times \mathbf{A}) = 0 \qquad ...(115)$$

Eq. (115) can be verified by calculating the divergence of curl of vector **A** with its components expressed in rectangular coordinates.

1.7. GREEN'S THEOREM

This theorem is a corollary of the divergence theorem and may be derived as follows :

Let $\mathbf{A} = \psi\, \overline{\nabla}\, \phi$

where ψ and ϕ are scalar functions continuous together with their partial derivatives of first and second orders. Then,

$$\overline{\nabla} \cdot \mathbf{A} = \overline{\nabla} \cdot (\psi\, \overline{\nabla}\, \phi)$$
$$= \psi\, \nabla^2\, \phi + \overline{\nabla}\, \psi \cdot \overline{\nabla}\, \phi$$

Following divergence theorem

$$\iiint_V (\overline{\nabla} \cdot \mathbf{A})\, dv = \iint_S \mathbf{A}\, ds, \text{ we have}$$

$$\iiint_V (\psi\, \nabla^2\, \psi + \overline{\nabla}\, \psi \cdot \overline{\nabla}\, \phi)\, dv = \iint_S \mathbf{A} \cdot d\mathbf{s}$$

$$= \iint_S \psi\, \nabla \cdot \phi\, d\mathbf{s} \qquad ...(116)$$

This is referred to as the *first form of Green's theorem.*

Moreover, if the functions ψ and ϕ are interchanged, this will lead to have

$$\iiint_V (\phi \nabla^2 \psi + \overline{\nabla}\, \phi \cdot \overline{\nabla}\, \psi)\, dv = \iint_S \phi \overline{\nabla}\, \psi \cdot d\mathbf{s}) \qquad ...(117)$$

Subtracting (2) from (1),

$$\iiint_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dv = \iint_S (\psi \overline{\nabla} \phi - \phi \overline{\nabla} \psi) \cdot d\mathbf{s} \qquad ...(118)$$

This is referred to as the *second form of Green's theorem.* This theorem is extremely useful in the solution of boundary value problems in electromagnetic field theory.

1.8. HELMHOLTZ'S THEOREM

Helmholtzs' Theorem is also known as the *fundamental theorem of vector analysis*. According to this theorem any vector field is considered, in general, as a sum of solenoidal (fields having zero divergence everywhere) and irrotational (field having zero curl everywhere) fields. In other words a vector field is completely specified by its divergence and curl. The divergence and curl constitute the source and the vortex source respectively of the field.

1.8A. Solenoidal fields

Any vector field which has zero divergence everywhere is called a *solenoidal field.* In such a field, all the flow lines are continuous and close upon themselves (*e.g.* magnetic field).

Thus, if **F** be a solenoidal field, then

$$\overline{\nabla} \cdot \mathbf{F} = 0$$

But $\overline{\nabla} \times \mathbf{F} \neq 0$ (in general). Let this non-zero curl be given by

$$\overline{\nabla} \times \mathbf{F} = \mathbf{J}_{(x,\, y,\, z)}$$

where vector function J constitute the vortex source for the field **F**.

Also, $\quad \text{div}\,(\text{curl}\, \mathbf{A}) = 0$

Hence $\quad \overline{\nabla} \cdot \mathbf{F} = 0$

will be satisfied by assuming

$$\mathbf{F} = \overline{\nabla} \times \mathbf{A}$$

$$\overline{\nabla} \times \mathbf{F} = \overline{\nabla} \times (\overline{\nabla} \times \mathbf{A})$$

$$= \overline{\nabla}\, (\overline{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mathbf{J} \qquad ...(119)$$

Now only the curl of **A** has been specified while its divergence may be choosen as desired.

Thus choosing **A** such that

$$\overline{\nabla} \cdot \mathbf{A} = 0$$

Eq. (119) reduces to

$$\nabla^2 \mathbf{A} = -\mathbf{J} \qquad ...(120)$$

Equation (120) gives the *vector Poisson's equation* and each component of **A** satisfy the scalar Poisson's equation given by

$$\left.\begin{aligned} \nabla^2 A_x &= -J_x \\ \nabla^2 A_y &= -J_y \\ \nabla^2 A_z &= -J_z \end{aligned}\right] \qquad \text{...(121)}$$

Thus knowing **A**, either by (120) or by (121), vector field **F** may be obtained from

$$\boxed{\mathbf{F} = \overline{\nabla} \times \mathbf{A}} \qquad \text{...(122)}$$

1.8B. Irrotational Field

A vector field **F** is said to be irrotational provided

$$\boxed{\overline{\nabla} \times \mathbf{F} = 0} \qquad \text{...(123)}$$

This does not imply (in general), that $\overline{\nabla} \cdot \mathbf{F} = 0$, otherwise the field would vanish everywhere.

Since divergence of a vector is scalar quantity hence it may be assumed that

$$\overline{\nabla} \cdot \mathbf{F} = \rho\,(x, y, z) \qquad \text{...(124)}$$

where ρ constitute a source function for the field **F**. Using the identity

$$\text{curl (grad } \phi) = \overline{\nabla} \times (\overline{\nabla} \cdot \phi) = 0 \qquad \text{...(125)}$$

It may be conveniently assumed that

$$\mathbf{F} = -\overline{\nabla}\,\phi \qquad \text{...(126)}$$

The significance of choosing minus sign will be made clear in the next chapter. In fact one may assume plus sign as well. Thus,

$$\overline{\nabla} \cdot \mathbf{F} = \overline{\nabla} \cdot (\overline{\nabla}\,\phi) = \rho$$

$$\therefore \qquad \boxed{\nabla^2 \phi = -\rho} \qquad \text{...(127)}$$

This equation is known as *Poisson's equation* and *function* ϕ *is called scalar potential.* ϕ is determined by solving the differential Eq. (1) and hence field **F** may be computed from

$$\mathbf{F} = -\overline{\nabla}\phi \qquad \text{...(128)}$$

1.8C. General vector field

Since any vector field may be resolved into a solenoidal and irrotational fields, hence this vector field may be derived by superposition of the two types of fields, that is, the solenoidal field obtained from

$$\mathbf{F} = \overline{\nabla} \times \mathbf{A}$$

where $\nabla^2 \mathbf{A} = \mathbf{J}$

and irrotational field obtained from

$$\mathbf{F} = -\overline{\nabla}\,\phi$$

where $\nabla^2 \phi = -\rho$

Example 1.25. *Verify whether the vector field*

$$\mathbf{F} = y^2 z\mathbf{a}_x + z^2 x\mathbf{a}_y + x^2 y\mathbf{a}_z$$

is irrotational, solenoidal or general.

Solution. For an irrotational field.

$$\overline{\nabla} \times \mathbf{F} = 0$$

$$\text{L.H.S.} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= \mathbf{a}_x\,[x^2 - 2zx] - \mathbf{a}_y\,[y^2 - 2xy] + \mathbf{a}_z\,[z^2 - 2yz] \neq 0$$

Hence **F** will not be an irrotational field.

Now, in order that **F** is a solenoidal field

$$\overline{\nabla} \cdot \mathbf{F} = 0$$

$$= \left[\frac{\partial}{\partial x}(y^2 z) + \frac{\partial}{\partial y}(z^2 x) + \frac{\partial}{\partial z}(x^2 y)\right] = 0$$

since the divergence of the field does not exist hence the field is solenoidal.

Example 1.26. *Find* $\nabla^2\mathbf{E}$, *where* $\mathbf{E} = (x + y)\,\mathbf{a}_x + (x^2 + y^2)\mathbf{a}_y$.

Solution. Using the value of ∇^2 in the cartesian coordinates we have,

$$\nabla^2\mathbf{E} = \frac{\partial^2}{\partial x^2}\mathbf{E} + \frac{\partial^2}{\partial y^2}\mathbf{E} + \frac{\partial}{\partial z^2}\mathbf{E}$$

$$= \frac{\partial^2}{\partial x^2}[\mathbf{a}_x\,\mathbf{E}_x + \mathbf{a}_y\mathbf{E}_y + \mathbf{a}_z\mathbf{E}_z] + \frac{\partial^2}{\partial y^2}[\mathbf{a}_x\mathbf{E}_x + \mathbf{a}_y\mathbf{E}_y + \mathbf{a}_z\mathbf{E}_z]$$

$$+ \frac{\partial^2}{\partial z^2}[\mathbf{a}_x\mathbf{E}_x + \mathbf{a}_y\mathbf{E}_y + \mathbf{a}_z\mathbf{E}_z]$$

thus, $$\nabla^2\mathbf{E} = \mathbf{a}_x\left(\frac{\partial^2\mathbf{E}_x}{\partial x^2} + \frac{\partial^2\mathbf{E}_x}{\partial y^2} + \frac{\partial^2\mathbf{E}_x}{\partial z^2}\right) + \mathbf{a}_y\left(\frac{\partial^2\mathbf{E}_y}{\partial x^2} + \frac{\partial^2\mathbf{E}_y}{\partial y^2} + \frac{\partial^2\mathbf{E}_y}{\partial z^2}\right)$$

$$+ \mathbf{a}_z\left(\frac{\partial^2\mathbf{E}_z}{\partial x^2} + \frac{\partial^2\mathbf{E}_z}{\partial z^2} + \frac{\partial^2\mathbf{E}_z}{\partial z^2}\right) \quad ...(i)$$

But $\quad \mathbf{E} = (x + y)\,\mathbf{a}_x + (x^2 + y^2)\,\mathbf{a}_y,$

so that $\quad E_x = x + y,\ \ Ey = x^2 + y^2,\ E_z = 0$

Then, $\quad \dfrac{\partial E_x}{\partial x} = 1;\quad \dfrac{\partial^2 E_x}{\partial x^2} = 0$

$$\frac{\partial E_x}{\partial y} = 1; \quad \frac{\partial^2 E_x}{\partial y^2} = 0$$

$$\frac{\partial E_x}{\partial z} = 0\,; \quad \frac{\partial^2 E_r}{\partial z^2} = 0$$

and
$$\frac{\partial E_y}{\partial x} = 2x\,; \quad \frac{\partial^2 E_y}{\partial x^2} = 2$$

$$\frac{\partial E_y}{\partial y} = 2y\,; \quad \frac{\partial^2 E_y}{\partial y^2} = 0$$

$$\frac{\partial E_y}{\partial z} = 0\,; \quad \frac{\partial^2 E_y}{\partial z^2} = 0$$

and also since $E_z = 0$, thus

$$\frac{\partial E_z}{\partial x} = \frac{\partial E_z}{\partial y} = \frac{\partial E_z}{\partial z} = \frac{\partial^2 E_2}{\partial x^2} = \frac{\partial^2 E_z}{\partial y^2} = \frac{\partial^2 E_z}{\partial z^2} = 0.$$

Substituting these results in Eq. (*i*), we have

$$\nabla \mathbf{E} = (2 + 2)\, \mathbf{a}_y = 4\, \mathbf{a}_y.$$

1.9. COORDINATE SYSTEMS AND VECTOR RELATION

In order to describe a vector accurately, some specific lengths, directions, angles, projections or components must be given. There are three simple methods of doing this, but about eight or ten other methods are also available which are used in very special cases. The three useful methods are cartesian (rectangular), cylindrical (polar) and spherical coordinate systems. We will first discuss these three systems of coordinates and then the general curvilinear coordinate system will be described in order to have the important vector relations in the three coordinate systems.

1.9A. The Rectangular (Cartesian) Coordinate System

(*a*) In the cartesian coordinate system the three coordinate axes namely x, y and z are mutually at right angles to each other.

(*b*) In order that the cartesian system is right-handed coordinate system, the three axes are so chosen that a rotation (through the smaller angle) of the x-axis towards the y-axis would cause a right handed screw to progress in the direction of the z-axis. Using the right hand, the thumb, forefinger and middle finger may then be identified respectively as the x, y and z axes.

$$\boxed{\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z} \qquad ...(129)$$

(c) It is well known that the intersection of two surfaces is a line ; the intersection of three surfaces is a point. Thus the coordinates of a point may be given by stating three parameters, each of which defines a coordinate surface.

Let in the rectangular coordinate system, there be three planes

$$x = x_1 \; ; y = y_1 \; ; z = z_1$$

These three planes intersect at a point which is designated by the coordinates (x_1, y_1, z_1) [Fig. 1.27].

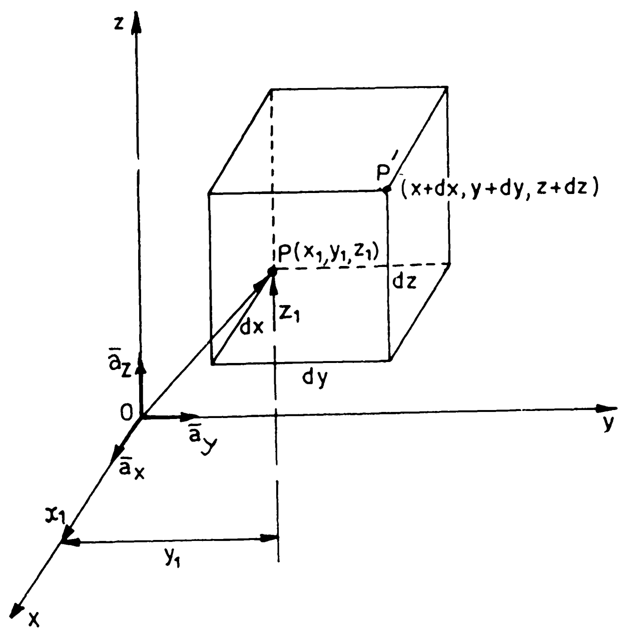


Fig. 1.27. The cartesian co-ordinate system.

Another method of locating a point is by giving its x, y and z coordinates. These are, respectively, the distances from the origin to the intersection of a perpendicular dropped from the point to the x, y and z axes.

(d) If P has coordinates x, y and z and $P', x + dx,\; y + dy,$ and $z + dz$. These six planes define a rectangular parallelopiped whose volume is $dv = dx\, dy\, dz$; the surface have differential areas ds of $dx\, dy$, $dydz$, and $dzdx$. The distance dL from P to P' is the diagonal of the parallelopiped and has a length of [Fig. 1.27].

$$dL = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \qquad \text{...(130)}$$

(*e*) The unit vectors are $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ along the *x*, *y* and *z* axes respectively. The position vector **OP** or $\mathbf{r}_p$ is given by $\overrightarrow{OP} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

$$= |\overrightarrow{OP}|\ \mathbf{a}_r$$

where $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$ and $\mathbf{a}_r$ is a unit vector along the line joining the origin towards point *P*.

1.9B. The Circular, Cylinderical (Polar) Coordinate System

The circular cylindrical coordinate system is the three-dimensional version of the polar coordinates of analytic geometry. In the two-dimensional polar coordinates, a point is located in a plane by assigning its distance *r* from the origin, and the angle ϕ between the line from the point to the origin and an arbitrary radial line taken as $\phi = 0$. A three-dimensional coordinate system, circular cylindrical coordinates is obtained by also specifying the distance *z* of the point from an arbitrary $z = 0$ reference plane which is perpendicular to the line $r = 0$.

For simplicity, the circular cylindrical coordinate system is usually referred as cylindrical coordinate system. The other similar systems are elliptic cylindrical coordinates, hyperbolic cylindrical coordinates, parabolic cylindrical coordinates and others.

In the case of a circular cylindrical coordinates we have the following important features :

(*a*) The three mutually perpendicular coordinate axes are taken as *r* ϕ and *z*. The *r*, ϕ and *z* coordinates are known as the radius, the azimuthal angle, and the distance along the axes respectively (1.28*a*).

(*b*) The unit vectors along the positive direction of *r*, ϕ and *z* are $\mathbf{a}_r$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$ respectively.

The unit vector $\mathbf{a}_r$ at any point (r_1, ϕ_1, z) is directed radially outward, normal to the cylindrical surface $r = r_1$. It lies in the planes $P = \phi_1$, and $z = z_1$. The unit vector $\mathbf{a}_\phi$, is normal to the plane $\phi = \phi_1$, points in the direction of increasing ϕ, lies in the plane $z = z_1$ and is tangent to the cylindrical surface $r = r_1$. The unit vector $\mathbf{a}_z$ is normal to the plane *z*-constant and points in the direction of increasing *z*.

The unit vectors $\mathbf{a}_r$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$ are mutually perpendicular and are so chosen as to give a right handed coordinate system. Thus

$$\boxed{\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z} \qquad ...(131)$$

shows that the forefinger, the middle finger and the thumb of the right hand should represent respectively the unit vectors $\mathbf{a}_r$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$. This is illustrated in Fig. 1.28*b*.

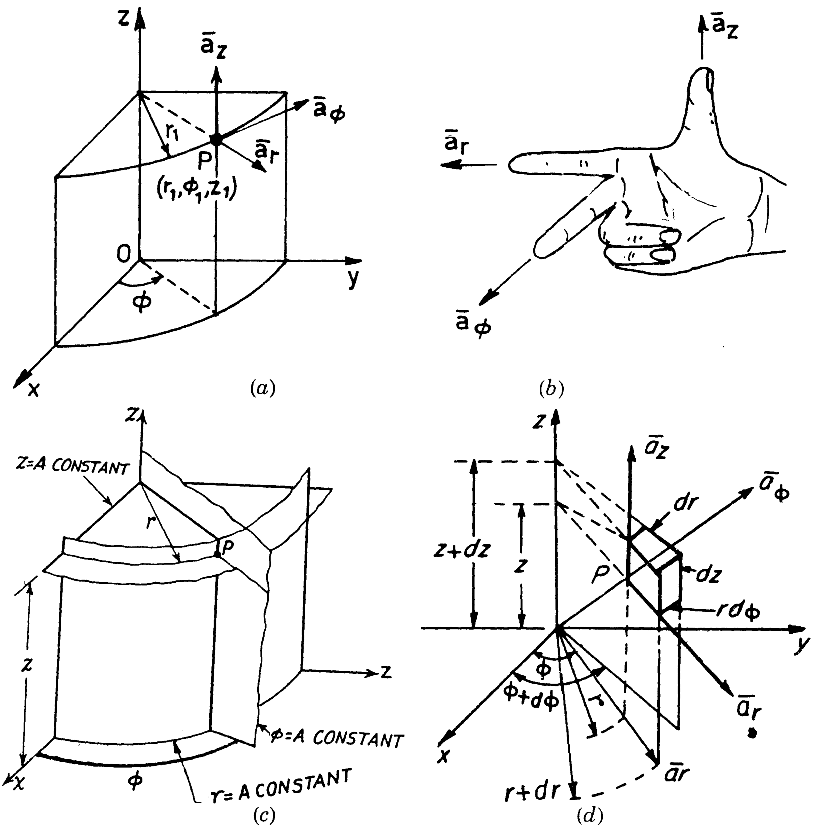


Fig. 1.28. A right handed circular cylindrical coordinate system.

(*c*) In circular cylindrical coordinate system the coordinate surfaces corresponding to

(*i*) a set of circular cylinders ; r = constant

(*ii*) a set of planes all passing through the axis ϕ = constant, and

(*iii*) a set of planes normal to the axis z = constant are used to locate a point.

(*d*) A differential volume element in cylindrical coordinates may be obtained by increasing r, ϕ, and z by the differential increments, dr, $d\phi$ and dz. The two cylinders of radius r and $r + dr$, the two radial planes at angles ϕ and $\phi + d\phi$, and the two "horizontal planes at elevations" z and $z + dz$ now enclose a small volume shown in Figs. 1.28 *c* and 1.28 *d*, having the shape of a truncated wedge. As the volume element becomes very small, its shape approaches that of a rectangular parallelopiped having sides of length dr, $rd\phi$, and dz. It should be noted that dr and dz are dimensionally lengths, but $d\phi$ is not, however $rd\phi$ is the curved length.

The surfaces have areas of $rdrd\phi$, $drdz$, and $rd\phi\, dz$, and the volume element whose edges are the elements of length corresponding infinitesimal increments in the coordinates at the point P becomes $rdrd\,\phi\, dz$.

(*e*) Elements of length dl corresponding to infinitesimal changes in the coordinates of a point are important. If the coordinates ϕ and z of the point P in Fig. 1.28*d* are kept constant while r is allowed to increase by dr, P is displaced by an amount $d\mathbf{l} = \mathbf{a}_r\, dr$. On the other hand, if r and z are held constant while ϕ is allowed to increase by $d\phi$, then P is displaced by $d\mathbf{l} = \mathbf{a}_\phi\, rd\phi$. Finally, if r and ϕ are held constant while z is allowed to increase by dz, then $d\mathbf{l} = \mathbf{a}_z\, dz$. For arbitrary increments dr, $d\phi$ and dz,

$$d\mathbf{l} = \mathbf{a}_r d_r + \mathbf{a}_\phi r d\phi + \mathbf{a}_z\, d_z \qquad ...(132)$$

and

$$dl = [(dr)^2 + (rd\phi)^2 + (dz)^2]^{1/2}$$

1.9C. The Spherical Coordinate System

The salient features of a spherical coordinate system are :

(*a*) A spherical coordinate system is constructed on the three cartesian axes by first defining the distance from the origin to any point as 'r'. A surface r = constant represents a sphere in spherical coordinates. The second coordinate is the angle θ between the z-axis and the line drawn from the origin to the point P.

The surface corresponding to θ = constant, is a cone, and the two surfaces, cones and sphere, are everywhere perpendicular. The intersection of the cone and sphere is a circle of radius $r \sin\theta$. The coordinate θ corresponds to latitude except that latitude is measured from the equator and θ is measured from the "North Pole".

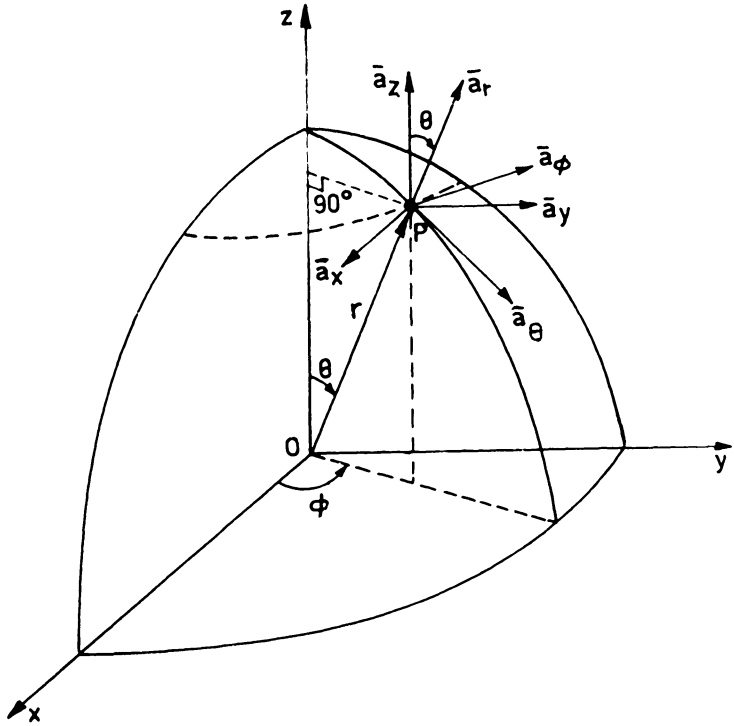
The third coordinate ϕ is also an angle and in this case is exactly the same as the angle ϕ of cylindrical coordinates. It is the angle between the x-axis and the projection in the $z = 0$ plane of the line drawn from the origin to the point P. It corresponds to the angle of longitude but the angle ϕ increases to the "east". The surface ϕ = constant is a plane passing through the $\theta = 0$ line (or the z-axis).

(*b*) The point of intersection of the three mutually perpendicular surfaces namely,

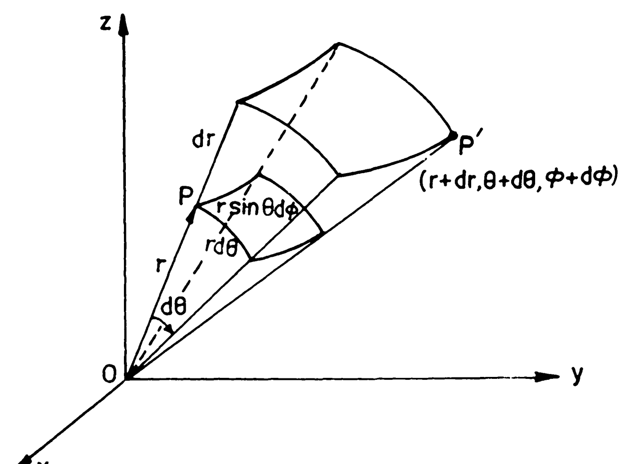
(*i*) a set of spheres (radius r from the origin = constant),

(*ii*) a set of cones about the axis (θ = constant), and

(*iii*) a set of planes passing through the polar axis (ϕ = constant) is used to define the coordinates of a point.



(a)



(b)

Fig. 1.29. The spherical co-ordinate system.

For example, the intersection of sphere $r = r_1$, cone $\theta = \theta_1$, and plane $\phi = \phi_1$, gives a point whose coordinates are said to be

(r_1, θ_1, ϕ_1), where r_1 is the radius, θ is the polar angle or colatitude, and ϕ is the azimuth angle or longitude (Fig. 1.29*a*).

(*c*) The three unit vectors may be defined such that each unit vector is perpendicular to one of the three mutually perpendicular surfaces and oriented in that direction in which the coordinate increases.

The unit vector $\mathbf{a}_r$ is directed radially outward, normal to the sphere r = constant, and lies in the cone θ = constant and the plane ϕ = constant. The unit vector $\mathbf{a}_\theta$ is normal to the conical surface, lies in the plane, and is tangent to the sphere. It is directed along a line of "longitude" and points "south". The third vector $\mathbf{a}_\phi$ is the same as in cylindrical coordinates, being normal to the plane and tangent to both the cone and sphere. It is directed to the "east".

The three unit vectors are mutually perpendicular and a right-handed coordinate system is defined by choosing the directions for $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ such that

$$\boxed{\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi} \qquad ...(133)$$

our system is right-handed, as an inspection of Fig. 1.29*a* will show, on application of the definition of the cross product. The right-hand rule serves to identify the thumb, forefinger, and middle finger with the directions of increasing z, θ and ϕ, respectively.

(*d*) A differential volume element may be constructed in spherical coordinates by increasing r, θ and ϕ, by dr, $d\theta$, and $d\phi$, as shown in Fig. 1.29 *b*. The distance between the two spherical surfaces of radii r and $r + dr$ is dr ; the distance between the two cones having generating angles of θ and $\theta + d\theta$ is $rd\theta$; and the distance between the two radial planes at angles ϕ and $\phi + d\phi$ is found to be $r \sin\theta \, d\phi$. The surfaces have areas to $rdrd\theta$, $r \sin\theta \, drd\phi$, and $r^2 \sin\theta d\theta d\phi$, and the volume is $r^2 \sin\theta dr \, d\theta \, d\phi$.

(*e*) The distance element $d\mathbf{l}$ corresponding to arbitrary increments of the coordinates is

$$d\mathbf{l} = \mathbf{a}_r \, dr + \mathbf{a}_\theta r d\theta + \mathbf{a}_\phi r \sin\theta d\phi \qquad ...(134)$$

and

$$dl = [(d_r)^2 + (rd\theta)^2 + r^2 \sin^2\theta \, (d\phi)^2]^{1/2}$$

1.9D. Transformation Between Coordinate Systems

In the preceeding sections, we have discussed the three important and most commonly used coordinate systems. Either of these can be used to express a vector quantity but in certain more difficult problems it is some times felt necessary to tackle the problem as a first step in cartesian coordinates and then to have the answer in the desired cylindrical or spherical coordinates. Sometimes, it has been found that a vector if expressed in cartesian coordinate system will provide sufficient simplification in the solution of the electric and magnetic field problems. Further, in some other problems, the transformation between coordinate systems in the reverse direction is wanted.

However, there may be cases in which it may be advantageous to use a mixed coordinate system and hence the transformation between coordinate systems becomes an essential aspect.

(a) **Transformation between Certain and Cylindrical Systems**

The transformation of a vector in cartesian coordinates into one in cylindrical coordinates is fundamentally a two step problem involving a change in variables as well as a change in components.

Let us consider a vector **A** in cartesian coordinate system given by :

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z \qquad ...(135)$$

where, A_x, A_y and A_z are functions of x, y and z. This vector **A** is desired to be expressed in cylindrical coordinates. Let **A** in cylindrical coordinates be :

$$\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z \qquad ...(136)$$

where A, A_ϕ and A_z are the functions of r, ϕ and z.

In order to carry out the transformation ; we will proceed in two steps as explained below :

Ist Step : Change in the Variables. If we set up the two coordinate systems as shown in Fig. 1.30, so that the $z = 0$ planes coincide, and the $y = 0$ plane is the $\phi = 0$ plane, then the following relationships exist between the variables, at a general point P (Fig. 1.30).

$$\left.\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \\ z &= z \end{aligned}\right\} \qquad ...(137)$$

From the above three equations, we get

$$\left.\begin{aligned} r^2 &= x^2 + y^2 \\ \tan \phi &= \frac{y}{x} \\ z &= z \end{aligned}\right\} \qquad ...(138)$$

The first set of equations [Eq. (137)], gives x, y and z in terms of r, ϕ and z ; and the second set of equations [Eq. (138)] gives r, ϕ and z in terms of x, y and z. It is seen that no change is involved in the variable z, and that the unit vector $\mathbf{a}_z$ is same in each system.

IInd Step : Change in the Components. In general, we have the three cartesian components A_x, A_y and A_z of the vector **A** [Eq. (135)] and desire to get the three cylindrical components A_r, A_ϕ and A_z. Immediately, we realize that the A_z components are same, since the unit vectors have the same significance in each case.

To find the A_r components, we know that a component in a desired direction may be obtained by taking the dot product of the vector and a unit vector in the desired direction.

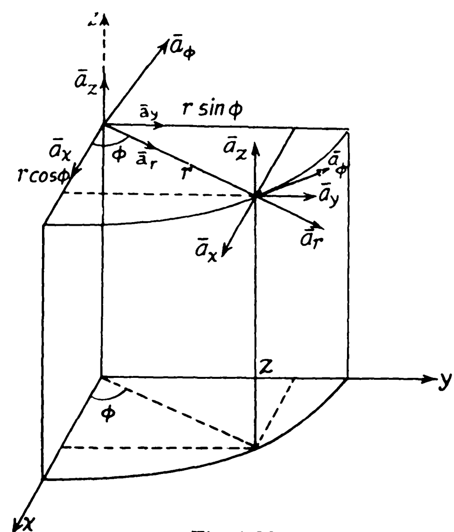


Fig. 1.30

Hence, $$A_r = \mathbf{A} \cdot \mathbf{a}_r \quad \text{...(139)}$$

$$A_\phi = \mathbf{A} \cdot \mathbf{a}_\phi \quad \text{...(140)}$$

Expanding these dot products, we have,

$$A_r = (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) \cdot (\mathbf{a}_r)$$

$$= A_x (\mathbf{a}_x \cdot \mathbf{a}_r) + A_y (\mathbf{a}_y \cdot \mathbf{a}_r) + A_z (\mathbf{a}_z \cdot \mathbf{a}_r)$$

$$= A_x (\mathbf{a}_x \cdot \mathbf{a}_r) + A_y (\mathbf{a}_y \cdot \mathbf{a}_r) \quad \text{...(141)}$$

and $$A_\phi = \mathbf{A} \cdot \mathbf{a}_\phi$$

$$= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) \cdot \mathbf{a}_\phi$$

$$= A_x (\mathbf{a}_x \cdot \mathbf{a}_\phi) + A_y (\mathbf{a}_y \cdot \mathbf{a}_\phi) + A_z (\mathbf{a}_z \cdot \mathbf{a}_\phi)$$

$$= A_x (\mathbf{a}_x \cdot \mathbf{a}_\phi) + A_y (\mathbf{a}_y \cdot \mathbf{a}_\phi) \quad \text{...(142)}$$

since $\mathbf{a}_z \cdot \mathbf{a}_r$ and $\mathbf{a}_x \cdot \mathbf{a}_\phi$ are zero.

Referring to Fig. 1.30, we have

$$\mathbf{a}_x \cdot \mathbf{a}_r = 1.1 \cos\phi = \cos\phi \quad \text{...(143)}$$

$$\mathbf{a}_y \cdot \mathbf{a}_r = \sin \phi \qquad ...(144)$$

$$\mathbf{a}_x \cdot \mathbf{a}_\phi = 1.1 \cos [\angle(\mathbf{a}_x, \mathbf{a}_r) + \angle(\mathbf{a}_r, \mathbf{a}_\phi)]$$

$$= \cos (\phi + 90°) = -\sin \phi \qquad ...(145)$$

and $$\mathbf{a}_y \cdot \mathbf{a}_\phi = 1.1 \cos [90 - (90 - \phi)] = \cos \phi \qquad ...(146)$$

Introducing (143) and (144) in (141) and, (145) and (146) in (142) we have

$$A_r = (A_x \cos \phi + A_y \sin \phi) \qquad ...(147)$$

and $$A_\phi = -A_x \sin \phi + A_y \cos \qquad ...(148)$$

The aforesaid steps can then be utilized to obtain the result. Thus substituting Eqs. (147) and (148) in Eq. (136), we have the final expression for vector **A** in cylindrical coordinate

$$\mathbf{A} = (A_x \cos \phi + A_y \sin \phi)\, \mathbf{a}_r + (-A_x \sin \phi + A_y \cos \phi)\, \mathbf{a}_\phi + A_z\, \mathbf{a}_z \qquad ...(149)$$

Example 1.27. *Transform the vector to cylindrical coordinates :*

$$\mathbf{F} = \mathbf{a}_x z + \mathbf{a}_y (1 - x) + \mathbf{a}_z \frac{y}{x}.$$

Solution. We will follow the following steps to make the transformation.

Ist Steps : Change of Variables. Substituting

$$x = r \cos \phi$$

$$y = r \sin \phi$$

and $$z = z$$

in the components F_x, F_y, F_z we have

$$F_x = z = z$$

$$F_y = (1 - x) = 1 - r \cos \phi$$

$$F_z = \frac{y}{x} \tan \phi$$

Therefore, $$\mathbf{F} = z\, \mathbf{a}_x + (1 - r \cos \phi)\, \mathbf{a}_y + \tan \phi\, \mathbf{a}_z.$$

IInd Step : The change of components. We have,

$$F_r = F_x \cos \phi + F_y \sin \phi$$

$$= z \cos \phi + (1 - r \cos \phi) \sin \phi \qquad ...(i)$$

and $$F_\phi = -F_x \sin \phi + F_y \cos \phi \qquad ...(ii)$$

$$= -z \sin \phi + 1\,(1 - r \cos \phi) \cos \phi$$

$$F_z = \tan \phi \qquad ...(iii)$$

substituting Eqs. (*i*), (*ii*) and (*iii*) in the expression

$$\mathbf{F} = F_r\,\mathbf{a}_r + F_\phi\,\mathbf{a}_\phi + F_z\,\mathbf{a}_z,$$

we have the transformed vector **F** in cylindrical coordinate systems

$$\begin{aligned}\mathbf{F} &= (F_x \cos\phi + F_y \sin\phi)\,\mathbf{a}_r + (-F_x \sin\phi + F_y \cos\phi)\,\mathbf{a}_\phi + F_z\,\mathbf{a}_z \\ &= [z\cos\phi + (1 - r\cos\phi)\sin\phi)]\,\mathbf{a}_r + [-z\sin\phi \\ &\qquad + (1 - r\cos\phi)\cos\phi]\,\mathbf{a}_\phi + \tan\phi\,\mathbf{a}_z\end{aligned}$$

Basically the transformation between coordinate systems is simple and is accomplished by following the steps : first change variable and then change components. The informations listed in Table 1.1 will make the problem easier.

Table 1.1

Cartesian to Cylindrical	*Cylindrical to Cartesian*
$x = r\cos\phi$	$r = \sqrt{x^2 + y^2}$
$y = r\sin\phi$	$\phi = \tan^{-1}\left(\frac{y}{x}\right)$
z	z
$A_r = A_x\cos\phi + A_y\sin\phi$	$A_x = A_r\frac{x}{\sqrt{x^2+y^2}} - A_\phi\frac{y}{\sqrt{x^2+y^2}}$
$A_\phi = -A_x\sin\phi + A_y\cos\phi$	$A_y = A_r\frac{y}{\sqrt{x^2+y^2}} + A_\phi\frac{x}{\sqrt{x^2+y^2}}$
$A_z = A_z$	$A_z = A_z$
$\mathbf{A} = A_x\,\mathbf{a}_r + A_\phi\,\mathbf{a}_\phi + A_z\,\mathbf{a}_z$	$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\mathbf{a}_y + A_z\,\mathbf{a}_z$

Transformation from cylindrical to cartesian form

$$\mathbf{A} = A_r\,\mathbf{a}_r + A_\phi\,\mathbf{a}_\phi + A_z\,\mathbf{a}_z$$

in cylindrical form and the desired form is

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z.$$

Following dot product

$$\begin{aligned}A_x &= \mathbf{A}\cdot\mathbf{a}_x = (A_r\,\mathbf{a}_r + A_\phi\,\mathbf{a}_\phi + A_z\,\mathbf{a}_z)\cdot\mathbf{a}_x \\ &= A_r\,\mathbf{a}_r\cdot\mathbf{a}_x + A_\phi\,\mathbf{a}_\phi\cdot\mathbf{a}_x \\ &= A_r\cos\phi + A_\phi(-\sin\phi) \\ &= \left(A_r\frac{x}{\sqrt{x^2+y^2}} - A_\phi\frac{y}{\sqrt{x^2+y^2}}\right) \qquad \text{...(150)}\end{aligned}$$

and

$$\begin{aligned}A_y &= \mathbf{A}\cdot\mathbf{a}_y = (A_r\,\mathbf{a}_r + A_\phi\,\mathbf{a}_\phi + A_z\,\mathbf{a}_z)\cdot\mathbf{a}_y \\ &= A_r\,\mathbf{a}_r\cdot\mathbf{a}_y + A_\phi\,\mathbf{a}_\phi\cdot\mathbf{a}_y + A_z\,\mathbf{a}_z\cdot\mathbf{a}_y \\ &= A_r\sin\phi + A_\phi\cos\phi \qquad \text{...(151)} \\ &= \left(A_r\frac{y}{\sqrt{x^2+y^2}} + A_\phi\frac{x}{\sqrt{x^2+y^2}}\right)\end{aligned}$$

and $$A_z = A_z \qquad ...(152)$$

substituting Eqs. (150), (151) and (152) for A_x, A_y and A_z respectively in Eq. (136), we have the desired result that is,

$$\mathbf{A} = \left(A_r \frac{x}{\sqrt{x^2+y^2}} - A_\phi \frac{y}{\sqrt{x^2+y^2}}\right)\mathbf{a}_x + \left(A_r \frac{y}{\sqrt{x^2+y^2}} + A_x \frac{x}{\sqrt{x^2+y^2}}\right)\mathbf{a}_y + A_z\,\mathbf{a}_z \qquad ...(153)$$

(*b*) **Transformation of a vector in cartesian coordinates to a vector in spherical coordinates and vice-versa.**

The transformation of a vector from cartesian coordinates to spherical coordinates, or the reverse, is accomplished by following the same general procedure as discussed above. The relationship between the variables are somewhat more complicated but are readily obtained with the aid of Fig. (1.32).

From the figure, it is seen that

$$\rho = r \sin\theta$$

Thus, $$\left.\begin{aligned} x &= \rho\cos\phi = r\sin\theta\cos\phi \\ y &= \rho\sin\phi = r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right] \qquad ...(154)$$

These give $$\left.\begin{aligned} r^2 &= x^2+y^2+z^2 \\ \cos\theta &= \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \tan\phi &= \frac{y}{x} \end{aligned}\right] \qquad ...(155)$$

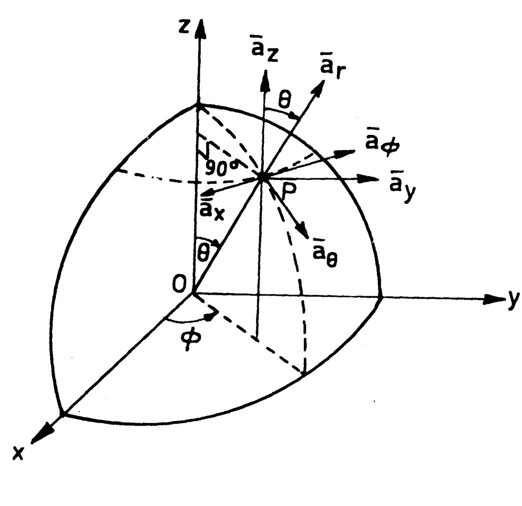


Fig. 1.31

Let the vector $\mathbf{A} = A_x\, \mathbf{a}_x + A_y\, \mathbf{a}_y + A_z\, \mathbf{a}_z$ ...(156)

is to be transformed into spherical system of coordinates, *viz*,

$$\mathbf{A} = A_r\, \mathbf{a}_r + A_\theta\, \mathbf{a}_\theta + A_\phi\, \mathbf{a}_\phi \quad ...(157)$$

Ist Step : Change of Variables. This step is accomplished by putting

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = z \cos\theta$$

in the components A_x, A_y and A_z of vector $\mathbf{A}$.

IInd Step : Change of Components. This step involves the determination of the various components, namely A_r A_θ and A_ϕ of the spherical coordinate system. This is approached as given below :

$$A_r = \mathbf{A} \cdot \mathbf{a}_r$$
$$= (A_x\, \mathbf{a}_x + A_y\, \mathbf{a}_y + A_z\, \mathbf{a}_z) \cdot \mathbf{a}_r$$
$$= A_x\, (\mathbf{a}_x \cdot \mathbf{a}_r) + A_y\, (\mathbf{a}_y \cdot \mathbf{a}_r) + A_z\, (\mathbf{a}_z \cdot \mathbf{a}_r) \quad ...(158)$$
$$A_\theta = \mathbf{A} \cdot \mathbf{a}_\theta$$
$$= A_x\, (\mathbf{a}_x \cdot \mathbf{a}_\theta) + A_y\, (\mathbf{a}_y \cdot \mathbf{a}_\theta) + \mathbf{a}_x\, (\mathbf{a}_z \cdot \mathbf{a}_\phi) \quad ...(159)$$
$$A_\phi = \mathbf{A} \cdot \mathbf{a}_\phi$$
$$= A_x\, (\mathbf{a}_x \cdot \mathbf{a}_\phi) + A_y\, (\mathbf{a}_y \cdot \mathbf{a}_\phi) + A_z\, (\mathbf{a}_z \cdot \mathbf{a}_\phi) \quad ...(160)$$

The dot product relations between the unit vectors are :

(*i*) the dot-produced with $\mathbf{a}_z$

$$\mathbf{a}_r \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_r = 1 \cdot 1 \cdot \cos\theta = \cos\theta \quad ...(161)$$
$$\mathbf{a}_\theta \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_\theta = \cos(90 + \theta) = -\sin\theta \quad ...(162)$$

Since $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$, so $\mathbf{a}_r$ and $\mathbf{a}_\theta$ will be in one plane at right angles and hence the net angle between $\mathbf{a}_z$ and $\mathbf{a}_\theta$ is $(90 + \theta)$

$$\mathbf{a}_\phi \cdot \mathbf{a}_z = \mathbf{a}_z\, \mathbf{a}_\phi = \cos 90° = 0 \quad ...(163)$$

since $\mathbf{a}_\phi$ is at right angle to $\mathbf{a}_z$.

(*ii*) **The dot-product with $\mathbf{a}_x \cdot \mathbf{a}_r \cdot \mathbf{a}_z$** is obtained by finding first, the projection of $\mathbf{a}_r$ on the *x*-*y* plane, that is $|\mathbf{a}_r| \sin\theta$, then noting the angle between $|\mathbf{a}_r| \sin\theta$ and $\mathbf{a}_x$ vector, we have

$$\mathbf{a}_r \cdot \mathbf{a}_x = \mathbf{a}_x \cdot \mathbf{a}_r = \sin\theta \cos\phi \quad ...(164)$$
$$\mathbf{a}_\theta \cdot \mathbf{a}_x = \mathbf{a}_x \cdot \mathbf{a}_\theta = \cos\theta \cos\phi \quad ...(165)$$
$$\mathbf{a}_\phi \cdot \mathbf{a}_x = \mathbf{a}_x \cdot \mathbf{a}_\phi$$

Here, $\mathbf{a}_\phi$ is in the same plane as $\mathbf{a}_x$ and $\mathbf{a}_y$ but the angle between $\mathbf{a}_\phi$ and $\mathbf{a}_x$ is $(90 + \phi)$. Therefore,

$$\mathbf{a}_\phi \cdot \mathbf{a}_x = \cos(90 + \phi) = -\sin\phi \qquad ...(166)$$

(*iii*) Similarly, it can be deduced that

$$\mathbf{a}_r \cdot \mathbf{a}_y = \sin\theta \sin\phi \qquad ...(167)$$

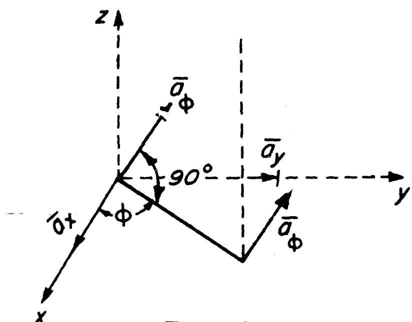


Fig. 1.32

$$\mathbf{a}_\theta \cdot \mathbf{a}_y = \cos\theta \sin\phi = \mathbf{a}_y \cdot \mathbf{a}_\theta \qquad ...(168)$$

$$\mathbf{a}_\theta \cdot \mathbf{a}_\phi = \cos[90 - (90 - \phi)]$$

$$= \cos\phi = \mathbf{a}_\phi \cdot \mathbf{a}_\phi \qquad ...(169)$$

$$\therefore \quad A_r = A_x \sin\theta \cos\phi + A_\phi \sin\phi \sin\phi + A_z \cos\phi \qquad ...(170)$$

$$A_\theta = A_x \cos\theta \cos\phi + A_\phi \cos\theta \sin\phi - A_z \sin\theta \qquad ...(171)$$

and

$$A_\phi = -A_x \sin\phi + A_\phi \cos\phi \qquad ...(172)$$

substituting Eqs. 170—172 in Eq. (157), we get the desired result.

Similarly, if $\mathbf{A} = A_r\,\mathbf{a}_r + A_\theta\,\mathbf{a}_\theta + A_\phi\,\mathbf{a}_\phi$

is given and the desired form be

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z$$

then,

$$A_x = \mathbf{A} \cdot \mathbf{a}_x = A_r\,\mathbf{a}_r \cdot \mathbf{a}_x + A_\theta\,\mathbf{a}_\theta \cdot \mathbf{a}_x + A_\phi\,\mathbf{a}_\phi \cdot \mathbf{a}_x$$

$$= A_r \cdot \frac{x}{\sqrt{x^2 + y^2 + z^2}} + A_\theta \frac{xz}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}}$$

$$- A_\phi \frac{y}{\sqrt{x^2 + y^2}} \qquad ...(173)$$

$$A_y = \mathbf{A} \cdot \mathbf{a}_y$$

$$= A_r \cdot \frac{y}{\sqrt{x^2 + y^2 + z^2}} + A_\theta \frac{yz}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}}$$

$$+ A_\phi \frac{x}{\sqrt{x^2 + y^2}} \qquad ...(174)$$

and

$$A_z = \mathbf{A} \cdot \mathbf{a}_z$$

$$= A_r \frac{z}{\sqrt{x^2 + y^2 + z^2}} - A_\phi \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \qquad \text{...(175)}$$

Introducing A_x, A_y and A_z from equations (173), (174) and (175), in equation (156) ; we get the transformed vector in cartesian coordinate system from that in spherical coordinate systems.

The relations for the transformation between cartesian and spherical forms are tabulated below in Table 1.2.

Table 1.2

Cartesian to spherical	*Spherical to cartesian*
$x = r \sin\theta \cos\phi$	$r = \sqrt{x^2 + y^2 + z^2}$
$y = r \sin\theta \sin\phi$	$r = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}$
$z = r \cos\phi$	$\phi = \tan^{-1}\left(\frac{y}{x}\right)$
$A_r = A_x \sin\theta \cos\phi$	$A_x = A_r \cdot \frac{x}{\sqrt{x^2 + y^2 + z^2}}$
$+ A_y \sin\theta \sin\phi + A_z \cos\theta$	$+ A_\theta \frac{xz}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}} - A_\phi \frac{y}{\sqrt{x^2 + y^2}}$
$A_\theta = A_x \cos\theta \cos\phi$	$A_\phi = A_r \frac{y}{\sqrt{x^2 + y^2 + z^2}}$
$+ A_y \cos\theta \sin\theta - A_7 \sin\theta$	$+ A_\theta \frac{yz}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}} + A_\phi \frac{x}{\sqrt{x^2 + y^2}}$
$A_\phi = -A_x \sin\theta + A_y \cos\phi$	$A_z = A_r \frac{z}{\sqrt{x^2 + y^2 + z^2}} - A_\theta \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$

Example 1.28. *Loc te on a sketch and give the cylindrical coordinates of the point whose cartesian coordinates are $x = 1$, $y = 2$, $z = 3$.*

Solution. Given $x = 1, y = 2, z = 3$. From Fig. 1.33, it is evident that

$$x = r \cos\phi = 1$$

$$y = r \sin\phi = 2$$

Then $$r = \sqrt{1 + 4} = \sqrt{5} = 2.24$$

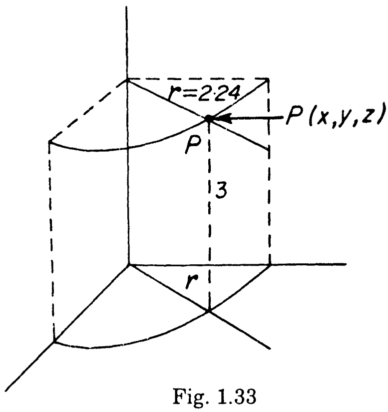


Fig. 1.33

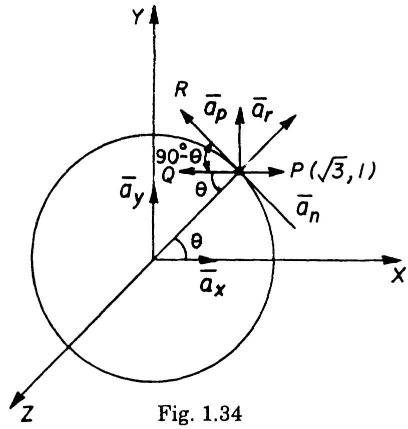


Fig. 1.34

and $\tan \phi = \frac{y}{x} = \frac{2}{1} = 2$

Therefore $\phi = \tan^{-1}(2) = 63.4°$

and $z = 3.$

Thus the point P shown in Fig. 1.34 has the cylindrical coordinates (2.24, 63.4°, 3).

Example 1.29. *A circle, centered at the origin with a radius of two units, lies in the xy plane. Determine the unit vector in (a) Cartesian coordinates (b) polar coordinates, which lies in the xy plane, is tangent to the circle at (√3, 1, 0), and is in the general direction of increasing values of y. Refer to Figure 1.34.*

Solution. Since the circle is centred at the origin, lies in the xy plane, and has point $P\,(x = \sqrt{3}, y = 1)$, the problem becomes a two dimensional one (Fig. 1.34).

From the given coordinates of point P, we have

$$x = \sqrt{3} = r \cos \theta \qquad ...(i)$$

$$y = 1 = r \sin \theta \qquad ...(ii)$$

where r is the radius of the cylinder and is given by

$$r = \sqrt{3 + 1} = 2 \qquad ...(iii)$$

and $$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30° \qquad ...(iv)$$

Referring now to Fig. 1.34, we have

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$$

$$= |\overrightarrow{PQ}|\ (-\mathbf{a}_x) + |\overrightarrow{QR}|\ (\mathbf{a}_y)$$

$= (\text{component of } \vec{\mathbf{a}}_p \text{ along } x\text{-axis}) (-\mathbf{a}_x)$

$+ (\text{component of } \vec{\mathbf{a}}_s \text{ along } x\text{-axis}) (\mathbf{a}_y)$

$= |\mathbf{a}_\rho \cos(90 - \theta)| (-\mathbf{a}_x) + |\mathbf{a}_\rho \sin(90 - \theta)| (\mathbf{a}_y)$

where $\mathbf{a}_\rho$ is a unit vector along $\vec{PR}$, thus

$$\vec{PR} = -\sin\theta\, \mathbf{a}_x + \cos\theta\, \mathbf{a}_y \qquad ...(v)$$

Substituting the value of sin θ and cos θ for θ = 30°, we have

$$\vec{PR} = -\frac{1}{2}\mathbf{a}_x + \frac{\sqrt{3}}{2}\mathbf{a}_y. \qquad ...(vi)$$

since the magnitude of $\vec{PR}$ is unity, hence we can write

$$\vec{PR} = \mathbf{a}_\rho = -\frac{1}{2}\mathbf{a}_x + \frac{\sqrt{3}}{2}\mathbf{a}_y.$$

(*b*) To determine unit vector $\mathbf{a}_\rho$ at *P* in cylindrical coordinates :

The unit vector $\mathbf{a}_\rho$ in cartesian coordinate is

$$\mathbf{a}_\rho = -\frac{1}{2}\mathbf{a}_x + \frac{\sqrt{3}}{2}\mathbf{a}_y.$$

thus $\mathbf{a}_\rho$ in polar coordinates will be given by

$$\mathbf{a}_\rho = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta \qquad ...(vii)$$

where

$$\mathbf{A}_r = \mathbf{a}_\rho \cdot \mathbf{a}_r$$
$$= \left(-\frac{1}{2}\mathbf{a}_x + \frac{\sqrt{3}}{2}\mathbf{a}_y\right)\cdot \mathbf{a}_r$$
$$= -\frac{1}{2}\mathbf{a}_x\cdot\mathbf{a}_r + \frac{\sqrt{3}}{2}\mathbf{a}_y\cdot\mathbf{a}_r$$
$$= -\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$$
$$= -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0. \qquad ...(viii)$$

$$A_\theta = \mathbf{a}_\rho\cdot\mathbf{a}_\theta = \left(-\frac{1}{2}\mathbf{a}_x + \frac{\sqrt{3}}{2}\mathbf{a}_y\right)\cdot\mathbf{a}_\theta$$
$$= -\frac{1}{2}\mathbf{a}_x\cdot\mathbf{a}_\theta + \frac{\sqrt{3}}{2}\mathbf{a}_y\cdot\mathbf{a}_\theta$$
$$= -\frac{1}{2}\cos(90+\theta) + \frac{\sqrt{3}}{2}\sin(90+\theta)$$
$$= +\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = \frac{1}{4} + \frac{3}{4} = 1 \qquad ...(ix)$$

Substituting (*viii*) and (*ix*) in (*vii*), we have,

$$\mathbf{a}_\rho = A_r\mathbf{a}_r + A_\theta\cdot\mathbf{a}_\theta = 1\cdot\mathbf{a}_\theta \qquad ...(x)$$

This shows and also is evident from Fig. 1.34, that $\mathbf{a}_\rho$ is in the increasing direction of θ and has no component in the radial direction, since it is tangential to the circle at P.

Example 1.30. *Transform the vector from cylindrical coordinates to cartesian coordinates.*

$$\mathbf{A} = r\mathbf{a}_r + r\mathbf{a}_\phi.$$

Solution. $\mathbf{A} = r\mathbf{a}_r + r\mathbf{a}_\phi$ in cylindrical coordinates

$$= A_r\,\mathbf{a}_r + A_\phi \mathbf{a}_\phi + A_z\,\mathbf{a}_z \qquad \text{(general form)}$$

On comparison $\quad A_r = r$

$$A_\phi = r \quad \text{and} \quad A_z = 0$$

Now $\quad r = \sqrt{x^2 + y^2}$

$$r \cos\phi = x, \quad r \sin\phi = y$$

and $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

Following Table 1.1

$$A_x = A_r \frac{x}{\sqrt{x^2+y^2}} - A_\phi \frac{y}{\sqrt{x^2+y^2}} = x - y$$

$$A_y = A_r \cdot \frac{y}{\sqrt{x^2+y^2}} + A_\phi \frac{x}{\sqrt{x^2+y^2}} = y + x$$

$$A_z = 0$$

Hence $\quad \mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z = (x - y)\,\mathbf{a}_x + (x + y)\,\mathbf{a}_y$

Example 1.31. *Transform the vector* **A** *to spherical coordinates*

$$\mathbf{A} = \frac{yz}{x}\,\mathbf{a}_z$$

Solution. Given $\quad \mathbf{A} = \frac{yz}{x}\,\mathbf{a}_z$

in cartesian coordinates. Comparing it with the general form

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

We get $\quad A_x = A_y = 0 \quad$ and $A_z = \frac{yz}{x}$

Following Table 1.2

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$$

$$= \frac{yz}{x}\cos\theta$$

$$= \frac{r\sin\theta\sin\phi \cdot r\cos\theta}{r\sin\theta\cos\phi} \cdot \cos\theta$$

$$= r\cos^2\theta \tan\phi \qquad ...(i)$$

$$A_\theta = A_x \cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$$

$$= -\frac{r\sin\theta\sin\phi\, r\cos\phi}{r\sin\theta\cos\phi}\sin\theta$$

$$= -r\tan\phi\sin\theta\cos\theta \qquad ...(ii)$$

and $$A_\phi = -A_x\sin\phi + A_y\cos\phi = 0. \qquad ...(iii)$$

Then, the required vector in spherical coordinates will be obtained by substituting Eqs. (*i*), (*ii*) and (*iii*) in the equation

$$\mathbf{A} = A_r\,\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi$$

Thus, $$\mathbf{A} = r\tan\phi\cos^2\theta\,\mathbf{a}_r - r\tan\phi\sin\theta\cos\theta\,\mathbf{a}_\theta.$$

Example 1.32. *Transform the vector from spherical coordinates to cartesian coordinates :*

$$\mathbf{A} = \frac{1}{r\sin\theta\cos\phi}\,\mathbf{a}_\theta$$

Solution. $$\mathbf{A} = \frac{1}{r\sin\theta\cos\phi}\,\mathbf{a}_\theta$$

in spherical coordinates comparing it with the general form,

$$\mathbf{A} = A_r\,\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\,\mathbf{a}_\phi$$

we get $$A_r = A_\phi = 0$$

and $$A_\theta = \frac{1}{r\sin\theta\cos\phi}$$

Now, following Table 1.2

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

and $$A_x = A_r\frac{x}{\sqrt{x^2+y^2+z^2}} + A_\theta\frac{xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - A_\phi\frac{y}{\sqrt{x^2+y^2}}$$

$$= 0 + A_\theta\frac{xz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + 0$$

$$= \left(\frac{1}{\sqrt{x^2+y^2+z^2}\;\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\;\dfrac{x}{\sqrt{x^2+y^2}}}\right)\frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$$

$$= \frac{z}{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2+z^2}} \qquad ...(i)$$

$$A_y = A_r \frac{y}{\sqrt{x^2+y^2+z^2}} + A_\theta \frac{yz}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + A_\phi \frac{x}{\sqrt{x^2+y^2}}$$

$$= 0 + \frac{yz}{x\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + 0 \qquad ...(ii)$$

and
$$A_z = \frac{A_r z}{\sqrt{x^2+y^2+z^2}} - \frac{A_\theta \sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$= -\frac{\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2+z^2}} \qquad ...(iii)$$

Then the required vector in the cartesian coordinates will be obtained from

$$\mathbf{A} = A_x\,\mathbf{a}_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z$$

By substituting A_x, A_y and A_z from Eqs. (*i*), (*ii*) and (*iii*), we have

$$\mathbf{A} = \frac{xz\mathbf{a}_x + yz\mathbf{a}_y - \sqrt{x^2+y^2}\,\mathbf{a}_z}{x\sqrt{(x^2+y^2)(x^2+y^2+z^2)}}$$

Example 1.33. Of the components of vector **V** are

$$V_x = \frac{x}{x^2+y^2},\ V_y = \frac{y}{x^2+y^2},\ V_z = 0$$

Compute the divergence and curl using cylindrical coordinates.

Solution. With components

$$V_x = \frac{x}{x^2+y^2},\ V_y = \frac{y}{x^2+y^2}$$

and $V_z = 0$,

vector **V** in cartesian will be

$$\mathbf{V} = \left(\frac{x}{x^2+y^2}\right)\mathbf{a}_x + \left(\frac{y}{x^2+y^2}\right)\mathbf{a}_y \qquad ...(i)$$

First we will transform **V** into cylindrical coordinates.

The first step to be followed is to change the variables by putting in Eq. (*i*)

$$x = r\cos\phi,\quad y = r\sin\phi,\quad z = z$$

Thus,
$$V = \frac{r\cos\phi}{r^2}\,\mathbf{a}_x + \frac{r\sin\phi}{r^2}\,\mathbf{a}_y$$

$$= \frac{\cos\phi}{r}\,\mathbf{a}_x + \frac{\sin\phi}{r}\,\mathbf{a}_y \qquad ...(ii)$$

The next step will be to change the components. This will be approached as given below

The vector in cylindrical coordinates is

$$\mathbf{V} = V_r\mathbf{a}_r + V_\phi\,\mathbf{a}_\phi + V_z\mathbf{a}_z \qquad ...(iii)$$

Multiplying (*ii*) by the unit vector $\mathbf{a}_r$ we get

$$\mathbf{V}\cdot\mathbf{a}_r = \frac{\cos\phi}{r}(\mathbf{a}_x\cdot\mathbf{a}_r) + \frac{\sin\phi}{r}(\mathbf{a}_y\cdot\mathbf{a}_r)$$

But from Eq. (*iii*), $\mathbf{V}\cdot\mathbf{a}_r = V_r$

So, $$V_r = \frac{\cos\phi}{r}(\mathbf{a}_x\cdot\mathbf{a}_r) + \frac{\sin\phi}{r}(\mathbf{a}_y\cdot\mathbf{a}_r) \qquad ...(iv)$$

similarly, we can deduce this,

$$V_\phi = \frac{\cos\phi}{r}(\mathbf{a}_x\cdot\mathbf{a}_\phi) + \frac{\sin\phi}{r}(\mathbf{a}_y\cdot\mathbf{a}_\phi) \qquad ...(v)$$

Further,
$$\mathbf{a}_x\cdot\mathbf{a}_r = \cos\phi$$
$$\mathbf{a}_y\cdot\mathbf{a}_r = \sin\phi$$
$$\mathbf{a}_x\cdot\mathbf{a}_\phi = -\sin\phi$$
and
$$\mathbf{a}_y\cdot\mathbf{a}_\phi = \cos\phi$$

Introducing these in Eqs. (*iv*) and (*v*), we have

$$A_r = \left(\frac{\cos^2\phi}{r} + \frac{\sin^2\phi}{r}\right) = \frac{1}{r}$$

$$A_\phi = -\frac{\cos\phi\sin\phi}{r} + \frac{\cos\phi\sin\phi}{r} = 0$$

and
$$A_z = 0 \qquad ...(vi)$$

Substituting A_r, A_ϕ and A_z from Eq. (*vi*) in (*iii*), we have the vector in cylindrical coordinate,

$$\mathbf{V} = \frac{1}{r}\,\mathbf{a}_r$$

Now, $(\overline{\nabla}\cdot\mathbf{V})$ in cylindrical coordinate

$$\overline{\nabla}\cdot\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rVr) + \frac{1}{r}\frac{\partial}{\partial\phi}V_\phi + \frac{\partial V_z}{\partial_z}$$

$$= \frac{1}{r}V_r + \frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\phi}{\partial\phi}$$

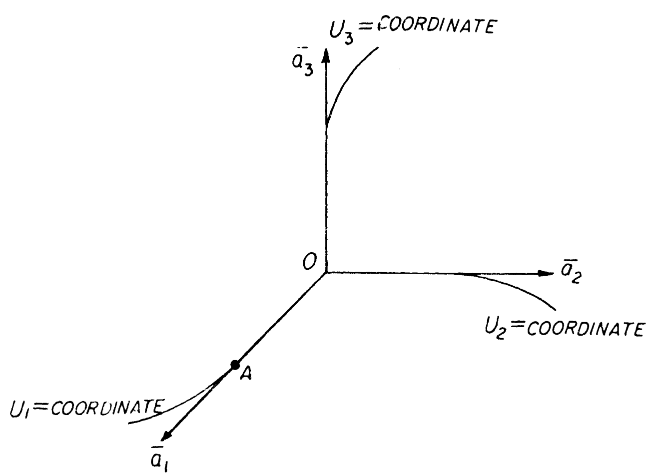
$$= \frac{1}{r^2} - \frac{1}{r^2} = 0$$

and also
$$\bar{\nabla} \times \mathbf{V} = \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{1}{r}\dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ \dfrac{1}{r} & 0 & 0 \end{vmatrix}$$
$$= \mathbf{a}_\phi \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right) - 0 \right] + \mathbf{a}_z \left[-\frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\right) \right] = 0.$$

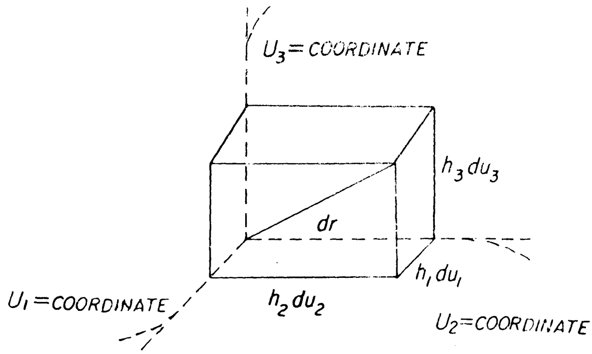
1.9E. General Curvilinear Coordinate System

Each of the three coordinate systems, discussed earlier and many others utilized in mathematical physics, are orthogonal coordinate systems. In orthogonal coordinate system, the three coordinate surfaces intersect at right angles to one another at any given point. Because of the simplicity that results from such a choice, orthogonal coordinates are used almost exclusively. Since the fundamental definitions of gradient, divergence and curl do not involve a particular coordinate system, it is possible to develop general expressions for these in terms of a general coordinate system. This system is oftenly found useful in examples with symmetrical fields specially in cases of cylindrical and spherical configurations.

Consider coordinate curves as shown in Fig. 1.35 where u_1, u_2, u_3 are orthogonal coordinate. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ be unit vectors tangent to the respective coordinate curves at any point P. The unit vectors are chosen as positive in the direction in which the corresponding coordinates increase.



(a)



(b)

Fig. 1.35

Now let, dl_1, dl_2, and dl_3 be differential component arc lengths measured as positive along the coordinate curves in the positive directions of the coordinates. The derivative of these arcs with respect to the coordinates shall be designated as

$$\frac{dl_1}{du_1} = h_1 \quad ...(176)$$

$$\frac{dl_2}{du_2} = h_2 \quad ...(177)$$

and $$\frac{dl_3}{du_3} = h_3 \quad ...(178)$$

in which h_1, h_2, and h_3 are, in general, a function of the coordinates, u_1, u_2, and u_3 and are called scalar factors. Fig. 1.35*b* shows a differential volume in terms of the differential components of dl. In the limits of differential space, this element of volume has the edges of

$$dl_1 = h_1\, du_1\ ;\ dl_2 = h_2\, du_2\ ;\ dl_3 = h_3\, du_3 \quad ...(179)$$

and volume, given by the product of these edges oriented respectively in the directions of the unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$,

$$dv = dl_1\, dl_2\, dl_3$$

substituting dl_1, dl_2, dl_3 from Eq. (179) in the above equation, we have

$$dv = h_1\, h_2\, h_3\, du_1\, du_2\, du_3 \quad ...(180)$$

The orientation of the three co-ordinates of the curvilinear co-ordinate system is chosen such that it follows a right-handed coordinate system. That is, if $\mathbf{a}_1$ is rotated into $\mathbf{a}_2$ (in anticlockwise direction), the resultanting direction of advance of a right-handed screw thread establishes the direction of the third unit vector $\mathbf{a}_3$ as

$$\boxed{\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2} \qquad ...(181)$$

(*a*) Cartesian Co-ordinate System

In cartesian system of co-ordinates the unit vectors may be designated as

$\mathbf{a}_x$, unit vector in the $+x$ direction.

$\mathbf{a}_y$, unit vector in the $+y$ direction.

$\mathbf{a}_z$, unit vector in the $+z$ direction.

These unit vectors are illustrated in Fig. 1.36. The differential components of the general arc dl are identical with the differentials of the co-ordinates.

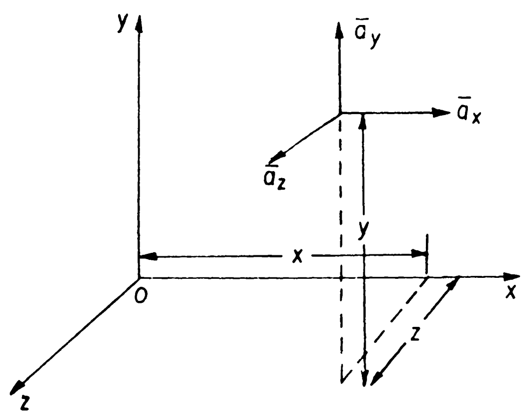


Fig. 1.36

Consequently, $u_1 = l_1 = x; \quad u_2 = l_2 = y \ ; \quad u_3 = l_3 = z$ ...(182)

and hence $du_1 = dl_1$ ...(183)

$du_2 = dl_2$ ...(184)

$du_3 = dl_3$ ...(185)

Consequently $h_1 = h_2 = h_3 = 1$ ...(186)

and the element of volume is

$$dv = dx\, dy\, dz \qquad ...(187)$$

(*b*) Cylindrical Co-ordinate System

A radial distance, an angular displacement, and a linear (or axial) displacement may also be used to describe a cylindrical co-ordinate system that is also an orthogonal system. Fig. 1.37, illustrates the cylindrical co-ordinates r, φ and z, and $\mathbf{a}_r$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$ the unit vectors in the positive direction of the respective co-ordinates. It may be noted

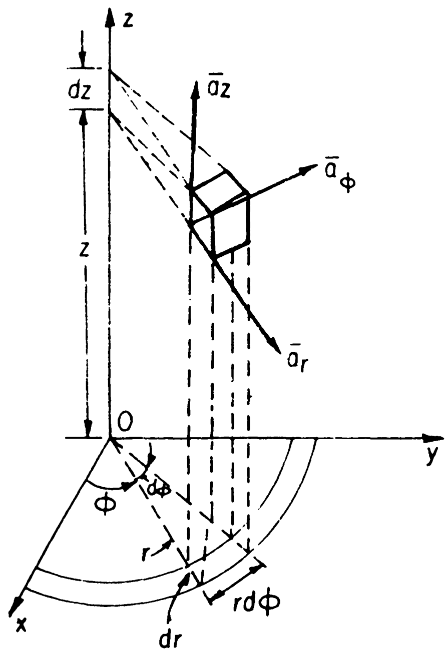


Fig. 1.37

that $\mathbf{a}_z$ is still the z-co-ordinate unit vector and hence is carried over from the $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ notation of the cartesian co-ordinate system.

Reviewing Fig. 1.37, we have

$$dl_1 = dr \; ; \; dl_2 = rd\phi \; ; \; dl_3 = dz \qquad \text{...(188)}$$

also

$$u_1 = r \; ; \; u_2 = \phi \; ; \; u_3 = z \qquad \text{...(189)}$$

and hence

$$du_1 = dr \; ; \; du_2 = d\phi$$

$$du_3 = dz \qquad \text{...(190)}$$

Consequently

$$\left.\begin{aligned} h_1 &= \frac{dl_1}{du_1} = \frac{dr}{dr} = 1 \\ h_2 &= \frac{dl_2}{du_2} = \frac{rd\phi}{d\phi} = r \\ h_3 &= \frac{dl_3}{du_3} = \frac{dz}{dz} = 1 \end{aligned}\right] \qquad \text{...(191)}$$

and the element of volume is

$$dv = dl_1 \, dl_2 \, dl_3 = rdrd\phi dz \qquad \text{...(192)}$$

(c) Spherical Co-ordinate System

A radial displacement and two angular displacements may be used to establish a spherical co-ordinate system that is also orthogonal. Fig. 1.38 illustrates the spherical co-ordinates, r, θ and ϕ, and the $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ unit vectors chosen in the positive directions of the respective co-ordinates. Furthermore, $\mathbf{a}_r$ and $\mathbf{a}_\phi$ are still associated with the r and ϕ co-ordinates and hence are carried over from

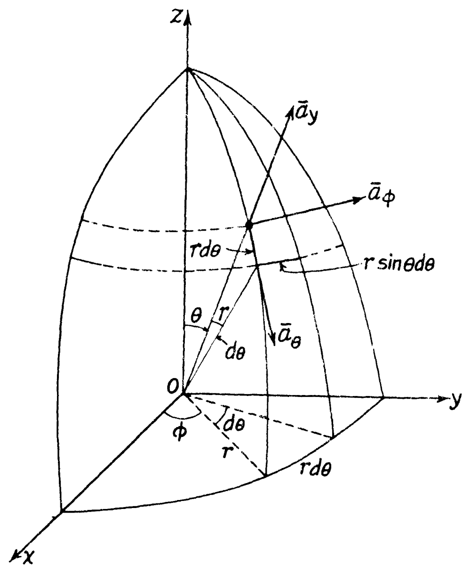


Fig. 1.38

the $\mathbf{a}_r$, $\mathbf{a}_\phi$, $\mathbf{a}_z$ notation of cylindrical co-ordinates.

Reviewing Fig. 1.38, indicates that

$$dl_1 = dr \; ; \; dl_2 = r d\theta$$
$$dl_3 = r \sin\theta \, d\phi \qquad \ldots(193)$$

Also $\quad u_1 = r \; ; \; u_2 = \theta$

and $\quad u_3 = \phi \qquad \ldots(194)$

and hence $\quad du_1 = dr \; ; \; du_2 = d\theta$

$$du_3 = d\phi \qquad \ldots(195)$$

Consequently

$$\left.\begin{aligned} h_1 &= \frac{dl_1}{du_1} = \frac{dr}{dr} = 1 \\ h_2 &= \frac{dl_2}{du_2} = \frac{r d\theta}{d\theta} = r \\ h_3 &= \frac{dl_3}{du_3} = \frac{r \sin\theta \, d\phi}{d\phi} = r \sin\theta \end{aligned}\right] \qquad \ldots(196)$$

and the element of volume is

$$dv = r^2 \sin\theta \, dr d\theta \, d\phi \qquad \text{...(197)}$$

The elemental area's are

$$dl_1 dl_2,\ dl_2 dl_3, \text{ and } dl_3 dl_1$$

and therefore, for the surface with constant radial distance '*r*' is (as depicted in Fig. 1.38) is

$$r d\theta \, r \sin\theta \, d\phi = r^2 \sin\theta \, d\theta d\phi \qquad \text{...(198)}$$

(*d*) **Scalar and Vector Products**

The fundamental definitions of vector multiplications, namely the 'dot' product and 'cross' product, can also be used to have general relations pertaining to orthogonal coordinates. Thus, if **A** and **B** are two vectors expressed in curvilinear coordinates by

$$\mathbf{A} = \mathbf{a}_1 A_1 + \mathbf{a}_2 A_2 + \mathbf{a}_3 A_3 \qquad \text{...(199)}$$

and

$$\mathbf{B} = \mathbf{a}_1 B_1 + \mathbf{a}_2 B_2 + \mathbf{a}_3 B_3 \qquad \text{...(200)}$$

then, since $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are mutually perpendicular so that

$$\mathbf{a}_1 \cdot \mathbf{a}_1 = \mathbf{a}_2 \cdot \mathbf{a}_3 = \mathbf{a}_3 \cdot \mathbf{a}_1 = 1\ ;$$

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_2 \cdot \mathbf{a}_3 = \mathbf{a}_3 \cdot \mathbf{a}_1 = 0\ ;$$

$$\mathbf{a}_1 \times \mathbf{a}_1 = \mathbf{a}_2 \times \mathbf{a}_2 = \mathbf{a}_3 \times \mathbf{a}_3 = 0\ ;$$

and

$$\mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{a}_3$$

$$\mathbf{a}_2 \times \mathbf{a}_3 = \mathbf{a}_1$$

$$\mathbf{a}_3 \times \mathbf{a}_1 = \mathbf{a}_2$$

Consequently,

$$\mathbf{A} \cdot \mathbf{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 \qquad \text{...(201)}$$

and

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \qquad \text{...(202)}$$

It should be noted that, one of these vectors if replaced by the 'del' operator (discussed in section 1.9.5.5), the above expressions do not hold.

(*e*) **The $\overline{\nabla}$-Del) operator**

Vectors may, in general, be functions of both space and time. The $\overline{\nabla}$-operator is a vector space-function operator and is defined through the partial derivatives with respect to space variables. The use of partial derivatives is simply a method of mathematically holding time fixed (if the vector is a function of time) and investigating the behaviour of a vector spatially in the region at that particular instant of time.

In any orthogonal coordinate system the ∇ vector operator is

$$\overline{\nabla} = \mathbf{a}_1 \cdot \frac{\partial}{\partial l_1} + \mathbf{a}_2 \frac{\partial}{\partial l_2} + \mathbf{a}_3 \frac{\partial}{\partial l_3} \qquad ...(203)$$

or in terms of the coordinates u_1, u_2 and u_3, since in general the spatial differential $dl = hdu$

$$\overline{\nabla} = \mathbf{a}_1 \frac{1}{h_1} \frac{\partial}{\partial u_1} + \mathbf{a}_2 \frac{1}{h_2} \frac{\partial}{\partial u_2} + \mathbf{a}_3 \frac{1}{h_3} \frac{\partial}{\partial u_3} \qquad ...(204)$$

In cartesian coordinates where

$\mathbf{a}_1 = \mathbf{a}_x : \mathbf{a}_2 = \mathbf{a}_y ; \mathbf{a}_3 = \mathbf{a}_z$

$u_1 = x ; u_2 = y ; u_3 = z$

and $h_1 = h_2 = h_3 = 1,$

the del operator is

$$\overline{\nabla} = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \qquad ...(205)$$

In the cylindrical coordinate system where

$\mathbf{a}_1 = \mathbf{a}_r ; \mathbf{a}_2 = \mathbf{a}_\phi ; \mathbf{a}_z = \mathbf{a}_z$

$u_1 = r ; u_2 = \phi, u_3 = z$

and $h_1 = 1 ; h_2 = r ; h_3 = 1$

the del operator is

$$\overline{\nabla} = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\phi \frac{1}{r} \frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z} \qquad ...(206)$$

Finally, in the spherical coordinate system where

$\mathbf{a}_1 = \mathbf{a}_r ; \mathbf{a}_2 = \mathbf{a}_\theta ; \mathbf{a}_3 = \mathbf{a}_\phi$

$u_1 = r ; u_2 = \theta ; u_3 = \phi$

and $h_1 = 1 ; h_2 = r ; h_3 = r \sin \theta$

the del operator is

$$\overline{\nabla} = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \qquad ...(207)$$

(*f*) Gradient of a Scalar Function

For a complete meaning in the scence of an equation, the $\overline{\nabla}$ operator must be applied to or be permitted to operate directly upon some other function. The gradient results when this operation occurs upon a scalar function. In this section we will first find an expression for the gradient of a scalar function in curvilinear orthogonal coordinate system from the definition of gradient and then the result so obtained will be extended to other orthogonal coordinate systems.

Referring to Fig. 1.35*a* let $\phi\,(u_1, u_2, u_3)$ be a scalar function. The component of gradient of ϕ along the u_1 axis, from the definition, is

$$(\text{grad}\,\phi)_1 = \lim_{du_1 \to 0} \frac{\phi\,(A) - \phi(O)}{dl_1}$$

$$= \lim_{du_1 \to 0} \frac{\phi\,(A) - \phi\,(O)}{(h_1 du_1)}$$

$$= \frac{1}{h_1} \frac{\partial\phi}{du_1} \qquad \text{...(208)}$$

where $\partial\phi$ is an increment in the value of function ϕ between points *A* and *O* corresponding to a change in length $\partial l_1 = h_1\,\partial u_1$ along the u_1 axis.

This follows from the definition that the gradient of any scalar function ϕ will be a vector whose component in any direction is given by the change of ϕ for a change in distance along that direction. Similarly,

$$(\text{grad}\,\phi)_2 = \frac{1}{h_2} \frac{\partial\phi}{\partial u_2} \qquad \text{...(209)}$$

and

$$(\text{grad}\,\phi)_3 = \frac{1}{h_3} \frac{\partial\phi}{\partial u_3} \qquad \text{...(210)}$$

Multiplying Eqs. (208), (209) and (210) by $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ respectively, the resultant expression for gradient of function ϕ *i.e.* grade ϕ becomes

$$\boxed{\text{grad}\,\phi = \mathbf{a}_1 \frac{1}{h_1} \frac{\partial\phi}{\partial u_1} + \mathbf{a}_2 \frac{1}{h_2} \frac{\partial\phi}{\partial u_2} + \mathbf{a}_3 \frac{1}{h_3} \frac{\partial\phi}{\partial u_3}} \qquad \text{...(211)}$$

Introducing Eq. (204), we have

$$\text{grad}\,\phi = \overline{\nabla}\,\phi \qquad \text{...(212)}$$

where

$$\boxed{\overline{\nabla} = \mathbf{a}_1 \frac{1}{h_1} \frac{\partial}{\partial u_1} + \mathbf{a}_2 \frac{1}{h_2} \frac{\partial}{\partial u_2} + \mathbf{a}_3 \frac{1}{h_3} \frac{\partial}{\partial u_3}}$$

From Eq. (211), we have for

(*i*) Cartesian coordinates :

$$\boxed{\overline{\nabla}\,\phi = \mathbf{a}_x \frac{\partial\phi}{\partial x} + \mathbf{a}_y \frac{\partial\phi}{\partial y} + \mathbf{a}_z \frac{\partial\phi}{\partial z}} \qquad \text{...(213)}$$

(*ii*) Cylindrical coordinates :

$$\boxed{\overline{\nabla}\,\phi = \mathbf{a}_r \frac{\partial\phi}{\partial r} + \mathbf{a}_\phi \frac{1}{r} \frac{\partial\phi}{\partial\phi} + \mathbf{a}_z \frac{\partial\phi}{\partial z}} \qquad \text{...(214)}$$

(*iii*) Spherical coordinates :

$$\boxed{\overline{\nabla}\phi = \mathbf{a}_r \frac{\partial\phi}{\phi r} + \mathbf{a}_\theta \frac{1}{r} \frac{\partial\phi}{\partial\theta} + \mathbf{a}_\phi \frac{1}{r \sin\theta} \frac{\partial\phi}{\partial\phi}} \qquad \text{...(215)}$$

If the function ϕ is the sum of two components, as

$$\phi = \phi_1 + \phi_2 \qquad ...(216)$$

then the gradient of the sum of the two components can be shown to be the sum of the gradients of the components, as

$$\overline{\nabla}(\phi_1 + \phi_2) = \overline{\nabla}\phi_1 + \overline{\nabla}\phi_2 ; \qquad ...(217)$$

since from Eq. (211) each term of the partial derivative can be differential as

$$\frac{\partial(\phi_1 + \phi_2)}{\partial u_n} = \frac{\partial \phi_1}{\partial u_n} + \frac{\partial \phi_2}{\partial u_n} \qquad ...(218)$$

If the function ϕ is the product of two scalar functions ω_1 and ω_2 as

$$\phi = \omega_1 \omega_2 \qquad ...(219)$$

then, the gradient of the product of two components can be shown to be

$$\overline{\nabla}(\omega_1\omega_2) = \omega_1 \overline{\nabla}\omega_2 + \omega_2 \overline{\nabla}\omega_1 ; \qquad ...(220)$$

since each term resulting from Eq. (211) can be differentiated as

$$\frac{\partial(\omega_1\omega_2)}{\partial u_n} = \omega_1 \frac{\partial \omega_2}{\partial u_n} + \omega_2 \frac{\partial \omega_1}{\partial u_n} \qquad ...(221)$$

and a collection of all the terms will confirm Eq. (220)

(*g*) **Divergence of a Vector**

The divergence of a vector field **B** may be computed from divergence (Gauss's) theorem which states that

$$\iiint_v (\overline{\nabla} \cdot \mathbf{B})\, dv = \iint_S \mathbf{B} \cdot ds \qquad ...(222)$$

In the limit $dv \to 0$,

$$\overline{\nabla} \cdot \mathbf{B} = \text{div } \mathbf{B} = \lim_{dv \to 0} \frac{\iint_S \mathbf{B} \cdot ds}{dv} \qquad ...(223)$$

It, therefore, follows that the divergence of a vector field may be determined by finding the net outflow per unit volume in the limit the volume approaches zero.

Let us now consider a vector flux field in which a vector **B** is specified. In any orthogonal coordinate system the differential element of volume is (dl_1, dl_2, dl_3), *i.e.*

$$dv = h_1 h_2 h_3\, du_1 du_2 du_3 \qquad ...(224)$$

Let the vectoı **B** be

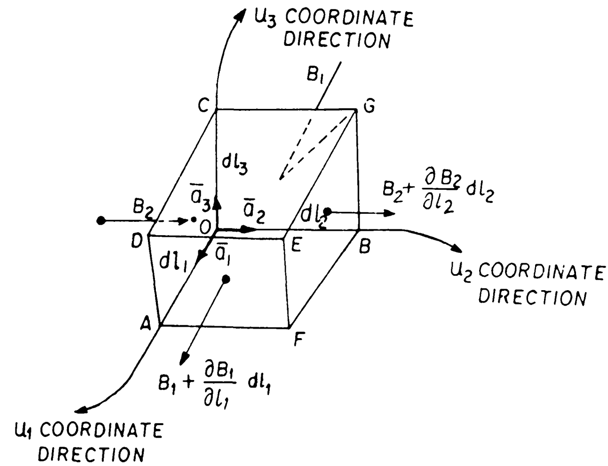


Fig. 1.39

$$\mathbf{B} = \mathbf{a}_1 B_1 + \mathbf{a}_2 B_2 + \mathbf{a}_3 B_3 \qquad ...(225)$$

where B_1 is assumed to be the average value of the component of **B** along u_1 axis and at the plane $u_1 = 0$.

Now, the surface integral $\iint_S \mathbf{B} \cdot d\mathbf{s}$ gives the net out flow of flux (differential flux emanating) through the surface s.

Referring to Fig. 1.39 and using the symbol ψ to indicate the flux, the area of face *OADCO* is

$$dl_1 dl_3 = h_1 h_3 du_1 du_3$$

the flux entering (that –ve flux) the face (*OADCO*) is

$$d\psi_{(u2)} = B_2 h_1 h_3 du_1 du_3 \qquad ...(226)$$

According to Taylor's expansion formula, the flux at face *BFEGB* from the value of **B** givèn by

$$B_2 \cdot \frac{\partial B_2}{\partial l_2} dl_2 + \frac{\partial^2 B_2}{\partial l_2^{\,2}} \frac{(dl_2)^2}{2\,!} + ...$$

If dl_2 is very small, the square and higher order terms may be neglected, so that at the right face *BFEGB*, the normal component of B_2 becomes

$$B_2 + \frac{\partial B_2}{\partial l_2} dl_2 = B_2 + \frac{\partial B_2}{h_2 \partial u_2} h_2 du_2 \qquad ...(227)$$

Eq. (226) gives the component flux density B_2 in terms of coordinates u_1, u_2, u_3 and directed in the number 2 co-ordinate direction. Over the differential surface at the right of the volume where the u_2 co-ordinate takes on the value of $u_2 + du_2$, the density is in general changed, since it may be a function of position along the u_2 co-ordinate. The differential flux over the right surface at $u_2 + du_2$ then is

$$d\psi\,(u_2 + du_2) = (B_2 h_1 h_3)\, du_1 du_3 + \frac{\partial(B_2 h_1 h_3)\,(du_1 du_3)}{\partial u_2}\, du_2 \quad ...(228)$$

In general h_1 and h_3 may contain some function of the coordinate u_2 and hence must remain within the partial derivative.

The net differential flux gain of the right over the left surface is Eq. (228) less Eq. (227)

or $$d\psi\,(u_2 + du_2) - d\psi\,(u_2) = \frac{\partial\,(B_2 h_1 h_3)}{\partial u_2}\, du_1 du_2 du_3 \quad ...(229)$$

Similarly fluxes in the directions u_3, and u_1 are respectively

$$\frac{\partial\,(B_3 h_2 h_1)}{\partial u_3}\, du_1 du_2 du_3 \quad ...(230)$$

and $$\frac{\partial(B_1 h_3 h_2)}{\partial u_1}\, du_1 du_2 du_3 \quad ...(231)$$

Adding (229–231), and using

$$\text{Div}\,\mathbf{B} = \lim_{dv \to 0} \frac{\iint_S \mathbf{B} \cdot d\mathbf{s}}{dv} = \overline{\nabla} \cdot \mathbf{B}$$

We have $$= \lim_{dv \to 0} \frac{\iint_S \mathbf{B} \cdot d\mathbf{s}}{dl_1 dl_2 dl_3}$$

$$\therefore \quad \overline{\nabla} \cdot \mathbf{B} = \frac{1}{h_1 h_2 h_3 du_1 du_2 du_3} \left[\frac{\partial(B_1 h_2 h_3)}{\partial u_1} du_1 du_2 du_3 + \frac{\partial(B_2 h_3 h_1)}{\partial u_2} du_1 du_2 du_3 + \frac{\partial(B_3 h_1 h_2)}{\partial u_3} du_1 du_2 du_3 \right]$$

$$\boxed{\overline{\nabla} \cdot \mathbf{B} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial\,(B_1 h_2 h_3)}{\partial u_1} + \frac{\partial(B_2 h_3 h_1)}{\partial u_2} + \frac{\partial(B_3 h_1 h_2)}{\partial u_3} \right]} \quad ...(232)$$

This expression is valid for any orthogonal co-ordinates system. Eq. (232) simplifies to the following forms for the three systems as given below :

Cartesian $\overline{\nabla} \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial_z}$ ...(233)

Cylindrical $\overline{\nabla} \cdot \mathbf{B} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_\phi}{\partial \phi} + \frac{\partial}{\partial z}(rB_z)\right]$

$$= \frac{B_r}{r} + \frac{\partial B_r}{\partial r} + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} \qquad \text{...(234)}$$

Spherical $\overline{\nabla} \cdot \mathbf{B} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\, B_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, B_\theta) + \frac{\partial}{\partial \phi}(rB\phi)\right]$

$$= \frac{2B_r}{r} + \frac{\partial B_r}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\cot\theta}{r}B_\theta + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi} \qquad \text{...(235)}$$

(*h*) **Curl of a Vector**

Consider a space field in which a vector **A** is specified. The curl of **A** is a vector determined by the vector sum of three-line integrals of the scalar, or dot, products of the vector **A**, taken over the perimeters of the three mutually orthogonal elementary areas and divided by these areas. The curl of **A** is symbolized by $\overline{\nabla} \times \mathbf{A}$ and the *n*th component (n = 1, 2, or 3) is expressed as

$$\left|\overline{\nabla} \times \mathbf{A}\right|_n = \lim_{\Delta s \to 0} \frac{\int_{cn} \mathbf{A} \cdot \mathbf{dl}}{\Delta s_n} \qquad \text{...(236)}$$

Fig. 1.40 illustrates some of the variables to be used in evaluating the first component of the curl in general orthogonal coordinates ($n = 1$). This curl component is normal to the coordinate surface for which u_1 is a constant and is directed in the positive sense of the u_1 coordinate. The line integral is to be taken in the direction around the elementary area in the order *oabc*, to confirm with the convention of the advance of a right-hand thread in the positive u_1 co-ordinate direction. This direction of circulation is indicated in the figure by the arrow symbol encircling the small circle with the dot in the centre (right hand thumb rule), which represents the unit vector $\mathbf{a}_1$ pointing out of the paper.

Let the vector field **A** be of the form

$$\mathbf{A} = \mathbf{a}_1 A_1 + \mathbf{a}_2 A_2 + \mathbf{a}_3 A_3 \qquad \text{...(237)}$$

Then the closed integral along the path *oabco* is

$$\int_{c1} \mathbf{A} \cdot \mathbf{dl} = \int_0^a \mathbf{A} \cdot \mathbf{dl} + \int_a^b \mathbf{A} \cdot \mathbf{dl} + \int_b^c \mathbf{A} \cdot \mathbf{dl} + \int_c^0 \mathbf{A} \cdot \mathbf{dl} \qquad \text{...(238)}$$

The line integral from *o* to *a* is $\mathbf{A}_2 \cdot \mathbf{dl}_2$ or in terms of the coordinate u_2, the magnitude is $A_2 h_2 du_2$. The arc length in general or-

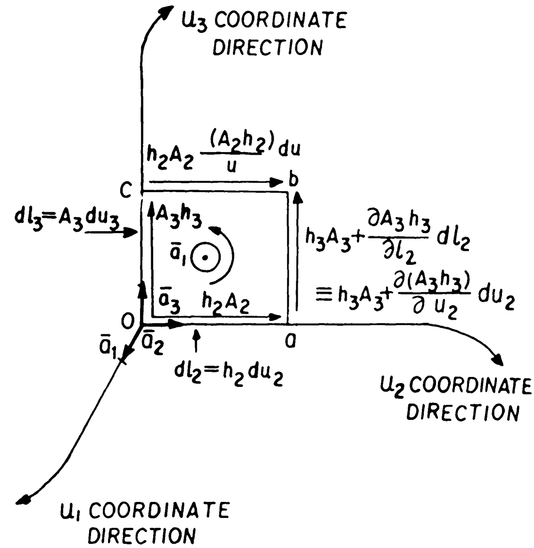


Fig. 1.40. Line integral used in evaluation of (curl A)$_1$.

thogonal co-ordinates is dl_2, but the differential variable is du_2. The h_2 factor, even though it originates from dl_2, must now be carried with A_2 as h_2A_2 when considering how this combination changes as the variable is shifted to the section cb of the line integral. Because, in general, both the vector and the respective h's are functions of the space coordinates along the section cb, the combination h_2A_2 takes on a value of

$$h_2A_2 + \frac{\partial\,(h_2A_2)}{du_3}\,du_3 \qquad \text{...(239)}$$

when differential of higher order are neglected. A similar procedure determines that the h_3A_3 combination takes on a value of

$$h_3A_3 + \frac{\partial\,(h_3A_3)}{\partial u_2}\,du_2 \qquad \text{...(240)}$$

along the direction of $\mathbf{a}_3$, in the section ab.

Now, writing the total line integral of $\mathbf{A} \cdot \mathbf{dl}$ or $(h_3\mathbf{A}_n)\,du_n$ around the total path $oabco$ yields the following relations, written in magnitude form with minus signs included where the scalar, or dot, product is negative,

$$[h_2A_2]\,du_2 + \left[h_3A_3 + \frac{\partial(h_3A_3)}{\partial u_2}\,du_2\right]du_3 - \left[h_2A_2 + \frac{\partial\,(h_2A_2)}{\partial u_3}\,du_3\right]du_2 - (h_3A_3]\,du_3$$

$$\equiv \frac{\partial(h_3A_3)}{\partial u_2}\,du_2du_3 - \frac{\partial(h_2A_2)}{\partial u_3}\,du_2du_3 \qquad ...(241)$$

Now considering that the area dl_2dl_3 $(= h_2h_3du_2du_3)$ shrinks to a point, the component of the curl in the u_1 direction, from Eq. (236) is

$$(\overline{\nabla} \times \mathbf{A})_1 = \mathbf{a}_1 \frac{\dfrac{\partial\,(h_3A_3)}{\partial u_2} - \dfrac{\partial\,(h_2A_2)}{\partial u_3}}{h_2h_3} \qquad ...(242)$$

In a similar manner the second and third components of the curl can be evaluated. The vector sum of these three components in general orthogonal co-ordinates is

$$\overline{\nabla} \times \mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\mathbf{a}_1h_1\left[\frac{\partial\,(h_3A_3)}{\partial u_2} - \frac{\partial\,(h_2A_2)}{\partial u_3}\right] + \mathbf{a}_2h_2\left[\frac{\partial\,(h_1A_1)}{\partial u_3} - \frac{\partial\,(h_3A_3)}{\partial u_1}\right] + \mathbf{a}_3h_3\left[\frac{\partial\,(h_2A_2)}{\partial u_1} - \frac{\partial\,(h_1A_1)}{\partial u_2}\right]\right\} \qquad ...(243)$$

In determinant form, the curl of **A** in general orthogonal coordinates is

$$\overline{\nabla} \times \mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \mathbf{a}_1h_1 & \mathbf{a}_2h_2 & \mathbf{a}_3h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \qquad ...(244)$$

which in cartesian coordinates is

$$\overline{\nabla} \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad ...(245)$$

The cylindrical co-ordinates it is

$$\overline{\nabla} \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_{\phi_r} & \mathbf{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \qquad ...(246)$$

In spherical coordinates it is

$$\overline{\nabla} \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\theta r & \mathbf{a}_\phi r \sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA\theta & r \sin A_\phi \end{vmatrix} \quad ...(247)$$

Further, Eq. (244) can also expressed as

$$\overline{\nabla} \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ \dfrac{1}{h_1}\dfrac{\partial}{\partial u_1} & \dfrac{1}{h_2}\dfrac{\partial}{\partial u_2} & \dfrac{1}{h_3}\dfrac{\partial}{\partial u_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

(*i*) **Scalar Laplacian-Divergence of the gradient of a scalar function.** It has been established that the divergence of vector **A**, in curvilinear coordinates is

$$\overline{\nabla} \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right] \quad ...(248)$$

If vector **A** is the gradient of some scalar function ϕ, that is

$$\mathbf{A} = \overline{\nabla}\phi$$

$$= \mathbf{a}_1 \frac{1}{h_1}\frac{\partial \phi}{\partial u_1} + \mathbf{a}_2 \frac{1}{h_2}\frac{\partial \phi}{\partial u_2} + \mathbf{a}_3 \frac{1}{h_3}\frac{\partial \phi}{\partial u_3} \quad ...(249)$$

Now, comparing (240) with the general expression

$$\mathbf{A} = \mathbf{a}_1 A_1 + \mathbf{a}_2 A_2 + \mathbf{a}_3 A_3$$

we obtain

$$A_1 = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}$$

$$A_2 = \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}$$

$$A_3 = \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}$$

Introducing these in Eq. (248), yields

$$\overline{\nabla} \cdot \mathbf{A} = \overline{\nabla} \cdot (\overline{\nabla}\phi) = \overline{\nabla}^2 \phi$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial u_3}\right)\right] \quad ...(250)$$

The symbol $\overline{\nabla} \cdot \overline{\nabla}$, indicating a successive operation upon a scalar function, could always be used, but precedent has established the symbol ∇^2 as a notation. The combined form has been given the name of the Laplacian operator.

Writing (250) for the three commonly used coordinate systems gives

Cartesian : $$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \quad ...(251)$$

Cylindrical : $$\nabla^2\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial\phi}{\partial\phi}\right) + \frac{1}{\partial z}\left(r\frac{\partial\phi}{\partial z}\right)\right]$$

$$= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\phi^2} + \frac{\partial^2\phi}{\partial z^2} \quad ...(252)$$

Spherical : $$\nabla^2\phi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\phi}\right)\right]$$

$$= \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2} \quad ...(253)$$

It should be noted that two forms are given for each of the expressions for the cylindrical and spherical coordinates. One form may be more convenient to use than the other in a specific situation.

Example 1.34. *Compute the work done by the force vector*

$$\mathbf{F} = y\mathbf{a}_x - x\mathbf{a}_y$$

around the closed path *abcdefa* shown in Fig. 1.41.

Solution. The work done by the force vector **F** is given by $\int \mathbf{F}\cdot d\mathbf{l}$.

$$abcdefa = \int_a^b \mathbf{F}\cdot d\mathbf{l} + \int_b^c \mathbf{F}\cdot d\mathbf{l} + \int_c^d \mathbf{F}\cdot d\mathbf{l} + \int_d^f \mathbf{F}\cdot d\mathbf{l} + \int_f^a \mathbf{F}\cdot d\mathbf{l}.$$

These five integrals will be determined independently.

Before we find these integrals, we see that

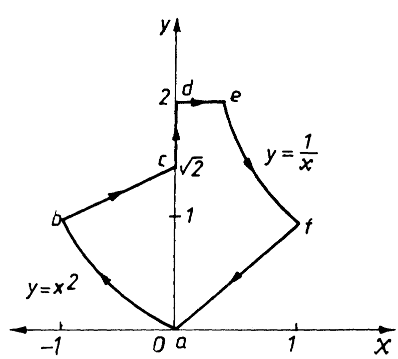


Fig. 1.42

$$\mathbf{F} \cdot d\mathbf{l} = (y\mathbf{a}_x - x\mathbf{a}_y) \cdot (dx\mathbf{a}_x + dy\ \mathbf{a}_y)$$
$$= y\ dx - x\ dy$$

(*a*) Along path *ab*, $y = x^2$, $dy = 2x\ dx$

and $$\mathbf{F} \cdot d\mathbf{l} = x^2\ dx - 2x^2 dx = -x^2\ dx$$

$$\therefore \quad \int_a^b \mathbf{F} \cdot d\mathbf{l} = -\int_{x=0}^{x=-1} x^2\ dx = \frac{1}{3} \qquad ...(i)$$

(*b*) Along path *bc*, $y = (\sqrt{2} - 1)\,x + \sqrt{2}$, then $= dy = (\sqrt{2} - 1)\ dx$.

Then, $$\mathbf{F} \cdot d\mathbf{l} = y\ dx - x\ (\sqrt{2} - 1)\ dx$$
$$= [(\sqrt{2} - 1)\,x + \sqrt{2}]\ dx - x\ (\sqrt{2} - 1)\ dx$$
$$= \sqrt{2}\ dx$$

$$\therefore \quad \int_b^c \mathbf{F} \cdot d\mathbf{l} = \int_{x=-1}^{0} \sqrt{2}\ dx = \sqrt{2} \qquad ...(ii)$$

(*c*) Along path *cd*, $x = 0$, $dx = 0$, $\mathbf{F} \cdot d\mathbf{l} = 0$

$$\therefore \quad \int_c^d \mathbf{F} \cdot d\mathbf{l} = 0 \qquad ...(iii)$$

(*d*) Along path *de*, $y = 2$, $dy = 0$

$$\mathbf{F} \cdot d\mathbf{l} = 2\ dx$$

and $$\int_d^e \mathbf{F} \cdot dl = \int_{x=0}^{1/2} 2\ dx = 1 \qquad ...(iv)$$

(*e*) Along path *ef*, $y = 1/x$, $dy = -\frac{1}{x^2}\ dx$

and $$\mathbf{F} \cdot d\mathbf{l} = \left(\frac{2}{x}\right) dx. \quad \text{Then}$$

$$\int_e^f \mathbf{F} \cdot d\mathbf{l} = \int_{x=1/2}^{1} \frac{2}{x}\ dx = 2\log_e 2 \qquad ...(v)$$

(*f*) Along path *fa*, $y = x$, $dy = dx$ and

$$\mathbf{F} \cdot d\mathbf{l} = x\ dx - x\ dx = 0$$

$$\therefore \quad \int_f^a \mathbf{F} \cdot d\mathbf{l} = 0 \qquad ...(vii)$$

Adding Eqs. (*i*) - (*vii*), we obtain the total work done

$$\int \mathbf{F} \cdot d\mathbf{l} = \frac{1}{3} + \sqrt{2} + 0 + 1 + 2\log_e 2 + 0 \approx 4.134.$$

Note that $\mathbf{F} = y\mathbf{a}_x - x\mathbf{a}_y = -r\mathbf{a}_\phi$, *i.e.*, the force is everywhere tangential to the circle with the centre at the origin and since paths *cd* and fa are radial to the origin, $\mathbf{F} \cdot d\mathbf{l} = 0$ for these paths. Thus $\int \mathbf{F} \cdot d\mathbf{l}$ is zero for the paths cd and fa.

Example 1.35. *Show by direct integration that the surface area of a sphere of radius R is $4\pi R^2$ and that the enclosed volume is $\frac{4}{3}\pi R^3$.*

Solution. The surface area is given by

$$S = \int dS_r = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} R^2 \sin\theta \, d\theta \, d\phi$$

$$= R^2 \int_{\phi=0}^{2\pi} \int_{\theta}^{\pi} \sin\theta \, d\theta \, d\phi = 4\pi R^2$$

The volume is similarly obtained as

$$V = \int dv = \int_{r=0}^{R} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{4\pi}{3} R^3.$$

Example 1.37. *Find the (smallest) angle between the two vectors*

$$\mathbf{A} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$$
$$\mathbf{B} = -3\mathbf{a}_x + 4\mathbf{a}_y - \mathbf{a}_z$$

and also determine a unit vector perpendicular to the plane containing **A** *and* **B**.

Solution. The angle between the two vectors can be found by using either the dot product or the cross product. Using the dot product, we have

$$\cos\theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{|A|\,|B|}$$

Forming

$$\mathbf{A} \cdot \mathbf{B} = (\mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z) \cdot (-3\mathbf{a}_x + 4\mathbf{a}_y - \mathbf{a}_z)$$
$$= -3 - 12 - 2 = -17$$
$$|A| = \sqrt{1+9+4} = \sqrt{14} = 3.74$$
$$|B| = \sqrt{9+16+1} = \sqrt{26} = 5.1$$

Then $\cos\theta_{AB} = \dfrac{-17}{3.74 \times 5.1} = -0.89$

or $\theta_{AB} = 153°$

from the cross product we may also obtain

$$\sin \theta_{AB} = \frac{|\mathbf{A} \times \mathbf{B}|}{|A|\ |B|}$$

forming

$$\mathbf{A} \times \mathbf{B} = -5\,\mathbf{a}_x - 5\mathbf{a}_y - 5\mathbf{a}_z$$

$\therefore$ $$|\mathbf{A} \times \mathbf{B}| = 5\sqrt{1+1+1} = 8.66$$

Then $$\sin \theta_{AB} = \frac{8.66}{3.74 \times 5.1} = 0.45$$

$\therefore$ $$\theta_{AB} = 27° \quad \text{or} \quad 153°$$

From the dot product, the ambiguity in angle is resolved. There are two unit vectors perpendicular to the plane containing **A** and **B**.

$$\mathbf{a}_n = \frac{\mathbf{A} \times \mathbf{B}}{|A|\ |B| \sin \theta_{AB}}$$

$$= -\frac{5\,(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{3.74 \times 5.1 \times 0.45}$$

$$= -0.58\,(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$$

or $$\mathbf{a}_n = 0.58\,(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$$

The second result is obtained as the negative of the first one.

EXERCISES

1. Prove, by vector methods that the sum of the squares of diagonals of a parallelogram is equal to the sum of the squares of its sides.

[**Hint :** Let **a** and **b** denote two sides of the parallelogram in magnitude and direction, and let $\mathbf{d}_1$ and $\mathbf{d}_2$ denote the two diagonals. Then $\mathbf{d}_1 = \mathbf{a} + \mathbf{b}$ and $\mathbf{d}_2 = \mathbf{a} - \mathbf{b}$. It follows that

$$\mathbf{d}_1 \cdot \mathbf{d}_1 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$$

and $$\mathbf{d}_2 \cdot \mathbf{d}_2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$

Add these to have the result].

2. If **A** and **B** are the sides of a parallelogram **C** and **D** are the diagonals, and θ is the angle between **A** and **B**, show that

$$(C^2 + D^2) = 2\,(A^2 + B^2)$$

and that $$C^2 - D^2 = 4\,AB \cos \theta$$

3. If $$\mathbf{r} \cdot d\mathbf{r} = 0$$

show that $$r = \text{constant.}$$

4. If $$\mathbf{A} = 2\,\mathbf{a}_x + 2\,\mathbf{a}_y - \mathbf{a}_z$$

and $$\mathbf{B} = 6\mathbf{a}_x - 3\,\mathbf{a}_y + 2\mathbf{a}_z.$$

Find (*i*) $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$

(*ii*) a unit vector perpendicular to the vectors **A** and **B**

$$\left[\frac{\mathbf{a}_x - 10\,\mathbf{a}_y - 18\,\mathbf{a}_z}{5\sqrt{17}}\right]$$

5. If $$\mathbf{A} = 4\,\mathbf{a}_x + 3\,\mathbf{a}_y + \mathbf{a}_z$$

and $$\mathbf{B} = 2\,\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z.$$

Find : (*i*) a unit vector $\mathbf{a}_n$ perpendicular to vectors **A** and **B** such that **A, B** and $\mathbf{a}_n$ form a right-handed system, and

(*ii*) the angle between the vectors **A** and **B**.

$$\left[\textbf{Ans.}\ \frac{7\,\mathbf{a}_x - 6\mathbf{a}_y - 10\mathbf{a}_z}{\sqrt{185}}\ ;\ 62°\ 40'\right]$$

6. If **a** and **b** are two unit vectors lying in the xy-plane and vectors **a** and **b** makes angles α and β with the x-axis respectively, such that

$$\mathbf{a} = \mathbf{a}_x \cos\alpha + \mathbf{a}_y \sin\alpha$$

and $$\mathbf{b} = \mathbf{a}_x \cos\beta + \mathbf{a}_y \sin\beta.$$

show that the trigonometric relations for the sine and cosine of the sum and difference of two angles follow from the interpretation of $(\mathbf{a}\cdot\mathbf{b})$ and $(\mathbf{a}\times\mathbf{b})$.

7. (*a*) Determine 't' so that the vectors $2\mathbf{a}_x + t\,\mathbf{a}_y + \mathbf{a}_z$ and $4\mathbf{a}_x - 2\mathbf{a}_y - 2\mathbf{a}_z$ are perpendicular.

(*b*) If $$\mathbf{A} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$$
$$\mathbf{B} = -\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$$
and $$\mathbf{C} = 3\mathbf{a}_x + \mathbf{a}_y.$$

Find 't' such that $\mathbf{A} + t\,\mathbf{B}$ is perpendicular to **C**. [$t = 5$]

8. Prove that

(*i*) $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

(*ii*) $$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

(*iii*) $$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$$

(*iv*) $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = \begin{vmatrix} \mathbf{A}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{C} \\ \mathbf{A}\cdot\mathbf{D} & \mathbf{B}\cdot\mathbf{A} \end{vmatrix}$$

(*v*) $(\mathbf{A}\cdot\mathbf{B})\cdot\{(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A})\} = \{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\}^2$

9. If **A, B** and **C** are three vectors with components A_x, A_y, A_z etc., show that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

10. If **r** be the radius vector from the origin of coordinates to any point, and **A** be a constant vector, then show that

$$\overline{\nabla}\cdot(\mathbf{A}\cdot\mathbf{r}) = \mathbf{A}$$

11. If **A** is a constant vector and **r** is the vector from origin to the point (x, y, z), show that

(*a*) $(\mathbf{r} - \mathbf{A})\cdot\mathbf{A} = 0$, is the equation of a plane

(*b*) $(r - \mathbf{A})\cdot\mathbf{r} = 0$, is the equation of a sphere

(*c*) div $\mathbf{r} = 3$; curl $\mathbf{r} = 0$; grad $(\mathbf{A}\cdot\mathbf{r}) = A$

(*d*) for any arbitrary function $f(r)$,

$$\text{grad}\, f(r) = \frac{\mathbf{r}}{r}\frac{df}{dr}$$

12. Find the gradient of a scalar function ϕ, where (*i*) $\phi(x, y, z) = x^2y + e^z$, at the point (1, 5, – 2)

(*ii*) $\phi = \log_e r$,

where $r = \sqrt{x^2 + y^2 + z^2}$

$$\left[\textbf{Ans. } \overline{\nabla}\phi = 10\,\mathbf{a}_x + \mathbf{a}_y + 0.136\mathbf{a}_z \left(r = \frac{\mathbf{a}_z}{e^2}\right)\overline{\nabla}\phi = \frac{\mathbf{r}}{r^2}\right]$$

13. Find the divergence and curl of the vector **A**, where

(*i*) $\mathbf{A} = x^2y\mathbf{a}_x - 2\,xz\mathbf{a}_y + 2\,yz\mathbf{a}_z$

(*ii*) $\mathbf{A} = xyz\mathbf{a}_x + 3x^2y\mathbf{a}_y + (xz^2 - y^2z)\,\mathbf{a}_z$

14. Prove that

(*i*) $\overline{\nabla} \cdot (\phi\mathbf{A}) = (\overline{\nabla}\phi) \cdot \mathbf{A} + \phi\,(\nabla \cdot \mathbf{A})$

(*ii*) $\overline{\nabla}\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \overline{\nabla})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \times (\overline{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\overline{\nabla} \times \mathbf{A})$

(*iii*) curl **A** = grad div $\mathbf{A} - \nabla^2\mathbf{A}$

(*iv*) div curl **A** = 0.

(*v*) If $\mathbf{F} = (x + y + 1)\,\mathbf{a}_x + \mathbf{a}_y - (x + y)\,\mathbf{a}_z$, then

$$\mathbf{F} \cdot \text{curl}\,\mathbf{F} = 0.$$

15. (*a*) Show that the vector field **A**, where

$$\mathbf{A} = (x^2 - y^2 + x)\,\mathbf{a}_x \quad 2\,xy + y)\,\mathbf{a}_z$$

is irrotational, and find a scalar function ϕ such that

$$\mathbf{A} = \text{grad}\,\phi.$$

$$\left[\textbf{Ans. } \frac{1}{3}x^3 - xy^2 + \frac{1}{2}x^2 - \frac{1}{2}y^2\right]$$

(*b*) A vector field is given by

$$\mathbf{A} = (x^2 + xy^2)\,\mathbf{a}_x + (y^2 + x^2y)\,\mathbf{a}_y$$

show that the field is irrotational, and find the scalar potential.

$$\left[\frac{1}{2}x^2y^2 + \frac{1}{3}(x^3 + y^3)\right]$$

16. Show that the vector field

$$\mathbf{V} = (x + 3y)\,\mathbf{a}_x + (y - 3z)\,\mathbf{a}_y + (x - 2z)\,\mathbf{a}_z$$

is solenoidal.

17. If $\mathbf{V} = \dfrac{x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$.

show that $\overline{\nabla} \cdot \mathbf{V} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$ and $\overline{\nabla} \times \mathbf{V} = 0$

18. Verify that the vector field

$$\mathbf{A} = yz\mathbf{a}_x + zx\mathbf{a}_y + xy\mathbf{a}_z,$$

is both irrotational and also solenoidal.

19. Find the divergence and curl of the vector field

$$\mathbf{F} = \frac{x^2}{r}\,\mathbf{a}_x$$

where $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$.

$$\left[\textbf{Ans.}\quad \bar{\nabla}\cdot\mathbf{F} = \frac{x^3 + 2x\,(y^2+z^2)}{r^3},\ \bar{\nabla}\times\mathbf{F} = \frac{x^2\,(\mathbf{a}_z y - \mathbf{a}_y z)}{r^3}\right]$$

20. Given $\mathbf{a} = \mathbf{a}_x \cos\alpha - \mathbf{a}_y \sin\alpha$

and $\mathbf{b} = \mathbf{a}_x \cos\beta + \mathbf{a}_y \sin\beta$,

using the above vectors derive the formula for expansion of sin $(\alpha + \beta)$. Illustrate by a figure.

21. A circular disc rotates with angular velocity ω about its axis of symmetry, which is taken to the z-axis. The direction of rotation is related to that of the z-axis by the right-hand screw rule. Find the velocity $\mathbf{v}$ of a point on the disc, and show that

$$\bar{\nabla}\times\mathbf{v} = 2\omega\mathbf{a}_z.$$

If now the disc is assumed to be nonrigid, such that ω is a function of the radius r, show that $\bar{\nabla}\times\mathbf{v} = 0$ if $\omega = \text{const}/r^2$.

22. A vector field is defined by $\mathbf{A} = f(r)\,\mathbf{r}$. Show that $f(r) = \text{const}\ /r^3$ if $\bar{\nabla}\cdot\mathbf{A} = 0$. Show that $\bar{\nabla}\times\mathbf{A} = 0$.

23. The vector $\mathbf{A} = 3x\,\mathbf{a}_x + y\mathbf{a}_y + 2\,z\mathbf{a}_z$, and $f = x^2 + y^2 + z^2$. Show that $\bar{\nabla}\cdot f\mathbf{A}$ at the point (2, 2, 2) is 120 by finding the vector $f(\mathbf{A})$ and taking its divergence.

Make the same calculation by first finding the vector $\bar{\nabla} f$ and using the identity.

$$\bar{\nabla}\cdot(f\mathbf{A}) = f(\bar{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot\bar{\nabla} f$$

If x, y and z are measured in centimeters, what are the units of $\bar{\nabla}\cdot f\mathbf{A}$?

24. The components of a vector $\mathbf{A}$ are

$$A_x = y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y},\ A_y = z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}$$

$$A_z = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}$$

where f is a function of x, y and z. Show that

$$\mathbf{A} = \mathbf{r}\times\bar{\nabla} f,\ \mathbf{A}\cdot\mathbf{r} = 0$$

and $$\mathbf{A}\cdot\bar{\nabla} f = 0.$$

25. Show that a field of force is conservative if the force exerted on a body is always directed toward a fixed centre and is only a function of r. Such a field is called a central force field. Find the potential energy at a distance r from a centre of attraction if the force varies at $1/r^2$. Set the potential energy equal to zero at infinity.

26. Show, by differentiating the appropriate expressions for $\mathbf{r}$, that the velocity $\dot{\mathbf{r}}\left(\text{or } \frac{dr}{dt}\right)$ is given by

$$\dot{\mathbf{r}} = \dot{r}\mathbf{a}_x + r\dot{\phi}\mathbf{a}_y + \dot{z}\mathbf{a}_z$$

in cylindrical coordinates and by

$$\dot{\mathbf{r}} = \dot{r}\mathbf{a}_x + r\dot{\theta}\mathbf{a}_y + r\sin\theta\,\dot{\phi}\,\mathbf{a}_z$$

in spherical coordinates.

27. The vector **A** is such that it is everywhere perpendicular to, and directed away from, a given straight line ; that is, in cylindrical coordinates, $A_z = A_\phi = 0$. Calculate the outgoing flux for a volume element, and show that

$$\overline{\nabla} \cdot \mathbf{A} = \frac{A_r}{r} + \frac{\partial A_r}{\partial r}.$$

28. The vector **r** is directed from P' (x', y', z') to P (x, y, z). If the point P is fixed and the point P' is allowed to move, show that the gradient of $(1/r)$ under these conditions is given by

$$\overline{\nabla}'\left(\frac{1}{r}\right) = \frac{\mathbf{r}_1}{r^2}$$

where $\mathbf{r}_1$ is the unit vector along **r**. Show that the above expression gives the maximum rate of change of $1/r$.

Show similarly that, if P' is fixed and P is allowed to move,

$$\overline{\nabla}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}_1}{r^2}$$

29. Show that

$$\int_v (\overline{\nabla} \times \mathbf{A})\, dv = -\int_S \mathbf{A} \times d\mathbf{s}$$

where **A** is an arbitrary vector and s is the surface bounding the volume v.

30. From the definition of the divergence, obtain an expression for div **F** in cylindrical coordinates.

$$\left[\textbf{Ans. } \text{div. } \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r_r F) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}\right]$$

31. A vector field is given by the expression

$$\mathbf{A} = \left(\frac{x}{r}\right)\mathbf{a}_x + \left(\frac{y}{r}\right)\mathbf{a}_y + \left(\frac{z}{r}\right)\mathbf{a}_z$$

where $\qquad r = \sqrt{x^2 + y^2 + z^2}$.

Transform the vector to cylindrical coordinates.

32. Explain the transformation of a vector from cylindrical coordinates into cartesian co-ordinates.

Transform the following vectors from cylindrical coordinates to cartesian coordinates

(*a*) $\frac{1}{r}\,\mathbf{a}_r$

(*b*) $\mathbf{a}_\phi \sec\phi$.

33. (*a*) Transform the following vector to spherical coordinates

(*i*) $\mathbf{A} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

(*ii*) $\frac{yz}{x}\,\mathbf{a}_z$

(*b*) Transform the following vector from spherical coordinates to cartesian coordinates.

(*i*) $\mathbf{A} = \frac{1}{r \sin\theta}\,\mathbf{a}_\phi$

(*ii*) $\mathbf{A} = \frac{1}{r \sin\theta \cos\phi}\,\mathbf{a}_\theta$.

34. Determine the cartesian components at (1, 1, 1) of the vector field $\mathbf{C} = \frac{1}{r}\,\mathbf{a}_r$ if the field is expressed in (*a*) cylindrical co-ordinates ; (*b*) spherical coordinates.

[**Ans.** (*a*) $\mathbf{F}\,(1, 1, 1) = \frac{1}{2}\,(\mathbf{a}_x + \mathbf{a}_y)$

(*b*) $\mathbf{F}\,(1, 1, 1) = \frac{1}{3}\,(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$]

35. Given the vector field

$$\mathbf{A} = \mathbf{a}_r \cos\phi \sin\theta + \mathbf{a}_\theta \cos\phi \cos\theta + \mathbf{a}_\phi \sin\phi$$

(*a*) Show that the magnitude of **A** is everywhere the same ;

(*b*) Show that $\mathbf{A} = \mathbf{a}_x \cos 2\phi + \mathbf{a}_y \sin 2\phi$.

36. (*a*) Prove that $\nabla^2 \left(\frac{1}{r}\right) = 0$, with the usual meaning of r.

(*b*) Transform the vector field $E = (1/r)\,\mathbf{a}\phi$, given in cylindrical co-ordinates, to spherical coordinates.

37. (*a*) If $A = x^2y\,\mathbf{a}_x - 2xz\,\mathbf{a}_y + 2yz\,\mathbf{a}_z$, where $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, represent unit vectors in the x, y, and z directions respectively, Find $\overline{\nabla} \times (\overline{\nabla} \times \mathbf{A})$. [**Ans.** $(2x + 2)\,\mathbf{a}_y$]

(*b*) Derive an expression for the divergence and curl of a vector field in curvilinear coordinates. Also, deduce expressions for grad ϕ and $\nabla^2\phi$ for the commonly used orthogonal systems.