1 Force Systems

1.1 Definitions

Mechanics. It is the science which deals with the action of forces on bodies.

Body. A body is a portion of matter occupying finite space. It is thus limited in every direction.

Rigid Body. A rigid body is that which does not change its shape or size when subjected to external forces.

Particle. It is a portion of matter which is indefinitely small in size.

Mass. The mass of a body is the quantity of matter it contains.

Force. It is a pull or push, which acting on a body changes or tends to change, the state of rest or of uniform motion of the body.

Force System. When a number of forces act on a body, they are called a force system or a system of forces.

Plane. A plane is a surface such that if two points are taken on the surface then the line joining these two points lies wholly in the surface.

Coplanar Forces. When all the forces acting on a body lie in the same plane, they are called coplanar forces.

Collinear Forces. When systems of coplanar forces have a common line of action, they are called coplanar forces.

Concurrent Forces. When system of forces intersect at a common point, then they are called concurrent forces.

Parallel Forces. When the lines of action of a system of forces are parallel, they are called parallel forces.

Equilibrium. If a number of forces acting on a body keep it at rest, the forces are said to be in equilibrium.

Statics. It is a science which deals with the action of forces on bodies, the forces being so arranged that they are in equilibrium.

Scalar Quantity. A quantity which has magnitude but no definite direction in space.

Vector Quantity. A quantity which has magnitude and a definite direction in space.

Composition of Forces. The combination of several forces into a single force, is called composition of forces.

Resolution of forces. The process of splitting a force into two or more components is called resolution of forces.

Resultant and Components. If two or more forces F_1, F_2, F_3,... etc. act upon a rigid body and if a single force F can be found whose effect upon the body is the same as that of all the forces F_1, F_2, F_3,..., then this force F is called the resultant of the forces F_1, F_2, F_3,... and the forces F_1, F_2, F_3,..., are called the components of F.

Weight. The weight of a body is the force with which the earth attracts the body towards its centre.

Unit of Force. In the MKS system, the unit of force is the weight of one kilogram and is written as kgf. In the SI system of units, the unit of force is Newton (N) (1 Newton = 0.102 kgf).

1.2 Representation of a Force

In order to completely specify a force, we require

(*a*) its magnitude,
(*b*) its point of application,
(*c*) the line along which it acts, and
(*d*) the direction along the line of action.

Thus we can completely represent a force by a straight line OA drawn through the point of application, O, along the line of action of the force, the length of the line OA representing the magnitude of the force to some arbitrarily chosen scale and the order of the letters O, A specifying the direction of the force. It is a customary and convenient practice to indicate the direction of the force by putting an arrow on the line in the direction of the force. For example, a force of 5 N acting at a point O towards right as

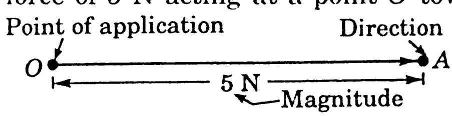


Fig. 1.1 Representation of a force.

shown in Fig. 1.1 can be represented by a straight line OA directed from the point O towards the right to a conveniently chosen scale. The line $\overrightarrow{OA}$ represents a force of 5 N in magnitude, direction and sense.

1.3 Force Systems

Force system. When a number of forces act on a body, they are said to form a force system or system of forces.

Classification of force systems. The force system may be classified according to the line of action and the arrangement of forces acting on a body, as explained below :

1. *Coplanar Forces.* When all the forces acting on a body lie in the same plane, they are called coplanar forces (see Fig. 1.2 (*a*)).

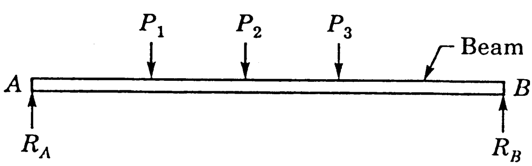
2. *Concurrent Forces.* When the system of forces intersect at a common point, they are called concurrent forces (see Fig. 1.2 (*b*)).

3. *Non-concurrent Forces.* When the system of forces do not have a common point of application, they are called non-concurrent forces. (see. Fig. 1.2 (*c*)).

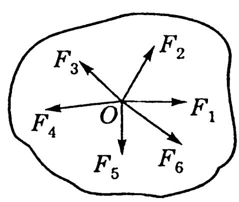
4. *Collinear Forces.* When all the forces of a force system are acting along the same line, they are called collinear forces (see Fig. 1.2 (*d*)).

5. *Non-coplanar concurrent forces.* A system of forces which do not lie in the same plane but have the same point of application are called non-coplaner concurrent forces. For example, the tripod stand for lifting a load shown in Fig. 1.2 (e) represents such a system of forces.

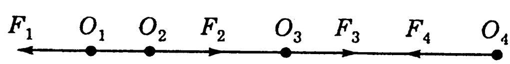
6. *Spatial forces.* These forces lie in space, as the forces in a three-dimensional truss.



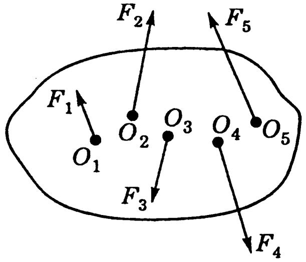
(*a*) Coplanar forces



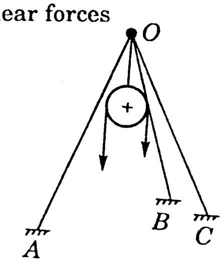
(*b*) Concurrent forces



(*d*) Collinear forces



(*c*) Non-concurrent forces



(*e*) Non-coplanar concurrent forces

Fig. 1.2 Various force systems

Two-Dimensional Force System. When a body is subjected to a coplanar force system then the forces can be resolved along two mutually perpendicular directions, say x-and y-axes. These two forces are said to form a two-dimensional force system.

Three-Dimensional Force System. When a body is subjected to a spatial force system then the forces can be resolved along three mutually perpendicular directions, say x-, y- and z- axes. These three forces are said to form a three dimensional force system.

1.4 Laws of Mechanics

There are six fundamental laws of mechanics as listed below :

1. Newton's three laws of motion.
2. Newton's law of gravitation.
3. Principle of transmissibility of force.
4. Parallelogram law of forces.

1. Newton's three laws of motion.

(*i*) *Newton's first low.* It states that everybody continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by an external force impressed on it. This law helps us to define a force as the external ageney which changes or tends to change the state of rest or of uniform linear motion of the body. Inertia is the tendency of a body to continue in its state of rest or of motion. Therefore, first law of motion may be considered as the law of inertia.

(*ii*) *Newton's second law.* It states tht the time rate of change of momentum of a body is directly proportional to the impressed force and takes place in the direction of the straight line in which the force is acting on it. Thus

Force, $F \propto$ Rate of change of momentum

Now momentum = mass × velocity = $m \times v$

$$\therefore \quad F \propto \frac{d}{dt}(mv)$$

$$\propto m\frac{dv}{dt} \text{ if } m \text{ is constant.}$$

$$\propto \text{mass} \times \text{acceleration}$$

$$\propto ma = kma$$

where k = constant of proportionality.

If $F = 1\text{N}$, $m = 1$ kg. and $a = 1 \text{ m/s}^2$, then $k = 1$

$\therefore \quad F = ma$

This law helps us to measure force quantitatively.

(*iii*) *Newton's third law.* It states that to every action there is an equal and opposite reaction. It means that the forces of action and reaction between two bodies are equal in magnitude and opposite in direction.

2. Newton's law of gravitation

It states that the force of attraction between two bodies is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. Let m_1 and m_2 be the masses of two bodies and r the distance between them, then as shown in Fig. 1.3.

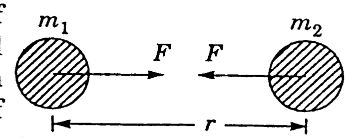


Fig. 1.3

$$\text{Force of attraction, } F \propto \frac{m_1 \times m_2}{r^2} = G\frac{m_1 m_2}{r^2}$$

where G = constant of proportionality and is called the universal constant of gravitation.

Consider a body of mass m lying on the surface of the Earth of mass M and radius R. Then

$$F = G\frac{mM}{R^2}$$

$$= \text{weight of the body, } W = mg$$

$$\therefore \quad g = \frac{GM}{R^2}$$

where g = acceleration due to gravity = 9.80665 m/s^2.

3. Parallelogram law of forces

This law states that if two forces acting at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram then their resultant is represented in magnitude and direction by the diagonal of the parallelogram. The diagonal passes through the point of intersection of the two sides representing the forces. If P and Q are the two forces acting at a point O and represented by the sides OA and OB of a parallelogram respectively, as shown in Fig. 1.4, then their resultant R is represented by the diagonal OC.

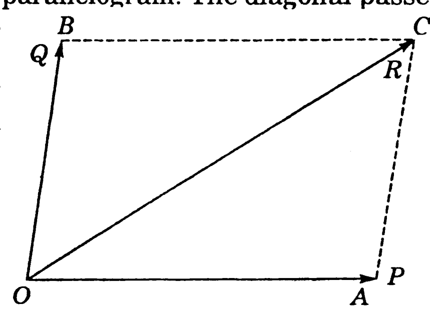


Fig. 1.4

4. Principle of transmissibility of force

This principle states that the state of rest or of motion of a rigid body remains unaltered if the point of application of the force acting on the rigid body is transmitted to act at any other point along the line of action of force.

Let F be the force acting on a rigid body acting along line ab at point A, as shown in Fig. 1.5 (a). According to the principle of

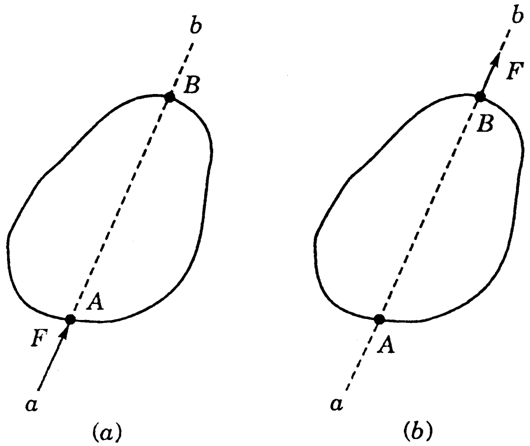


Fig. 1.5

transmissibility of force, the force F may be deemed to act at point B on straight line ab, as shown in Fig. 1.5 (b). This will not change the effect of force F on the rigid body. Let us apply two equal and opposite forces F_1 and F_2 such that $F_1 = F_2 = F$, at another point B located on the body along the line of action of F. The forces F_1 and F_2 will not disturb the equilibrium of the body being equal and opposite, cancel each other. Now forces F and F_1 being equal and opposite cancel each other, leaving a force $= F_2 = F$ at the point B. This proves the principle of transmissibility of a force.

1.5 S.I. System of Units

S.I. is the abbreviation for "The System International Units', also called the International System of Units. S.I. was formally recognised by the Eleventh General conference of weights and measures in 1960. This system is now being adopted throughout the world. It consists of 7 base units, 2 supplementary units and a number of derived units.

The base and supplementary units are given as follows in Table 1.1.

Table 1.1 Base and supplementary units

	Quantity	*Unit Name*	*S.I. Symbol*	*Other permissible units*
(*a*)	Base Units			
1.	Length (l)	metre	m	—
2.	Mass (m)	kilogram	kg	—
3.	Time (t)	second	s	—
4.	Temperature (T)	Kelvin	K	°C (celsius)
5.	Electric current (I)	Ampere	A	—
6.	Luminous intensity (Iv)	Candela	cd	—
7.	Amount of substance (n)	mole	mol	—
(*b*)	Supplementary units			
1.	Plane angle (α, β, θ, ϕ)	Radian	rad	—
2.	Solid angle (Ω)	Steradian	or	

The derived units used in Engineering Mechanics are given below in Table 1.2.

Table 1.2 Derived units used in Engineering Mechanics

	Physical Quantity	*Units*	*Symbol*
1.	Acceleration (a)	metre/second^2	m/s^2
2.	Angular speed (ω)	radian/second^2	rad/s^2
3.	Angular acceleration (α)	radian/second^2	rad/s^2
4.	Area (A)	square metre	m^2
5.	Density (ρ)	Kilogram/metre^3	kg/m^3
6.	Energy (E)	Joule	J(N.m)
7.	Force (F, P)	Newton	N
8.	Length (L)	metre	m
9.	Moment of a force (M)	newton-metre	N.m
10.	Power (P)	Watt	W(J/s)
11.	Pressure (p)	Pascal	Pa(N/m^2)
12.	Stress (σ)	newton/metre^2	Pa(N/m^2)
13.	Torque (T)	netwon/metre	N.m
14.	Velocity (v)	metre/second	m/s
15.	Volume (V)	Cubic metre	m^3
16.	Work (W)	Joule	J(N.m)

Prefixes used in S.I. units are given below in Table 1.3.

Table 1.3 Prefixes used in S.I. Units.

Prefix	*tera*	*giga*	*mega*	*kilo*	*hecto*	*deca*	*deci*	*centi*	*milli*	*micro*	*nano*	*pico*
Symbol	T	G	M	k	h	da	d	c	m	μ	n	p
Standard form	10^{12}	10^{9}	10^{6}	10^{3}	10^{2}	10^{1}	10^{-1}	10^{-2}	10^{-3}	10^{-6}	10^{-9}	10^{-12}

1.6 Parallelogram Law of Forces

This law states that if two forces acting at a point be represented in magnitude and direction by the adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.

Suppose that two forces F_1 and F_2 (Fig. 1.6) acting at a point O be represented in magnitude and direction by the sides OA and OB respectively of a parallelogram, their resultant F, is then represented completely by the diagonal OC of the parallelogram $OACB$.

In the vector notation form the parallelogram law of forces can be written as :

$$\vec{OA} + \vec{OB} = \vec{OC}$$

or $$\vec{F_1} + \vec{F_2} = \vec{F}$$

or $$F_1 + F_2 = F$$

where $\rightarrow$ stands for the vectorial sum.

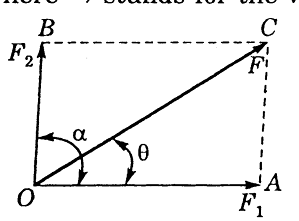


Fig. 1.6 Parallelogram law of forces

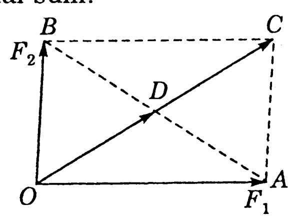


Fig. 1.7

Cor. 1 : Let the diagonal AB meet the diagonal OC in D. Then D is the middle point of OC. The resultant F is therefore represented by $2OD$ (Fig. 1.7).

Hence the resultant of two forces F_1, F_2 represented in magnitude and direction by the sides OA and OB of a triangle OAB is represented in magnitude and direction by $2OD$, where D is the middle point of the side AB.

Cor 2 : If $F_1 = F_2$, then $OA = OB$ and OD becomes the bisector of the angle AOB.

Hence the resultant of two equal forces bisects the angle between them.

1.7 Resultant of Two Forces Acting at a Point

Let two forces F_1 and F_2 acting at an angle α at a point O of a rigid body be represented in magnitude and direction by the two sides OA and OB of the parallelogram $OACB$, then the resultant F is represented by the diagonal OC.

To calculate the resultant F analytically, draw CD perpendicular to OA produced as shown in Fig. 1.8.

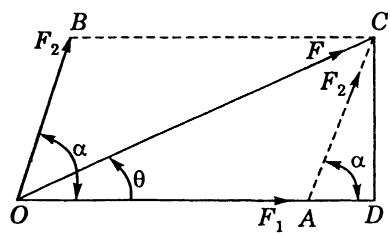


Fig. 1.8 Resultant of two forces.

Now $\quad OD = OA + AD = OA + AC \cos\alpha = F_1 + F_2 \cos\alpha$

$$CD = AC \sin\alpha = F_2 \sin\alpha$$

$$(OC)^2 = (OD)^2 + (DC)^2$$

$$F^2 = (F_1 + F_2 \cos\alpha)^2 + (F_2 \sin\alpha)^2$$

$$= F_1^2 + F_2^2 + 2F_1 F_2 \cos\alpha$$

or

$$F = \sqrt{F_1^2 + F_2^2 + 2F_1 F_2 \cos\alpha} \qquad \text{...(1.1)}$$

If θ is the angle which the resultant F makes with F_1, then we have

$$\tan\theta = \frac{CD}{OD} = \frac{CD}{OA + AD} = \frac{F_2 \sin\alpha}{F_1 + F_2 \cos\alpha}$$

$$\theta = \tan^{-1}\left(\frac{F_2 \sin\alpha}{F_1 + F_2 \cos\alpha}\right) \qquad \text{...(1.2)}$$

Cor. 1 : If the forces F_1 and F_2 are perpendicular to each other, *i.e.*, $\alpha = 90°$, then

$$F = \sqrt{F_1^2 + F_2^2} \text{ and } \tan\theta = \frac{F_2}{F_1}$$

Cor. 2 : If the forces F_1 and F_2 are equal, then

$$F = \sqrt{F_1^2 + F_1^2 + 2F_1^2 \cos \alpha}$$
$$= F_1 \sqrt{2\,(1 + \cos \alpha)}$$
$$= F_1 \sqrt{4 \cos^2 (\alpha/2)} = 2F_1 \cos \alpha/2$$

$$\tan \theta = \frac{F_1 \sin \alpha}{F_1 + F_1 \cos \alpha}$$
$$= \frac{2 \sin \alpha/2 \cos \alpha/2}{2 \cos^2 \alpha/2} = \tan \alpha/2$$

$$\therefore \qquad \theta = \frac{\alpha}{2}.$$

Hence the resultant of two equal forces acting at an angle bisects the angle between the forces.

Cor. 3 : If the two forces F_1 and F_2 act in the same direction, *i.e.*, $\alpha = 0°$, then

$$F = F_1 + F_2 \ \text{ and } \ \theta = 0$$

Thus the resultant of two forces acting along the same direction is equal to their sum and acts in the direction of the forces. This is also the greatest value of the resultant of two forces.

Cor. 4 : If the two forces F_1 and F_2 act in opposite directions, *i.e.*, $\alpha = 180°$, then

$$F = F_1 - F_2 \text{ (if } F_1 > F_2) \quad \text{ or } \quad F_2 - F_1 \text{ (if } (F_1 < F_2)$$

and $\qquad \theta = 0.$

Thus the resultant of two forces acting in the opposite directions is equal to their difference and acts in the direction of the bigger force. This is also the least value of the resultant of two forces.

Example 1.1 *The greatest possible resultant of two forces F_1 and F_2 is m times the least. Show that the angle α between them when their resultant is half their sum, is given by*

$$\cos \alpha = \frac{m^2 + 2}{2\,(m^2 - 1)}$$

Solution. The greatest possible resultant of F_1 and $F_2 = F_1 + F_2$.

The least possible resultant of F_1 and $F_2 = F_1 - F_2$ $(F_1 > F_2)$

Thus $\quad F_1 + F_2 = m\,(F_1 - F_2)$

or
$$F_1 = \left(\frac{m+1}{m-1}\right) F_2$$

when the angle between F_1 and F_2 is α, the resultant is $\frac{1}{2}(F_1 + F_2)$. Hence

$$[\tfrac{1}{2}(F_1 + F_2)]^2 = F_1^2 + F_2^2 + 2F_1 F_2 \cos\alpha$$

$$\cos\alpha = \frac{\frac{1}{4}(F_1^2 + F_2^2 + 2F_1 F_2) - (F_1^2 + F_2^2)}{2F_1 F_2}$$

$$= \frac{2F_1 F_2 - 3(F_1^2 + F_2^2)}{8F_1 F_2}$$

$$= \frac{2\left(\frac{m+1}{m-1}\right) F_2^2 - 3\left(\frac{m+1}{m-1}\right)^2 F_2^2 - 3F_2^2}{8\left(\frac{m+1}{m-1}\right) F_2^2}$$

$$= \frac{2(m^2-1) - 3(m+1)^2 - 3(m-1)^2}{8(m^2-1)}$$

$$= \frac{-4m^2 - 8}{8(m^2-1)} = -\frac{m^2+2}{2(m^2-1)}$$

Example 1.2 *Two forces of 40 N and 50 N act at a point at an angle of 60°. Determine their resultant in magnitude and direction.*

Solution. Here $F_1 = 40$ N, $F_2 = 50$ N, $\alpha = 60°$

Now
$$F = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$
$$= \sqrt{(40)^2 + (50)^2 + 2 \times 40 \times 50 \times \cos 60°}$$
$$= \sqrt{1600 + 2500 + 2000}$$
$$= \sqrt{6100} = 78.10 \text{ N}$$
$$\theta = \tan^{-1}\left[\frac{F_2 \sin\alpha}{F_1 + F_2\cos\alpha}\right]$$
$$= \tan^{-1}\left(\frac{50 \sin 60°}{40 + 50\cos 60°}\right) = \tan^{-1} 0.66617$$
$$\theta = 33.67°$$

Example 1.3 *Two forces act along the sides OA and OB of a triangle OAB, their magnitudes being proportional to cos* α *and cos* β. *Prove that their resultant is proportional to sin* γ *and its direction divides the angle* γ *into two parts* $\frac{1}{2}(\gamma + \beta - \alpha)$, $\frac{1}{2}(\gamma + \alpha - \beta)$.

Solution. Here $F_1 = k \cos \alpha$, $F_2 = k \cos \beta$, where k is a constant. Let θ be the angle which the resultant makes with OA. Then from Fig. 1.9, we have

$$F^2 = F_1^{\ 2} + F_2^{\ 2} + 2F_1 F_2 \cos \gamma$$

where $\gamma = \angle AOB$

$$F^2 = k^2 (\cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta \cos \gamma)$$

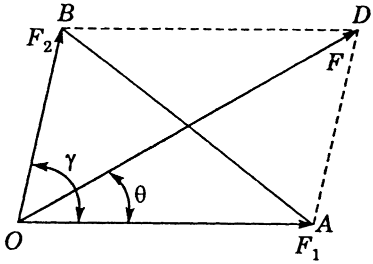


Fig. 1.9

Now $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 - 2 \cos \alpha \cos \beta \cos \gamma$

$\therefore \quad F_2 = k^2 (1 - \cos^2 \gamma) = k^2 \sin^2 \gamma$

Hence $\quad F \propto \sin \gamma$

Now
$$\tan \theta = \frac{k \cos \beta \sin \gamma}{k \cos \alpha + k \cos \beta \cos \gamma}$$
$$= \frac{\cos \beta \sin \gamma}{\cos \alpha + \cos \beta \cos \gamma}$$
$$= \frac{\cos \beta \sin \gamma}{\cos [\pi - (\beta + \gamma)] + \cos \beta \cos \gamma}$$
$$= \frac{\cos \beta \sin \gamma}{- \cos (\beta + \gamma) + \cos \beta \cos \gamma}$$
$$= \frac{\cos \beta \sin \gamma}{- \cos \beta \cos \gamma + \sin \beta \sin \gamma + \cos \beta \cos \gamma}$$
$$= \cot \beta = \tan \left(\frac{\pi}{2} - \beta \right)$$
$$= \frac{\pi}{2} - \beta$$
$$= \tfrac{1}{2} (\alpha + \beta + \gamma) - \beta$$
$$= \tfrac{1}{2} (\alpha + \gamma - \beta)$$
$$\angle BOD = \gamma - \tfrac{1}{2} (\alpha + \gamma - \beta)$$
$$= \tfrac{1}{2} (\beta + \gamma - \alpha)$$

Example 1.4 *A rigid body is subjected to two forces of 50 N and 40 N acting at 60°. Calculate their resultant in magnitude and direction.*

Solution. Given $P = 50$ N, $Q = 40$ N, $\alpha = 60°$

Resultant $$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$= \sqrt{50^2 + 40^2 + 2 \times 50 \times 40 \times \cos 60°} = 78.1 \text{ N}$$

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} = \frac{\sin\alpha}{\frac{P}{Q} + \cos\alpha}$$

$$= \frac{\sin 60°}{\frac{50}{40} + \cos 60°} = \frac{0.866}{1.25 + 0.5} = \frac{0.866}{1.75} = 0.4987$$

$$\theta = 26.33°.$$

Example 1.5 *The resultant of two forces P and Q is R. If one of the forces is reversed in direction, then the resultant becomes S. Prove that*

$$R^2 + S^2 = 2(P^2 + Q^2)$$

Solution. Let α = angle between the forces P and Q.

Then $$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

Let Q be reversed in direction

Then $$S = \sqrt{P^2 + (-Q)^2 + 2P \times (-Q) \times \cos\alpha}$$

$$= \sqrt{P^2 + Q^2 - 2PQ\cos\alpha}$$

$$R^2 + S^2 = (P^2 + Q^2 + 2PQ\cos\alpha) + (P^2 + Q^2 - 2PQ\cos\alpha) = 2(P^2 + Q^2)$$

Hence Proved.

Example 1.6 *The resultant of two forces 3P and 2P is R. If the first force is doubled the resultant is also doubled. Calculate the angle between the forces.*

Solution. Let α = angle between the forces

$$R = \sqrt{(3P)^2 + (2P)^2 + 2 \times 3P \times 2P \times \cos\alpha}$$

$$= \sqrt{9P^2 + 4P^2 + 12P^2\cos\alpha}$$

$$= P\sqrt{13 + 12\cos\alpha} \qquad \ldots(1)$$

$$2R = \sqrt{(6P)^2 + (2P)^2 + 2 \times 6P \times 2P \times \cos\alpha}$$

$$= \sqrt{36P^2 + 4P^2 + 24P^2\cos\alpha}$$

$$= P\sqrt{40 + 24\cos\alpha} = 2P\sqrt{10 + 6\cos\alpha}$$

or $$R = P\sqrt{10 + 6\cos\alpha} \qquad \ldots(2)$$

Comparing equation (1) and (2), we have

$$13 + 12\cos\alpha = 10 + 6\cos\alpha$$

$$6\cos\alpha = -3$$

$$\cos\alpha = -0.5$$

$$\alpha = 120°$$

Example 1.7 *Two equal forces act a point. If the square of their resultant is equal to three times of their product, then calculate the angle between them.*

Solution. Let each force = P and α the angle between them.

Resultant, $$R = \sqrt{P^2 + P^2 + 2P^2 \cos\alpha} = 2P \cos\left(\frac{\alpha}{2}\right)$$

$$R^2 = 3 \times P \times P$$

$$4P^2 \cos^2 \frac{\alpha}{2} = 3P^2$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{0.75} = 0.866$$

$$\frac{\alpha}{2} = 30°$$

$$\alpha = 60°$$

Example 1.8 *Two forces when acting at right angles produce a resultant of $\sqrt{10}$ N and when acting at 60° produce a resultant of $\sqrt{13}$ N. Find the two froces.*

Solution. Let the two forces be P and Q.

When $$\alpha = 90°, R = \sqrt{P^2 + Q^2}$$

$$10 = P^2 + Q^2 \quad ...(1)$$

When $$\alpha = 60°, R^2 = P^2 + Q^2 + 2PQ \cos 60°$$

$$13 = P^2 + Q^2 + 2PQ \cos 60° \quad ...(2)$$

Substituting equation (1) in (2), we have

$$13 = 10 + 2PQ \times 0.5$$

$$3 = PQ$$

Let $Q = 1$ N, then $P = 3$ N

Example 1.9 *Two forces equal to Pand 2P act on a rigid body. When the first force is increased by 100 N and the second force is doubled, the direction of the resultant remains unaltered. Determine the value of force P.*

Solution. Let α = angle between the forces

θ = angle between the resultant force and P

$$\tan\theta = \frac{2P \sin\alpha}{P + 2P\cos\alpha} = \frac{2\sin\alpha}{1 + 2\cos\alpha} \quad ...(1)$$

When P becomes $(P + 100)$ N and $2P$ becomes $4P$, then

$$\tan\theta = \frac{4P}{(P + 100) + (4P\cos\alpha)} \quad ...(2)$$

Comparing equation (1) and (2), we have

$$\frac{2\sin\alpha}{1 + 2\cos\alpha} = \frac{4P\sin\alpha}{(P + 100) + 4P\cos\alpha}$$

$$\frac{1}{1+2\cos\alpha} = \frac{2P}{(P+100)+4P\cos\alpha}$$

$$P + 100 + 4P\cos\alpha = 2P + 4P\cos\alpha$$

$$P + 100 = 2P$$

$$P = 100 \text{ N}$$

Example 1.10 *The resultant of two forces, one of which is double the other, is 300 N. If the direction of the larger force is reversed and the other remains unaltered, the resultant reduces to 150 N. Determine the magnitude of the forces and the angle between them.*

Solution. Let the smaller force be equal to P and the larger force equal to $2P$.

Also let α = angle between the forces. Then

Resultant force, $R^2 = P^2 + (2P)^2 + 2P + 2P\cos\alpha$

$$= P^2 + 4P^2 + 4P^2\cos\alpha$$

$$300 = P\sqrt{5 + 4\cos\alpha} \quad \ldots(1)$$

When the direction of larger force is reversed,

$$150 = \sqrt{P^2 + (-2P)^2 + 2P \times (-2P) \times \cos\alpha}$$

$$= P\sqrt{5 - 4\cos\alpha} \quad \ldots(2)$$

Dividing equation (1) by (2), we have

$$2 = \frac{\sqrt{5+4\cos\alpha}}{\sqrt{5-4\cos\alpha}}$$

or

$$4(5 - 4\cos\alpha) = 5 + 4\cos\alpha$$

$$20 - 16\cos\alpha = 5 + 4\cos\alpha$$

$$20\cos\alpha = 15$$

$$\cos\alpha = 0.75$$

$$\alpha = 41.4^\circ$$

From equation (1), we have

$$300 = P\sqrt{5 + 4 \times 0.75} = \sqrt{8}\,P$$

$$P = \frac{300}{\sqrt{8}} = 106.07 \text{ N}$$

1.8 Components of a Force in Two given Directions

Let OC represent the force F in magnitude and direction making an angle α with OX as shown in Fig. 1.10. Let OY be the other direction making an angle β with OC. Draw CA and CB parallel to OY and OX respectively to meet OX in A and OY in B. Then $OACB$ is a parallelogram and OA and OB are the required components.

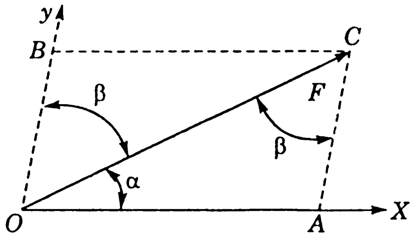


Fig. 1.10 Components of a force.

In ΔOAC, $\angle OCA = \beta$

Hence $\angle OAC = \pi - (\alpha + \beta)$

Using the sine formula, we get

$$\frac{OA}{\sin\beta} = \frac{AC}{\sin\alpha} = \frac{OC}{\sin[\pi - (\alpha + \beta)]}$$

Hence $$OA = \frac{F\sin\beta}{\sin(\alpha + \beta)} \quad ...(1.3)$$

$$OB = AC = \frac{F\sin\alpha}{\sin(\alpha + \beta)} \quad ...(1.4)$$

1.9 Resolved Part of a Force in a given Direction

The resolved part of a force in a given direction is the component in that direction, which with a component in a direction perpendicular to the given direction is equivalent to the given force.

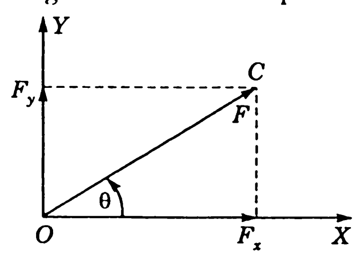


Fig. 1.11 Resolved parts of a force.

Consider a force F represented in magnitude and direction by OC making an angle θ with OX as shown in Fig. 1.11. Then the resolved part of the force F along OX is given by

$$F_x = F\cos\theta \quad ...(1.5)$$

The resolved part of F along OY which is perpendicular to OX is

$$F_y = F\sin\theta \quad ...(1.6)$$

Hence the resolved part of a force in a given direction is obtained by multiplying the given force by the cosine of the angle between the line of action of the given force and the given direction.

$$\tan\theta = \frac{AC}{OA} = \frac{F_y}{F_x}$$

and $F = \sqrt{F_x^2 + F_y^2}$

Similarly a force F acting on an inclined plane can be resolved into two rectangular components, as shown in Fig. 1.12.

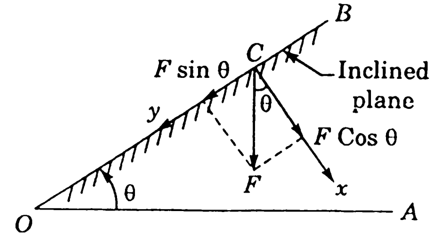


Fig. 1.12

Example 1.11 *A rigid body is subjected to two forces as shown in Fig. 1.13. Calculate their resultant and the angle made by the resultant with the x-axis.*

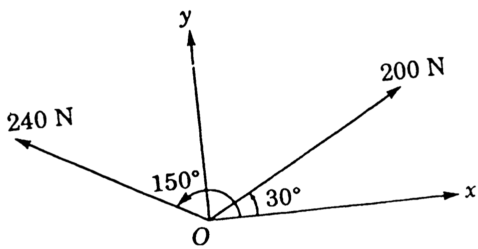


Fig. 1.13

Solution. Resolving the forces along x and y-axes, we have

$$\begin{aligned}\Sigma F_x &= 200\cos 30^\circ + 240\cos 150^\circ \\ &= 200 \times 0.866 + 240 \times (-0.866) \\ &= -34.641\text{ N.}\end{aligned}$$

$$\begin{aligned}\Sigma F_y &= 200\sin 30^\circ + 240\sin 150^\circ \\ &= 200 \times 0.5 + 240 \times 0.5 = 220\text{ N}\end{aligned}$$

Resultant, $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

$$= \sqrt{(-34.641)^2 + 220^2} = 227.71\text{ N}$$

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x} = \frac{220}{-34.641} = -6.35$$

$$\theta = -81.05^\circ$$

Angle made by resultant with x-axis $= 180^\circ - 81.05^\circ = 98.95^\circ$ ccw from x-axis.

1.10 Theorem of Resolved Parts

This theorem states that the algebraic sum of the resolved parts of any two forces F_1 and F_2 in any direction is equal to the resolved part of their resultant in that direction.

Let F_1 and F_2 be the two forces represented in magnitude and direction by OA and OB, respectively. Let OX be the direction along which the forces are to be resolved, as shown in Fig. 1.14

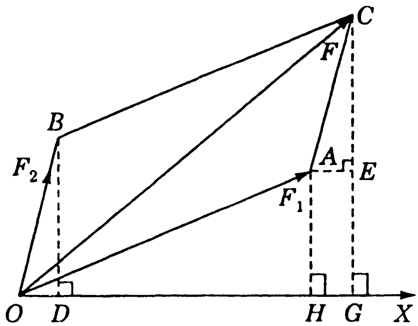


Fig. 1.14 Resolved parts of forces.

Complete the parallelogram $OACB$. The resultant of forces F_1 and F_2 is $OC = F$. Draw BD, AH and CG perpendicular on OX. Also draw AF perpendicular on CG.

Then OH, OD and OG are the resolved parts of F_1, F_2 and F respectively. The two triangles OBD and AEC being congruent,

$$OD = AE = HG$$

Hence $\quad OH + OD = OH + HG = OG$

1.10.1 Extension of the Theorem of Resolved Parts

The algebraic sum of the resolved parts of any number of on current forces in any direction is equal to the resolved parts of their resultant in that direction.

1.11 Resultant of Any Number of Forces

Let F_1, F_2, F_3, ... F_n be the n-forces acting on a body at a point O in directions $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ with OX. Let F be their resultant making an angle θ with OX (Fig. 1.15). Resolving the forces horizontally and vertically, we get

$$F_H = F \cos\theta = F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + \ldots + F_n \cos\alpha_n$$
$$F_V = F \sin\theta = F_1 \sin\alpha_1 + F_2 \sin\alpha_2 + \ldots + F_n \sin\alpha_n$$

Squaring and adding, we get

$$F = \sqrt{F_H^2 + F_V^2}$$
$$\tan\theta = F_V/F_H$$

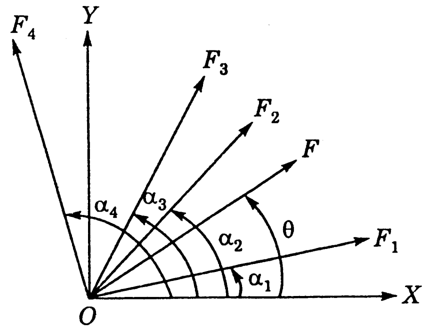


Fig. 1.15 Resultant of many forces.

Thus, the resultant force can be determined in magnitude and direction.

$$\text{Now } F^2 = (F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \ldots + F_n \cos \alpha_n)^2 + (F_1 \sin \alpha_1 + F_2 \text{ sir } \alpha_2 + \ldots + F_n \sin \alpha_n)^2$$

$$= F_1^2 \cos^2 \alpha_1 + F_2^2 \cos^2 \alpha_2 + \ldots + 2F_1 F_2 \cos \alpha_1 \cos \alpha_2 + \ldots + F_1^2 \sin^2 \alpha_1 + F_2^2 \sin^2 \alpha_2 + \ldots + 2F_1 F_2 \sin \alpha_1 \sin \alpha_2 + \ldots$$

$$= F_1^2 + F_2^2 + \ldots + F_n^2 + 2F_1 F_2 \cos (\alpha_2 - \alpha_1) + 2F_1 F_3 \cos (\alpha_3 - \alpha_1) + \ldots$$

$$= \sum_{i,j=1}^{n} [F_i^2 + 2F_t F_j \cos (\alpha_j - \alpha_i)]$$

$$F = \left\{ \sum_{i,j=1}^{n} [F_i^2 + 2F_i F_j \cos (\alpha_j - \alpha_i)] \right\}^{1/2}$$

Example 1.12 *The forces acting at a point O are as shown in Fig. 1.16. Determine their resultant in magnitude and direction.*

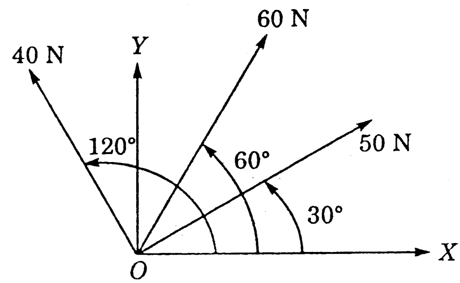


Fig. 1.16

Solution. (*a*) Analytical Method—Resolving the forces horizontally and vertically, we get

$$F_H = 50 \cos 30° + 60 \cos 60° + 40 \cos 120°$$
$$= 50 \sqrt{\frac{3}{2}} + 60 \times \frac{1}{2} - 40 \times \frac{1}{2}$$
$$= 43.30 + 30 - 20 = 53.3 \text{ N}$$
$$F_V = 50 \sin 30° + 60 \sin 60° + 40 \sin 120°$$
$$= 50 \times \frac{1}{2} + 60 \times \frac{3}{2} + 40 \times \sqrt{\frac{3}{2}}$$
$$= 25 + 51.96 + 34.64 = 111.60 \text{ N}$$
$$F = \sqrt{F_H^2 + F_V^2} = \sqrt{(53.30)^2 + (111.60)^2}$$
$$= \sqrt{2840.89 + 12454.78} = \sqrt{15295.67} = 123.67 \text{ N}$$
$$\tan \theta = \frac{F_V}{F_H} = \frac{111.60}{53.60} = 2.09380$$
$$\theta = 64.47°$$

(*b*) Graphical Method (Fig. 1.17)—Take a scale of 1 cm = 10 N. Draw OA = 5 cm making an angle of 30° with OX. At A draw AB = 6 cm at 60° with horizontal at A. At B draw BC = 4 cm at 120° with horizontal at B. Join OC. Then OC represents the resultant force in magnitude and direction. On measurement, we get OC = 12.4 cm. Hence F = 124 N. $\angle XOC = \theta = 65°$

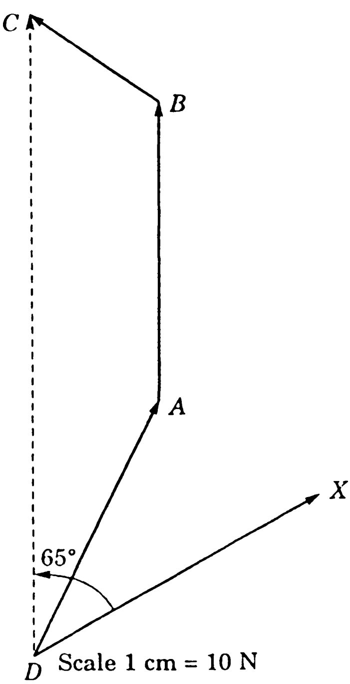


Fig. 1.17

Example 1.13 *Three forces acting at a point are parallel to the sides of a triangle ABC, taken in order, and are proportional to the consines of the opposite angles. Show that their resultant is proportional to*

$$(1 - 8 \cos A \cos B \cos C)^{1/2}$$

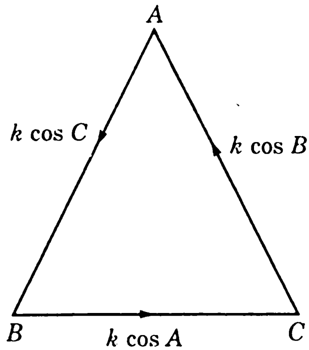


Fig. 1.18

Solution. The forces acting along the sides of the triangle are shown in Fig. 1.18, where k is a constant. Resolving along and perpendicular to BC, we get

$$F_H = k \cos A - k \cos B \cdot \cos C - k \cos C \cdot \cos B$$
$$= k (\cos A - 2 \cos B \cos C)$$
$$= k [\cos A - \{\cos (C - B) + \cos (C + B)\}]$$
$$= k [2 \cos A - \cos (C - B)] \qquad [\because \quad C + B = \pi - A]$$
$$F_V = k \cos B \sin C - k \cos C \cdot \sin B$$
$$= k (\cos B \sin C - \cos C \sin B) = k \sin (C - B)$$
$$F = \sqrt{F_H^2 + F_V^2}$$
$$= k [\{2 \cos A - \cos (C - B)\}^2 + \sin^2 (C - B)]^{1/2}$$
$$= k [\{4 \cos^2 A + \cos^2 (C - B) - 4 \cos A \cos (C - B)\} + \sin^2 (C - B)]^{1/2}$$
$$= k [1 + 4 \cos^2 A - 4 \cos A \cos (C - B)]^{1/2}$$
$$= k [1 + 4 \cos A \{\cos A - \cos (C - B)\}]^{1/2}$$
$$= k [1 + 4 \cos A \{- \cos (c + B) - \cos (C - B)\}]^{1/2}$$
$$= k [1 - 4 \cos A \{2 \cos C \cos B\}]^{1/2}$$
$$= k [1 - 8 \cos A \cos B \cos C]^{1/2}$$

Hence $F \propto (1 - 8 \cos A \cos B \cos C)^{1/2}$

Example 1.14 *Calculate the resultant in magnitude and direction of the system of concurrent and coplanar forces shown in Fig. 1.19.*

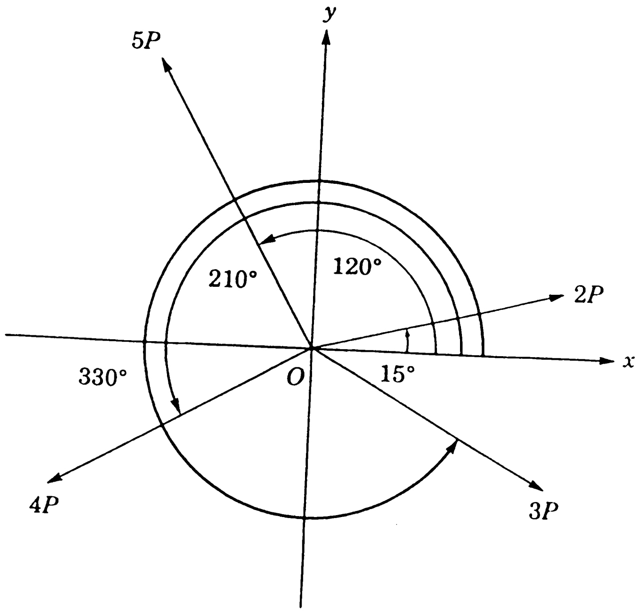


Fig. 1.19

Solution. Given : $F_1 = 2P$, $F_2 = 5P$, $F_3 = 4P$, $F_4 = 3P$

$$\alpha_1 = 45°, \alpha_2 = 120°, \alpha_3 = 210°, \alpha_4 = 330°$$

$$\sum F_x = \sum_{i=1}^{4} F_i \cos \alpha_i$$

$$= P\,[2 \cos 45° + 5 \cos 120° + 4 \cos 210° + 3 \cos 330°]$$

$$= -\,1.952\,P$$

$$\sum F_y = \sum_{i=1}^{4} F_i \sin \alpha_i$$

$$= P\,[2 \sin 45° + 5 \sin 120° + 4 \sin 210° + 3 \sin 330°]$$

$$= 2.24\,P$$

Resultant force,

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$= P\sqrt{(-\,1.952)^2 + (2.244)^2} = 2.974\,P$$

$$\theta = \tan^{-1}\left[\frac{\Sigma F_y}{\Sigma F_x}\right] = \tan^{-1}\left[\frac{2.244}{-\,1.952}\right] = \tan^{-1}(-\,1.1496)$$

$$= -\,48.98° \text{ or } 131.02° \text{ ccw from } x\text{-axis}$$

Example 1.15 *Figure 1.20 shows the forces acting on a block resting on an inclined plane. Determine the value of θ for which the resultant of the forces is directed parallel to the inclined plane.*

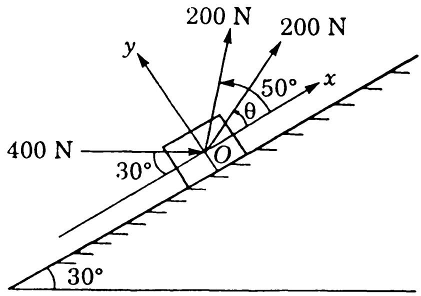


Fig. 1.20

Solution. As the resultant is directed parallel to the x-axis, so the resolved part of the forces along y-axis is zero.

$$\sum F_y = \sum_{i=1}^{3} F_i \sin \alpha_i = 0$$

$$200 \sin \theta + 200 \sin 50° - 400 \sin 30° = 0$$

$$\sin \theta + \sin 50° - 2 \sin 30° = 0$$

$$\sin \theta = 2 \times 0.5 - 0.766 = 0.234$$

$$\theta = 13.53°$$

Example 1.16 *Five forces are acting on a regular hexagon at an angular point as shown in Fig. 1.21. Calculate their resultant in magnitude and direction.*

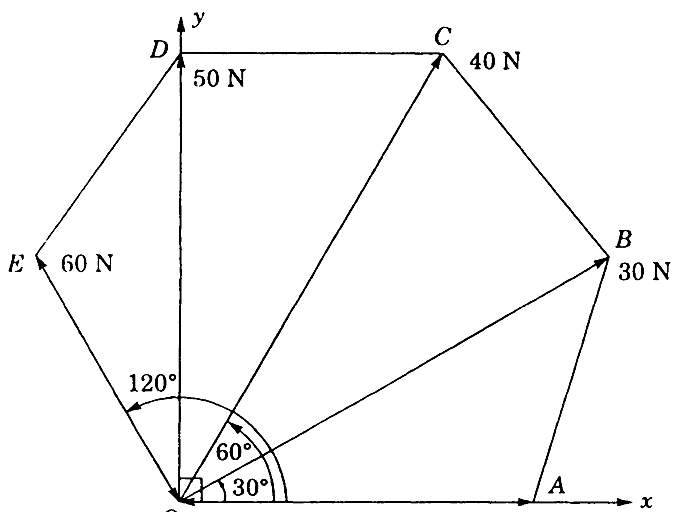


Fig. 1.21

Solution. Let $OABCDE$ be the regular hexagon.

Given : $F_1 = 20$ N, $F_2 = 30$ N, $F_3 = 40$ N, $F_4 = 50$ N, $F_5 = 60$ N

$$\alpha_1 = 0°, \alpha_2 = 30°, \alpha_3 = 60°, \alpha_4 = 90°, \alpha_5 = 120°$$

$$\Sigma F_x = \sum_{i=1}^{5} F_i \cos \alpha_i$$
$$= 20 \cos 0° + 30 \cos 30° + 40 \cos 60° + 50 \cos 90° + 60 \cos 120°$$
$$= 35.98 \text{ N}$$
$$\Sigma F_y = \sum_{i=1}^{5} F_i \sin \alpha_i$$
$$= 20 \sin 0° + 30 \sin 30° + 40 \sin 60° + 50 \sin 90° + 60 \sin 120°$$
$$= 151.60 \text{ N}$$

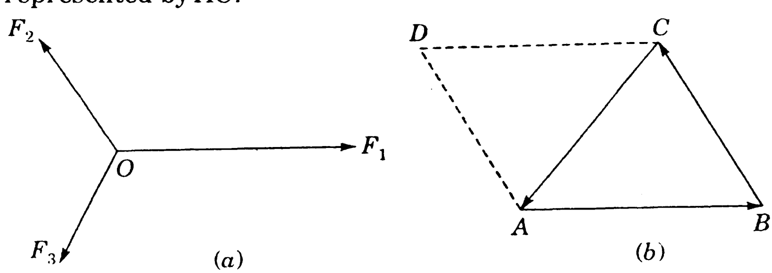
Resultant force,

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{(35.98)^2 + (151.60)^2} = 155.8 \text{ N}$$
$$\theta = \tan^{-1}\left[\frac{\Sigma F_y}{\Sigma F_x}\right] = \tan^{-1}\left[\frac{151.60}{35.98}\right] = \tan^{-1} 4.21345$$
$$\theta = 76.648°$$

1.12 Triangle Law of Forces

This law states that if three forces acting at a point be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.

Let the forces F_1, F_2 and F_3 acting at the point O be represented in magnitude and direction by the sides AB, BC and CA respectively of a triangle ABC. Complete the parallelogram $ABCD$ (Fig. 1.22). By the parallelogram law of forces, the resultant of AB and AD is represented by AC.



(a) (b)
Fig. 1.22 Triangle law of forces.

The resultant of AB, BC and CA

= the resultant of AB, AD and CA
= the resultant of AC and CA
= zero

Hence the forces F_1, F_2, F_3 are in equilibrium.

Cor. 1 : The resultant of forces AB and BC

= the resultant of AB and $AD = AC$.

Hence, if two concurrent forces acting at a point be represented in magnitude and direction by the sides of a triangle taken in the same order, their resultant is represented by the third side of the triangle in magnitude and direction taken in the opposite order, *i.e.*,

$$AB + BC = AC$$

1.12.1 Converse of the Triangle Law of Forces

If three concurrent forces acting at a point be in equilibrium, they can be represented in magnitude and direction by the sides of any triangle taken in order, which is drawn so as to have its sides respectively parallel to the directions of the forces.

1.13 Lami's Theorem

This theorem states that if three coplanar forces acting at a point are in equilibrium, each is proportional to the sine of the angle between the other two forces

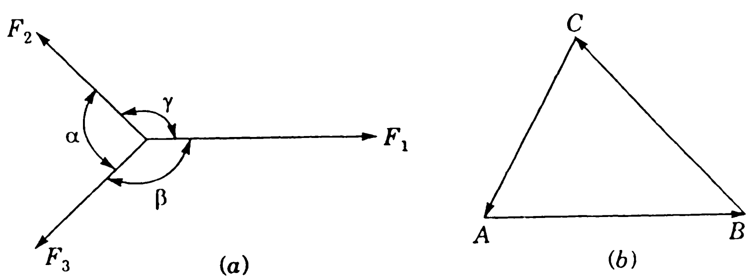


Fig. 1.23 Lami's theorem.

Let the forces $F_1, F_2\, F_3$ acting at a point O as shown in Fig. 1.23 be in equilibrium. Draw a $\Delta\, ABC$ whose sides AB, BC and CA are respectively parallel to the forces F_1, F_2 and F_3. These sides represent the three forces in magnitude and direction.

Now $\quad \angle ABC = \pi - \gamma$

$\angle BCA = \pi - \alpha$

$\angle CAB = \pi - \beta$

Using the sine formula, we have

$$\frac{AB}{\sin BCA} = \frac{BC}{\sin CAB} = \frac{CA}{\sin ABC}$$

or $$\frac{F_1}{\sin(\pi-\alpha)} = \frac{F_2}{\sin(\pi-\beta)} = \frac{F_3}{\sin(\pi-\gamma)}$$

$\therefore$ $$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma} \qquad \text{...(1.7)}$$

Hence each force is proportional to the sine of the angle between the other two forces.

1.13.1 Converse of Lami's Theorem

If three coplanar forces acting at a point are such that each is proportional to the sine of the angle between the other two forces, the forces are in equilibrium.

1.14 Polygon Law of Forces

This law states that if any number of forces acting at a point be represented in magnitude and direction by the sides of a polygon taken in order, the forces shall be in equilibrium.

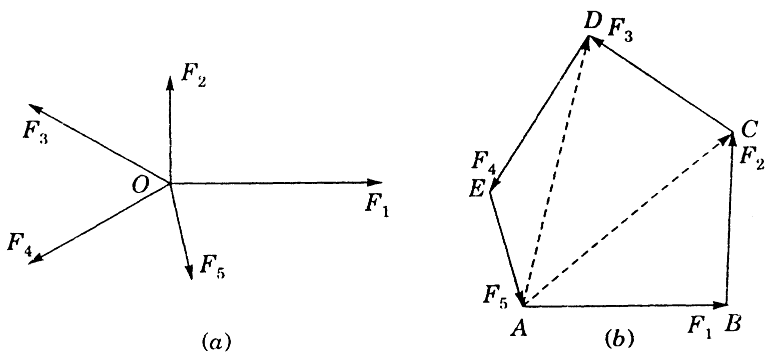


Fig. 1.24 Polygon law of forces.

Let the forces $F_1, F_2 ..., F_5$ act at a point O as shown in Fig. 1.24 (*a*). Let the sides AB, BC, CD, DE and EA represent these forces in magnitude and direction. Join AC and AD as shown in Fig. 1.24 (*b*).

Then $$AB + BC = AC$$

$\therefore$ $$AB + BC + CD = AC + CD = AD$$

$\therefore$ $$AB + BC + CD + DE + EA = AD + DE + EA$$
$$= AE + EA = 0$$

Hence, the forces are in equilibrium.

Cor. 1 : If any number of concurrent forces acting at a point be represented in magnitude and direction by the sides of a polygon taken in order, then the resultant is represented in magnitude and direction by the closing side of the polygon taken in the reverse order.

Example 1.17 *A weight of 5 N hangs by a string from a fixed point. The string is drawn out of the vertical by applying a force of 2.5 N to the weight. In what direction must this force be applied in order that, in equilibrium, the deflection of the string from the vertical may have its greatest value. What is the amount of the greatest deflection ? Find also the tension in the string.*

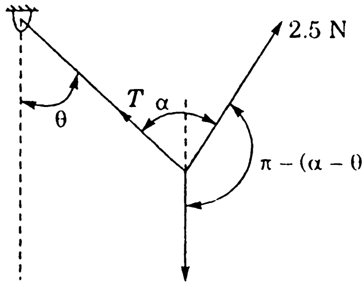


Fig. 1.25

Solution. Let the force of 2.5 N be applied in a direction making an angle α with the string. Let the deflection of the string from the vertical be θ and T be the tension in the string as shown in Fig. 1.25

Using Lami's theorem, we get

$$\frac{T}{\sin[\pi - (\alpha - \theta)]} = \frac{5}{\sin \alpha} = \frac{2.5}{\sin(\pi - \theta)}$$

$$\frac{T}{\sin(\alpha - \theta)} = \frac{5}{\sin \alpha} = \frac{2.5}{\sin \theta}$$

Hence $\qquad \sin \theta = \frac{1}{2} \sin \alpha$

θ will be maximum when α is maximum, which is so when $\alpha = 90°$

Hence $\qquad \sin \theta = \frac{1}{2}$

or $\qquad \theta = 30°$

Also $\qquad T = \frac{5 \sin(\alpha - \theta)}{\sin \alpha} = 5 \sin(90° - \theta)$

$$= 5 \sin 60° = 5 \times \frac{\sqrt{3}}{2} = 4.33 \text{ N}$$

Example 1.18 *A light rod BC of length 'a' is suspended from a fixed point A by means of two light strings AB and AC of lengths c and b respectively. If two weights each equal to W, are suspended at B and C, show that in the position of equilibrium the tensions of the strings are proportional to their lengths and that the thrust in the rod is of magnitude*

$$\frac{aW}{\sqrt{2b^2 + 2c^2 - a^2}}$$

Solution. In the position of equilibrium, let the rod be inclined at an angle θ to the vertical, as shown in Fig. 1.26. Let the tensions in AC and AB be T_1 and T_2 respectively and let T be the thrust in the rod BC.

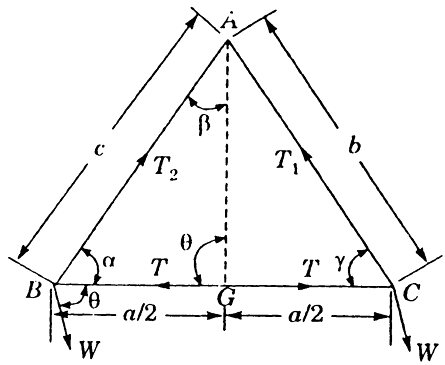


Fig. 1.26

For the equilibrium of point B, using Lami's theorem, we get

$$\frac{T}{\sin(\theta+\alpha)} = \frac{T_2}{\sin\theta} = \frac{W}{\sin\alpha}$$

$$T = \frac{W\sin(\theta+\alpha)}{\sin\alpha} = \frac{\sin(\pi-\beta)}{\sin\alpha} = W\frac{\sin\beta}{\sin\alpha}$$

Hence $\quad T = W\dfrac{BG}{AG}$

Now $\quad BG = \frac{1}{2}a$

and $\quad 2(AG)^2 + 2(BG)^2 = (AB)^2 + (AC)^2$

$\therefore \quad AG = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Hence $\quad T = W\dfrac{\frac{1}{2}a}{\frac{1}{2}\sqrt{2b^2+2c^2-a^2}} = \dfrac{aW}{\sqrt{2b^2+2c^2-a^2}}$

Also $\quad T_2 = \dfrac{W\sin\theta}{\sin\alpha}$

Similarly $T_1 = \dfrac{W\sin\theta}{\sin\gamma}$

$\therefore \quad \dfrac{T_1}{T_2} = \dfrac{\sin\alpha}{\sin\gamma} = \dfrac{b}{c}$

Hence, the tensions in the strings are proportional to their lengths.

Example 1.19 *Two small smooth rings each of weight w are free to slide along a fixed circular wire of radius 'a' in a vertical plane. The rings are joined by a light inextensible string of length 2b (< 2a), on which slides a smooth ring of weight W. Prove that when the system is in equilibrium, with the two rings on opposite sides of the vertical diameter, the distance of the ring W from the centre of the wire must be equal to*

$$\left[\frac{w(a^2-b^2)}{w+W}\right]^{1/2}$$

Solution. The equilibrium position of the system is shown in Fig. 1.27. The weight W will be vertically above the centre O and A, B will be at equal heights.

Let the tension in the string be T and the reaction of the wire be R. Let the string be inclined to the vertical at an angle θ and let $\angle COB = \phi$, $OC = y$. Also $OA = OB = a$ and $AC = BC = b$.

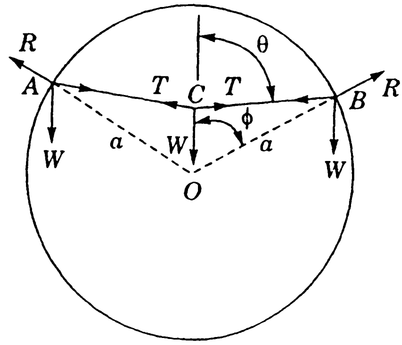


Fig. 1.27

In ΔBOC $\quad \dfrac{a}{\sin\theta} = \dfrac{b}{\sin\phi} = \dfrac{y}{\sin(\theta-\phi)}$

For the equilibrium of point C, we have

$$W = 2T\cos\theta$$

For the equilibrium of ring B, we have by Lami's theorem

$$\frac{w}{\sin(\theta-\phi)} = \frac{T}{\sin\phi}$$

$\therefore$
$$T = \frac{w\sin\phi}{\sin(\theta-\phi)}$$

Hence
$$W = \frac{2w\sin\phi\cos\theta}{\sin(\theta-\phi)} = \frac{2wb}{y}\cos\theta$$

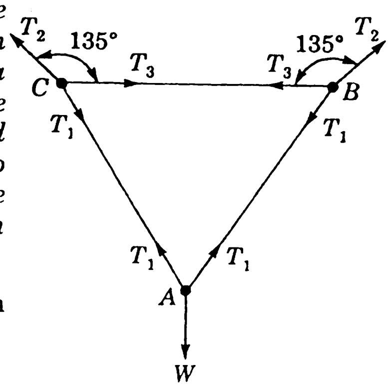
$$= \frac{2wb}{y}\left[-\frac{b^2+y^2-a^2}{2by}\right]$$

$$y^2W = -w(b^2+y^2-a^2)$$

$$(w+W)y^2 = w(a^2-b^2)$$

$$y = \left[\frac{w(a^2-b^2)}{w+W}\right]^{1/2}$$

Example 1.20 *Three equal strings of no sensible weight are knotted together to form an equilateral triangle ABC and a weight W is suspended from A. If the triangle and weight be supported with BC horizontal, by means of two strings at B and C, each at an angle of 135° with BC, show that tension in BC is* $\frac{1}{6}W(3-\sqrt{3})$.



Solution. Let the tensions in the strings be as shown in Fig. 1.28

For the equilibrium of point A, we have

$$\frac{T_1}{\sin 150°} = \frac{W}{\sin 60°}$$

$$T_1 = W\frac{\sin 150°}{\sin 60°} = \frac{W}{\sqrt{3}}$$

For the equilibrium of point B, we have

$$\frac{T_3}{\sin 165°} = \frac{T_1}{\sin 135°}$$

$$T_3 = T_1\frac{\sin 165°}{\sin 135°} = \frac{W}{\sqrt{3}}\cdot\frac{\frac{\sqrt{(3)}-1}{2\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= \frac{W(\sqrt{3})-1)}{2\sqrt{3}} = \frac{W(3-\sqrt{(3)})}{6}$$

Example 1.21 *ABCD is a string suspended from points A and D and carries a weight of 5 N at B and a weight of W N at C. The inclination to the vertical of AB and CD are 45° and 30° respectively and angle ABC is 165°. Find W and the tensions in the different parts of the string.*

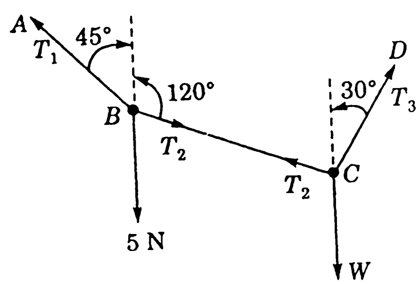


Fig. 1.29

Solution. Let T_1, T_2 and T_3 be the tensions in the parts AB, BC and CD respectively, as shown in Fig. 1.29. For the equilibrium of point B, we have

$$\frac{T_1}{\sin 60°} = \frac{T_2}{\sin 135°} = \frac{5}{\sin 165°}$$

$$T_1 = 5\frac{\sin 60°}{\sin 165°} = \frac{5\times 0.86602}{0.25882} = 16.73 \text{ N}$$

$$T_2 = 5\frac{\sin 135°}{\sin 165°} = \frac{5\times 0.70710}{0.25882} = 13.66 \text{ N}$$

For the equilibrium of point C, we have

$$\frac{T_2}{\sin 150°} = \frac{T_3}{\sin 120°} = \frac{W}{\sin 90°}$$

$$T_3 = T_2\frac{\sin 120°}{\sin 150°} = 13.66\times\frac{\sqrt{(3)}}{2}\times\frac{2}{1} = 23.66 \text{ N}$$

$$W = \frac{T_2 \sin 90°}{\sin 150°} = \frac{13.66\times 1}{0.5} = 27.32 \text{ N}$$

Example 1.22 *Four smooth pegs A, B, C, D are fixed in a vertical wall. ABCD is a square with two sides horizontal. An endless string of length l (> 4a, where a is the length of one side of the square) passes over the pegs and a weight W is free to slide on the lowest part of the string. Show that the tension in the string is*

$$\frac{W(l-3a)}{2\sqrt{l^2-6al+8a^2}}$$

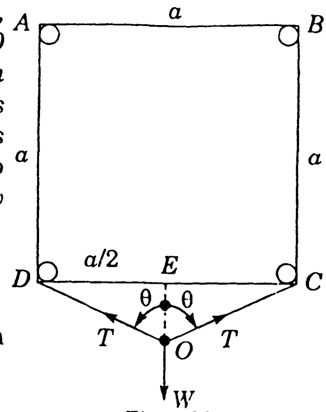


Fig. 1.30

Solution. The weight W will be in equilibrium, as shown in Fig. 1.30

$$DO = \frac{1}{2}(l-3a)$$

$$EO = \sqrt{(DO)^2-(DE)^2} = \sqrt{\tfrac{1}{4}(l-3a)^2-\tfrac{1}{4}a^2}$$

$$= \frac{1}{2}\sqrt{l^2+8a^2-6al}$$

Let T be the tension in the string. For the equilibrium of point O, we have

$$W = 2T\cos\theta = 2T\frac{EO}{DO} = 2T\frac{\tfrac{1}{2}\sqrt{l^2+8a^2-6al}}{\tfrac{1}{2}(l-3a)}$$

$$T = \frac{W(l-3a)}{2\sqrt{l^2-6al+8a^2}}$$

Example 1.23 *A fine string ABCDE whose extremity A is fixed has weights W_1 and W_2 attached to it at B and C and passes over a smooth pulley at D carrying a weight of 20 N at the free end E. If in the position of equilibrium, BC is horizontal and AB, CD make angles 60° and 30° respectively with the vertical, find*

(*a*) *tensions in the portions AB, BC, CD and DE*

(*b*) *the value of the weights W_1 and W_2, and*

(*c*) *the pressure on the pulley axis.*

Solution. Since the string passes over a smooth pulley at D, the tension in CD portion of string is 20 N. Let the tensions in AB and BC be T_1 and T_2 respectively, as shown in Fig. 1.31

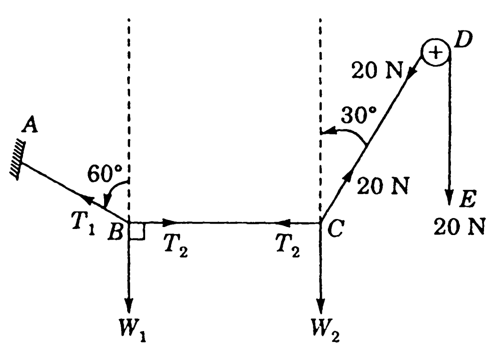


Fig. 1.31

For the equilibrium of point B, we have

$$\frac{T_1}{\sin 90^\circ} = \frac{T_2}{\sin 120^\circ} = \frac{W}{\sin 150^\circ}$$

and for the equilibrium of point C,

$$\frac{T_2}{\sin 150^\circ} = \frac{20}{\sin 90^\circ} = \frac{W_2}{\sin 120^\circ}$$

Hence $W_2 = 20 \times \dfrac{\sin 120^\circ}{\sin 90^\circ} = 20 \times \dfrac{\sqrt{3/2}}{2} = 17.32$ N

$T_2 = 20 \times \dfrac{\sin 150^\circ}{\sin 90^\circ} = 20 \times \dfrac{1}{2} = 10$ N

Thus $T_1 = T_2 \times \dfrac{\sin 90^\circ}{\sin 120^\circ} = \dfrac{10 \times 2}{\sqrt{3}} = 11.55$ N

$W_1 = T_2 \times \dfrac{\sin 150^\circ}{\sin 120^\circ} = 10 \times \dfrac{1}{2} \times \dfrac{2}{\sqrt{3}} = 5.77$ N

Pressure on the pulley

$$F = \sqrt{(20)^2 + (20)^2 + 2 \times 20 \times 20 \times \cos 30^\circ}$$

$$= 20\sqrt{2 + 2 \times \sqrt{3/2}} = 20\sqrt{2 + \sqrt{3}} = 38.6 \text{ N}$$

Example 1.24 *A steel ball of weight 250 N rests upon a smooth horizontal table. Two ropes AB and AC are attached to the centre of the ball and pass over frictionless pulleys at B anc C and carry loads of 150 N and 200 N as shown in Fig. 1.32. If the rope AB is horizontal, find (a) the angle θ that the rope AC makes with the horizontal, and (b) the pressure between the ball and the table.*

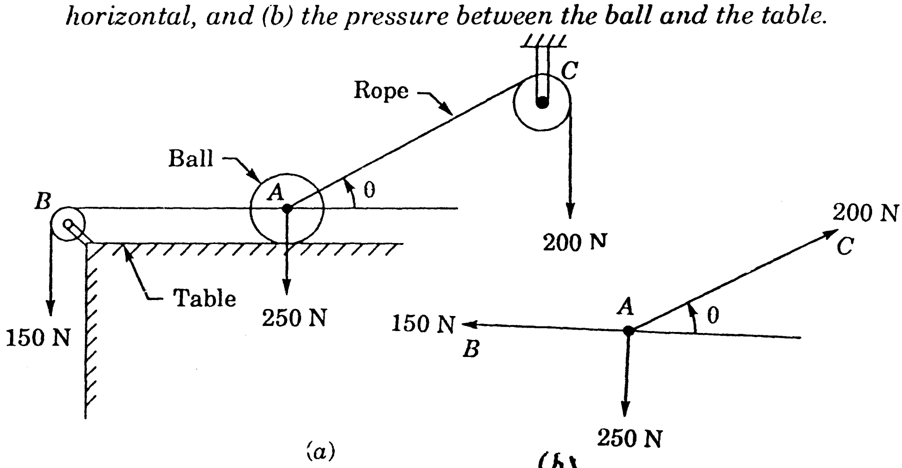


Fig. 1.32

Solution. (*a*) The forces acting on the ball are shown in Fig. 1.32 (*b*).

Resolving the forces horizontally, we have

$$200 \cos \theta = 150$$
$$\cos \theta = 0.75$$
$$\theta = 41.4°$$

(*b*) Let R = pressure between the ball and table

Resolving the forces on ball vertically, we have

$$R = 250 - 200 \sin 41.4° = 117.74 \text{ N}$$

Example 1.25 *A string of length 60 cm is tied to two supports at the same level and 40 cm apart. A smooth ring of weight 400 N is tied to the string at 40 cm from the left end and pulled by a horizontal force P. Determine the magnitude of force P, assuming that tensions in the string on both sides of ring are same.*

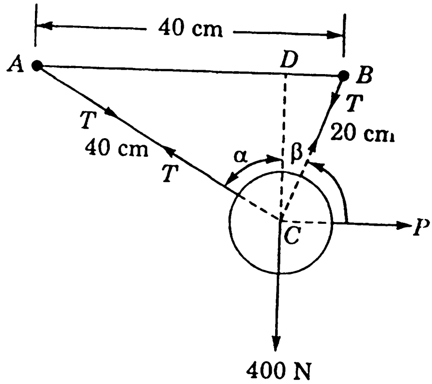


Fig. 1.33

Solution. The ring string system is shown in Fig. 1.33.

$$AD^2 = AC^2 - CD^2$$

or $$CD^2 = AC^2 - AD^2$$

and $$CD^2 = BC^2 - BD^2$$

$$\therefore \quad AC^2 - AD^2 = BC^2 - BD^2$$

$$40^2 - AD^2 = 20^2 - (AB - AD)^2$$

$$1600 - AD^2 = 400 - (40 - AD)^2$$

$$= 400 - 1600 - AD^2 - 80 \times AD$$

$$AD = \frac{2800}{80} = 35 \text{ cm}$$

$$CD = \sqrt{40^2 - 35^2} = 19.365 \text{ cm}$$

$$BD = 40 - 35 = 5 \text{ cm}$$

$$\tan \alpha = \frac{AD}{CD} = \frac{35}{19.365} = 1.807$$

$$\alpha = 61.04°$$

$$\tan \beta = \frac{BD}{CD} = \frac{5}{19.365} = 0.258$$

$$\beta = 14.48°$$

$$\alpha + \beta = 61.04 + 14.48 = 75.52°$$

Resolving forces at C horizontally, we have

$$P + T\cos(90^\circ - \beta) = T\cos(90^\circ - \alpha)$$

$$P + T\cos 75.52^\circ = T\cos 28.96^\circ$$

$$P = T(0.87496 - 0.25004) = 0.62492\,T$$

Resolving forces vertically at C,

$$T(\cos\alpha + \cos\beta) = 400$$

$$T(\cos 61.04^\circ + \cos 14.48^\circ) = 400$$

$$T = \frac{400}{1.4524} = 275.4 \text{ N}$$

$$P = 0.62492 \times 275.4 = 172.1 \text{ N}$$

Example 1.26 *A weight of 25 N is suspended by two strings 5 m and 10 m long. The other ends of the strings are fastened to the ends of a rod of 12 m length. The rod is kept in such a position that the weight hangs immediately below the middle point of the rod. Determine the tensions in the strings.*

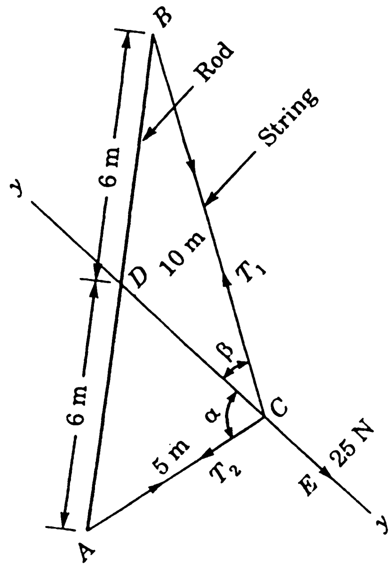


Fig. 1.34

Solution. Given :

$AB = 12$ m

$AD = DB = 6$ m

$AC = 5$ m, $BC = 10$ m

Let

T_1 = tension in string BC

T_2 = tension in string AC

Draw the Fig. 1.34 as follows :

1. Draw a vertical line y-y.
2. Choose a point C on y-y.
3. Draw two arcs with radii 5 m and 10 m to a convenient scale.
4. Draw a straight line AB by selecting a point A on 5 m arc and point B on 10 m arc such that point D is on line y-y and $AD = DB = 6$ m.
5. Join A and C and B and C.

6. Measure $\angle ACD = \alpha = 75°$ and $\angle BCD = \beta = 30°$.

Then $\angle ACB = \alpha + \beta = 105°$

$\angle ACE = 105°$, $\angle BCE = 150°$

Applying Lami's theorem, we have

$$\frac{25}{\sin 105°} = \frac{T_1}{\sin 105°} = \frac{T_2}{\sin 105°}$$

$\therefore$ $T_1 = 25$ N

$$T_2 = 25 \times \frac{\sin 150°}{\sin 105°} = 12.94 \text{ N}$$

Example 1.27 *A ladder of weight 200 N rests against a smooth vertical wall at a height of 5 m above the ground. The foot of the ladder is 2 m from the wall. Assuming the ground to be rough, determine the pressure due to the wall and the ground.*

Solution. The ladder system is shown in Fig. 1.35 (a).

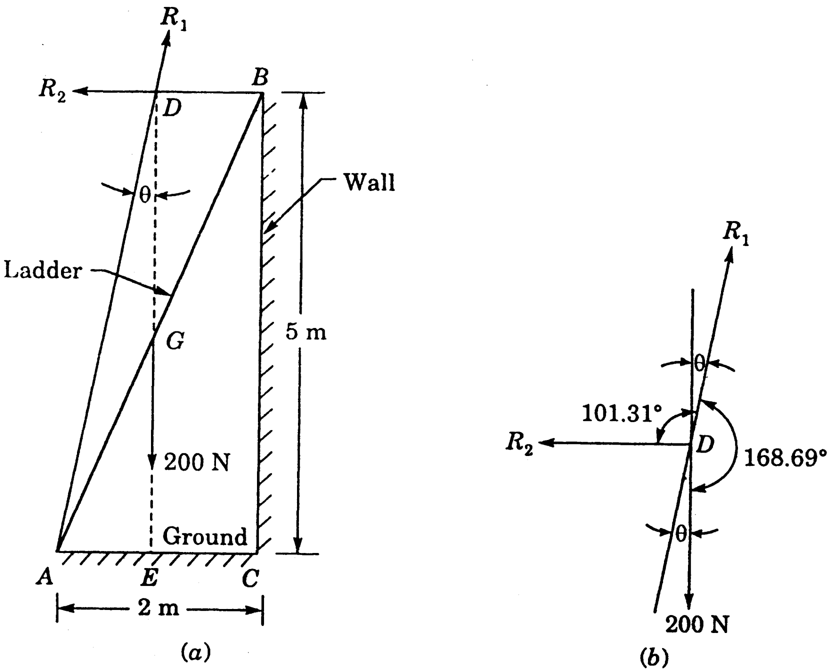


Fig. 1.35

Length of ladder, $AB = \sqrt{AC^2 + BC^2}$

$= \sqrt{2^2 + 5^2} = \sqrt{29} = 5.385$ m

$AE = EC = 1$ m

$$\tan\theta = \frac{AE}{DE} = \frac{AE}{BC} = \frac{1}{5} = 0.2$$

$\theta = 11.31°$

Applying Lami's theorem at point D (Fig. 1.35 b) we have

$$\frac{200}{\sin 101.31°} = \frac{R_1}{\sin 90°} = \frac{R_2}{\sin 168.69°}$$

$$R_1 = 200 \times \frac{\sin 90°}{\sin 101.31°} = 203.96 \text{ N}$$

$$R_2 = 200 \times \frac{\sin 168.69°}{\sin 101.31°} = 40 \text{ N}$$

1.15 Applied and Non-applied Forces

The various forces which may act on a rigid body can be classified into two groups as given below :

1. Applied forces
2. Non-applied forces

1. Applied Forces. These are the forces applied externally to a rigid body. They are also called external forces or action. They arise due to the contact with the body. Depending upon the type of contact with the body, the applied forces may be classified as follows :

(i) Point forces (ii) Distributed forces.

(i) *Point force.* It is the force which has make point contact with the body. Practically there is no force which may have point contact with the body. However, when the contact area is so small as compared to the dimensions of the body, it may be considered as a point contact for solving the problem. Some examples of point forces are :

(a) A sphere lying on a table.
(b) A man standing on a ladder.
(c) A fan hanging from the ceiling.
(d) A weight being lifted by a crane hook.

(ii) *Distributed Forces.* These forces are distributed over a line, surface area or volume. Correspondingly they are known as linear, surface and body forces.

(a) *Linear Forces.* A linear force acts along a line on the body. It is usually represented with abscissa representing the position on the body and ordinate representing the magnitude of the load. It is generally said to be a uniformly distributed load (udl). A common example of such a force is a beam subjected to udl. The force on an elementary length dl is $dF = w.dl$, where $w = udl$ per unit length.

(*b*) *Surface Force.* A force acting on the surface area of a body is called a surface force. It arises due to the contact between two bodies. The stress in a body is the outcome of surface force. The surface froce is defined as the force per unit area. The weight of a block lying on a table is a surface force. A man sitting on a chair gives rise to surface force on the chair. The hydrostatic pressure on a dam is the surface force.

Surface force, $dF = \sigma \times dA$ where σ = stress.

(*c*) *Body Force.* This force is defined either per unit mass or per unit volume. Such a force acts on the whole body. Some examples of body force are : gravitational force, electrostatic force, magnetic force, etc.

Body force, $dF = \gamma \times dV$

where γ = force per unit volume.

2. **Non-Applied Forces.** There are two types of non-applied forces :

(*i*) Self weight, and (*ii*) Reactions

(*i*) *Self Weight.* The weight of a body is due to the gravitational pull of the Earth. The self weight of a body is given by :

W = mass of the body,

$m \times$ acceleration due to gravity, g

where $g = 9.81$ m/s^2 on the Earth's surface.

The self weight of a body acts in a vertically downward direction and is considered to act through the centre of gravity of the body.

(*ii*) *Reactions.* These are self-adjusting forces developed by the other bodies which come in contact with the body under consideration. According to Newton's third law of motion, the action and reaction are equal and opposite. Therefore, reactions are the consequences or outcome of Newton's third law of motion. The reactions are also called *internal forces.*

If the surface of contact is smooth, the direction of the reaction is normal to the surface of contact. If the surface of contact is not smooth, apart from the normal reaction, there will be frictional reaction also. Hence, the resultant reaction will not be normal to the surface of contact.

The reactions may be either tensile or compressive in nature, as shown in Fig. 1.36.

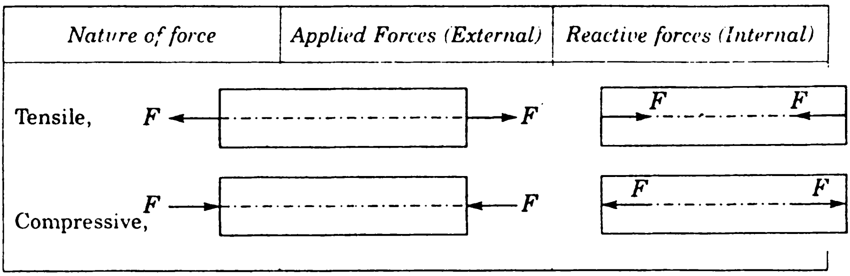
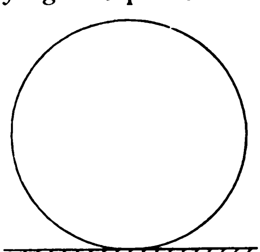
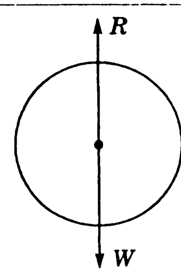


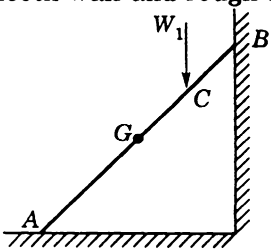
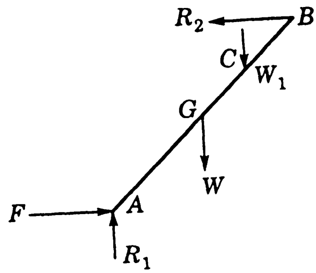
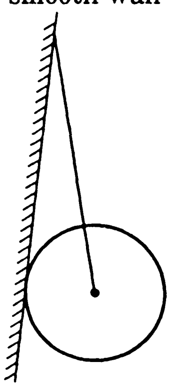
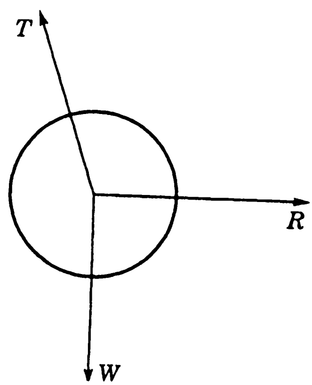
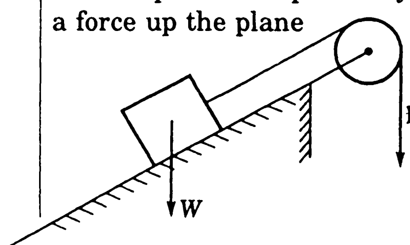
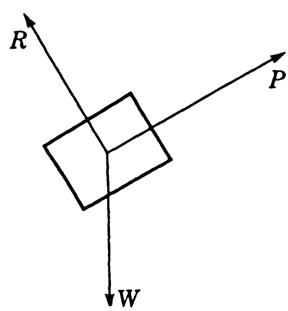
Fig. 1.36 Representation of external and internal forces.

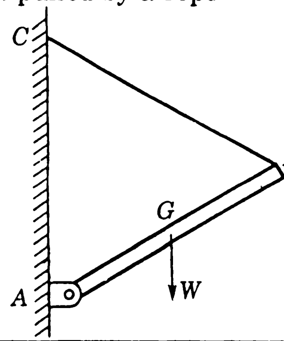
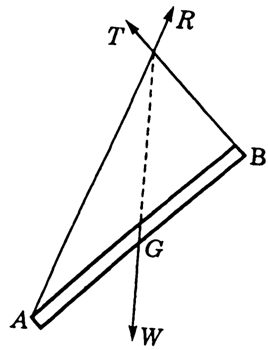
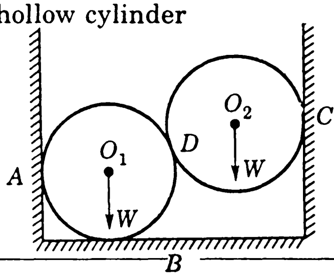
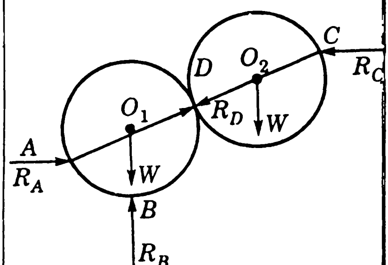
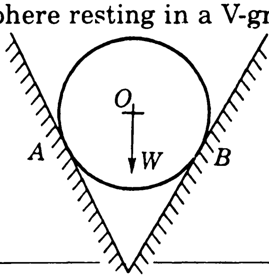
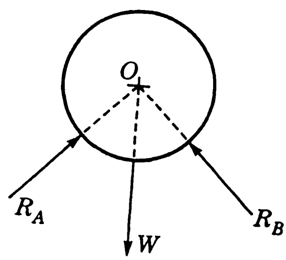
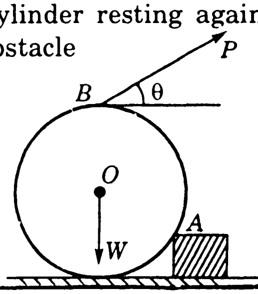
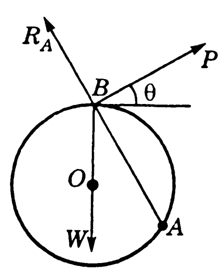
1.16 Free Body Diagrams

Free Body Diagram. In order to solve a problem, it is essential to isolate the body under consideration from the other bodies in contact and draw all the external (or applied) and internal (or reactive and self weight) forces acting on the body. *Such a diagram of the body in which the body under consideration is freed from all the contact surfaces and all the forces acting on it (both applied and non-applied) are drawn, is called a free body diagram.* In such a diagram, all the supports like walls, floors, hinges, etc. are removed and replaced by the reactions which these supports exert on the body. The free body diagrams for some of the cases are shown in the Table 1.4

Table 1.4 Free-body diagrams for some cases.

	Physical system	*Free body diagram*
1.	Ball lying on a plane surface 	

	Physical system	*Free body diagram*
2.	Ladder resting against a smooth wall and rough floor 	
3.	Ball tied by a string resting against a smooth wall 	
4.	Block resting on an inclined smooth plane and pulled by a force up the plane 	

	Physical system	*Free body diagram*
5.	Rod hinged against a wall and pulled by a rope 	
6.	Two spheres resting in a hollow cylinder 	
7.	Sphere resting in a V-groove 	
8.	Cylinder resting against an obstacle 	

Example 1.28 *ABC is a uniform rod of weight W. It is supported with its end A against a smooth vertical wall AD by means of a string BD, DC being horizontal and BD inclined to the wall at an angle of 30°. Find the tension in the string and the reaction of the wall. Also prove that AB = $\frac{1}{3}$AC.*

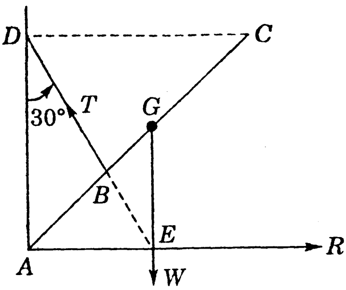


Fig. 1.37

Solution. Let the reaction of the smooth wall at A be R and perpendicular to the wall. Let the line of action of the weight W of the rod acting at the midpoint of AC meet the line of action of R at E. The three forces R, T and W must meet at E (Fig. 1.37). Using Lami's theorem, we get

$$\frac{T}{\sin 90^\circ} = \frac{R}{\sin 150^\circ} = \frac{W}{\sin 120^\circ}$$

$$T = W \cdot \frac{\sin 90^\circ}{\sin 120^\circ} = \frac{2W}{\sqrt{3}}$$

$$R = W \cdot \frac{\sin 150^\circ}{\sin 120^\circ} = \frac{W}{\sqrt{3}}$$

The triangles ABD and EBG are similar. Hence

$$\frac{BG}{AB} = \frac{GE}{AD} = \frac{1}{2}$$

Adding 1 to both sides, we get $1 + \frac{BG}{AB} = 1 + \frac{1}{2}$

$$\frac{AB + BG}{AB} = \frac{3}{2}$$

$$\frac{AG}{AB} = \frac{3}{2}$$

$$2AG = 3AB \quad \text{or} \quad AC = 3AB$$

$$\therefore \quad AB = \tfrac{1}{3}AC$$

Example 1.29 *A uniform bar AB of weight W and length l is hinged at its upper end A and a horizontal force is applied to the end B so that the bar is in equilibrium with B at a distance d from the vertical through A. Show that the reaction at the hinge is*

$$\frac{1}{2} W \left(\frac{4l^2 - 3d^2}{l^2 - d^2}\right)^{1/2}$$

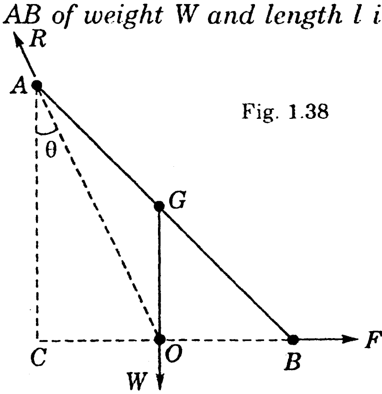


Fig. 1.38

Solution. Let the force F at B and weight W meet at O, then the reaction at A must pass through O. Let R make an angle θ with the vertical (Fig. 1.38)

Now $\quad AB = l, \quad BC = d$

$\therefore \quad AC = \sqrt{l^2 - d^2}$

Also $\quad CO = OB \quad \text{as} \quad AG = GB$

$$\therefore \quad OA = \sqrt{(AC)^2 + (OC)^2} = \sqrt{l^2 - d^2 + \frac{d^2}{4}} = \frac{1}{2}\sqrt{4l^2 - 3d^2}$$

Resolving the forces vertically, we get

$$R \cos\theta = W$$

$$R = \frac{W}{\cos\theta}$$

$$\text{Now } \cos\theta = \frac{AC}{AO} = \frac{\sqrt{l^2 - d^2}}{\frac{1}{2}\sqrt{4l^2 - 3d^2}} \quad \therefore \quad R = \frac{1}{2} W \frac{\sqrt{4l^2 - 3d^2}}{\sqrt{l^2 - d^2}}$$

Example 1.30 *Two smooth spheres of weights W_1 and W_2 rest upon two smooth inclined planes and against each other. If α, β be the inclinations of the planes of the horizontal and θ that of the line joining the centres of the spheres, prove that*

$$\tan\theta = \frac{W_1 \cos\beta - W_2 \cot\alpha}{W_1 + W_2}$$

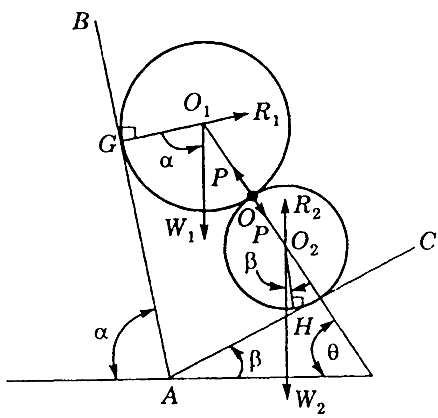


Fig. 1.39

Solution. Let the spheres touch the planes AB and AC at G and H respectively, as shown in Fig. 1.39. Let P be the pressure between the spheres at the point of contact O and R_1 and R_2 the normal reactions at G and H respectively. Then for the sphere of weight W_1,

$$\frac{P}{\sin\alpha} = \frac{W_1}{\sin(90^\circ + \alpha - \theta)}$$

and for the sphere of weight W_2,

$$\frac{P}{\sin\beta} = \frac{W_2}{\sin(90^\circ + \theta + \beta)}$$

$$\text{Hence} \quad \frac{W_1 \sin\alpha}{\cos(\alpha - \theta)} = \frac{W_2 \sin\beta}{\cos(\theta + \beta)}$$

$$W_1 \sin\alpha\,(\cos\theta\,\beta - \sin\theta \sin\beta)$$

$$= W_2 \sin\beta\,(\cos\alpha \cos\theta + \sin\alpha \sin\theta)$$

$$\cos\theta\,(W_1 \sin\alpha \cos\beta - W_2 \sin\beta \cos\alpha) = \sin\theta\,(W_2 \sin\beta \sin\alpha + W_1 \sin\alpha \sin\beta)$$

$$\tan\theta = \frac{W_1 \sin\alpha \cos\beta - W_2 \sin\beta \cos\alpha}{W_1 \sin\alpha \sin\beta + W_2 \sin\beta \sin\alpha} = \frac{W_1 \cot\beta - W_2 \cot\alpha}{W_1 + W_2}$$

Example 1.31 *A smooth circular cylinder of radius 2 m is lying in a triangular groove, one side of which makes an angle of 10° and the other an angle of 30° with the horizontal, as shown in Fig. 1.40. Find the reactions at the surface of contact if there is no friction and the weight of the cylinder is 50 N.*

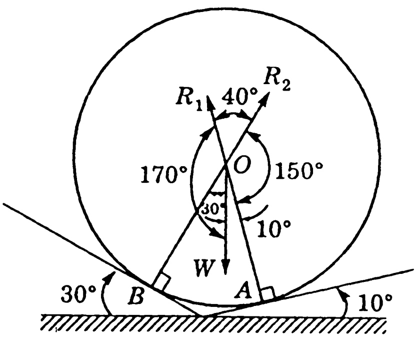


Fig. 1.40

Solution. Let R_1 and R_2 be the reactions of the 10° and 30° planes respectively.

Using Lami's theorem, we get

$$\frac{W}{\sin 40°} = \frac{R_1}{\sin 150°} = \frac{R_2}{\sin 170°}$$

$$R_1 = W\frac{\sin 150°}{\sin 40°} = W \times \frac{0.5}{0.64278} = 0.778\text{ W}$$

$$= 0.778 \times 150 = 116.6\text{ N}$$

$$R_2 = W\frac{\sin 170°}{\sin 40°} = 150 \times \frac{0.17365}{0.64278} = 40.52\text{ N}$$

Example 1.32 *Two smooth spheres of weight W and radius r each are in equilibrium in a horizontal channel of width (b (< 4r) and vertical sides, as shown in Fig. 1.41. Find the three reactions from the sides of the channel which are all smooth. Also find the force exerted by each sphere on the other. Calculate these values if r = 25 cm, b = 90 cm and W = 100 N.*

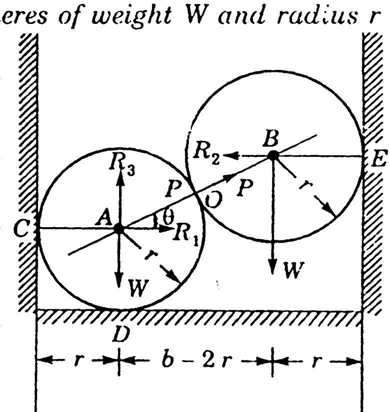


Fig. 1.41

Solution. Let R_1, R_2 and R_3 be the reactions at C, E and D respectively. Also let P be the force exerted by one sphere on the other at the point of contact O. Then

$$\cos\theta = \frac{b-2r}{2r}$$

The forces acting at the point A are $(R_3 - W)$, R_1 and P. Using Lami's theorem, we get

$$\frac{R_3 - W}{\sin\theta} = \frac{R_1}{\sin(90° - \theta)} = \frac{P}{\sin 90°}$$

$$\therefore \quad P = \frac{R_3 - W}{\sin\theta} \; ; \; R_1 = (R_3 - W)\cot\theta$$

The forces acting at the point B are W, R_2 and P. Again using Lami's theorem

$$\frac{R_2}{\sin(90° - \theta)} = \frac{P}{\sin 90°} = \frac{W}{\sin\theta}$$

$$R_2 = W\cot\theta \; ; \; P = \frac{W}{\sin\theta}$$

For $r = 25$ cm and $b = 90$ cm

$$\cos\theta = \frac{90 - 2\times 25}{2\times 25} = \frac{40}{50} = 0.8,$$

$$\theta = 36.87°$$

$$\therefore \quad R_2 = 100\cot 36.87° = 133.33 \text{ N}$$

$$P = \frac{100}{\sin 36.87°} = 166.66 \text{ N}$$

$$R_3 = P\sin\theta + W = 166.66\sin 36.87° + 100 = 200 \text{ N}$$

$$R_1 = (200 - 100)\cot 36.87° = 133.33 \text{ N}.$$

Example 1.33 *A uniform beam of length 2a rests in equilibrium against a smooth vertical wall and over a peg, at a distance b from the wall. If* θ *be the inclination of the beam to the vertical, show that sin* $\sin^3\theta = b/a$.

Solution. $AB = 2a$ is the beam and P is the peg. AD is the wall. R and S are the reactions at the wall and peg respectively (Fig. 1.42). $PE = b$. $AG = GB = a$. R, S and W must meet at the same point O, being in equilibrium.

$\angle BAD = \angle AGO = \angle APO = \theta$.

In ΔAPE, $\sin\theta = PE/AP = b/AP$ ...(1)

In ΔAPO, $\sin\theta = AP/OA$ ...(2)

In ΔAGO, $\sin\theta = OA/AG = OA/a$ ...(3)

Multiplying (1), (2) and (3), we get

$$\sin^3\theta = b/a$$

Hence proved.

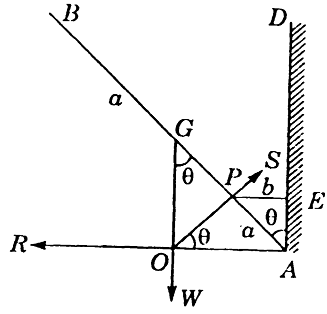


Fig. 1.42

Example 1.34 *ABCD is a square whose side is 2 units. Forces P, Q, R, S act along the sides AB, BC, CD, DA taken in order and forces $\sqrt{2p}$, $\sqrt{2q}$ act along AC and BD respectively. Show that if $p - q = R - P$ and $p + q = S - Q$, the forces are equivalent to a couple of moment $P + Q + R + S$.*

Solution. Let O be the centre of the square. Distance of O from each side is one unit. Sum of the resolved parts of the forces along AB (Fig. 1.43) is

$$= P - R + \sqrt{2p}\cos 45^\circ - \sqrt{2q}\cos 45^\circ$$

$$= P - R + p - q \qquad ...(1)$$

Similarly, along AD

$$= Q - S + \sqrt{2p}\cos 45^\circ + \sqrt{2q}\cos 45^\circ$$

$$= Q - S + p + q \qquad ...(2)$$

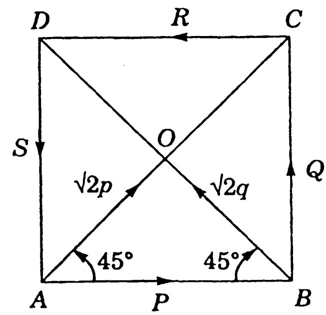


Fig. 1.43

If the system reduces to a couple, then (1) and (2) must vanish separately.

$$p - q = R - P$$

$$p + q = S - Q$$

Moment of the couple = Algebraic sum of the moments of the forces about any point in their plane, say, point O.

$$= P * 1 + Q * 1 + R * 1 + S * 1$$

$$= P + Q + R + S$$

Example 1.35 *Two weights P and Q are suspended from a fixed point O by strings OA, OB and are kept apart by a light rod AB. If G is the point on the rod vertically below O, determine the thrust in the rod AB.*

Solution.

Let $\quad T_1$ = tension in OA

T_2 = tension in OB

T = thrust in the rod AB

Applying Lami's theorem at A (Fig. 1.44), we have

$$\frac{T_1}{\sin\theta} = \frac{P}{\sin\alpha} = \frac{T}{\sin(\alpha+\theta)}$$

$$T = \frac{P\sin(\alpha+\theta)}{\sin\alpha}$$

Similarly at B,

$$\frac{T_2}{\sin\theta} = \frac{Q}{\sin\beta} = \frac{T}{\sin(180^\circ - \theta + \beta)}$$

$$T = \frac{Q\sin(\theta-\beta)}{\sin\beta}$$

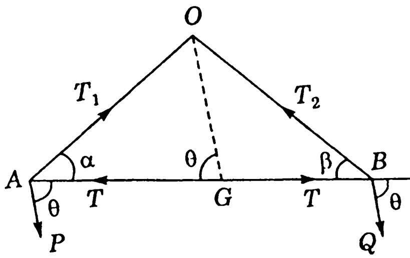


Fig. 1.44

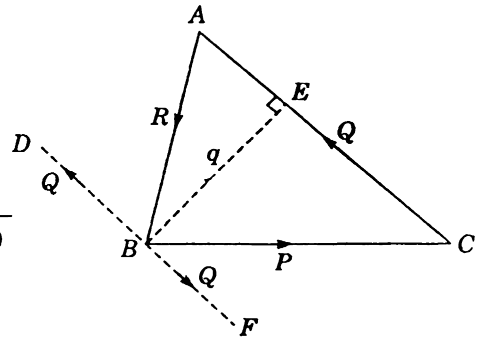


Fig. 1.45

Example 1.36 *Prove that if three forces acting upon a rigid body are represented in magnitude, direction and line of action by the sides of a triangle taken in order, they are equivalent to a couple whose moment is represented by twice the area of the triangle.*

Solution. Let ABC be the triangle and P, Q, R the forces acting along the sides BC, CA and AB respectively (Fig. 1.45). Draw DBF parallel to AC. Introduce two equal and opposite forces each equal to Q at B. The forces P, Q, R acting at B along BC, BD and AB respectively are in equilibrium. We are thus left with a force Q along CA and an equal and opposite force Q along BF. These two forces form a couple of moment Qq, where q is the length of the perpendicular BE on AC from B. Also $Qq = CA \cdot BE = 2$ area of triangle ABC.

Example 1.37 *A small ring is free to slide on a smooth circular wire of radius r, fixed to a vertical plane. It is attached by a string of length l $(\angle\sqrt{2}a)$ to a point of the wire at the same height as the centre. If W be the weight of the ring, show that the tension of the string is*

$$T = \frac{(2r^2 - l^2)\,W}{r\,(4r^2 - l^2)^{1/2}}$$

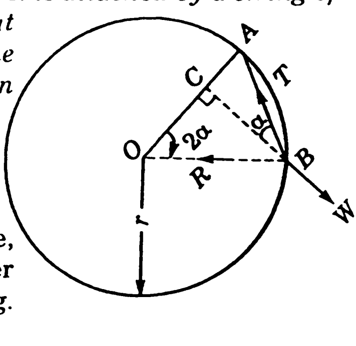


Fig. 1.46

Solution. Since $l \angle\sqrt{2}\,a$, therefore, the ring is at rest at B, which is higher than the lower point of the wire (Fig. 1.46).

Let R be the reaction of the wire and T the tension in the string. $\angle AOB = 2\alpha$, $AB = l$, $OB = r$.

$$(AB)^2 = (OA)^2 + (OB)^2 - 2 \cdot OA \cdot OB \cdot \cos 2\alpha$$
$$l^2 = r^2 + r^2 - 2\,r^2 \cos 2\alpha$$

$$\cos 2\alpha = \frac{2r^2 - l^2}{2r^2}$$

Now $$\angle AOB = \angle OBA = \frac{180° - 2\alpha}{2} = 90° - \alpha$$

$$\angle OBA = 90° - 2\alpha$$
$$\angle ABC = \angle OBA - \angle OBC$$
$$= 90° - \alpha - (90° - 2\alpha) = \alpha$$

Applying Lami's theorem at B, we have

$$\frac{T}{\sin(90° - 2\alpha)} = \frac{W}{\sin(90° - \alpha)}$$

$$\frac{T}{\cos 2\alpha} = \frac{W}{\cos \alpha}$$

$$T = W\frac{\cos 2\alpha}{\cos \alpha} = W\left(\frac{2r^2 - l^2}{2r^2}\right)\frac{1}{\cos \alpha}$$

Now $$\cos 2\alpha = 2\cos^2 \alpha - 1$$

$$\cos \alpha = \sqrt{\frac{1 + \cos 2\alpha}{2}} = \sqrt{\frac{1 + (2r^2 - l^2)/2r^2}{2}}$$

$$= \frac{\sqrt{4r^2 - l^2}}{2r}$$

$$T = W\left(\frac{2r^2 - l^2}{2r^2}\right)\cdot\left(\frac{2r}{\sqrt{4r^2 - l^2}}\right)$$

$$T = \frac{W(2r^2 - l^2)}{r\sqrt{4r^2 - l^2}}$$

Example 1.38 *ABC is a triangle and D, E, F are the middle points of the sides. Forces represented by AD, $\frac{2}{3}$ BE and $\frac{1}{3}$ CF act on a particle at the point where AD and BE meet. Show that the resultant is represented in magnitude and direction by $\frac{1}{2}$ AC and that its line of action divides BC in the ratio 2 : 1.*

Solution. Let AD, BE and CF meet at G, the centroid of ΔABC. Then (Fig. 1.47).

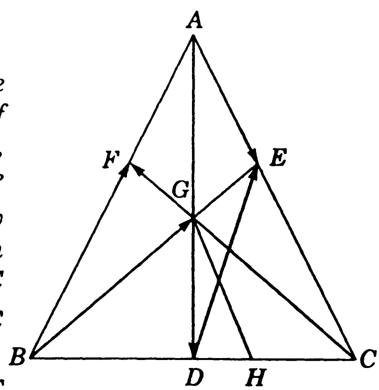


Fig. 1.47

$$AD + \tfrac{2}{3}BE + \tfrac{1}{3}CF = AD + BG + GF$$
$$= AD + BF = AD + DE = AE$$
$$= \tfrac{1}{2}AC$$

Draw. GH parallel to AC, then GH is the line of action of the resultant. In triangle BCE

$$\frac{BH}{HC} = \frac{BG}{GE} = \frac{2}{1}$$

Hence H divides BC in the ratio of 2 : 1.

Example 1.39 *A light rod AB = 50 cm is hung from a peg O by two light strings OA and OB which are 40 cm and 30 cm long respectively. A weight W_1 is fixed to the rod at A and a weight W_2 at B. If in the position of equilibrium, the angle between the rod and the vertical is θ, show that*

$$\tan\theta = \frac{12\,(W_1 - W_2)}{16\,W_1 - 9\,W_2}$$

Solution. Let T_1 and T_2 be the tensions in the strings OA and OB respectively and R the thrust in the rod AB (Fig. 1.48). Applying Lami's theorem at A, we have

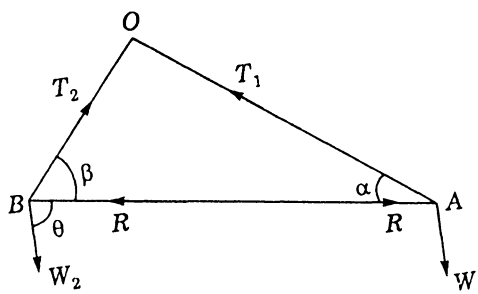


Fig. 1.48

$$\frac{R}{\sin(180^\circ - \theta + \alpha)} = \frac{W_1}{\sin\alpha}$$

$$R = \frac{W_1 \sin(\theta - \alpha)}{\sin\alpha}$$

Similarly, at B, $$R = \frac{W_2 \sin(\theta + \beta)}{\sin\beta}$$

Hence $$\frac{W_1 \sin(\theta - \alpha)}{\sin\alpha} = \frac{W_2 \sin(\theta + \beta)}{\sin\beta}$$

Now $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$, $\sin\beta = \frac{4}{5}$, $\cos\beta = \frac{3}{5}$

$$\frac{5}{3} W_1 [\sin\theta\cos\alpha - \cos\theta\sin\alpha] = \frac{5}{4} W_2 [\sin\theta\cos\beta - \cos\theta\sin\beta]$$

$$\frac{5}{3} W_1 \left[\frac{4}{5} \sin \theta - \frac{3}{5} \cos \theta\right] = \frac{5}{4} W_2 \left[\frac{\frac{3}{5} \sin 8 - 4}{5 \cos \theta}\right]$$

$$\frac{1}{3} W_1 [4 \sin \theta - 3 \cos \theta] = \frac{1}{4} W_2 [3 \sin \theta - 4 \cos \theta]$$

$$16W_1 \sin \theta - 12 W_1 \cos \theta = 9W_2 \sin \theta - 12W_2 \cos \theta$$

$$(16 W_1 - 9W_2) \sin \theta = (12 W_1 + 12W_2) \cos \theta$$

$$\tan \theta = \frac{12 (W_1 + W_2)}{16 W_1 - 9 W_2}$$

Example 1.40 *A weight W is hung from a weightless ring which can slip over a smooth circular wire fixed in a vertical plane. The ring is also tied tu a string which passing as a chord of the fixed peg at the top of the circle, sustains a given weight P. If θ be the angle that the radius to the ring makes with the vertical, in the position of rest, then show that*

$$\sin (\theta/2) = \tfrac{1}{2} \cdot (P/W)$$

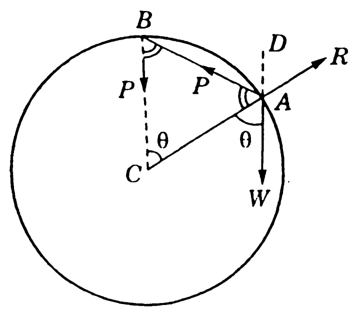


Fig. 1.49

Solution. Let A be the position of the ring and B that of the peg (Fig. 1.49).

$$\angle CAB = \angle CBA = 90° - \theta/2 = \angle BAD$$

Since the peg is smooth, therefore, tension in the string AB is P. For the equilibrium of A, we have

$$\frac{P}{\sin \theta} = \frac{W}{\sin (90° - \theta/2)}$$

$$P \cos (\theta/2) = W \sin \theta$$

$$= 2 W \sin (\theta/2) \cdot \cos (\theta/2)$$

$$\sin (\theta/2) = \tfrac{1}{2} P/W.$$

Example 1.41 *Two forces P, Q and their resultant R act at a point A. If their directions meet a transversal at L, M, N respectively, prove that P/AL + Q/AM = R/AN.*

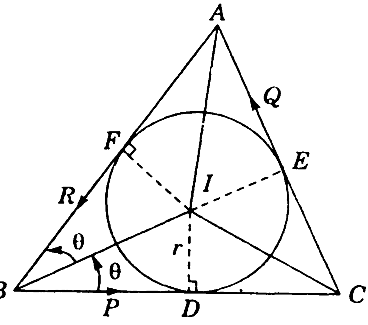


Fig. 1.50

Solution. The forces P, Q and R are represented by the sides AB, AD and AC respectively of the parallelogram $ABCD$. Draw BE and DF parallel to LMN, then (Fig. 1.50).

$$\frac{AB}{AL} = \frac{AE}{AN}$$

$$\frac{AD}{AM} = \frac{AF}{AN}$$

$$\frac{AB}{AL} + \frac{AD}{AM} = \frac{AE}{AN} + \frac{AF}{AN}$$

$$= \frac{(AE + AF)}{AN} = \frac{(AE + EC)}{AN} \qquad [\because \; AF = EC]$$

$$= \frac{AC}{AN}$$

$$\frac{P}{AL} + \frac{Q}{AM} = \frac{R}{AN}.$$

Example 1.42 *The resultant of two forces P and Q is at right angles to P. The resultant of P and Q′ acting at the.same angle is at right angles to Q′. Prove that P is the geometric mean between Q and Q*

Solution.

Let θ = angle between P and Q

α = angle between P and R

$$\tan \alpha = \frac{(Q \sin \theta)}{(P + Q \cos \theta)}$$

For $\alpha = 90°$, $P + Q \cos \theta = 0$ ...(1)

Similarly, $Q' + P \cos \theta = 0$ ...(2)

Hence $P^2 - QQ' = 0$

or $P^2 = QQ'$

Hence P is the geometric mean between Q and Q'.

Example 1.43 *A uniform wheel of 0.5 m diameter and weighing 1.5 kN rests against a rectangular block 0.2 m high lying on a horizontal plane, as shown in Fig. 1.51 (a). It is to be pulled over this*

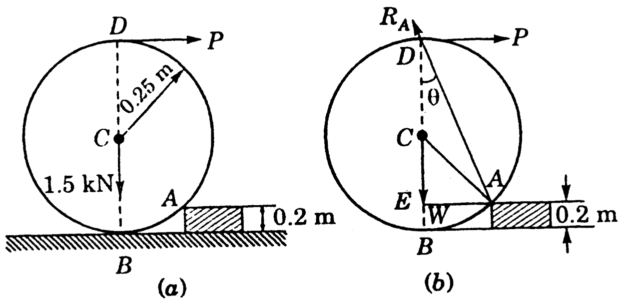


Fig. 1.51

block by a horizontal force P applied to the end of a string wound round the circumference of the wheel. Find the force P when the wheel is just about to roll over the block.

Solution.

Let W = weight of wheel

R_A = reaction on the wheel at A

The three forces P, W and R_A are in equilibrium. Since P and W meet at D, therefore R_A must pass through D [Fig. 1.51 (*b*)]. Using Lami's theorem, we have

$$\frac{P}{\sin(180° - \theta)} = \frac{W}{\sin(90° + \theta)}$$

$$\frac{P}{\sin\theta} = \frac{W}{\cos\theta}$$

$$P = W\tan\theta$$

$$\tan\theta = \frac{AE}{DE}$$

$$AE = \sqrt{AC^2 - CE^2} = \sqrt{(0.25)^2 - (0.25 - 0.2)^2}$$

$$= \sqrt{0.0625 - 0.0025} = \sqrt{0.06} = 0.245 \text{ m}$$

$$DE = DC + CE = 0.25 + 0.05 = 0.3 \text{ m}$$

$$\tan\theta = \frac{0.245}{0.3} = 0.8165$$

$$\theta = 39.23°$$

$$P = 1.5 \times 0.8165 = 1.225 \text{ kN}$$

Example 1.44 *A smooth circular cylinder of weight W = 600 N and radius r = 0.5 m rests in a V-shaped groove whose sides are inclined at angles α = 30° and β = 60° to the horizontal, as shown in Fig. 1.52. Find the reactions R_A and R_B at the points of contact.*

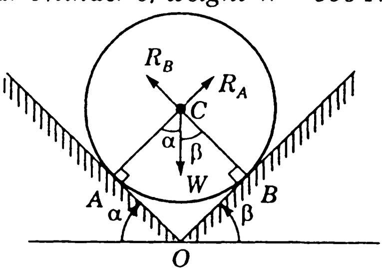


Fig. 1.52

Solution. The forces R_A, R_B and W are in equilibrium, and therefore, must meet at the same point C. Using Lami's theorem we have

$$\frac{R_A}{\sin(180° - \beta)} = \frac{R_B}{\sin(180° - \alpha)} = \frac{W}{\sin(\alpha + \beta)}$$

$$\frac{R_A}{\sin\beta} = \frac{R_B}{\sin\alpha} = \frac{600}{\sin 90°}$$

$$R_A = 600 \times \sin 60° = 600 \times \frac{\sqrt{3}}{2} = 519.6 \text{ N}$$

$$R_B = 600 \sin 30° = 600 \times \frac{1}{2} = 300 \text{ N}$$

Example 1.45 *Two rollers of weights W_1 and W_2 are connected by a flexible string AB. The rollers rest on two mutually perpendicular planes DE and EF, as shown in Fig. 1.53. Find the tension in the string and the angle θ that it makes with the horizontal when the system is in equilibrium. Take $W_1 = 60$ N, $W_2 = 120$ N and $\alpha = 30°$*

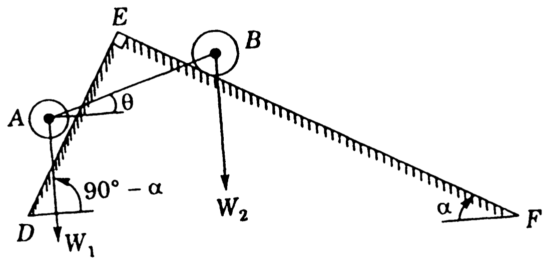


Fig. 1.53

Solution. Let R_A and R_B be the reactions on the planes at A and B respectively and T the tension in the string AB. These forces are shown in Fig. 1.54.

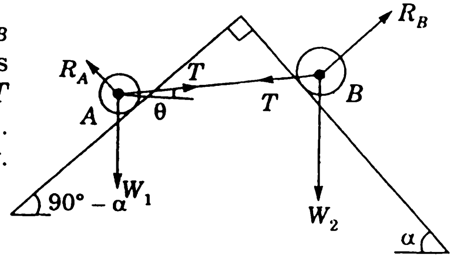


Fig. 1.54 (*a*)

Roller A [Fig. 1.54 (*b*)]

Applying Lami's theorem at A, we have

$$\frac{T}{\sin (90° + \alpha)} = \frac{W_1}{\sin \{180° - (\alpha + \theta)\}}$$

$$\frac{T}{\cos \alpha} = \frac{W_1}{\sin (\alpha + \theta)} \qquad ...(1)$$

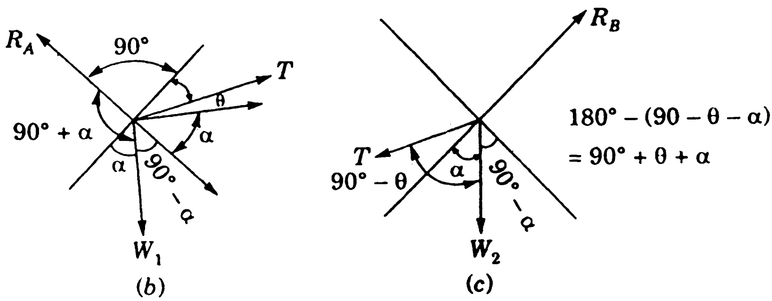


Fig. 1.54

Roller B [Fig. 1.54 (*c*)]

$$\frac{T}{\sin(180° - \alpha)} = \frac{W_2}{\sin\{90° + (\theta + \alpha)\}}$$

$$\frac{T}{\sin\alpha} = \frac{W_2}{\cos(\theta + \alpha)} \qquad \ldots(2)$$

$$\sin(\theta + \alpha) = \frac{W_1 \cos\alpha}{T}$$

$$\cos(\theta + \alpha) = \frac{W_2 \sin\alpha}{T}$$

$$\tan(\theta + \alpha) = \frac{W_1}{W_2} \cdot \cot\alpha$$

$$= \frac{60}{120} \times \cot 30° = 0.866$$

$$\theta + \alpha = 40.89°$$

$$\theta = 40.89° - 30° = 10.89°$$

$$T = \frac{W_1 \cos\alpha}{\sin(\alpha + \theta)} = \frac{60 \cos 30°}{\sin 40.89°} = 79.38 \text{ N}$$

Example 1.46 *A uniform beam AB of length 2l rests in equilibrium with one end resting against a smooth vertical wall and with a point C of its length resting upon a smooth horizontal peg which is parallel to the wall and at a distance b from it. Show that the inclination of the beam with the vertical is* $\sin^{-1}(b/l)^{1/3}$

Solution. The forces acting on the beam under equilibrium are (Fig. 1.55).

1. Weight W of the beam.
2. Reaction S of the wall at A.
3. Reaction R at the peg C.

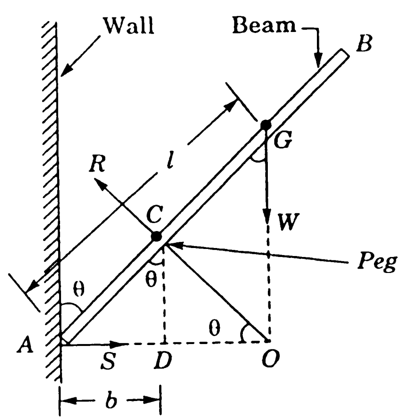


Fig. 1.55

In ΔACD, $\sin\theta = \dfrac{AD}{AC}$

In ΔAOC, $\sin\theta = \dfrac{AC}{AO}$

In ΔAOG, $\sin\theta = \dfrac{AO}{AG}$

Multiplying, we have

$$\sin^3\theta = \frac{AD}{AG} = \frac{b}{l}$$

$$\theta = \sin^{-1}(b/l)^{1/3}$$

Example 1.47 *A smooth sphere of radius r and weight W hangs by a light string of length l, as shown in Fig. 1.56. Determine the reaction of the wall and the tension in the string.*

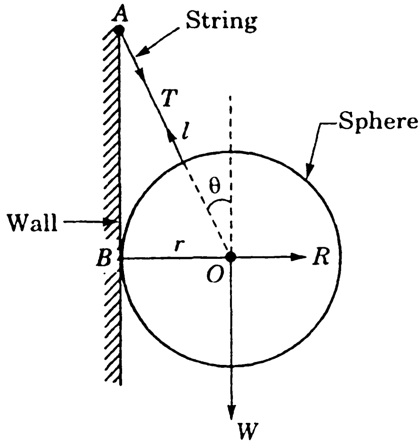


Fig. 1.56

Solution. The sphere is in equilibrium under the action of following forces :

1. Weight W of the sphere.
2. Normal reaction R of the wall.
3. Tension T in the string.

The three forces meet at O. Resolving the forces horizontally and vertically, we get

$$T \cos \theta = W$$
$$T \sin \theta = R$$
$$\tan \theta = R/W$$

Now
$$\sin \theta = \frac{r}{r + l}$$
$$\cos \theta = \frac{\sqrt{l^2 + 2rl}}{r + l} \quad \tan \theta = \frac{r}{\sqrt{l^2 + 2rl}}$$
$$R = \frac{Wr}{\sqrt{l^2 + 2rl}}$$
$$T = \frac{W(l + r)}{\sqrt{l^2 + 2rl}}$$

Example 1.48 *A string 1.5 m long is tied to the ends of a rod weighing 100 N. The rod is 1 m long. The string passes over a nail such that the rod hangs horizontally. Find the tension in the string.*

Solution. $\sin \alpha = \frac{AD}{AC} = \frac{0.5}{0.75} = 0.667$

$$\alpha = 41.81°$$

The forces acting on the rod are shown in Fig. 1.57 (*b*).

Applying Lami's theorem, we have

$$\frac{100}{\sin 2\alpha} = \frac{T}{\sin (180° - \alpha)}$$

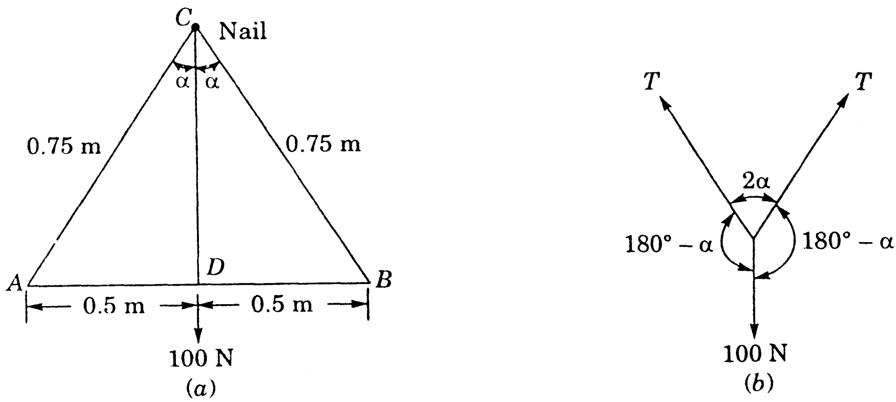


Fig. 1.57

$$\frac{100}{\sin 83.62^\circ} = \frac{T}{\sin 138.19^\circ}$$

$$T = 67.08 \text{ N}$$

Example 1.49 *A cylindrical roller of weight 600 N is resting on a smooth inclined plane having incline of 30°. The roller is held by a rope as shown in Fig. 1.58. Find the tension in the rope and reaction at the point of contact between roller and plane.*

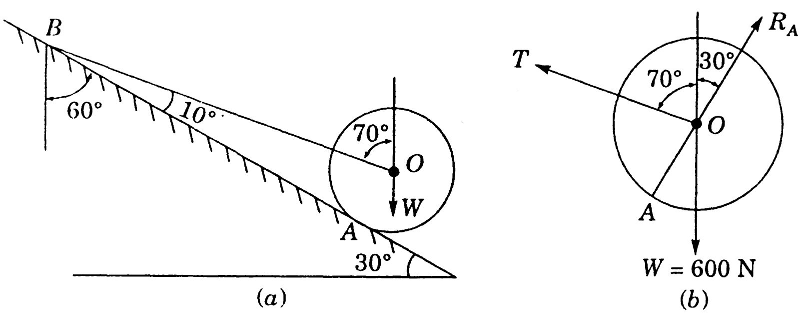


Fig. 1.58

Solution. The free body diagram of roller is shown in Fig. 1.58 (*b*).

Applying Lami's theorem, we have

$$\frac{600}{\sin 100^\circ} = \frac{T}{\sin 150^\circ} = \frac{R_A}{\sin 110^\circ}$$

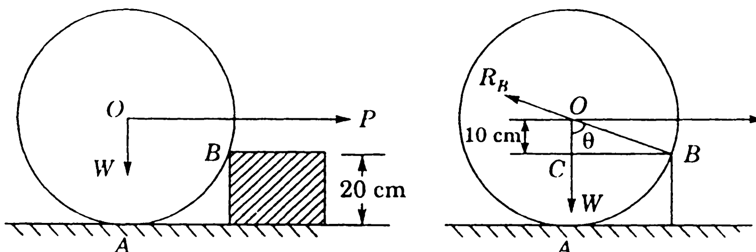
$$T = 600 \times \frac{\sin 150^\circ}{\sin 100^\circ} = 304.63 \text{ N}$$

$$R_A = 600 \times \frac{\sin 110^\circ}{\sin 100^\circ} = 572.51 \text{ N}$$

Example 1.50 *A wheel of 60 cm diameter and 1.5 kN weight rests against a rectangular obstacle of thickness 2 cm. Find the least horizontal pull to be applied (a) through the centre of the wheel, and (b) through a point on the vertical diameter of the wheel at the top to turn over the wheel over the obstacle.*

Solution.

(*a*) The wheel resting against the obstacle is shown in Fig. 1.59 (*a*). The free body diagram of wheel is shown in Fig. 1.59 (*b*).

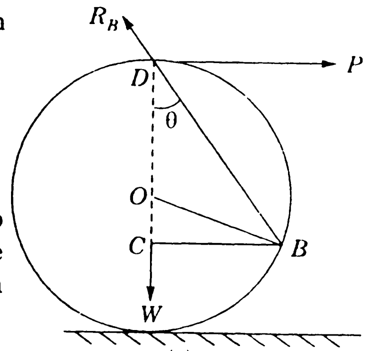


(*a*) (*b*)

$OB = 30$ cm, $OC = 30 - 20 = 10$ cm

$$\cos\theta = \frac{OC}{OB} = \frac{10}{30} = 0.333$$

$$\theta = 70.53°$$



(*c*)

Fig. 1.59

When the wheel is just about to turn over the obstacle, it will lose contact with the floor and the reaction at *A* shall no longer be there.

Angle between W and $R_B = 180° - \theta = 180° - 70.53° = 109.47°$

Angle between R_B and $P = 90° + \theta = 90° + 70.53° = 160.53°$

Applying Lami's theorem, we have

$$\frac{1500}{\sin 160.53°} = \frac{R_B}{\sin 90°} = \frac{P}{\sin 109.47°}$$

$$R_B = 1500 \times \frac{\sin 90°}{\sin 160.53°} = 4500.3 \text{ N}$$

$$P = 1500 \times \frac{\sin 109.47°}{\sin 160.53°} = 4242.93 \text{ N}$$

(*b*) The free body diagram for the wheel is shown in Fig. 1.59 (*c*).

$$BC = \sqrt{OB^2 - OC^2} = \sqrt{30^2 - 10^2} = 28.284 \text{ cm.}$$

$$CD = 30 + 10 = 40 \text{ cm}$$

$$\tan\theta = \frac{BC}{CD} = \frac{28.284}{40} = 0.707$$

$$\theta = 35.26°$$

Angle between W and $R_B = 180° - \theta = 180° - 35.26° = 144.74°$

Angle between R_B and $P = 90° + \theta = 90° + 35.26 = 125.26°$

Applying Lami's theorem.

$$\frac{W}{\sin 125.26°} = \frac{R_B}{\sin 90°} = \frac{P}{\sin 144.74°}$$

$$R_B = \frac{1500 \times \sin 90°}{\sin 125.26°} = 1837 \text{ N}$$

$$P = \frac{1500 \times \sin 144.74°}{\sin 125.26°} = 1060.5 \text{ N}$$

Example 1.51 *A weight of 100 N hangs from a point C by means of two strings AC and BC as shown in Fig. 1.60. Using Lami's theorem determine the forces in the strings.*

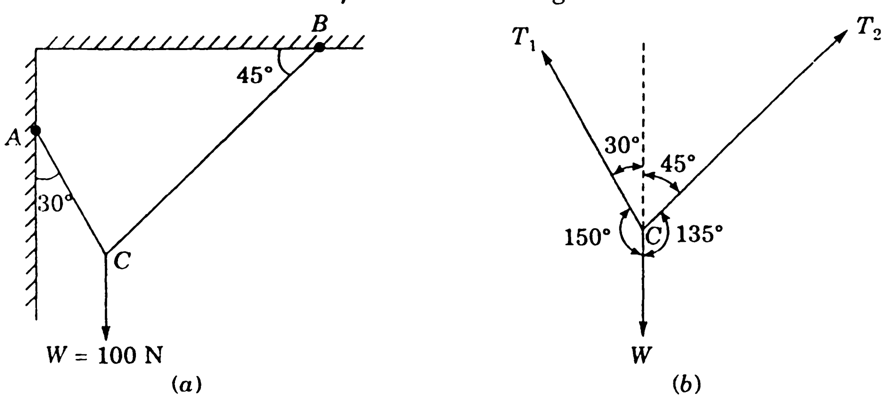


Fig. 1.60

Solution. Let T_1 and T_2 be the tensions in the strings AC and BC respectively. The free body diagram for the strings is shown in Fig. 1.60 (*b*).

$$\frac{W}{\sin 75°} = \frac{T_1}{\sin 135°} = \frac{T_2}{\sin 150°}$$

$$T_1 = 100 \times \frac{\sin 135°}{\sin 75°} = 73.2 \text{ N}$$

$$T_2 = 100 \times \frac{\sin 150°}{\sin 75°} = 51.76 \text{ N}$$

Example 1.52 *A string ABCDE whose end A is fixed has weights W_1 and W_2 attached to it at B and C, and passes over a smooth peg at D carrying a weight of 1000 N at the free end E, as shown in Fig. 1.61. In the equilibrium position BC makes an angle of 20° with the horizontal and AB and CD make angles of 150° and 120° respectively with BC. Determine*

(*a*) *The tensions in portion AB, BC, CD and DE.*

(*b*) *The values of weights W_1 and W_2*

(*c*) *The pressure on the peg D.*

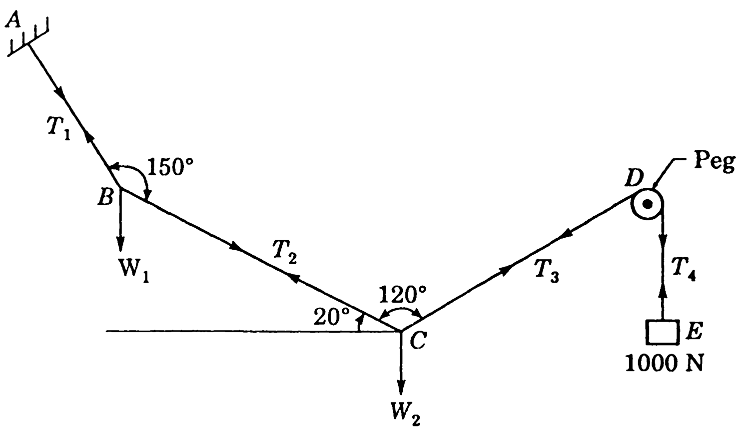


Fig. 1.61

Solution. Let T_1, T_2, T_3 and T_4 be the tensions in *AB, BC, CD,* and *DE* respectively. For the equilibrium of peg, $T_3 = T_4 = 1000$ N

Applying Lami's theorem at point *C*, we have

$$\frac{T_3}{\sin 110°} = \frac{T_2}{\sin 40°} = \frac{W_2}{\sin 120°}$$

$$T_2 = 1000 \times \frac{\sin 40°}{\sin 110°} = 684 \text{ N}$$

$$W_2 = 1000 \times \frac{\sin 120°}{\sin 110°} = 921.6 \text{ N}$$

Applying Lami's theorem at *B*, we have

$$\frac{T_2}{\sin 140°} = \frac{T_1}{\sin 70°} = \frac{W}{\sin 150°}$$

$$T_1 = 684 \times \frac{\sin 70°}{\sin 140°} = 1000 \text{ N}$$

$$W_1 = 864 \times \frac{\sin 150°}{\sin 140°} = 532 \text{ N}$$

Example 1.53 *A heavy spherical ball of weight 150 N rests in a V-shaped block whose sides are inclined at 30° and 45° to the horizontal. Find the pressure exerted on each side of the block.*

Solution. The sphere resting in the *V*-block is shown in Fig. 1.62 (*a*). The reactions at *A* and *B* are perpendicular to the sides of the block and pass through the centre of sphere *O*. The free body diagram is shown in Fig. 1.62 (*b*).

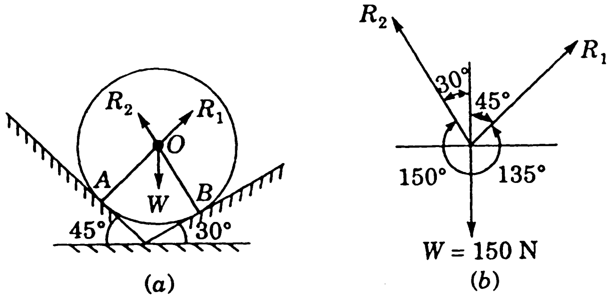


Fig. 1.62

Applying Lami's theorem, we have

$$\frac{150}{\sin 75°} = \frac{R_1}{\sin 150°} = \frac{R_2}{\sin 135°}$$

$$R_1 = 150 \times \frac{\sin 150°}{\sin 75°} = 77.64 \text{ N}$$

$$R_2 = 150 \times \frac{\sin 135°}{\sin 75°} = 109.8 \text{ N}$$

Example 1.54 *Two smooth balls each of radius 20 cm and weighing 400 N are lying in a vertical cylinder of diameter 70 cm. Determine the pressure exerted on the wall and base of the cylinder by the balls.*

Solution. The balls in the cylinder are shown in Fig. 1.63 (*a*) along with the reactions of cylinder walls at the points of contact. The free body diagram is shown in Fig. 1.63 (*b*).

$$O_1O_2 = 40 \text{ cm}$$

Horizontal distance between the centres of balls $= 70 - 2 \times 20 = 30$ cm

$$\cos \alpha = \frac{30}{40} = 0.75$$

$$\alpha = 41.4''$$

Applying Lami's theorem at O_2, we have

$$\frac{W}{\sin (180° - \alpha)} = \frac{R_C}{\sin (90° + \alpha)} = \frac{R_D}{\sin 90°}$$

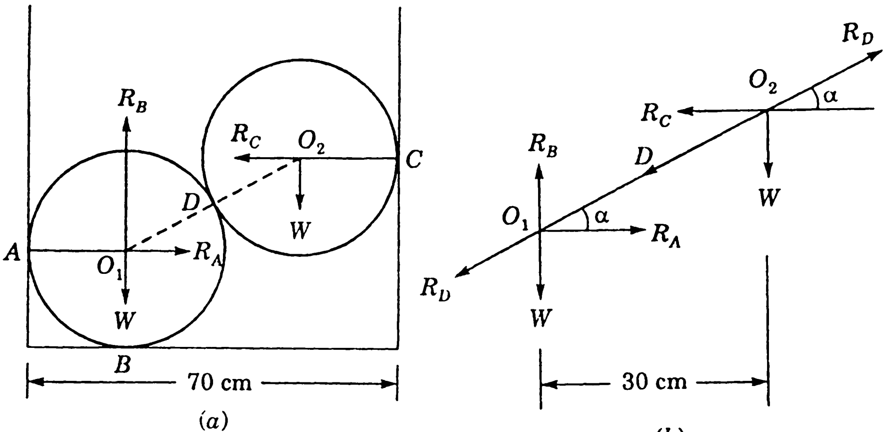


Fig. 1.63

$$R_C = 400 \times \frac{\cos \alpha}{\sin \alpha} = \frac{400}{\tan \alpha} = \frac{400}{\tan 41.4^\circ} = 453.7 \text{ N}$$

$$R_D = \frac{400}{\sin 41.4^\circ} = 604.86 \text{ N}$$

Applying Lami's theorem at O_1, we have

$$\frac{R_B - W}{\sin (180^\circ - \alpha)} = \frac{R_A}{\sin (90^\circ + \alpha)} = \frac{R_D}{\sin 90^\circ}$$

$$\frac{R_B - 400}{\sin \alpha} = \frac{R_A}{\cos \alpha} = \frac{604.86}{1}$$

$$R_A = 604.86 \cos 41.4^\circ = 453.7 \text{ N}$$

$$R_B - 400 = 604.86 \sin 41.4^\circ = 400$$

$$R_B = 800\ N$$

1.17 Parallel Forces

Parallel Forces. A set of forces whose lines of action are parallel to each other are called parallel forces.

Types of parallel forces. The various types of parallel forces are :

1. **Like parallel forces.** When the two parallel forces act in the same direction they are called like parallel forces, as shown in Fig. 1.64 (*a*). These forces can be equal or unequal in magnitude.
2. **Unlike unequal parallel forces.** When the two parallel forces act. in opposite directions and are unequal in magnitude, as shown in Fig. 1.64 (*b*).

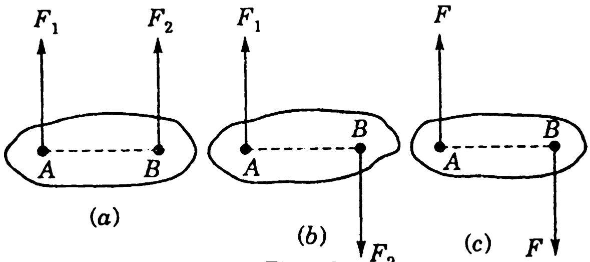


Fig. 1.64

3. **Unlike equal parallel forces.** When the two parallel forces act in opposite directions and are equal in magnitude, as shown in Fig. 1.64 (*c*).

1.17.1 Like Parallel Forces

Two parallel forces are said to be like when they act in the same direction. In order to determine the resultant of two like parallel forces, consider two parallel forces F_1 and F_2 acting at points A and B as shown in Fig. 1.65. Join AB and at A and B introduce two equal and opposite forces, each equal to F, as shown in Fig. 1.65.

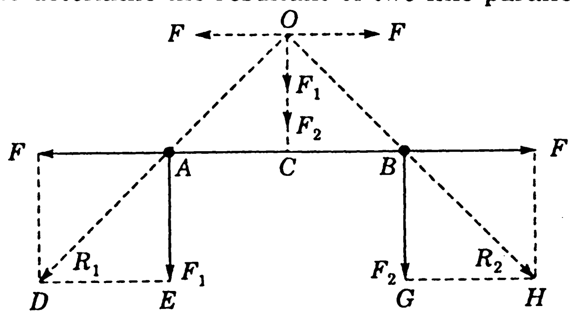


Fig. 1.65 Resultant of like parallel forces.

These two forces being equal and opposite, balance each other and do not affect the resultant of F_1 and F_2.

The forces F_1 and F at A have a resultant R_1 and the forces F_2 and F at B have a resultant R_2. The points of application of R_1 and R_2 can be transferred to meet at O.

Resolve the force R_1 at O into its components F and F_1 and R_2 into F and F_2. The two forces F, F at O being equal and opposite balance each other and we are left with two forces F_1 and F_2 both acting along OC. Hence the resultant of F_1 and F_2 is $(F_1 + F_2)$ acting along OC, which is parallel to the forces F_1 and F_2.

Thus the resultant of two like parallel forces is equal to their sum and acts parallel to the forces.

Now Δs AOC and ADE are similar, hence

$$\frac{OC}{CA} = \frac{AE}{DE} = \frac{F_1}{F}$$

$$\therefore \qquad F_1 \times CA = F \times OC$$

Similarly from Δs BOC and BGH, we get

$$F_2 \times CB = F \times OC$$

Hence $\qquad F_1 \times CA = F_2 \times CB$

or $\qquad \frac{F_1}{F_2} = \frac{CB}{CA}$

Hence the resultant of two like parallel forces divides the distance between the forces internally in the inverse ratio of the forces.

Therefore, we obtain the important result that the resultant of two like parallel forces is equal to their sum, acts parallel to the forces and divides the distance between the forces internally in the inverse ratio of the forces.

1.17.2 Unlike Parallel Forces

Two parallel forces are said to be unlike when they act in opposite directions. In order to determine the resultant of two unlike parallel forces, consider two forces F_1 and F_2 acting at A and B respectively as shown in Fig. 1.66. Join AB and introduce two equal and opposite forces each equal to F at A and B. The resultant of F_1 and F at A is R_1 and that of F_2 and F at B is R_2. The points of application of R_1 and R_2 can be transferred to meet at O.

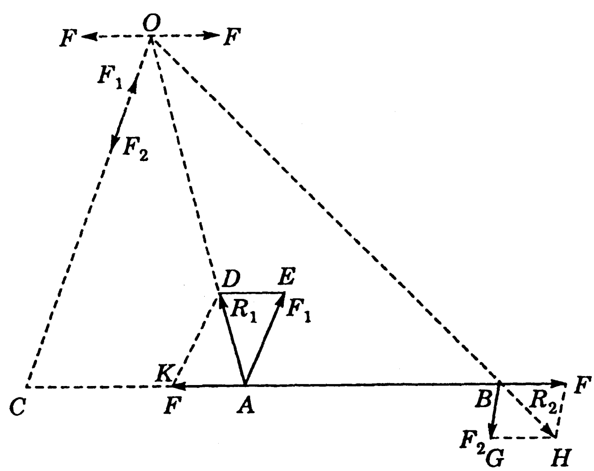


Fig. 1.66 Resultant of unlike parallel forces.

Resolve the forces R_1 and R_2 at O into their original components. The components F and F at O are equal and opposite and cancel each other and we are left with two forces F_1 and F_2 acting along CO and OC respectively.

Hence the resultant of two unlike parallel forces F_1 and F_2 is $(F_1 - F_2)$ and acts along CO which is parallel to F_1 and F_2.

Thus, the resultant of two unlike parallel forces is equal to their difference, acts parallel to the forces and is in the direction of the greater force.

Now the Δs AOC and AFD are similar.

Hence $$\frac{OC}{CA} = \frac{DF}{AF} = \frac{F_1}{F}$$

or $$F \times OC = F_1 \times CA$$

Similarly Δs OCB and BGH are similar and we get,

$$\frac{OC}{CB} = \frac{BG}{GH} = \frac{F_2}{F}$$

or $$F \times OC = F_2 \times CB$$

Thus $$F_1 \times CA = F_2 \times CB$$

or $$\frac{F_1}{F_2} = \frac{CB}{CA}$$

Therefore the resultant of two unlike parallel forces divides the distance between the forces externally in the inverse ratio of the forces.

Example 1.55 *Two uniform rods AB and BC are rigidly joined together at B such that ABC is a right angle, hang freely in equilibrium from the end A. The lengths of the rods are a and b and their weights are W_1 and W_2. Show that if θ be the inclination of AB to the vertical,*

$$\tan \theta = \frac{bW_2}{a\,(W_1 + 2W_2)}$$

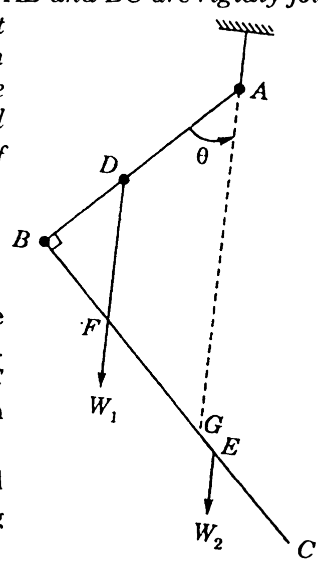


Fig. 1.67

Solution. Let D and E be the mid-points of AB and BC respectively. The verticles from A and D meet BC in F and G respectively, as shown in Fig. 1.57.

W_1 and W_2 are two like parallel forces and their resultant acts along AG.

Hence $\quad W_1 \times FG = W_2 \times GE$

Now $\quad BG = AB \tan\theta = a\tan\theta$

$$BE = \frac{b}{2}$$

$$\therefore \quad GE = BE - BG = \frac{b}{2} - a\tan\theta$$

$$BF = \frac{a}{2}\tan\theta$$

$$\therefore \quad FG = BG - BF = a\tan\theta - \frac{a}{2}\tan\theta = \frac{a}{2}\tan\theta$$

Hence $W_1 \times \frac{a}{2}\tan\theta = W_2 \times \left(\frac{b}{2} - a\tan\theta\right)$

$$\left(W_1 \times \frac{a}{2} + W_2 \times a\right)\tan\theta = W_2 \times \frac{b}{2}$$

$$\tan\theta = \frac{bW_2}{a\,(W_1 + 2W_2)}$$

Example 1.56 *Two parallel forces of magnitude 5 kN and 3 kN are acting at two points on a line 4m. Find the magnitude, direction and location of the resultant force, when (a) the forces are like, and (b) forces are unlike.*

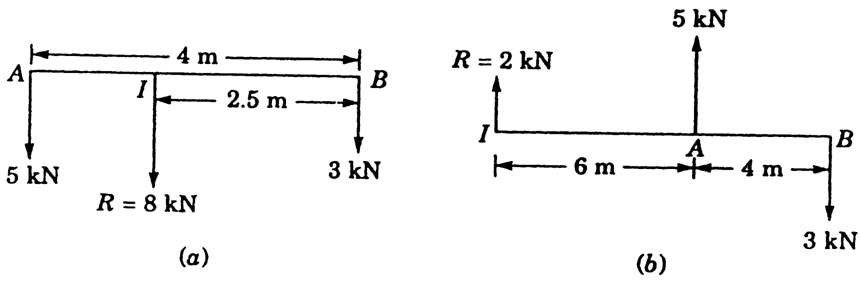


Fig. 1.68

Solution.

(*a*) Like force (Fig. 1.68 (*a*)) :

$$R = P + Q = 5 + 3 = 8\ kN$$

$$\frac{IB}{IA} = \frac{P}{Q} = \frac{5}{3}$$

$$\frac{IB}{IA + IB} = \frac{5}{3+5} = \frac{5}{8}$$

$$IB = \frac{5}{8} \times AB = \frac{5}{8} \times 4 = 2.5\ \text{m}$$

(*b*) Unlike forces (Fig. 1.68 (*b*)) :

$$R = P - Q = 5 - 3 = 2\ kN$$

$$\frac{IB}{IA} = \frac{P}{Q} = \frac{5}{3}$$

$$\frac{IB - IA}{IA} = \frac{5-3}{3}$$

$$\frac{AB}{LA} = \frac{2}{3},\ IA = 4 \times \frac{3}{2} = 6 \text{ m}$$

1.18 Moment of a Force

The moment of a force about a point is defined as the product of the force and the perpendicular distance from the point on the line of action of the force.

Consider a force F acting on a rigid body as shown in Fig. 1.69. If we are interested to determine the moment of the force F about the point O, then drop a perpendicular from O on the line of action of force F. Let this distance be r. Then the moment (M) of the force F about point O, by definition is,

$$M = F \times r$$

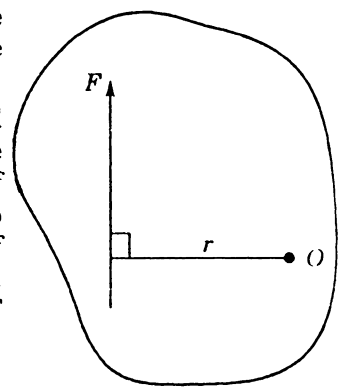


Fig. 1.69 Moment of a force.

The moment of a force about a point may be thought of as the tendency of the force to rotate the body about that point. The units of moment are Newton-metre (N.m).

If the tendency of the force to rotate the body about a point is in the clockwise direction then the moment is said to be clockwise moment (*e.g.*, $M_1 = F_1 \times r_1$ ↓). Likewise if the tendency of the force to rotate the body about a point is in the anticlockwise direction then the moment is said to be anticlockwise or counterclockwise (*e.g.*, $M_2 = F_2 \times r_2$ ↑) moment.

1.19 Varignon's Theorem or Principle of Moments

This theorem states that the algebraic sum of the moments of two unequal forces about any point in their plane is equal to the moment of their resultant about that point.

Proof :

(*a*) *Forces Acting at a Point*

Let two forces F_1 and F_2 acting on a body at the point O be represented in magnitude and direction by OA and OB respectively, as shown in Fig. 1.70. Complete the parallelogram $OACB$. Then the diagonal OC represents the resultant F of forces F_1 and F_2 in magnitude and direction. Let O_1 be the point in the plane of the body about which moments of F_1, F_2 and F are to be determined.

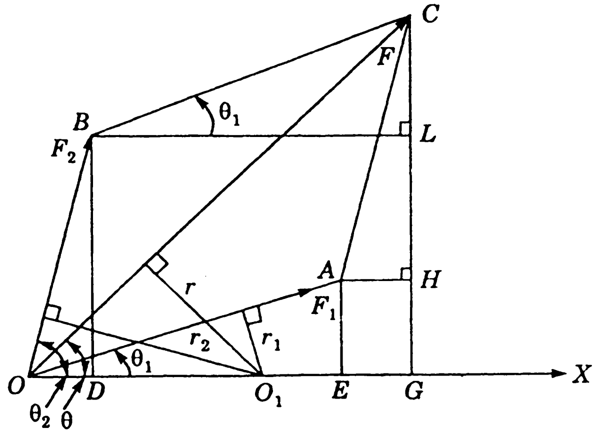


Fig. 1.70 Forces acting at a point.

From O_1 draw r_1, r_2 and r perpendiculars on F_1, F_2 and F respectively. From A, B and C draw perpendiculars on OX and from A and B on CG, as shown in Fig. 1.70. Let θ_1, θ_2 and θ be the angles which F_1, F_2 and F make with OX.

Moment of forces F_1 about O_1

$$\begin{aligned} &= F_1 \times r_1 = F_1 \times OO_1 \sin \theta_1 \\ &= (F_1 \sin \theta_1) \times OO_1 \\ &= EA \times OO_1 = GH \times OO_1 \end{aligned}$$

Moment of forces F_2 about O_2

$$\begin{aligned} &= F_2 \times r_2 = F_2 \times OO_1 \sin \theta_2 \\ &= (F_2 \sin \theta_2) \times OO_1 \\ &= DB \times OO_1 = GL \times OO_1 \end{aligned}$$

Moment of force F about O_1

$$\begin{aligned} &= F \times r = F \times OO_1 \sin \theta \\ &= (F \sin \theta) \times OO_1 = GC \times OO_1 \\ &= (LC + GL) \times OO_1 = (GH + GL) \times OO_1 \end{aligned}$$

Also $F_1 \times r_1 + F_2 \times r_2 = GH \times OO_1 + GL \times OO_1$

$$= (GH + GL) \times OO_1$$

Hence $\quad F \times r = F_1 \times r_1 + F_2 \times r_2$

Thus proved.

(*b*) *Like Parallel Forces*

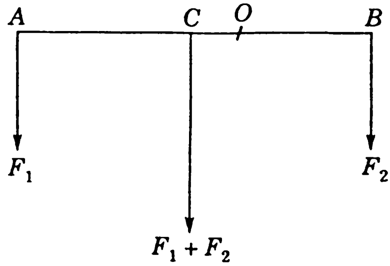


Fig. 1.71 Like parallel forces.

Let F_1 and F_2 be two like parallel forces acting at A and B respectively so that their resultant $(F_1 + F_2)$ is acting at C as shown in Fig. 1.71. Let O be the point about which moments are to be determined. From O draw a perpendicular to the lines of action of the forces. Then

$$F_1 \times AC = F_2 \times BC$$

Moment of F_1 about $O = F_1 \times AO \uparrow = F_1 \times (AC + CO) \downarrow$

Moment of F_2 about $O = F_2 \times BO \downarrow = F_2 \times (BC - CO) \downarrow$

Sum of the moments of F_1 and F_2 about O

$$= F_1 \times (AC + CO) - F_2 \times (BC - CO)$$
$$= F_1 \times AC + F_1 \times CO - F_2 \times BC + F_2 \times CO$$
$$= (F_1 + F_2) \times CO$$

= Moment of the resultant about O.

(*c*) *Unlike Parallel Forces*

Let F_1 and F_2 be two unlike forces acting at A and B respectively so that their resultant $F_1 - F_2$ $(F_1 > F_2)$ is acting at C, as shown in Fig. 1.72. Let O be the point about which moments are to be determined. From O draw a perpendicular to the lines of action of the forces. Then

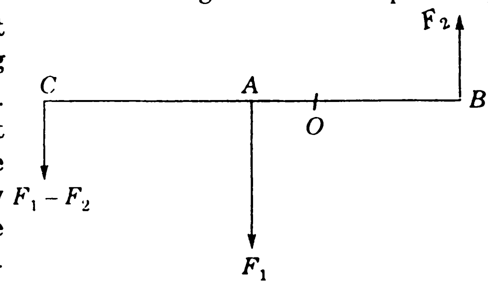


Fig. 1.72 Unlike parallel forces.

$$F_1 \times AC = F_2 \times BC$$

Moment of F_1 about $O = F_1 \times AO \uparrow = F_1 \times (CO - AC) \uparrow$

Moment of F_2 about $O = F_2 \times BO \downarrow = F_2 \times (BC - CO) \downarrow$

Sum of the moments of F_1 and F_2 about O

$$= F_1 \times (CO - AC) + F_2 \times (BC - CO)$$
$$= F_1 \times CO - F_1 \times AC + F_2 \times BC - F_2 \times CO$$
$$= (F_1 - F_2) \times CO$$

= Moment of the resultant about O.

1.20 Generalised Theorem of Moments

This theorem states that if any number of forces acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant.

If the sum of the moments of any number of coplanar forces about a point is zero, then either their resultant passes through that point or the resultant itself is zero.

Example 1.57 *A uniform rod of length 2l and weight W is lying across two pegs on the same level. If neither peg can withstand a stress greater than T, show that the length of the rod which can project beyond either peg cannot be greater than*

$$l - \frac{d\,(W - T)}{W}$$

where d is the distance between the pegs.

Solution. Let AB be the rod supported on two pegs at C and D as shown in Fig. 1.73. Let the rod project beyond the peg at D by an amount x and T be the thrust at D. Then the thrust on the peg at C will be $W - T$.

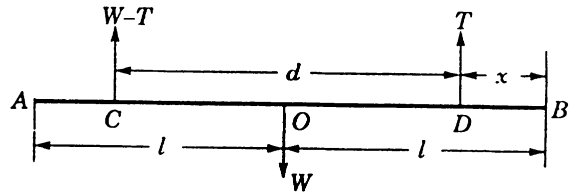


Fig. 1.73

Now
$$OD = l - x$$
$$OC = d - (l - x)$$

Taking moments about O, we get

$$(W - T)\,[d - (l - x)] = T\,(l - x)$$
$$Wd - W\,(l - x) - Td + T\,(l - x) = T\,(l - x)$$
$$Wd - WL + Wx - Td = 0$$
$$Wx = Wl + Td - Wd$$
$$x = l + \frac{Td}{W} - d$$

or
$$x = l - \frac{d\,(W - T)}{W}$$

Example 1.58 *A uniform rod AB is of weight w. When supported at a point C, it rests in a horizontal position with weights*

W_0 and W_1 suspended from A and B and also weights W_2 and W_0 suspended from A and B respectively. Show that the weight which suspended from B will keep it in a horizontal position is

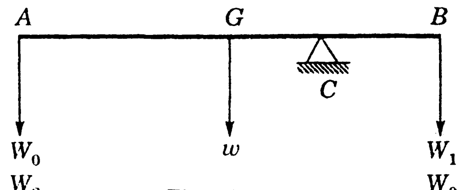


Fig. 1.74

$$\frac{W_0^2 - W_1 W_2}{W_0 - W_2}$$

Solution. Let $AB = 2l$, so that

$$AG = BG = l \text{ and } GC = x \qquad (\textit{See}. \text{ Fig. } 1.74)$$

With W_0 and W_1 suspended from A and B respectively, taking moments about C, we get

$$W_0 (l + x) + wx = W_1 (l - x) \qquad ...(1)$$

Similarly, when W_2 and W_0 are suspended from A and B respectively, then

$$W_2 (l + x) + wx = W_0 (l - x) \qquad ...(2)$$

Let W be the weight which suspended from B will keep the rod in horizontal position, then taking moments about C, we get

$$wx = W (l - x) \qquad ...(3)$$

Subtracting Eq. (2) from Eq. (1) and Eq. (3) from Eq. (2), we get

$$(W_0 - W_2)(l + x) = (W_1 - W_0)(l - x) \qquad ...(4)$$

$$W_2 (l + x) = (W_0 - W)(l - x) \qquad ...(5)$$

Dividing Eq. (4) by Eq. (5), we get

$$\frac{W_0 - W_2}{W_2} = \frac{W_1 - W_0}{W_0 - W}$$

$$(W_0 - W_2)(W_0 - W) = W_2 (W_1 - W_0)$$

$$W_0^2 - WW_0 - W_0W_2 + WW_2 = W_1W_2 - W_0W_2$$

$$W (W_2 - W_0) = W_1 W_2 - W_0^2$$

$$W = \frac{W_0^2 - W_1W_2}{W_0 - W_2}$$

Example 1.59 *The wire passing round a telegraph pole is horizontal and the two portions attached to the pole are inclined at an angle of 60° to each other. The pole is supported by a guy wire attached to the middle point of the pole and inclined at 60° to the* *horizontal. Show that the tension in this wire is $4\sqrt{3}$ times that of the telegraph wire.*

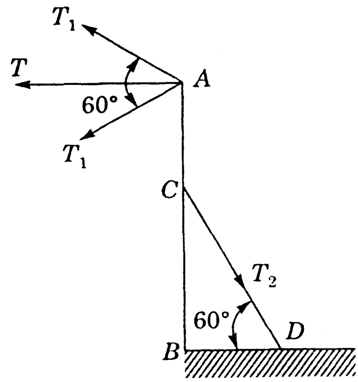


Fig. 1.75

Solution. Let the tension in the two portions of the telegraph wire be T_1 each and the tension in the guy wire be T_2, as shown in Fig. 1.75.

Then $T = 2T_1 \cos 30° = \sqrt{3}\, T_1$

Let $AC = BC = l$

Taking moments about B, we get

$$T \times 2l = T_2 \cos 60° \times l$$

$$T_2 = 2T_1 \times \sqrt{3} \times 2 = 4\sqrt{3}\, T_1$$

$$\frac{T_2}{T_1} = 4\sqrt{3}$$

1.21 Couple and its Moment

Two equal, unlike, parallel forces whose lines of action are not the same, form a couple.

The *arm of the couple* is the perpendicular distance between the lines of action of the forces forming the couple.

The *moment of a couple* is the product of one of the forces forming the couple and the arm of the couple.

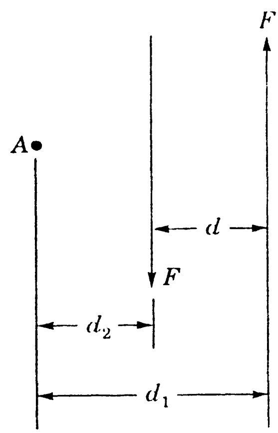


Fig. 1.76

Properties of a couple

1. A couple is said to be positive when its moment is clockwise and negative when it is anti-clockwise.
2. The algebraic sum of the moments of two forces forming a couple about any point in their plane is constant and equal to the moment of the couple, as shown in Fig. 1.76.

$$\sum M_A = F \times d_1 - F \times d_2 = F\,(d_1 - d_2) = Fd = C$$

3. The effect of a couple upon a rigid body is unaltered if it be transferred to any plane parallel to its own plane, the arm remaining parallel to its original position.

4. The effect of any number of couples in the same plane acting on a rigid body are equivalent to a single couple, whose moment is equal to the algebraic sum of the moments of the couples. Thus

$$C = \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} F_1 d_i$$

5. A couple can be balanced by an equal and opposite couple in the same plane.

Example 1.60 *A uniform rod AB of length 2m and weight 150 N is hinged at A and held by a cable at B in horizontal position. The cable is inclined at 30° to the horizontal. Determine the tension in the cable.*

Solution. The rod-cable system is shown in Fig. 1.77.

Let T = tension in the cable

Perpendicular distance of line of action of T from the hinge A.

$$AD = AB \sin 30°$$

$$= 2 \times \frac{1}{2} = 1 \text{ m}$$

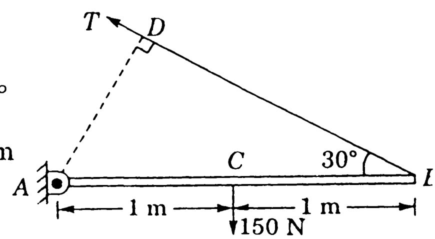


Fig. 1.77

Taking moments about A, we have

$$\downarrow T \times AD = 150° \times AC \downarrow$$

$$T = \frac{150 \times 1}{1} = 150 \text{ N}$$

Example 1.61 *A uniform wheel of weight 25 kN and of 50 cm diameter rests against a 15 cm thick rigid block as shown in Fig. 1.78 (a).* Determine (*a*) the least pull to be applied to a sting tied to the centre of the wheel to just turn the wheel over the corner of the bolck, and (*b*) the reaction of the block.

Solution. When the wheel is just about to turn over the corner of the block, its contact with the ground will be lost. Consequently, there shall be no reaction at A. The free body diagram for the wheel is shown in Fig. 1.78 (*b*).

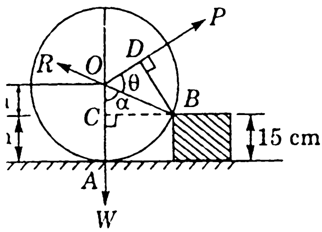
(*a*) $OC = OA - AC = 25 - 15 = 10$ cm

$OB = 25$ cm

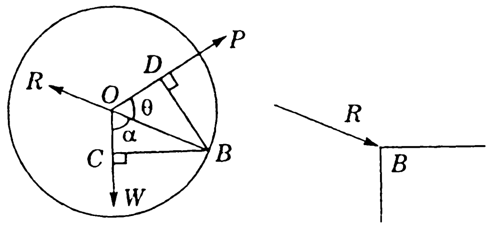
$BC = \sqrt{OB^2 - OC^2} = \sqrt{25^2 - 10^2} = 22.91$ cm

Let θ = angle between P and R.

$BD = OB \sin \theta$



(a)



(b) Free body diagrams

Fig. 1.78

Taking moments about point B, we have

$$P \times BD = W \times BC$$
$$P \times OB \sin \theta = 25 \times 22.91$$
$$P \times 25 \sin \theta = 25 \times 22.91$$
$$P = \frac{22.91}{\sin \theta} \text{ kN}$$

P will be minimum when $\sin \theta$ is maximum, *i.e.*, $\theta = 90°$. Therefore pull in the string is to be applied perpendicular to OB.

$\therefore \qquad P_{min} = 22.91$ kN

(*b*) Now $\qquad \cos \alpha = \frac{OC}{OB} = \frac{10}{25} = 0.4$

$$\alpha = 66.42°$$

Resolving forces parallel to OB when $\theta = 90°$, we have

$$R = W \cos \alpha = 25 \times 0.4 = 10 \text{ kN}$$

1.22 Resultant of a Force and a Couple

A single force and a couple acting in the same plane of a rigid body can not produce equilibrium, but are equivalent to a single force acting in a direction parallel to its original direction.

Let the couple Pd and the force F acting at A lie in the same plane, as shown in Fig. 1.79. Let the couple Pd be replaced by a couple Fa, one of whose forces act opposite to the given force F.

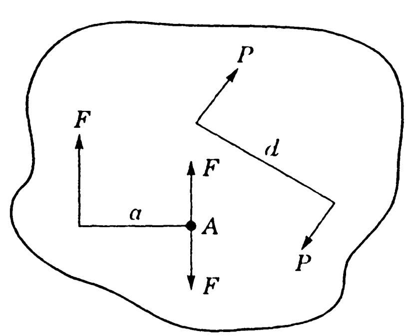


Fig. 1.79 Resultant of a force and a couple.

Now $\qquad Fa = Pd$

$\therefore \qquad a = \frac{Pd}{F}$

The two forces each equal to F at A being equal and opposite balance each other and we are left with a single force acting parallel to its original line of action at a distance of $\frac{Pd}{F}$ from it.

1.23 Resolution of a Force into a Force and a Couple

Consider a force F acting at a point A of a rigid body. Let B be any other point in the body at a distance d from A ,as shown in Fig. 1.80. At B introduce two equal and opposite forces each equal to F and parallel to the force at A. These two forces do not affect the body as they balance each other. The force F at A and the unlike, equal and parallel force at B form a couple whose moment is equal to Fd.

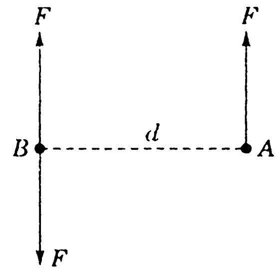


Fig. 1.80 Resolution of a force into a force and a couple.

Hence the force F at A is equivalent to an equal and parallel force F at B together with a couple of moment Fd.

Example 1.62 *ABCD is a rectangle such that* $AB = CD = a$ *and* $BC = AD = b$. *Forces* F_1 *act along AD and CB forces* F_2 *act along AB and CD, as shown in Fig. 1.81. Prove that the perpendicular distance between the resultant of the forces* F_1, F_2 *at A and the resultant of the forces* F_1, F_2 *at C is*

$$\frac{F_1a - F_2b}{\sqrt{F_1^2 + F_2^2}}$$

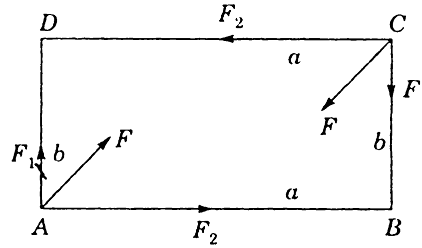


Fig. 1.81

Solution. The resultant of F_1, F_2 at A is $F = \sqrt{F_1^2 + F_2^2}$. Similarly, the resultant of F_1, F_2 at C is an equal, unlike and parallel force F. These two forces form a couple whose moment is Fd, where d is the perpendicular distance between the resultant force at A and C.

Now $$F \times d = F_1 \times a - F_2 \times b$$

$$\therefore \quad d = \frac{F_1a - F_2b}{F} = \frac{F_1a - F_2b}{\sqrt{F_1^2 + F_2^2}}$$

1.24 Graphical Method for Resultant of Parallel Coplanar Forces

1.24.1 Definitions

Space Diagram. A diagram shcwing the outline of a body and exact location of different forces acting on it is known as a space diagram.

Vector Diagram. A composite diagram in which magnitude, direction and sense of forces are represented is known as a vector or force diagram.

1.24.2 Funicular Polygon

In the Bow's notation for representation of a force, two alphabet letters are placed in space on either side of the force. The placing of letters is either done clockwise or anticlockwise. Therefore, the force between letters A and B will be called as force AB and that between B and C, the force BC and so on.

Let us consider the beam with the loading, as shown in Fig. 1.82 (*a*). The spaces between forces are indicated with the letters A to E. Thus the space around the beam from R_1 to F_1 is A, that between

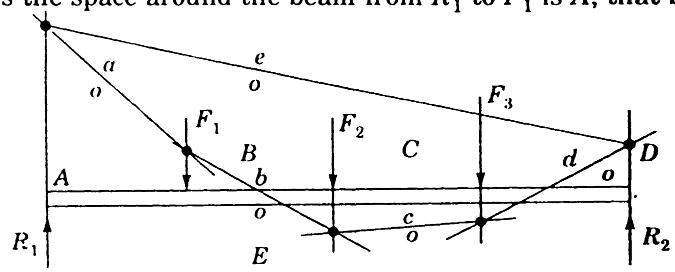


Fig. 1.82 (*a*) Resultant of parallel coplanar forces.

F_1 and F_2 is B and so on. The load line ad is laid out to scale, as shown in Fig. 1.82 (*b*), and lower case letters are used to identify the forces on this line. A pole o is chosen at any convenient position, and the rays ao, bo, co and do are drawn to give us the force polygon. The rays are components of the forces. Thus the components of ab are ao and ob, the components of bc are bo and oc.

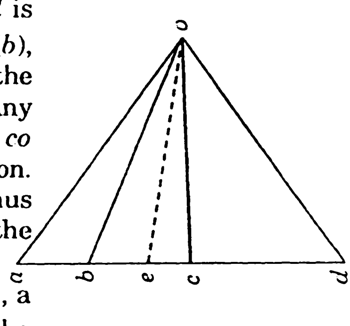


Fig. 1.82 (*b*) Polar diagram.

To determine the reactions R_1 and R_2, a point is chosen in the space figure on the action line of R_1 ; a line is drawn through this point parallel to ray ao and extended until it cuts the action line of force *AB*. These lines in the space figure are known as strings, and the polygon constructed of these strings, as funicular polygon or link polygon. The strings are labelled with appropriate lower case letters, as shown in Fig. 1.82 (*a*). From the point where string *ao* intersects the action line of force *AB*, a string *bo* is drawn parallel to ray *bo* and this line is extended until it cuts the line of action of force *BC*. The process is continued until string *do* intersects the action line of R_2, which is force *DE*. If the force system is in equilibrium, the funicular polygon must close. This condition establishes the direction of string *eo* since it must pass through the point where string *do* intersects the action line of force, *DE*, *i.e.*, R_2 and the point where string *ao* intersects the action line of force *EA*, *i.e.*, R_1. Having established the direction of string *eo*, we draw ray *eo* parallel to it, and thus locate *e* on the load line. The scaled value of R_1 is *ea* and that of R_2 is *de*.

Example 1.63 *A beam 12 m long is loaded as shown in Fig. 1.83.*

(*a*) *Compute the reactions graphically.*

Solution. Choose a scale of 1 cm = 1 kN. Draw the load line *AC* so that ab = 2 cm and bc = 3 cm. Take a pole *O* as shown in Fig. 1.83 (*b*). Join *ao*, *bo* and *co*. Extend the lines of action of forces *AB*. *BC*, *CD* and *DA* downwards. Draw a string *ao* parallel to ray *ao* to intersect the lines of action of forces *DA* and *AB*. Where the string *ao* intersects the line of action of force *AB*, draw the string *bo*, parallel to ray *bo*, where the string *bo* intersects the line of action of force *BC*, draw the string *co* parallel to ray *co* to intersect the line of action of force *CD* and locate the string *do*. Draw the ray *do* parallel to the string *do*. Then $da = R_1$ and $cd = R_2$. On meas-

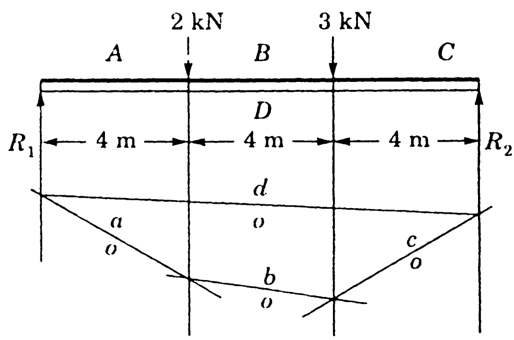


Fig. 1.83 (*a*)

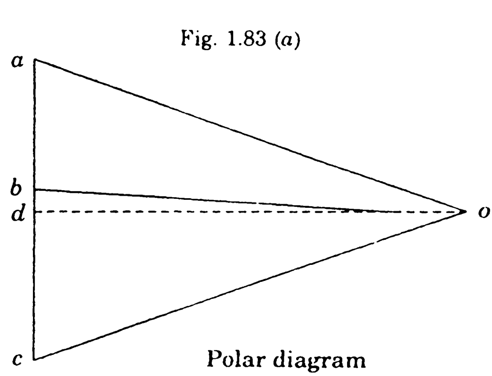


Fig. 1.83 (*b*)

urement da = 2.3 cm and cd = 2.7 cm. Hence R_1 = 2.3 kN and R_2 = 2.7 kN. By calculations R_1 = 2.33 kN and R_2 = 2.67 kN.

1.24.3 Position of The Resultant of Parallel Forces

Consider a number of parallel forces F_1, F_2, F_3 and F_4 as shown in Fig. 1.84 (a). The position of the resultant of these forces can be determined, as explained below :

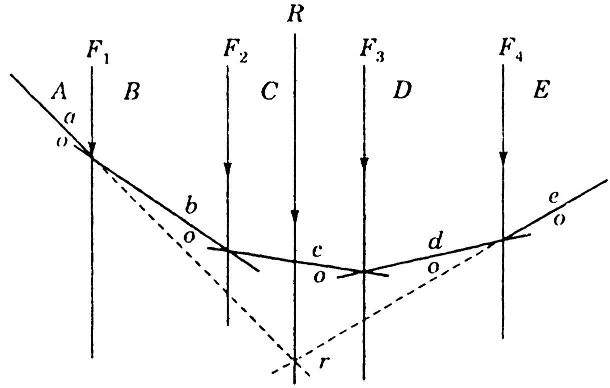


Fig. 1.84 (a) Position of resultant of parallel forces.

(a) Draw the load line ae to a suitable scale so that $ab = F_1$, $bc = F_2$, $cd = F_3$ and $de = F_4$.

(b) Select a suitable pole o and join ao, bo, co, do and eo, as shown in Fig. 1.84 (b).

(c) Select some point on the line of action of force F_1 and draw string ao parallel to ray ao. Where the string ao intersects the line of action of F_1, draw string bo parallel to ray bo and so on.

(d) Extend the strings ao and ea to intersect at point r. Through point r draw a line parallel to the load line ae, which gives the required position of the resultant R.

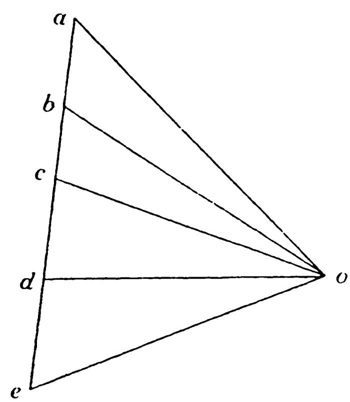


Fig. 1.84 (b) Polar diagram.

Example 1.64 *On a horizontal line PQRS 12 cm long, where PQ = QR = RS = 4 cm, forces of 1000, 1500, 1000 and 500 N, are acting at P, Q, R and S respectively, all downwards, their lines of action making angles of 90, 60, 45 and 30 degrees respectively with PS. Obtain the resultant of the system completely in magnitude, direction and position, graphically and check the answer analytically.*

Solution. Draw the space diagram, as shown in Fig. 1.85(a). To determine the magnitude, direction and position of resultant, the procedure, as explained may be followed :

Choose a scale of 1 cm = 500 N. Draw ab parallel to AB force and equal to 2 cm, bc parallel to BC force and equal to 3 cm, cd parallel to CD force and equal to 2 cm and de parallel to DE and equal to 1 cm [Fig. 1.85 (b)]. Join ea. Then ae = 7.55 cm gives the resultant force. Hence R = 3775 N.

Take a suitable pole o. Join ao, bo, co, do and eo. Draw string ao parallel to ray ao to intersect the line of action of force AB at a convenient point. At that point draw string bo parallel to ray bo to intersect the line of action of force BC. At that point draw string co parallel to ray co to intersect the line of action of force CD. At that point draw string do parallel to ray do to intersect the line of action of force DE. At that point draw string eo parallel to ray eo. Produce the strings ao and eo to meet at the point r. At r draw a line parallel to ae to intersect the line PS in T. Then PT = 4.2 cm and the resultant R makes an angle of 60° with PS as shown in Fig. 1.85 (a).

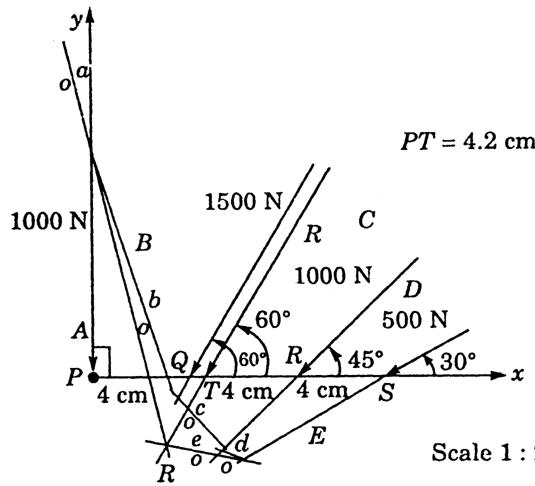


Fig. 1.85 (a)

Analytical Method

Force, N	F_x, N	F_y, N
1000	—	− 1000.00
1500	− 750.00	− 1299.04
1000	− 707.10	− 707.10
500	− 433.01	− 250.00

$\Sigma F_x = -1890.11$ $\quad$ $\Sigma F_y = -3256.14$

$$\text{Resultant } R = \sqrt{F_x^2 + F_y^2}$$

$$= \sqrt{(1890.11)^2 + (3256.14)^2}$$

$$= 3764.96 \text{ N}$$

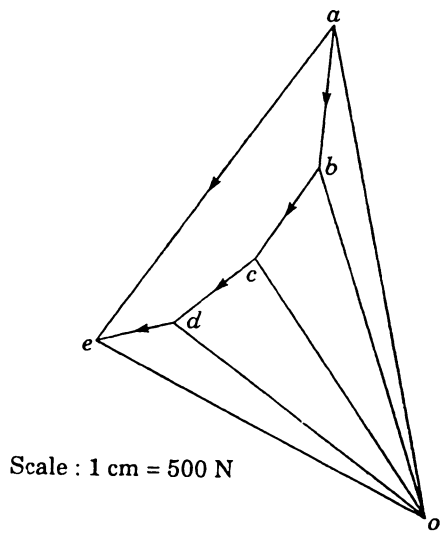


Fig. 1.85 (*b*)

$$\tan\theta = \frac{F_y}{F_x} = \frac{3256.14}{1890.11} = 1.72272$$

$$\theta = 59.866°$$

To determine the location of resultant force, taking moments about point *P*, we get

$$3256.14 \times x = 1299.04 \times 4 + 707.10 \times 8 + 250 \times 12$$

$$= 5196.16 + 5656.80 + 3000 = 13852.96$$

$$x = 4.25 \text{ cm}$$

Example 1.65 *Two halves of a round homogeneous cylinder are held together by a thread wrapped round the cylinder with two equal weights, P attached to its ends, as shown in Fig. 1.86. The complete cylinder weighs W N. The plane of contact of both of its halves is vertical. Determine the minimum value of P for which both halves of the cylinder will be in equilibrium on a horizontal plane.*

Solution. The free body diagram for the half cylinder is shown in Fig. 1.86 (*b*).

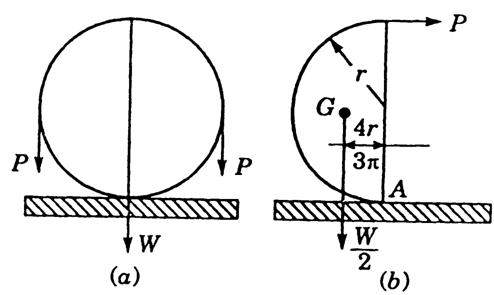


Fig. 1.86

Taking moments about A, we get

$$\frac{W}{2} \times \frac{4r}{3\pi} = P \times 2r$$

$$P = \frac{W}{3\pi} \text{ N}$$

Example 1.66 *Three forces P, Q and R act along the sides BC, CA, and AB of a triangle ABC, taken in order. Show that if their resultant passes through the*

(*i*) *centroid, then* $P + Q + R = 0$

(*ii*) *circumcentre, then* $P \cos A + Q \cos B + R \cos C = 0$

(*iii*) *orthocentre, then* $P \sec A + Q \sec B + R \sec C = 0$

(*iv*) *centroid, then* $P \operatorname{cosec} A + Q \operatorname{cosec} B + R \operatorname{cosec} C = 0$

Solution. (*i*) Let the internal bisectors of the angles of the triangle meet at I, the incentre (Fig. 1.87). The perpendiculars on the sides from I are all equal and each equal to r (say). Taking moments of forces about I, we get

$$Pr + Qr + Rr = 0$$

or $$P + Q + R = 0$$

(*ii*) Let the right bisectors of the sides of the triangle meet at 0, the circumcentre. Then $OA = OB = OC = R_1$ (say) (Fig. 1.88).

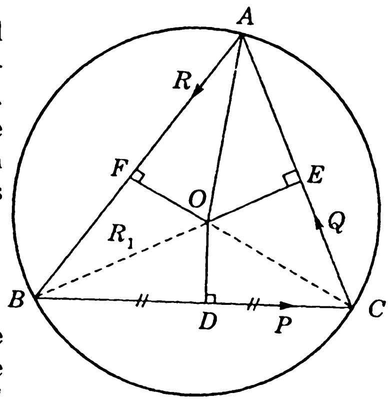


Fig. 1.87

$$\angle BOC = 2A, \quad \angle BOD = A$$

$$OD = OB \cos A = R_1 \cos A$$

Similarly, $$OE = R_1 \cos B \; ; \; OF = R_1 \cos C$$

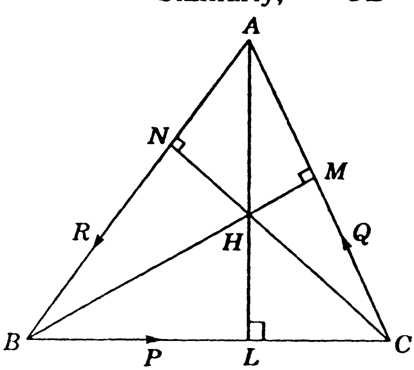


Fig. 1.88

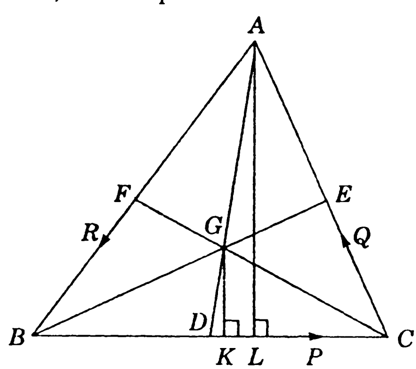


Fig. 1.89

Taking moments about O, we get

$$PR_1 \cos A + Q\,R_1 \cos B + R\,R_1 \cos C = 0$$

or $$P \cos A + Q \cos B + R \cos C = 0$$

(*iii*) Let the perpendiculars AL, BM, CN on the opposite sides meet in H, the orthocentre (Fig. 1.89). If O is the circumcentre and OD the perpendicular on BC, then

$$AH = 2OD = 2\,R_1 \cos A$$

$$HL = AL - AH = AB \sin B - 2R_1 \cos A$$

Now $AB = c$ and $2R_1 \sin C = c$

$$HL = c \sin B - c \cos A/\sin C$$

$$= \frac{c\,[\sin B \sin C - \cos A]}{\sin C}$$

$$= \frac{c\,[\sin B \sin C + \cos (B + C)]}{\sin C}$$

$$= \frac{c\,[\sin B \sin C + \cos B \cos C - \sin B \sin C]}{\sin C}$$

$$= c \cos B \cos C/\sin C = 2\,R_1 \cos B \cos C$$

Similarly, $HM = 2R_1 \cos C \cos A$

$$HN = 2\,R_1 \cos A \cos B$$

Taking moments about H, we get

$$P \cdot 2R_1 \cos B \cos C + Q \cdot 2R_1 \cos C \cos A + R \cdot 2R_1 \cos A \cos B = 0$$

or $$P \sec A + Q \sec B + R \sec C = 0$$

(*iv*) Let the medians AD, BE, CF meet at G, the centroid (Fig. 1.90). Draw AL and GK perpendiculars on BC. Then

$$GK = \frac{1}{3} AL = \frac{1}{3} c \sin B = \frac{2}{3} R_1 \sin C \sin B.$$

Similarly, the perpendiculars from G on CA and AB are :

$$\frac{2}{3} R_1 \sin A \sin C, \frac{2}{3} R_1 \sin A \sin B.$$

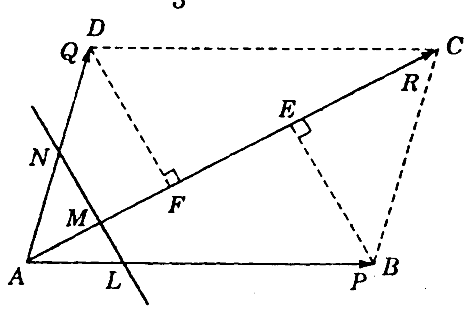


Fig. 1.90

Taking moments about G, we get

$$P \cdot \frac{2}{3} \cdot R_1 \sin B \sin C + Q \cdot \frac{2}{3} R_1 \sin A \sin C + R \cdot \tfrac{2}{3} R_1 \sin A \sin B = 0$$

or
$$\frac{P}{\sin A} + \frac{Q}{\sin B} + \frac{R}{\sin C} = 0$$

or
$$P \operatorname{cosec} A + Q \operatorname{cosec} B + R \operatorname{cosec} C = 0$$

Example 1.67 *A wheel of weight W and radius r is to be dragged over an obstacle of height h by a horizontal force P applied to the centre of the wheel. Show that the horizontal force should be at least*

$$P > \frac{W\sqrt{2rh - h^2}}{r - h}$$

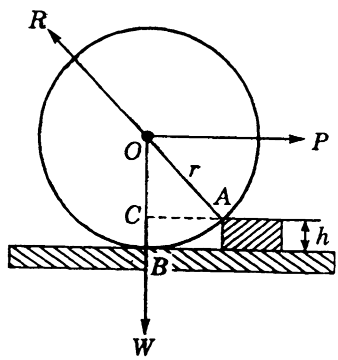


Fig. 1.91

Solution. When the wheel is about to be dragged over the obstacle it will not be resting over the surface at B. In that case, the forces acting on the wheel are (Fig. 1.91):

1. Weight W of the wheel
2. Horizontal force P.
3. Reaction R at A along AO.

Now $\quad OC = OB - CB = r - h$

$$AC = \sqrt{OA^2 - OC^2} = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$$

Taking moments about A, we have

$$P \times OC = W \times AC$$

$$P = \frac{W\sqrt{2rh - h^2}}{r - h}$$

Example 1.68 *A square wooden block of mass M is hinged at A and rests on a roller at B. It is pulled by means of a string attached at D and inclined at an angle of 30° with the horizontal. Determine the force P which should be applied to the string to just lift the block from the roller.*

Solution. The various forces acting on the wooden block are shown in Fig. 1.92 (b). Taking moments about A, when the block is just being lifted at roller B ($R_B = 0$), we have

$$P \cos 30° \times a + P \sin 30° \times a - Mg \times \frac{a}{2} = 0$$

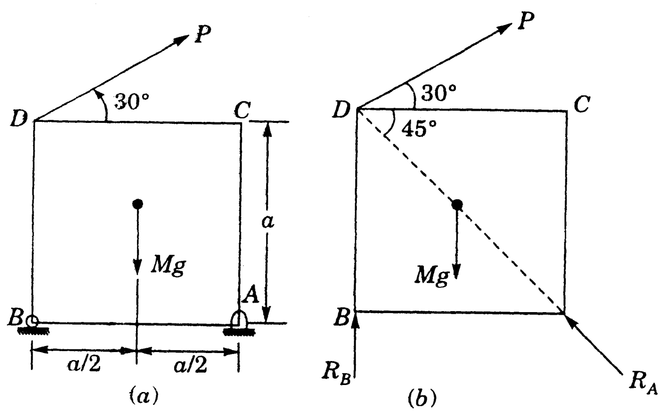


Fig. 1.92

$$P \times \frac{\sqrt{3}}{2} + P \times \frac{1}{2} - \frac{Mg}{2} = 0$$

$$2.732\, P = Mg$$

or $$P = 0.366\, Mg$$

Example 1.69 *A uniform wheel of 0.8 m diameter weighing 2 kN rests against a rectangular obstacle 0.25 m high (Fig. 1.93 (a)]. Find the least force required which when acting through the centre of the wheel will just turn the wheel over the corner of the block. Also, find the angle θ which this least force shall make with AC.*

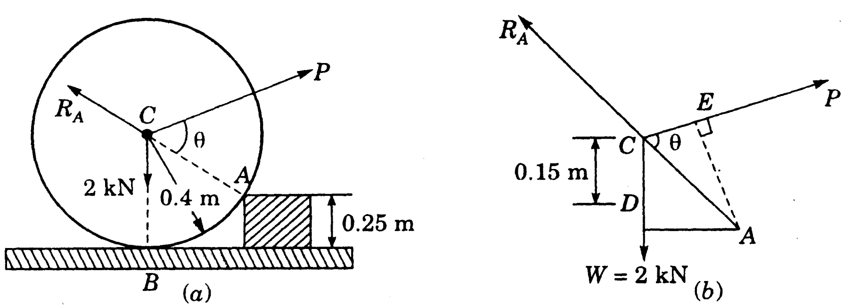


Fig. 1.93

Solution. The forces acting on the wheel are shown in Fig. 1.93 (*b*). At the time of lifting of wheel, $R_B = 0$. Taking moments about *A*, we have

$$2 \times AD - P \times AC \sin \theta = 0$$

$$2 \times \sqrt{(0.4)^2 - (0.15)^2} - P \times 0.4 \sin \theta = 0$$

$$2 \times 0.3708 - 0.4\, P \sin \theta = 0$$

$$P = \frac{1.854}{\sin\theta}$$

P is minimum when $\theta = 90°$

$$P_{min} = 1.854 \text{ kN}.$$

Example 1.70 *A uniform rod AB of negligible weight is hinged at the end A and supported at end B by a string, as shown in Fig. 1.94. Find the value of θ corresponding to the position of equilibrium of the bar if* $W_1 = W_2/2$.

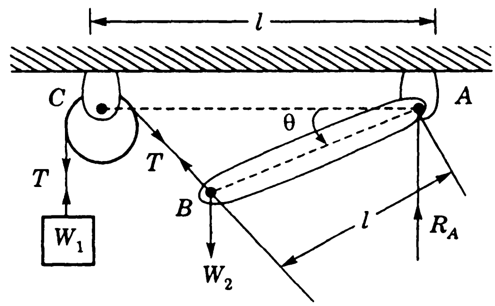


Fig. 1.94

Solution. The forces acting on the bar AB are :

1. Weight W_2 at B
2. Tension T in the string
3. Reaction R_A at A

Now $AB = AC$

$$\therefore \quad \angle ABC = \angle ACB = 90° - \frac{\theta}{2}$$

Taking moments about A, we have

$$W_2 \times l \cos\theta - T \times l \sin\left(90° - \frac{\theta}{2}\right) = 0$$

Now $T = W_1$

$$W_2 \cos\theta - W_1 \cos\frac{\theta}{2} = 0$$

$$W_2\left(2\cos^2\frac{\theta}{2} - 1\right) - W_1 \cos\frac{\theta}{2} = 0$$

$$2W_2 \cos^2\frac{\theta}{2} - W_1 \cos\frac{\theta}{2} - W_2 = 0$$

$$\cos\frac{\theta}{2} = \frac{W_1 \pm \sqrt{W_1^2 + 8W_2^2}}{4W_2}$$

$$= \frac{1}{4}\left(\frac{W_1}{W_2} \pm \sqrt{\left(\frac{W_1}{W_2}\right)^2 + 8}\right)$$

$$= \frac{1}{4}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + 8}\right) = 0.843$$

$$\theta = 65°$$

1.25 Resultant of Coplanar Non-Concurrent Forces

Let F_1, F_2 and F_3 constitute a system of forces acting on a body which are non-concurrent but coplanar, as shown in Fig. 1.95 (a). Each force can be replaced by a force of the same magnitude and acting in the same direction at point O and a couple of magnitude, $C_i = F_i d_i$, where d_i is the perpendicular distance between the line of action of force F_i and point O. Thus, the given system of forces shown in Fig. 1.95 (a) is equivalent to the system shown in Fig. 1.95 (b), where $\Sigma C_O = \Sigma C_i$, *i.e.*, algebraic sum of the couples of the given forces F_i about O. At O, the concurrent forces F_1, F_2 and F_3 can be combined as usual to get the resultant force R. Therefore, the system of forces is equivalent to resultant force R at O and a couple ΣC_O, as shown in Fig. 1.95 (c). The force R and couple ΣC_O shown in Fig. 1.95 (c) can be replaced by a single force R acting at a distance d from O such that the moment produced by R is equal to ΣC_O, as shown in Fig. 1.95 (d). Thus

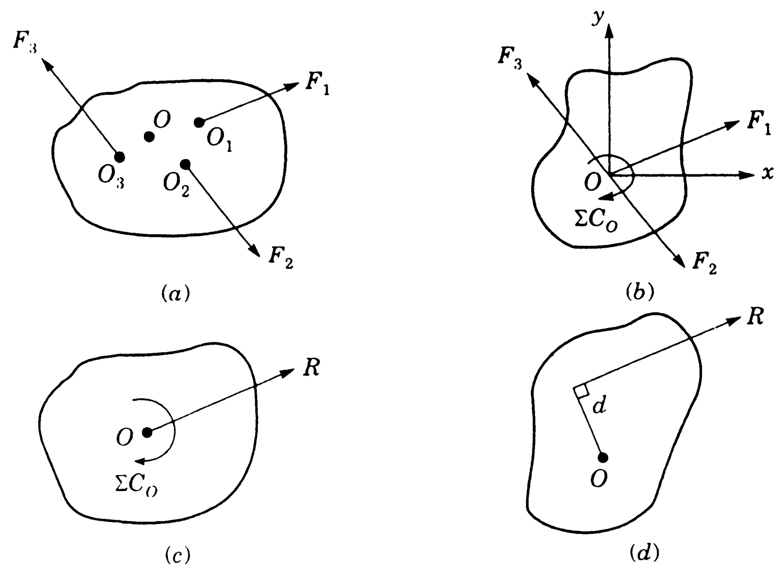


Fig. 1.95

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

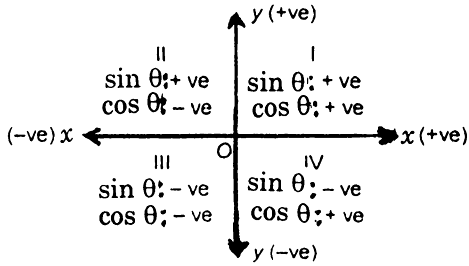
$$\tan \theta = \frac{\Sigma F_y}{\Sigma F_x}$$

The following procedure may be adopted to solve the problem :

1. Determine ΣF_x and ΣF_y
2. Determine, $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$
3. Find the direction of R w.r.t. x-axis by using the relation,

$$\tan \theta = \frac{\Sigma F_y}{\Sigma F_x}$$

Decide in which quadrant the resultant lies, depending upon the sign of ΣF_x and ΣF_y, where ΣF_y corresponds to $\sin \theta$ and ΣF_x to $\cos \theta$.



4. Calculate the algebraic sum of the moments of all forces about any given point O (say).
5. Mark the position of resultant such that it produces the same direction of moment about point O (see **Fig. 1.96** (*a*)).

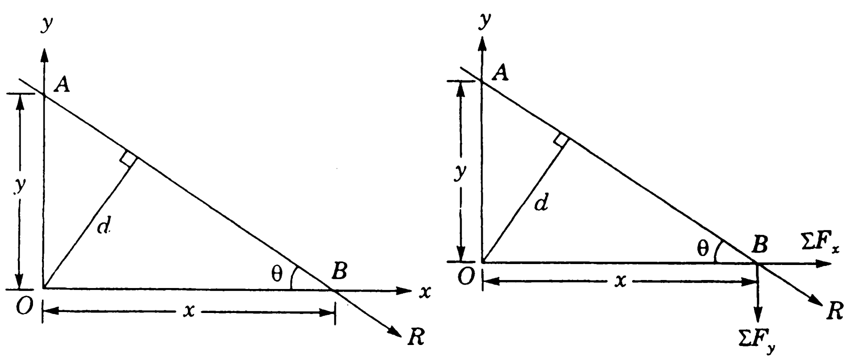


Fig. 1.96

6. Apply Varignon's theorem to find the exact position of resultant, *i.e.*,

$$\Sigma C_O = R \times d$$

7. Calculate the x and y intercepts of the line of action of resultant (Fig. 1.96 (b)).

$$x = \frac{d}{\sin\theta} \text{ and } y = \frac{d}{\cos\theta}$$

or
$$x = \frac{\Sigma C_O}{\Sigma F_y} \text{ and } y = \frac{\Sigma C_O}{\Sigma F_x}$$

Example 1.71 *A square ABCD is subjected to forces equal to P, 2P, 3P and 4P along the sides AB, BC, CD and DA. Determine the magnitude, direction and line of action of the resultant.*

Solution. The force system is shown in Fig. 1.97.

$$\Sigma F_x = P - 3P = -2P$$

$$\Sigma F_y = 2P - 4P = -2P$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{(-2P)^2 + (-2P)^2} = 2\sqrt{2}\,P$$

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x} = \frac{-2P}{-2P} = 1$$

$$\theta = 45^\circ$$

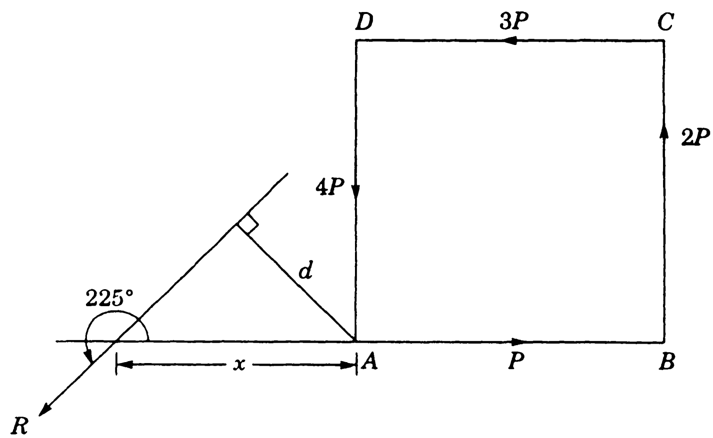


Fig. 1.97

Since numerator and denominator are both negative, therefore θ lies in the 3rd quadrant, *i.e.*,

$$\theta = 180^\circ + 45^\circ = 225^\circ$$

Let $\quad \alpha$ = length of side of the square

$$\Sigma M_A = -2P \times a - 3P \times a = -5Pa \text{ (ccw)}$$

The resultant R should lie as shown in Fig. 1.97, so that it can produce anticlockwise moment about point A.

If $\quad d$ = perpendicular distance of R from A, then

$$R \times d = \Sigma M_A$$

$$d = \frac{5Pa}{2\sqrt{2P}} = \frac{5a}{2\sqrt{2}}$$

The horizontal distance x of line of action of R from point A,

$$= \frac{d}{\sin\theta} = \left(\frac{5a}{2\sqrt{2}}\right) \times \frac{1}{\sin 45^\circ} = 2.5\,a$$

Example 1.72 *A beam is subjected to the system of loads as shown in Fig. 1.98. Determine the resultant completely.*

Solution.

$\Sigma F_x = 0$:

$$\Sigma F_x = 2\cos 60^\circ - 3\cos 30^\circ - 4\cos 45^\circ = -4.4265 \text{ kN}$$

$\Sigma F_y = 0$:

$$\Sigma F_y = -2\sin 60^\circ - 3\sin 30^\circ - 4\sin 45^\circ - 1.5 = -7.5605 \text{ kN}$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$= \sqrt{(-4.4265)^2 + (-7.5605)^2} = 8.761 \text{ kN}$$

$$\theta = \tan^{-1}\left[\frac{\Sigma F_y}{\Sigma F_x}\right] = \tan^{-1}\left[\frac{-7.5605}{-4.4265}\right] = \tan^{-1}(1.708)$$

$$\theta = 59.65^\circ$$

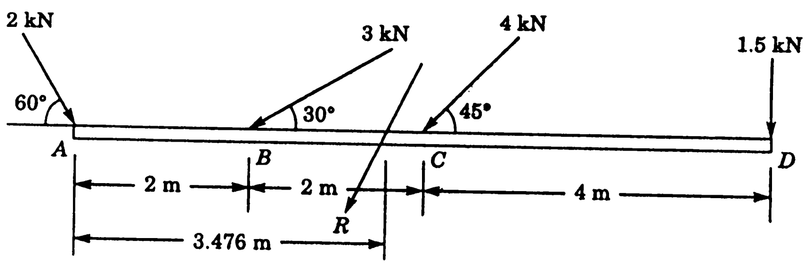


Fig. 1.98

Since numerator and denominator are both negative, therefore, R lies in the 3rd quadrant.

$$\Sigma M_A = 3\sin 30^\circ \times 2 + 4\sin 45^\circ \times 4 + 1.5 \times 8 = 26.314 \text{ kNm}$$

$$R \times d = \Sigma M_A$$

$$d = \frac{26.314}{8.761} = 3 \text{ m}$$

$$x = \frac{d}{\sin\theta} = \frac{3}{\sin 59.65^\circ} = 3.476 \text{ m}$$

The resultant is shown in Fig. 1.98 which produces clockwise moment about point A.

Ecample 1.73 *Find the magnitude and position of the resultant of the system of forces shown in Fig. 1.99*

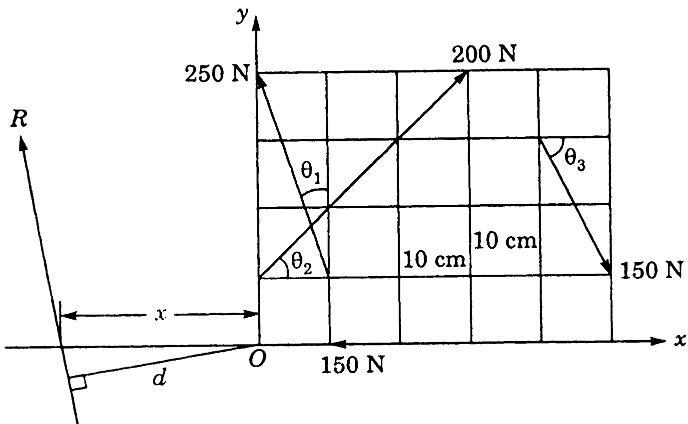


Fig. 1.99

Solution.

$$\theta_1 = \tan^{-1}\left(\frac{10}{30}\right) = 18.43°$$

$$\theta_2 = \tan^{-1}\left(\frac{30}{30}\right) = 45°$$

$$\theta_3 = \tan^{-1}\left(\frac{20}{10}\right) = 63.43°$$

$$\Sigma F_x = -250 \sin 18.43° + 200 \cos 45° + 150 \cos 63.43° - 150$$
$$= -20.52 \text{ N}$$

$$\Sigma F_y = 250 \cos 18.43° + 200 \sin 45° - 150 \sin 63.43°$$
$$= 244.44 \text{ N}$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{(-20.52)^2 + (244.44)^2} = 245.3 \text{ N}$$

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x} = \frac{244.44}{-20.52} = -11.912$$

$$\theta = -85.2°$$

The numerator is +ve and denominator is –ve, therefore θ lies in the 2nd quadrant.

$$\Sigma M_O = -250 \cos 18.43° \times 10 - 250 \sin 18.43° \times 10 + 200 \cos 45° \times 10 + 150 \cos 63.43° \times 30 + 150 \sin 63.43° \times 40$$

$$= 5631.2 \text{ N. cm}$$

$$d = \frac{\Sigma M_0}{R} = \frac{5631.2}{245.3} = 22.96 \text{ cm}$$

$$x = \frac{d}{\sin\theta} = \frac{22.96}{\sin 85.2°} = 23 \text{ cm}$$

$$y = \frac{d}{\cos\theta} = \frac{22.96}{\cos 85.2°} = 274.4 \text{ cm.}$$

Example 1.74 *Determine the resultant in magnitude and position of the forces shown in Fig. 1.100*

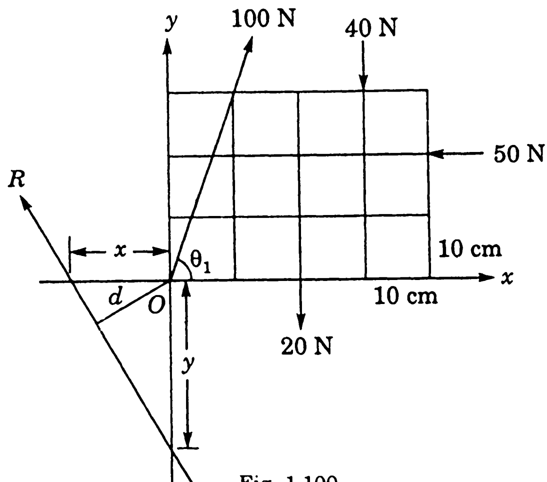


Fig. 1.100

Solution. $\theta_1 = \tan^{-1}\left(\frac{30}{20}\right) = 71.56°$

$\Sigma F_x = 0$: $\Sigma F_x = 100 \cos 71.56° - 50 = -18.37$ N

$\Sigma F_y = 0$: $\Sigma F_y = 100 \sin 71.56° - 40 - 20 = 34.86$ N

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$= \sqrt{(-18.37)^2 + (34.86)^2} = 39.4 \text{ N}$$

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x} = \frac{34.86}{-18.37} = -1.8976$$

$$\theta = -62.21°$$

The numerator is +ve and denominator is –ve, therefore, θ lies in the 2nd quadrant.

$\Sigma M_O = 0$: $\Sigma M_O = 20 \times 20 + 40 \times 30 - 50 \times 20 = 600$ N. cm

$$d = \frac{\Sigma M_O}{R} = \frac{600}{39.4} = 15.22 \text{ cm}$$

$$x = \frac{d}{\sin\theta} = \frac{15.22}{\sin 62.21°} = 17.2 \text{ cm}$$

$$y = \frac{d}{\cos\theta} = \frac{15.22}{\cos 62.21°} = 32.64 \text{ cm}$$

Example 1.75 *Find the resultant of coplanar force system acting on a lamina shown in Fig. 1.101.*

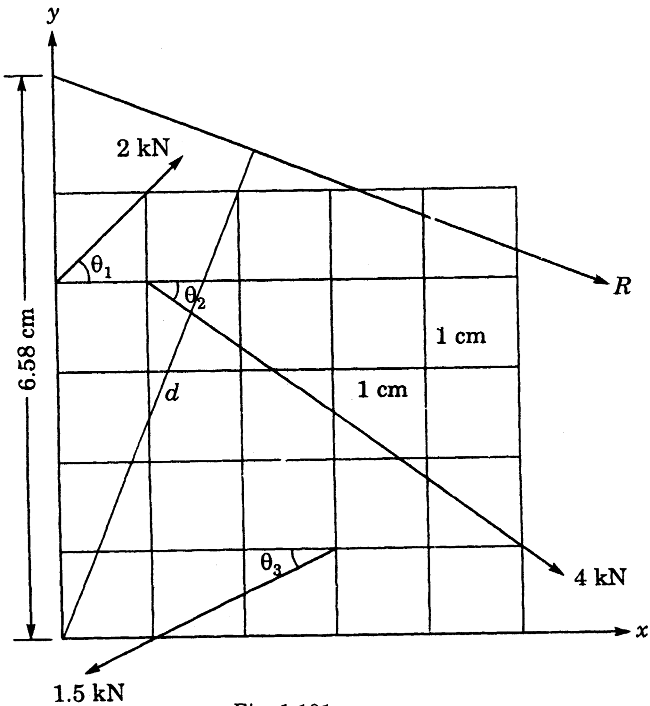


Fig. 1.101

Solution.

$$\theta_1 = \tan^{-1} 1 = 45°$$

$$\theta_2 = \tan^{-1} \frac{3}{4} = 36.87°$$

$$\theta_3 = \tan^{-1} \frac{1}{2} = 26.565°$$

$\Sigma F_x = 0$: $\quad \Sigma F_x = 2 \cos 45° + 4 \cos 36.87° - 1.5 \cos 26.565°$

$$= 3.272 \text{ kN}$$

$\Sigma F_y = 0$: $\quad \Sigma F_y = 2 \sin 45° - 4 \sin 36.87° - 1.5 \sin 26.565°$

$$= -1.657 \text{ kN}$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$= \sqrt{(3.272)^2 + (-1.657)^2} = 3.667 \text{ kN}$$

$$\theta = \tan^{-1}\left[\frac{\Sigma F_y}{\Sigma F_x}\right] = \tan^{-1}\left[\frac{-1.657}{3.272}\right]$$

$$= \tan^{-1}(-0.5064) = -26.85°$$

The numerator is –ve and denominator is +ve. Therefore θ lies in the 4th quadrant.

$$\Sigma M_O = 2\cos 45° \times 4 + 4\cos 36.87° \times 4 + 4\sin 36.87° \times 1 - 1.5\cos 26.565° \times 1 + 1.5\sin 26.565° \times 3$$

$$= 21.52 \text{ kN. cm}$$

$$d = \frac{\Sigma M_O}{R} = \frac{21.52}{3.667} = 5.87 \text{ cm}$$

$$x = \frac{d}{\sin\theta} = \frac{5.87}{\sin 26.85°} = 13 \text{ cm}$$

$$y = \frac{d}{\cos\theta} = \frac{5.87}{\cos 26.85°} = 6.58 \text{ cm}$$

Example 1.76 *The force system applied to an angle bracket is shown in Fig. 1.102. Determine the magnitude, direction and line of action of the resultant force.*

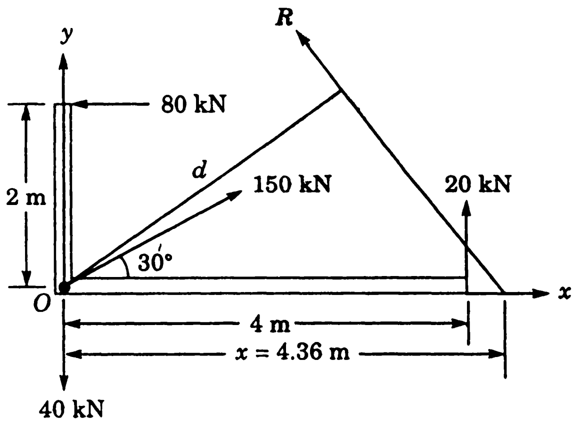


Fig. 1.102

Solution. $\Sigma F_x = 150\cos 30° - 80 = 49.9$ kN

$$\Sigma F_y = 150\sin 30° + 20 - 40 = 55 \text{ kN}$$

Resultant force,

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{(49.9)^2 + (55)^2} = 74.26 \text{ kN}$$

Inclination of R with horizontal,

$$\theta = \tan^{-1}\left[\frac{\Sigma F_y}{\Sigma F_x}\right] = \tan^{-1}\left[\frac{55}{49.9}\right] = \tan^{-1} 1.902$$

$$\theta = 47.78°$$

Both the numerator and denominator are positive. Therefore, the resultant lies in the first quadrant.

$$\Sigma M_O = 0 : \quad \Sigma M_O = -80 \times 2 - 20 \times 4 = \quad -240 \text{ kN. m}$$

$$d = \frac{\Sigma M_O}{R} = \frac{240}{74.26} = 3.23 \text{ m}$$

$$x = \frac{d}{\sin \theta} = \frac{3.23}{\sin 47.78^\circ} = 4.26 \text{ m}$$

$$y = \frac{d}{\cos \theta} = \frac{3.23}{\cos 47.78^\circ} = 4.8 \text{ m.}$$

REVIEW QUESTIONS

1. Define a force and give its characteristics.
2. What is a force system ? List various types of force systems.
3. State the laws of mechanics.
4. State the principle of transmissibility of force.
5. State the parallelogram law of forces.
6. What is the resultant of two forces P and Q inclined to each other at an angle α in megnitude and disection ?
7. Define resolved part of a force in a given direction and give its components.
8. State triangle law of forces.
9. State Lami's theorem.
10. What are the various types of forces to which a body may be subjected to ?
11. What is a free body diegram ?
12. Explain moment of a force, moment centre and moment arm.
13. State Varignon's theorem.
14. What are parallel forces ? State the laws governing parallel forces.
15. What is a couple and its moment.
16. What are the properties of a couple ?

MULTI-CHOICE QUESTIONS

1. The resultant of two forces $3P$ and $2P$ is R. If the first force is double the resultant is also doubled. Then the angle between two forces is
   (*a*) 30° (*b*) 60°
   (*c*) 120° (*d*) 150°
2. Three forces P, $2P$, and $3P$ are exerted along the direction of three sides of an equilateral triangle. Their resultant is
   (*a*) $\sqrt{3}P$ (*b*) $3\sqrt{3}\,P$
   (*c*) $3\sqrt{P}$ (*d*) $3P$
3. If α is the angle which the resultant of P and Q forces acting at an angle θ makes with force P then $\tan \alpha$ is equal to
   (*a*) $Q \cos \theta/(P + Q \cos \theta)$ (*b*) $Q \sin \theta/(P + Q \cos \theta)$
   (*c*) $Q/(P + Q \cos \theta)$ (*d*) $P/(Q + P \sin \theta)$.

4. If α is the angle between the forces P and Q then their resultant R is

(*a*) $[P^2 + Q^2 + 2PQ \cos \alpha]^{1/2}$ (*b*) $[P^2 + Q^2 + 2PQ \sin \alpha]^{1/2}$

(*c*) $[P^2 + Q^2 + PQ \cos \alpha]^{1/2}$ (*d*) $[P^2 + Q^2 + PQ \sin \alpha]^{1/2}$.

5. Three forces $\sqrt{3}P$, P and $2P$, acting on a particle are in equilibrium. If the angle between the first and second is 90 degree, then angle between the second and third will be

(*a*) 30° (*b*) 60°

(*c*) 120° (*d*) 150°.

6. Each of two equal forces is equal to P and the angle between them is α, then their resultant is equal to

(*a*) $2P \cos(\alpha/2)$ (*b*) $P \cos(\alpha/2)$

(*c*) $2P \sin(\alpha/2)$ (*d*) $P \sin \alpha$

7. The resultant of two forces, each equal to P, is also P. The angle between the two forces is

(*a*) 0° (*b*) 60°

(*c*) 90° (*d*) 120°.

8. Forces 7, 5 and 3 N acting on a particle are in equilibrium. The angle between the last pair of forces is

(*a*) 30° (*b*) 60°

(*c*) 60° (*d*) 90°.

9. Two equal forces act at a point. If the square of their resultant is equal to three times of their product then the angle between them is

(*a*) 30° (*b*) 45°

(*c*) 60° (*d*) 90°.

10. If a body is in equilibrium under the action of three coplanar forces, then

(*a*) they should act in a straight line

(*b*) they should meet at a point

(*c*) their horizontal and vertical components should be equal

(*d*) none of these is correct.

11. A body of 60 N rests in limiting equilibrium on an inclined plane whose slope is 30 degree. If the plane is raised to a slope of 60 degree then the force along the plane required to support it is

(*a*) 30 N (*b*) $20\sqrt{3}$ N

(*c*) $10\sqrt{3}$ N (*d*) $30\sqrt{3}$ N.

12. Two forces when acting at right angle produce a resultant of $\sqrt{10}$ N and when acting at 60 degree produce a resultant of $\sqrt{13}$ N. These forces are

(*a*) 2 N and $\sqrt{6}$ N (*b*) 3 N and 1 N

(*c*) $\sqrt{5}$ N and $\sqrt{2}$ N (*d*) 2 N and 5 N.

13. A force equal to 100 N is inclined at an angle of 60 degree to the horizontal. Its resolved part in a vertical direction is

(*a*) 50 N (*b*) 100 N

(*c*) $50\sqrt{3}$ N (*d*) $100\sqrt{3}$ N.

14. Two forces 130 N and $30\sqrt{3}$ N act on a particle at an angle θ and equal to a resultant force of 140 N, then the angle between the forces is

(*a*) 45 degree (*b*) 30 degree

(*c*) 60 degree (*d*) 90 degree.

15. A force which makes an angle of 30 degree with the horizontal has its horizontal resolved part as 100 $\sqrt{3}$ N, then the force is

(*a*) 100 N (*b*) 200 N

(*c*) $200\sqrt{3}$ N (*d*) $50\sqrt{3}$ N.

16. If three forces, acting on a rigid body be represented in magnitude and direction by the sides of a triangle taken in order, then they are

(*a*) in equilibrium

(*b*) equivalent to a couple

(*c*) equivalent to a force

(*d*) equivalent to a force and a couple.

17. The larger of two forces is 800 N. The angle between them is 120 degree. If their resultant is perpendicular to the smaller force, then the smaller force is

(*a*) 300 N (*b*) 400 N

(*c*) 500 N (*d*) 600 N.

18. Like parallel forces act at the vertices *A*, *B* and *C* of a triangle and are proportional to the lengths *BC*, *CA* and *AB* respectively. The centre of forces is at the

(*a*) centroid (*b*) circumcentre

(*c*) incentre (*d*) orthocentre.

19. *ABCD* is a square. Equal forces *P* are acting along *AB*, *CB*, *AD* and *DC*. Their resultant is a force 2*P* acting

(*a*) along *DC* (*b*) along *AB*

(*c*) along *AC*

(*d*) parallel to *AB* through the centre of the square.

20. The moment of a force (represented by a line *AB*) about a point *O* is

(*a*) half the area of the triangle *OAB*

(*b*) equal to the area of the triangle *OAB*

(*c*) two times the area of the triangle *OAB*

(*d*) three times the area of the triangle *OAB*.

21. If two forces *P* and *Q* are like, then their resultant is

(*a*) $\sqrt{2P+Q}$ (*b*) $\sqrt{P^2-Q^2}$

(*c*) $\sqrt{P+Q}$ (*d*) $P+Q$.

22. The resultant of two unlike parallel forces is 20 N and acts at a distance of 60 cm and 80 cm from them, the forces are

(*a*) 40 N, 20 N (*b*) 140 N, 120 N

(*c*) 50 N, 30 N (*d*) 80 N, 60 N

23. Two parallel forces not having the same line of action form a couple, if they are

(*a*) like and equal (*b*) like and unequal

(*c*) unequal and unlike (*d*) equal and unlike.

24. Two like forces *P* and 3*P* act on a rigid body at points *A* and *B* respectively. If the forces are interchanged in position, the position of application of the resultant will be displaced through a distance as

(*a*) $\frac{1}{2}AB$ (*b*) $\frac{1}{3}AB$

(*c*) $\frac{1}{4}AB$ (*d*) $\frac{3}{4}AB$.

25. According to the principle of transmissibility of forces, when a force acts upon a body, its effect is
(*a*) minimum when it acts at the C.G. of the body
(*b*) maximum when it acts at the C.G. of the body
(*c*) different at different points of the body
(*d*) same at every point in its line of action.

26. If two forces each equal to P in magnitude act at right angles, their effect may be neutralised by a third force acting along their bisector in opposite direction whose magnitude is equal to
(*a*) $2P$ (*b*) $\frac{1}{2}P$
(*c*) $\sqrt{2P}$ (*d*) $\frac{P}{2}$.

27. The resolved part of resultant of two forces inclined at an angle O in a given direction is equal to the
(*a*) algebraic sum of the resolved parts of the forces in the direction
(*b*) sum of the resolved parts of the forces in the direction
(*c*) difference of the forces multiplied by cos θ
(*d*) sum of the forces multiplied by sin θ.

ANSWERS

1. (*c*)	**2.** (*a*)	**3.** (*b*)	**4.** (*a*)	**5.** (*c*)
6. (*a*)	**7.** (*d*)	**8.** (*b*)	**9.** (*c*)	**10.** (*b*)
11. (*d*)	**12.** (*b*)	**13.** (*c*)	**14.** (*d*)	**15.** (*b*)
16. (*a*)	**17.** (*b*)	**18.** (*c*)	**19.** (*d*)	**20.** (*c*)
21. (*d*)	**22.** (*d*)	**23.** (*d*)	**24.** (*c*)	**25.** (*d*)
26. (*c*)	**27.** (*a*)			

EXERCISES

1.1 The resultant of two forces F_1 and F_2 is F. If F_2 be doubled F is doubled whilst, if F_2 be reversed, F is again doubled, show that $F_1 : F_2 : F :: \sqrt{2} : \sqrt{3} : 2$.

1.2 The line of action of the resultant of two forces divides the angle between them in the ratio 1 : 2. Determine the magnitude of the resultant. [**Ans.** $(F_1^2 - F_2^2)/F_2$]

1.3 The resultant of two forces P, Q acting at a certain angle is F, that of P, R acting at the same angle is also F and the resultant of Q, R again acting at the same angle is G. Prove that

$$P = \frac{QR\,(Q+R)}{Q^2 + R^2 - G^2} = \sqrt{(F^2 + QR)} \text{ and if } P + Q + R = O,\ G = F.$$

1.4 A weight W is hung from a weightless ring which can slip over a smooth circular wire fixed in a vertical plane, the ring is also tied to a string which passing as a chord of the circle over a fixed peg at the top of the circle sustains a given weight P. If θ be the angle that the radius to the ring makes with the vertical, in the position of rest, show that $\sin\dfrac{\theta}{2} = \dfrac{P}{2W}$

1.5 Two weights W_1 and W_2 rest on each of two smooth planes placed back-to-back of inclinations α and β, being connected by a string which runs horizontally from one plane to the other. Show that $W_1 \tan\alpha = W_2 \tan\beta$.

If the string passes over a smooth pulley at the top of the inclined planes, show that $W_1 \sin\alpha = W_2 \sin\beta$.

1.6 Two weights W_1 and W_2 are suspended from a fixed point O by strings OA and OB which are kept apart by a light rod AB. If the strings make angles α and β with the rod, show that the angle θ which the rod makes with the vertical is given by
$(W_1 + W_2)\cot\theta = W_2 \cot\beta - W_1 \cot\alpha$

1.7 A fine string AOB of length l is passed at O through a smooth ring of no appreciable weight and is attached at its extremities to two fixed points A and B at a distance d apart. A force P is applied to the ring in a direction making an angle α with BA. Show that in the position of equilibrium the two parts of the string are inclined to each other at an angle $2\sin^{-1}\left(\dfrac{d\sin\alpha}{l}\right)$ and the tension in the string is

$$\frac{lP}{2\sqrt{l^2 - d^2\sin^2\alpha}}$$

1.8 AB is a smooth straight wire fixed in a position inclined at an angle α to the vertical with B above A. A small ring of weight W capable of sliding freely on the wire is connected to B by a string which passes through a ring of equal weight hanging freely on the string. Show that the tension in the string is $\frac{1}{2} W(1 + 9\cot^2\alpha)^{1/2}$.

1.9 The ends of a piece of string are attached to two heavy rings P and Q of weight W_1 and W_2 respectively. The rings are free to slide on two smooth rods BA and BC respectively inclined at angles α and β to the horizontal and lying in the same vertical plane. Show that in the position of equilibrium the string is inclined to the horizontal at an angle.

$$\tan^{-1}\left(\frac{W_1\cot\beta - W_2\cot\alpha}{W_1 + W_2}\right)$$

1.10 A uniform circular disc of weight nW has a particle of weight W attached to a point on its rim. If the disc be suspended from a point A on its rim, B is the lowest point, also if suspended from B, A is the lowest point. Show that the angle subtended by AB at the centre of the disc is $2\sec^{-1} 2(n + 1)$.

1.11 A heavy carriage wheel of weight W and radius r is to be dragged over an obstacle of height h by a horizontal force F applied to the centre of the wheel. Show that F must be slightly greater than

$$\frac{W\sqrt{2rh - h^2}}{r - h}$$

1.12 A uniform rod AB of weight W and length $2l$ hangs from a fixed point O by a light string OA attached to the end A and a couple of moment M ($M < Wl$) is applied to the rod in a vertical plane. Find the tension in the string and the inclination of the rod to the vertical in the position of equilibrium. [**Ans.** W, $\sin^{-1}(M/Wl)$]

1.13 A circular disc of weight W and radius a is suspended horizontally by three equal vertical strings of length b attached symmetrically to its perimeter. Show that the magnitude of the horizontal couple required to keep it twisted through an angle θ is given by

$$\frac{Wa^2 \sin\theta}{\sqrt{b^2 - 4a^2 \sin^2 \theta/2}}$$

1.14 A uniform rod of length a hangs against a smooth vertical wall being supported by means of a string of length l tied to one end of the rod, the other end of the string being attached to a point in the wall. Show that the rod can rest inclined to the wall at an angle θ given by

$$\cos^2\theta = \frac{l^2 - a^2}{3a^2}$$

1.15 A beam whose centre of gravity divides it into two portions of lengths a and b rests in equilibrium with its ends resting on two smooth planes inclined at angles α and β respectively to the horizontal. Find the inclination of the beam to the horizontal and the reactions of the planes. [**Ans.** $(b \cot\beta - a \cot\alpha)/(a + b)$; $R_\alpha = W \sin\beta/\sin(\alpha + \beta)$
$R_\beta = W \sin\alpha/\sin(\alpha + \beta)$]

1.16 A uniform heavy rod of length $2a$ rests upon a smooth peg C and its upper end A is attached to a fixed point D situated in the same horizontal line as C by means of a string DA. If $AD = AC = b$, show that the inclination θ of the rod to the horizontal is

$$\theta = \tan^{-1}\left(\frac{a - b}{a + b}\right)^{1/2}$$

1.17 A sphere of weight W rests on two smooth planes inclined to the horizontal at angles α and β. Find the pressure on the planes.
[**Ans.** $W \sin\beta/\sin(\alpha + \beta)$, $W \sin\alpha/\sin(\alpha + \beta)$]

1.18 A smooth sphere is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being attached to a point in the wall. If the length of the string be equal to the radius of the sphere, find the inclination of the string to the vertical, the tension of the string and the reaction of the wall.
[**Ans.** 30°, $2/\sqrt{3}W$, $1/\sqrt{3}W$]

1.19 A rod rests wholly within a smooth hemispherical bowl of radius r, its centre of gravity dividing the rod into two portions of lengths a and b. Show that if θ be the inclination of the rod to the horizontal in the position of equilibrium, then $\sin\theta = \dfrac{b - a}{2\sqrt{r^2 - ab}}$. Also find the reactions between the rod and the bowl and prove that

$\tan\theta = \left(\dfrac{b-a}{b+a}\right)\tan\alpha$, where 2α is the angle subtended by the rod at the centre of the sphere.

[**Ans.** $W\sin(\alpha-\theta)/\sin 2\alpha$, $W\sin(\alpha+\theta)/\sin 2\alpha$]

1.20 A tricycle weighing 250 N has a small wheel symmetrically placed 100 cm behind the two large wheels, which are also 100 cm apart. If the centre of gravity of tricycle be at a horizontal distance of 25 cm behind the front wheels and that of the rider whose weight is 500 N, be 10 cm behind the front wheels, find the thrust on the ground under the three wheels.

[**Ans.** Rear wheel = 112.5 N ; Front wheels = 318.75 N]

1.21 Four forces equal to P, $2P$, $3P$ and $4P$ act along the sides AB, BC, CD and DA respectively of a square. Find the magnitude, direction and position of the resultant force.

[**Ans.** $2\sqrt{2}P$, 45°, at 2.5 a on CD produced (a = side of square)]

1.22 Two cylinders P and Q rest in a box with the base 20 cm wide, one side vertical and the other inclined at 60° to the horizontal base. The cylinder P has a diameter of 10 cm and weighs 150 N; whereas the cylinder Q weighs 300 N and has 20 cm diameter. If the cylinder Q is above the cylinder P, determine the pressures at all the four points of contact. [**Ans.** 203.8 N, 332.3 N, 273.5 N, 235.4 N]

1.23 Find the resultant of the forces shown in Fig. 1.103

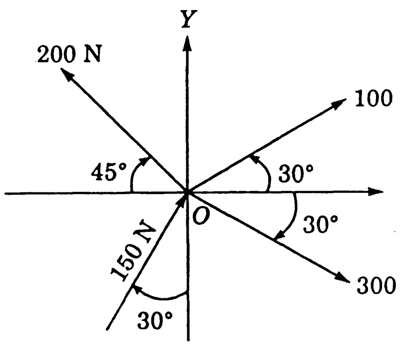


Fig. 1.103

[**Ans.** 318.98 N, 31.46°]

1.24 Calculate the thrust T against the cylinder wall in Fig. 1.104.

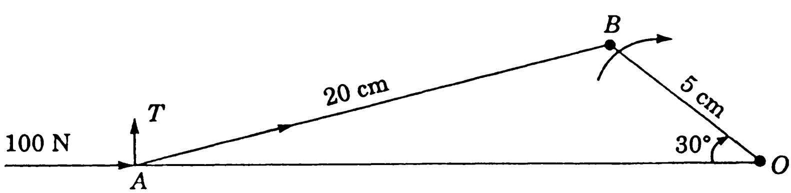


Fig. 1.104

[**Ans.** 12.6 N]

1.25 Find the magnitude and position of the resultant of the system of forces shown in Fig. 1.105. [**Ans.** 270 N, 19.18 cm from O]

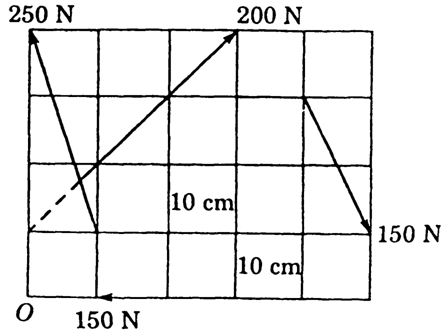


Fig. 1.105

1.26 Determine the resultant in magnitude and position of the forces shown in Fig. 1.106. [**Ans.** 35.92 N. 19.28 cm from O]

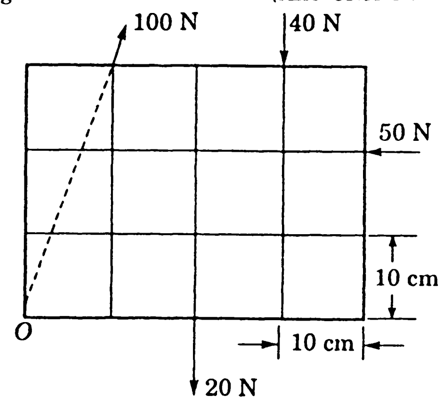


Fig. 1.106

1.27 Two forces $P + Q$ and $P - Q$ make an angle 2α with one another and their resultant makes an angle θ with the bisector of the angle between them. Show that $P \tan \theta = Q \tan \alpha$.

1.28 $ABCD$ is a quadrilateral. Forces represented by CA, CB, DA and DB act on a particle. Show that they are equivalent to a force represented by $4FE$, where E and F are the middle points of AB and CD respectively.

1.29 Forces 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 units respectively act at one of the angular points of a regular hexagon towards the five other points in order. Find the magnitude and direction of the resultant.

[**Ans.** 10 units towards the opposite angular point]

1.30 If forces P and Q acting at an angle θ be interchanged in position, show that the resultant turns through an angle ϕ such that

$$\tan(\phi/2) = \left[\frac{P - Q}{P + Q}\right] \tan \frac{\theta}{2}.$$

1.31 A small ring of weight W_1 is capable of sliding freely on a smooth circular hoop of radius r, fixed in a vertical plane. It is supported by a fine light string of length $l < 2r$, attaching it to the highest point of the hoop. A second string is attached to the ring and passes over a small smooth peg situated at the lowest point of the hoop supporting at its other extremity a mass of weight W_2. Find the tension in the first string and the reaction of the hoop on the wire.

[**Ans.** $W_2\, l/(4r^2 - l^2)^{1/2} + W_1 l/r$, $W_1 + 2r\, W_2/(4r^2 - l^2)^{1/2}$]

1.32 A hemisphere of radius r and weight W is placed with its curved surface on a smooth table and a string of length l ($< r$) is attached to a point on its rim and to a point on the table. Find the position of equilibrium and prove that the tension of the string is

$$\frac{3}{8} W \frac{(r - l)}{(2rl - l^2)^{1/2}}$$

1.33 A rod is movable in a vertical plane about a hinge at one end, and at the other end is fastened a weight equal to half the weight of the rod. This end is fastened by a string of length l to a point at a height h vertically over the hinge. Show that the tension of the string is lW/h.

1.34 One end of a uniform rod is attached to a hinge and the other end is supported by a string attached to the extremity of the rod. The rod and the string are inclined at the same angle θ to the horizontal. If W be the weight of the rod then show that the reaction at the hinge is

$$\frac{W}{4} \cdot (8 + \operatorname{cosec}^2 \theta)^{1/2}.$$

1.35 Two uniform beams AC and BC of two equal lengths and of unequal weights W_1 and W_2 respectively are pin-jointed at C and hinged at A and B to two fixed points at the same level. The beams are hung in a vertical plane, and are inclined at an angle α to the horizontal. Find the reaction at C and prove that the line of action is inclined to the vertical at an angle

$$\tan^{-1}\left\{\left[\frac{W_1 + W_2}{W_1 - W_2}\right] \cot \alpha\right\}$$