1

Introduction

1.1 GENERAL

Structural engineers are required to analyse and design various types of structures. The knowledge of displacements, strains and stresses is important to them to check serviceability and safety of the structures. There are different varieties of structures such as trusses, frames, buildings, bridges, irrigation structures, folded plates, box-girders etc. In the earlier days, analysis was carried out by introducing certain simplifying assumptions so as to bring the analysis within the reach of hand calculation methods.

One possible way of classifying varieties of structures in civil engineering field is to put them in two categories as

(*i*) Discrete element or skeletal structures

(*ii*) Continuum structures.

The discrete element structures are formed by joining discrete structural members at their ends forming joints. Each member may be straight or curved. For such members, sectional dimensions are small compared to the length dimension. Pin jointed trusses, frames, grids, arched portals and space frames are skeletal structures. Some of these are shown in Fig. 1.1. The members of the skeletal structures are simple members and their properties namely force-displacement relations can be found easily using standard structural mechanics formulae. The analysis of a skeletal structure is no more a difficult job. For large size structures, stiffness matrix method is now widely used tool of analysis.

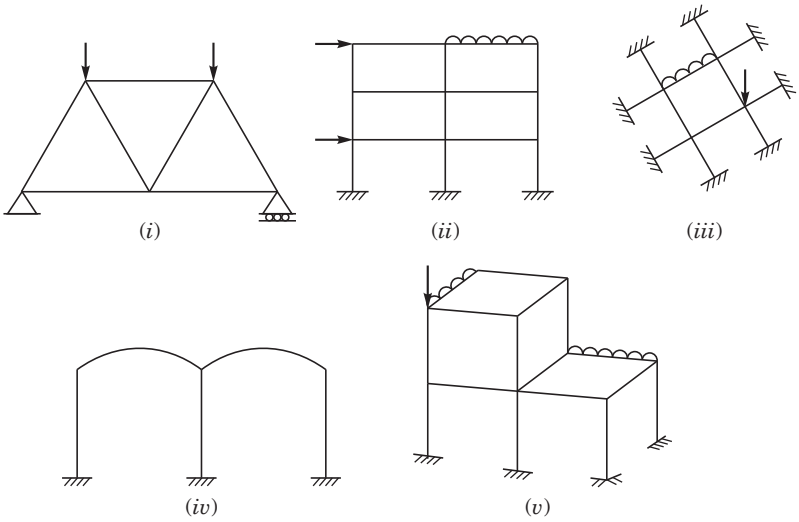


Fig. 1.1. Skeletal structures (*i*) Plane truss (*ii*) Plane frame (*iii*) Grid (*iv*) Arched portal (*v*) Space frame.

The continuum structures are formed from surface elements and solids. Plates, stiffened plates, shells, folded plates, box-girders are some examples of the continuum structures. Some of these are shown are in Fig. 1.2.

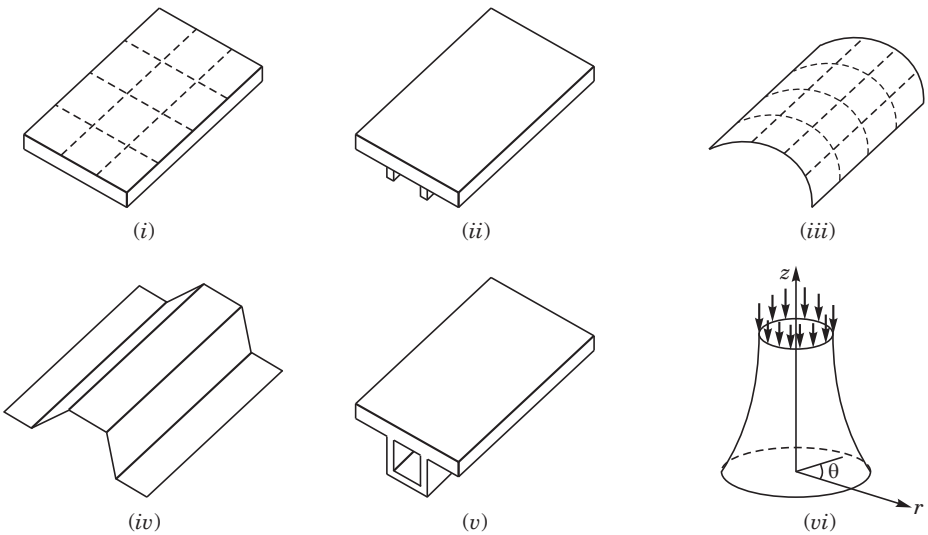


Fig. 1.2. Continum structures (*i*) Plate (*ii*) Stiffened (*iii*) Shell (*iv*) Folded plate (*v*) Box girder (*vi*) Axisymmetric solid.

The analysis of a continuum structure is a difficult job. To begin with, the behaviour of the structure is described in terms of a governing differential equation or equations. This is not so difficult. The general procedure is to express equilibrium of an infinitely small element of the structure in terms of stress-resultants and external forces acting on the element. The stress-resultants are written in terms of generalised strains and the strains are expressed as derivatives of displacements. Finally, combining all relations, it is generally possible to derive a governing differential equation in terms of displacements. This may be a single equation or a set of simultaneous differential equations. The analysts' aim is to obtain the solution of the governing differential equation(s) satisfying kinematic and force boundary conditions. Obviously, the first attempt is to obtain exact or analytical solution. In analytical solution, algebraic or trigonometric or hyperbolic expressions are obtained for field variable and other dependant variables. These expressions are continuous over the entire structure domain and it is possible to calculate the quantities of importance at any location in the structure using appropriate coordinates of the location.

An alternate approach of obtaining solution of governing differential equation is to express the field variable (w) as a combination of linearly independent trial or approximate functions satisfying boundary conditions such as $\overline{w} = \sum C_i \, \phi_i$. An error function $R(x)$ arises when $\overline{w}$ is substituted in differential equation. The values of the constants C_i associated with trial functions ϕ_i are obtained from the solution of simultaneous algebraic equations which are formed using any one of the following procedures :

(*i*) Equate residual or error function $R(x)$ to zero at selected locations known as collocation points (Point collocation).

(*ii*) Equate integral of the residual function $R(x)$ over sub-domains or zones of the structure to zero (Zone collocation).

(*iii*) Minimise the integral of the square of $R(x)$ over entire structure (Least square).

(*iv*) Equate integral of weighted residual over entire structure domain to zero. The method is known as Galerkin method if weighing functions are same as trial functions.

The details of these methods are discussed later.

Exact and other analytical methods of solving differential equations can be applied in a few cases of simple structures with simple boundaries and loading. There are difficulties in handling oblique and curved boundaries, patch loads and irregularly shaped structures.

With the advent of the computers, the emphasis of the researchers shifted to the numerical methods. Some of these are finite difference method, variational finite difference method (finite difference energy method), finite element method, discrete energy method and finite strip method. In numerical methods, the numerical values of the field variable are obtained at selected points or nodal points of the structure. The values of the variables at intermediate locations are to be obtained by suitable interpolation. The availability of the computers and ever increasing need of the engineers to analyse large complex structures have given tremendous boost to the rapid development of the numerical methods.

The various methods of analysis of continuum structures are divided in two groups as shown in Fig. 1.3. The methods which are based on solution of governing differential equation, either analytically or numerically, are included in group 1. For all these methods, it is necessary to know the governing differential equation. The methods included in group 2 are those methods which are based on minimisation of a functional. These are either analytical or numerical methods. For these methods, there is no need to know the governing differential equation. However, it is necessary to know the functional to be minimised. For structural engineering problems, in which displacements are treated as primary unknowns, the functional is the total potential energy of the structure.

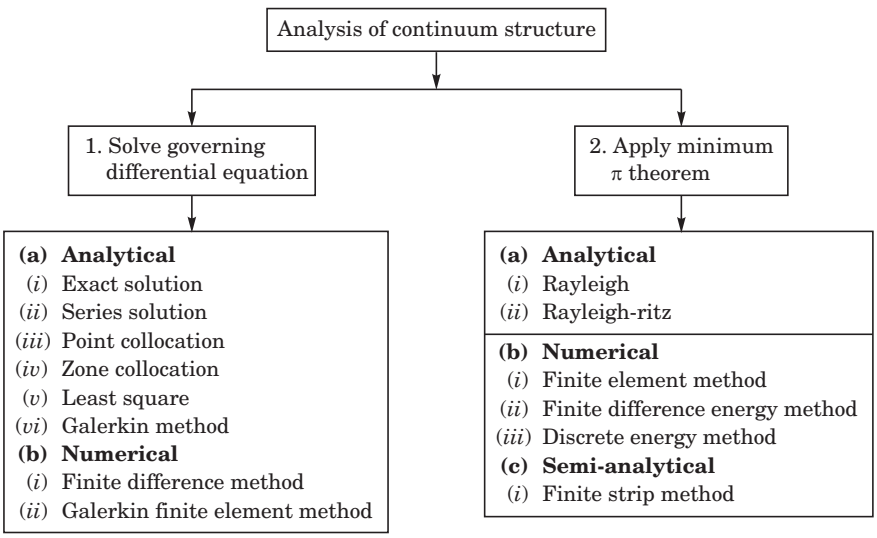


Fig. 1.3. Analysis of continuum structure-classification of methods.

From Fig. 1.3, it is noted that the finite element method is a numerical method used for analysis of continuum structures and is based on application of minimum potential energy theorem. The method has acquired immense popularity and has emerged as a powerful and versatile tool of analysis. The applications are not restricted to the structural engineering.

The problems related to soil engineering, hydraulics engineering, fluid flow, magnetic flux, seepage and heat flow, too, have been analysed using the method. The reason for the popularity of the method is it's ability to analyse any structure with oblique and curved boundaries, any loading, mixed boundaries and arbitrary shape. In structural engineering, the method has been applied for static analysis, frequency analysis, response analysis, geometric and material non-linear analysis and stability analysis.

Quite obviously, there is abundance of literature available dealing with various aspects of the finite element method. There are journals especially devoted to the method alone. Some well known books, often quoted, are listed in references [1-11].

The first paper which closely resembles application of finite elements to structural problem was published by Turner et al [17]. The term 'finite element' was first introduced by Clough in 1960 [18].

In finite element method, continuum structure is considered as an assemblage of elements. These elements are sub-domains or sub-regions or zones of the structure. The elements are treated as joined together at joints or nodes. The elements as well as nodes are not real, but are notional or imaginary. The displacement variation over each element region is prescribed in terms of displacements at nodes of that element using polynomial functions. Thus, the displacement variation over entire structure is prescribed, piecewise, separately over each element. This is in contrast to Rayleigh-Ritz method where displacement variation over entire structure is written in terms of continuous functions and each function is associated with a mathematical constant.

The total potential energy of an element can be expressed in terms of nodal parameters. The total potential energy of the entire structure is the summation of the energies of all elements and is the function of nodal parameters of all nodes of the structure. The conditions to minimise the total potential energy of the structure with respect to each of the nodal parameters result in a set of simultaneous equations. This set, after applying suitable kinematic boundary conditions, is solved to get numerical values of nodal displacements. Other dependant quantities can then be calculated using relevant relations.

The finite element method is a numerical method. It's application involves tremendous calculations. It is not possible to utilize the capacities of the method without use of computer. The steps involved in the application of the method are definite and are to be followed in a definite sequence. These steps can be concisely written in matrix notations and hence the method is amenable to computer usage. It is to be mentioned that the programming steps in finite element method are almost similar to those used in stiffness matrix method except for the details of calculating element stiffness matrices. As such, the stiffness method can be considered as a special case of finite element method. In recent books, a unified treatment of both methods is presented. It is rewarding to study stiffness matrix method and it's programming before taking up the study of finite element method.

There are two different views of looking at the finite element method. Many researchers, particularly mathematicians, describe the method as a tool to solve differential equation. This is true. However, in the present book, the finite element method is presented as a means of obtaining deformed state of the structure using minimum total potential energy theorem.

1.2 SOLUTION OF GOVERNING DIFFERENTIAL EQUATION

To explain various methods mentioned under group 1 of Fig. 1.3, examples of single span beams subjected to different loads and having different boundary conditions are considered. The governing differential equation of Euler-Bernoulli beam is:

$$\frac{d^4w}{dx^4} = \frac{q(x)}{EI} \qquad ...(1.1)$$

with usual meaning of the symbols. The dependant quantities are related to w as

$$\left.\begin{aligned} &\text{Slope of the tangent} = \frac{dw}{dx} \\ &\text{Bending moment} = -EI\frac{d^2w}{dx^2} \\ &\text{Shear force} = -EI\frac{d^3w}{dx^3} \end{aligned}\right\} \qquad ...(1.2)$$

Boundary conditions used are

$$\left.\begin{aligned} &\text{Simply supported} : w = \frac{d^2w}{dx^2} = 0 \quad (\text{since } EI \neq 0) \\ &\text{Clamped support} : w = \frac{dw}{dx} = 0 \\ &\text{Free end} : \frac{d^2w}{dx^2} = \frac{d^3w}{dx^3} = 0 \quad (\text{since } EI \neq 0) \end{aligned}\right\} \qquad ...(1.3)$$

The conditions on deflection and slope are known as geometric or kinematic or essential boundary conditions and those on bending moment and shear force are natural or force boundary conditions.

1.2.1 Exact Solution

Example 1.1. Figure 1.4 shows a propped cantilever subjected to distributed load of linearly varying intensity from A to B. The load is expressed as

$$q(x) = \frac{qx}{L} \qquad ...(1.4a)$$

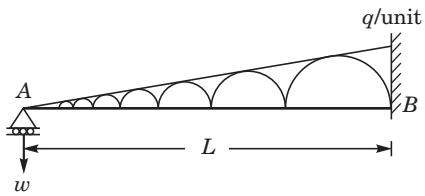


Fig. 1.4. Propped cantilever.

Hence, governing differential equation is

$$\frac{d^4w}{dx^4} = \frac{qx}{EIL} \qquad ...(1.4b)$$

Integrating successively,

$$\frac{d^3w}{dx^3} = \frac{qx^2}{2EIL} + C_1 \qquad ...(1.4c)$$

$$\frac{d^2w}{dx^2} = \frac{qx^3}{6EIL} + C_1x + C_2 \qquad ...(1.4d)$$

$$\frac{dw}{dx} = \frac{qx^4}{24EIL} + \frac{C_1x^2}{2} + C_2x + C_3 \qquad ...(1.4e)$$

$$w = \frac{qx^5}{120EIL} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4 \qquad ...(1.4f)$$

The boundary conditions are

$$\left.\begin{array}{lll} (i)\ w = 0 & \text{at} & x = 0 \\ (ii)\ \dfrac{d^2w}{dx^2} = 0 & \text{at} & x = 0 \\ (iii)\ w = 0 & \text{at} & x = L \\ (iv)\ \dfrac{dw}{dx} = 0 & \text{at} & x = L \end{array}\right\} \qquad ...(1.5)$$

Substituting Eq. 1.5 in Eq. 1.4

$$\left.\begin{aligned} 0 &= 0 + 0 + 0 + 0 + C_4 \\ 0 &= 0 + 0 + C_2 \\ 0 &= \frac{qL^5}{120EIL} + \frac{C_1L^3}{6} + \frac{C_2L^2}{2} + C_3L + C_4 \\ 0 &= \frac{qL^4}{24EIL} + \frac{C_1L^2}{2} + C_2L + C_3 \end{aligned}\right\} \qquad ...(1.6)$$

Solution of Eq. 1.6 gives

$$C_1 = -\frac{qL}{10EI}, \qquad C_2 = 0, \qquad C_3 = \frac{qL^3}{120EI}, \qquad C_4 = 0 \qquad ...(1.7)$$

Hence, solution for deflection w is

$$w(x) = \frac{1}{EI}\left[\frac{qx^5}{120L} - \frac{qLx^3}{60} + \frac{qL^3x}{120}\right] \qquad ...(1.8)$$

Expressions for dependant quantities can be found using Eq. 1.2.

1.2.2 Series Solution

Example 1.2. Figure 1.5 shows a simply supported beam carrying distributed load of linearly varying intensity from A to B. The load is expressed in half sine series as

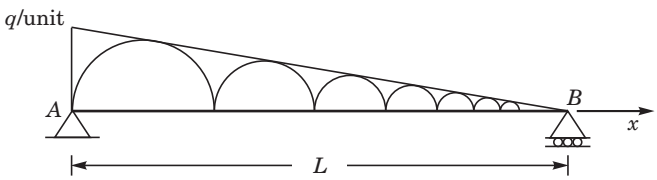


Fig. 1.5. Simply supported beam.

$$q(x) = q\left(1 - \frac{x}{L}\right) = \sum_{m=1,2,3}^{\infty} \frac{2q}{m\pi}\sin\left(\frac{m\pi x}{L}\right) \qquad ...(1.9)$$

Also, deflection $w(x)$ is expressed in terms of half sine series as

$$w(x) = \sum_{m=1,2,3}^{\infty} w_m \sin\left(\frac{m\pi x}{L}\right) \qquad ...(1.10)$$

Each term of assumed series for $w(x)$ satisfies all four boundary conditions. Substituting Eqs. 1.9 and 1.10 in governing differential equation (Eq. 1.1) and comparing coefficients:

$$\left.\begin{aligned} w_m\left(\frac{m^4\pi^4}{L^4}\right) &= \frac{2q}{E\,I\,m\pi} \\ w_m &= \frac{2qL^4}{E\,I\,m^5\pi^5} \end{aligned}\right\} \qquad ...(1.11)$$

Hence, solution for deflection $w(x)$ is

$$w(x) = \frac{2qL^4}{EI\pi^5}\sum_{m=1,2,3}^{\infty}\frac{1}{m^5}\sin\left(\frac{m\pi x}{L}\right) \qquad ...(1.12)$$

The deflection at $x = L/2$ using Eq. 1.12 is

$$w(at\ x = L/2) = \frac{2qL^4}{EI\pi^5}\left[\frac{1\times 1}{1^5} + 0 + \frac{-1\times 1}{243} + 0 + \frac{1\times 1}{3125}\right]$$
$$= 0.0065107qL^4/(EI)$$

Exact deflection at centre is $w = \dfrac{5}{768}\dfrac{qL^4}{EI}$ *i.e.*, $w = 0.0065104\ qL^4/(EI)$

1.2.3 Methods Based on Residual R(x)

Consider a differential equation

$$Lw - q = 0 \qquad ...(1.13)$$

where, L is a differential operator and w is field variable (deflection) for which solution is desired. An approximate (trial) function $\overline{w}(x)$ is assumed as a combination of functions satisfying boundary conditions. Each function is associated with an unknown constant. Thus,

$$\overline{w}(x) = \sum_{i=1}^{n} C_i\phi_i \qquad ...(1.14)$$

The differential equation will not, in general, be satisfied if $\overline{w}(x)$ is substituted in differential equation. An error function or residual $R(x)$ can be expressed as

$$R(x) = L\overline{w} - q \qquad ...(1.15)$$

If n functions are used to construct trial solution, then n simultaneous equations are obtained to get the values of $C_i (i = 1$ to $n)$. This is done by applying certain conditions on residual function as mentioned earlier.

1.2.4 Point Collocation Method

Example 1.3. Figure 1.6 shows a simply supported beam carrying uniformly distributed load on entire span.

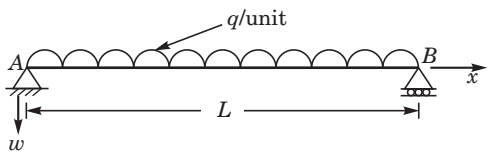


Fig. 1.6. Simply supported beam.

Approximate solution is assumed as

$$\overline{w}(x) = C_1\sin\left(\frac{\pi x}{L}\right) + C_3\sin\left(\frac{3\pi x}{L}\right) \qquad ...(1.16)$$

Each function independently satisfies all four boundary conditions. Residual $R(x)$ is obtained by substituting $\overline{w}(x)$ in governing differential equation (Eq. 1.1) as

$$R(x) = C_1 \frac{\pi^4}{L^4} \sin\left(\frac{\pi x}{L}\right) + C_3 \frac{81\pi^4}{L^4} \sin\left(\frac{3\pi x}{L}\right) - \frac{q}{EI} \quad ...(1.17)$$

In point collocation method, residual is equated to zero at selected points known as collocation points. For this example, the points chosen are at (*i*) $x = L/4$ and (*ii*) $x = L/2$. Substituting these values in Eq. 1.17 and simplifying:

$$\left.\begin{aligned} 0.707\, C_1 + 57.267\, C_3 &= k \\ C_1 + 81.0 C_3 &= k \end{aligned}\right\} \quad ...(1.18)$$

in which, $k = qL^4/(EI\pi^4)$

Solution of equations yields

$$C_1 = 1.207\, k, \qquad C_3 = 0.00256\, k \quad ...(1.19)$$

Hence, solution for deflection is

$$w(x) = \frac{qL^4}{EI\pi^4}\left[1.207 \sin\left(\frac{\pi x}{L}\right) + 0.00256 \sin\left(\frac{3\pi x}{L}\right)\right] \quad ...(1.20)$$

The deflection at centre ($x = L/2$) using Eq. 1.20 is

$$w(x = L/2) = 0.012365 qL^4/(EI)$$

The exact value of deflection is

$$w(exact) = \frac{5}{384}\frac{qL^4}{EI} = 0.013021\, qL^4/(EI)$$

1.2.5 Zone Collocation

Example 1.4. Simply supported beam shown in Fig. 1.6 is analysed. Approximate solution is assumed as

$$\overline{w}(x) = C_1 \sin\left(\frac{\pi x}{L}\right) \quad ... (1.21)$$

Substituting in governing differential equation (Eq. 1.1), the residual obtained is

$$R(x) = C_1 \frac{\pi^4}{L^4} \sin\left(\frac{\pi x}{L}\right) - \frac{q}{EI} \quad ...(1.22)$$

In zone collocation method, the integral of $R(x)$ over certain zones (sub-regions) of structure is equated to zero. Choosing the zone from to $x = L/4$ to $3L/4$:

$$\int_{L/4}^{3L/4} \left[C_1 \frac{\pi^4}{L^4} \sin\left(\frac{\pi x}{L}\right) - \frac{q}{EI}\right] dx = 0 \quad ...(1.23)$$

This gives

$$C_1 \frac{\sqrt{2}\,\pi^3}{L^3} - \frac{qL}{2EI} = 0$$

$$C_1 = \frac{qL^4}{2\sqrt{2}\, EI\pi^3}$$

Hence, expression for deflection $w(x)$ is

$$w(x) = \frac{qL^4}{2\sqrt{2}\, EI\, \pi^3} \sin\left(\frac{\pi x}{L}\right) \quad ...(1.24)$$

The deflection at centre ($x = L/2$) using Eq. 1.24 is obtained as

$$w(x = L/2) = 0.011403\, qL^4/(EI)$$

1.2.6 Least Square Method

Approximate (trial) function for field variable is assumed as a combination of functions satisfying boundary conditions as in Eq. 1.14. The integral of square of residual function is minimised with respect to constants C_i. This gives n simultaneous equations from which values of C_i are found. Thus,

$$\frac{\partial}{\partial C_i}\left[\int_0^L R^2(x)\,dx\right] = 0 \qquad ...(1.25)$$

This can also be written as

$$\int_0^L R(x)\frac{\partial R}{\partial C_i}\,dx = 0 \qquad ...(1.26)$$

Example 1.5. A simply supported beam carrying uniformly distributed load (Fig. 1.6) is analysed. Assume,

$$\bar{w}(x) = C_1 \sin\left(\frac{\pi x}{L}\right) + C_3 \sin\left(\frac{3\pi x}{L}\right) \qquad ...(1.27)$$

Each function independently satisfies all four boundary conditions. The residual is obtained by substituting Eq. 1.27 in Eq. 1.1.

$$R(x) = C_1\frac{\pi^4}{L^4}\sin\left(\frac{\pi x}{L}\right) + C_3\frac{81\pi^4}{L^4}\sin\left(\frac{3\pi x}{L}\right) - \frac{q}{EI} \qquad ...(1.28)$$

$$\left.\begin{aligned}\frac{\partial R}{\partial C_1} &= \frac{\pi^4}{L^4}\sin\left(\frac{\pi x}{L}\right)\\ \frac{\partial R}{\partial C_3} &= \frac{81\pi^4}{L^4}\sin\left(\frac{3\pi x}{L}\right)\end{aligned}\right\} \qquad ...(1.29)$$

Using Eq. 1.26 for C_1 and C_3

$$\left.\begin{aligned}\int_0^L \frac{\pi^4}{L^4}\sin\left(\frac{\pi x}{L}\right)R(x) = 0\\ \int_0^L \frac{81\pi^4}{L^4}\sin\left(\frac{3\pi x}{L}\right)R(x) = 0\end{aligned}\right\} \qquad ...(1.30)$$

Carrying out integrals

$$\left.\begin{aligned}C_1\left(\frac{\pi^4}{L^4}\right)^2\left(\frac{L}{2}\right) - \left(\frac{q}{EI}\right)\left(\frac{2L}{\pi}\right)\left(\frac{\pi^4}{L^4}\right) = 0\\ C_3\left(\frac{81\pi^4}{L^4}\right)^2\left(\frac{L}{2}\right) - \left(\frac{q}{EI}\right)\left(\frac{2L}{3\pi}\right)\left(\frac{81\pi^4}{L^4}\right) = 0\end{aligned}\right\} \qquad ...(1.31)$$

Solution gives

$$C_1 = \frac{4qL^4}{EI\pi^5}, \qquad C_3 = \frac{4qL^4}{243EI\pi^5} \qquad ...(1.32)$$

Hence, expression for $w(x)$ is written as

$$w(x) = \frac{4qL^4}{EI\pi^5}\left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{243}\sin\left(\frac{3\pi x}{L}\right)\right] \qquad ...(1.33)$$

Following standard mathematical formulae have been used to evaluate integrals in Eq. 1.31.

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \dfrac{L}{2} & \text{if} \quad m = n \end{cases}$$

$$\int_0^L \sin\left(\frac{m\pi\, x}{L}\right)dx = \begin{cases} 0 & \text{if} \quad m = 2, 4, 6..... \\ \dfrac{2L}{m} & \text{if} \quad m = 1, 3, 5..... \end{cases}$$

The deflection at the centre of beam using Eq. 1.33 is calculated as

$$w(x = L/2) = 0.01302qL^4/(EI)$$

Exact value is $\dfrac{5}{384}\dfrac{qL^4}{EI} = 0.01302qL^4/(EI)$

1.2.7 Galerkin Method (I)

Example 1.6. A simply supported beam AB of span L carrying distributed load of linearly varying intensity from A to B (Fig. 1.7) is analysed using Galerkin method.

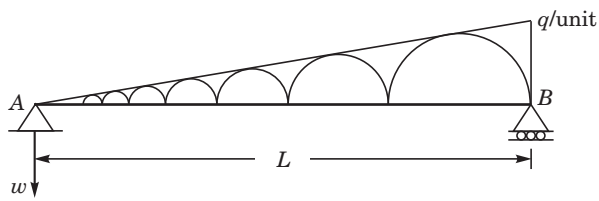


Fig. 1.7. Simply supported beam.

Loading on beam is expressed as

$$q(x) = \frac{qx}{L}$$

Approximate deflection function is assumed as

$$\bar{w}(x) = C_1 \sin\left(\frac{\pi x}{L}\right) + C_2 \sin\left(\frac{2\pi x}{L}\right) \qquad ...(1.34)$$

Each function satisfies boundary conditions. Residual function $R(x)$ is obtained by substituting Eq. 1.34 in governing differential equation (Eq. 1.1) as

$$R(x) = C_1 \sin\frac{\pi^4}{L^4}\sin\left(\frac{\pi x}{L}\right) + C_2\frac{16\pi^4}{L^4}\sin\left(\frac{2\pi x}{L}\right) - \frac{qx}{EIL} \qquad ...(1.35)$$

In Galerkin method, the integral of the weighted residual over problem domain is equated to zero. Thus,

$$\int_0^L \psi_i R(x)dx = 0 \qquad ...(1.36)$$

where, ψ_1 are weighing functions. In Galerkin method, the weighing functions are the functions used to construct the approximate $\bar{w}(x)$. Hence,

$$\left.\begin{aligned}\psi_1 &= \sin\left(\frac{\pi x}{L}\right)\\ \psi_2 &= \sin\left(\frac{2\pi x}{L}\right)\end{aligned}\right\} \quad ...(1.37)$$

Applying Eq. 1.36 for i = 1 and 2

$$\left.\begin{aligned}\int_0^L \sin\left(\frac{\pi x}{L}\right) R(x)dx = 0\\ \int_0^L \sin\left(\frac{2\pi x}{L}\right) R(x)dx = 0\end{aligned}\right\} \quad ...(1.38)$$

Carrying out integrals

$$\left.\begin{aligned}C_1\left(\frac{\pi^4}{L^4}\right)\left(\frac{L}{2}\right) - \frac{qL^2}{EI\pi L} = 0\\ C_2\left(\frac{16\pi^4}{L^4}\right)\left(\frac{L}{2}\right) - \left(\frac{-L^2}{2\pi}\right)\frac{q}{EIL} = 0\end{aligned}\right\} \quad ...(1.39)$$

This gives

$$\left.\begin{aligned}C_1 &= \frac{2qL^4}{EI\pi^5}\\ C_2 &= \frac{-2qL^4}{32EI\pi^5}\end{aligned}\right\} \quad ...(1.40)$$

Hence, expression for deflection is written as

$$w(x) = \frac{2qL^4}{EI\pi^5}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{32}\sin\left(\frac{2\pi x}{L}\right)\right] \quad ...(1.41)$$

The deflection at centre of beam using Eq. 1.41 is

$$w(x = L/2) = 0.0065355qL^4(EI)$$

Exact value of deflection at centre is $w(\text{exact}) = \dfrac{5}{2\times 384}\dfrac{qL^4}{EI} = 0.0065104qL^2(EI)$

Following formulae are used to calculate integrals in Eq. 1.38.

$$\int_0^L x\sin\left(\frac{\pi x}{L}\right)dx = \frac{L^2}{\pi}$$

$$\int_0^L x\sin\left(\frac{2\pi x}{L}\right)dx = \frac{-L^2}{2\pi}$$

1.2.8 Galerkin Method (II)

Example 1.7. A clamped beam carrying uniformly distributed load (Fig. 1.8) is analysed using polynomial function as trial function. It is difficult to imagine a function satisfying all boundary conditions. It is to be derived systematically as detailed below.

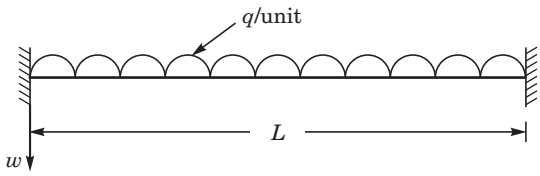


Fig. 1.8. Clamped beam.

Let
$$\bar{w}(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 \quad ...(1.42a)$$
$$\frac{d\bar{w}}{dx} = C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 \quad ...(1.42b)$$
$$\frac{d^2\bar{w}}{dx^2} = 2C_2 + 6C_3x + 12C_4x^2 \quad ...(1.42c)$$
$$\frac{d^3\bar{w}}{dx^3} = 6C_3 + 24C_4x \quad ...(1.42d)$$
$$\frac{d^4\bar{w}}{dx^4} = 24C_4 \quad ...(1.42e)$$

Using boundary conditions
$$\left.\begin{array}{llll} w = 0 & \text{at} & x = 0 & ; \quad C_0 = 0 \\ \dfrac{dw}{dx} = 0 & \text{at} & x = 0 & ; \quad C_1 = 0 \\ w = 0 & \text{at} & x = L & ; \quad C_2L^2 + C_3L^3 + C_4L^4 = 0 \\ \dfrac{dw}{dx} = 0 & \text{at} & x = L & ; \quad 2C_2L + 3C_3L^2 + 4C_4L^3 = 0 \end{array}\right\} \quad ...(1.43)$$

Last two equations involve three constants. Hence, their values can not be found. But, relation between them can be found. Eliminating C_2 from these equations –
$$C_3 = -2C_4L$$
and eliminating C_3 from these equations
$$C_2 = C_4L^2$$

Hence, trail function can be written as
$$\bar{w}(x) = C_4(L^2x^2 - 2Lx^3 + x^4) \quad ...(1.44)$$

This function satisfies all boundary conditions. Residual function is obtained by substituting Eq. 1.44 in Eq. 1.1.
$$R(x) = 24C_4 - \frac{q}{EI} \quad ...(1.45)$$

Integral of weighted residual over beam span is equated to zero.
$$\int_0^L (L^2x^2 - 2Lx^3 + x^4)\left(24C_4 - \frac{q}{EI}\right)dx = 0$$
$$\left(\frac{L^5}{30}\right)\left(24C_4 - \frac{q}{EI}\right) = 0$$

$$C_4 = \frac{q}{24EI}$$

Hence, $$w(x) = \frac{1}{EI}\left(\frac{qL^2x^2}{24} - \frac{qLx^3}{12} + \frac{qx^4}{24}\right) \quad ...(1.46)$$

This is the exact solution for clamped beam.

1.2.7 Finite Difference Method

Finite difference method gives numerical solution of a governing differential equation. The numerical values of the field variable are obtained at selected locations known as grid points or nodes. The differential equation is converted to a difference equation. This is done by expressing differentials of a function in terms of function values at neighbouring nodes. Fictitious outside nodes are introduced at both ends of the beam. The difference equation corresponding to a governing differential equation (Eq. 1.1) of a beam is written as

$$\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \end{bmatrix} w = \frac{q(x)h^4}{EI} \quad ...(1.47)$$

where, h is the distance between two nodes. The deflection at outside node is related to the deflection at inside node as

Simply supported end : $w_o = -w_i,\ w_b = 0$ ...(1.48*a*)

Clamped end : $w_o = +w_i,\ w_b = 0$ ...(1.48*b*)

Suffices i, o and b represent inside, outside and boundary node respectively. The difference equation is applied to the nodes where deflection is unknown. This results in a set of simultaneous equations. Thus, a problem of solving differential equation is converted to a problem of solving simultaneous algebraic equations.

Example 1.8. A propped cantilever of span L carrying uniformly distributed load on span (Fig. 1.9) is analysed. The span is divided in six equal parts giving $h = L/6$. Fig. 1.9 shows deflections at outside nodes as related to corresponding inside nodes. $w_A = w_B = 0$.

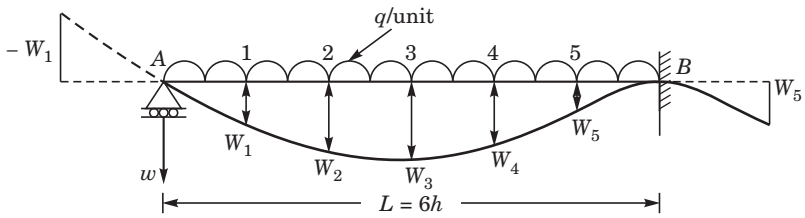


Fig. 1.9. Propped cantilever.

Applying difference equation to nodes 1 to 5 and simplifying, using boundary conditions, the equations obtained are written in matrix form as

$$\begin{bmatrix} 5 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 1 & -4 & 7 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{Bmatrix} = \begin{Bmatrix} k \\ k \\ k \\ k \\ k \end{Bmatrix} \quad ...(1.49)$$

where, $k = qh^4/(EI) = qL^4/(1296\ EI)$

Solution gives $w_1 = 4.5548k = 0.003515qL^4/(EI)$

$w_2 = 7.3288k = 0.005655qL^4/(EI)$

$w_3 = 7.5411k = 0.005819qL^4/(EI)$

$w_4 = 5.4110k = 0.004175qL^4/(EI)$

$w_5 = 2.1575k = 0.001665qL^4/(EI)$

The matrix of coefficients in finite difference method is not necessarily symmetric.

1.2.10 Galerkin Finite Element Method

This method is used to obtain numerical solution of governing differential equation. But, it is quite different from finite difference method discussed in art 1.2.9. In Galerkin finite element method, the structure is discretised into elements. The variation of field variable is prescribed piecewise over each element domain in terms of nodal values and derivatives of field variable, using shape functions. The shape functions should be sufficiently differentiable as required by the order of the differential equation. Further, if n^{th} order derivatives appear in differential equation then the function has to be such that it's $(n-1)$ derivatives are continuous at element boundary [1]. The weight functions are the shape functions. Consider a typical differential equation.

$$Lu + q = 0 \qquad ...(1.50)$$

The variation of u over element is

$$u = \sum_{j=1}^{r} N_j\, u_j \qquad ...(1.51)$$

The residual is obtained by substituting Eq. 1.51 in Eq. 1.50 as

$$R(x) = L(N_1u_1 + N_2u_2 + ... + N_ru_r) + q$$

where, r indicates total number of displacement parameters associated with the element. The integration of weighted residual is written as

$$\int_0^h N_i\,[L\,(N_1u_1 + N_2u_2 + ... + N_ru_r) + q] \qquad (i = 1 \text{ to } r) \qquad ...(1.52)$$

This gives the contribution of an element as

$$[A]\{\delta_e\} + \{q_e\} \qquad ...(1.53)$$

Combining such contributions from all elements and applying boundary condition, a set of equations is obtained in the form

$$[K]\{D\} + \{Q\} = \{0\} \qquad ...(1.54)$$

The values of nodal parameters are obtained by solution of equations in Eq. 1.54.

This approach is useful when functional associated with the problem is not known. This method is a numerical version of the analytical Galerkin method studied in art. 1.2.7.

To overcome the stringent requirement of $(n-1)$ continuity, an alternative approach is adopted in which the weighted residual function is converted to weak form by distributing the differential terms among weight function and field variable. The details of the alternate approach are discussed in art.13.1.

1.3 METHODS USING MINIMUM POTENTIAL ENERGY PRINCIPLE

For methods in this category, the knowledge of governing differential equation of the structure is not necessary. However, it is necessary to know the functional to be minimised. For problems in structural engineering (in which displacements are primary unknowns), the functional is the total potential energy of the structure.

The total potential energy of the structure π is expressed either in terms of unknown mathematical constants or in terms of nodal displacements. The conditions of minimising π with respect to constants or nodal displacements result in a set of simultaneous equations. The solution of these equations decides the deformed state of the structure. Other dependant quantities can then be found at various locations in the structure.

1.3.1 Rayleigh Method

In Rayleigh method, applied to beams and plates, the field variable (w) is assumed as

$$w = A\phi$$

where, A is a mathematical constant and does not have any physical meaning, and ϕ is a function (polynomial or trigonometric) which satisfies kinematic boundary conditions. The total potential energy of the structure is minimised with respect to unknown constant A.

Example 1.9. A cantilever beam AB of span L, flexural rigidity EI carries uniformly distributed load q/unit length on entire span (Fig. 1.10). Use Rayleigh method to analyse and comment on results.

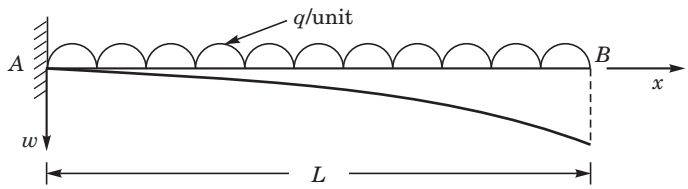


Fig. 1.10. Cantilever beam.

Assume,

$$\left.\begin{aligned} w(x) &= Ax^2 \\ \frac{dw}{dx} &= 2Ax \\ \frac{d^2w}{dx} &= 2A \end{aligned}\right\} \quad ...(1.55)$$

Assumed function satisfies kinematic boundary conditions as

(*i*) $w = 0$ at $x = 0$ (*ii*) $\dfrac{dw}{dx} = 0$ at $x = 0$

Hence, assumed function is acceptable. The strain energy in the beam is

$$\left.\begin{aligned} U = \frac{EI}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L (2A)^2\, dx \\ U = 2EIA^2L \end{aligned}\right\} \quad ...(1.56)$$

The potential of the distributed load is

$$\left.\begin{aligned} V = -\int_0^L q\, w\, dx = -q\int_0^L Ax^2\, dx \\ V = -qAL^3/3 \end{aligned}\right\} \quad ...(1.57)$$

Total potential energy of the beam is

$$\pi = U + V = 2EIA^2 - qAL^3/3 \quad ...(1.58)$$

Applying minimum π theorem

$$\frac{\partial \pi}{\partial A} = 4EIAL - qL^3/3 = 0$$

$$A = \frac{qL^2}{12EI}$$

Hence,
$$w(x) = \frac{qL^2x^2}{12EI} \qquad ...(1.59)$$

The deflection at free end ($x = L$) using Eq. 1.59 is $w(x = L) = qL^4/(12EI)$ as against exact value of $qL^4/(8EI)$.

In general, the results obtained by Rayleigh method are not satisfactory. We seek to find that deflected shape of the structure which gives absolute minimum total potential energy. For this, it is necessary to examine all possible kinematically admissible deflected shapes. But, Rayleigh method examines only those deflected shapes which are generated by assumed function by changing associated constant A. Obviously, the search is limited to the type of shapes generated by assumed function and the one giving minimum π is treated as true solution.

1.3.2 Rayleigh-Ritz Method

Example 1.10 In Rayleigh-Ritz method, the field variable is expressed as a linear combination of more than one functions. Thus,

$$w(x) = \sum A_i \phi_i \qquad ...(1.60)$$

Each function ϕ_i must satisfy kinematic boundary conditions of the problem. The cantilever beam shown in Fig. 1.10 is analysed.

Assume

$$\left.\begin{aligned} w(x) &= A_1x^2 + A_2x^3 + A_3x^4 \\ \frac{dw}{dx} &= 2A_1x + 3A_2x^2 + 4A_3x^3 \\ \frac{d^2w}{dx^2} &= 2A_1 + 6A_2x + 12A_3x^2 \end{aligned}\right\} \qquad ...(1.61)$$

The assumed trial function satisfies boundary conditions as

At $\quad x = 0, \quad w = 0 + 0 + 0$

At $\quad x = 0, \quad \dfrac{dw}{dx} = 0 + 0 + 0$

The strain energy in beam is

$$U = \frac{EI}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L (2A_1 + 6A_2x + 12A_3x^2)^2\, dx$$

$$U = \frac{EI}{2}\left[4A_1^2L + 12A_1A_2L^2 + 16A_1A_3L^3 + 12A_2^2L^3 + 36A_2A_3L^4 + \frac{144}{5}A_3^2L^5\right] \qquad ...(1.62)$$

Potential of distributed load is

$$\left.\begin{aligned} V &= -\int_0^L q(A_1x^2 + A_2x^3 + A_3x^4)dx \\ V &= -\frac{qA_1L^3}{3} - \frac{qA_2L^4}{4} - \frac{qA_3L^5}{5} \\ \pi &= U + V \end{aligned}\right\} \qquad ...(1.63)$$

Applying minimum π theorem

$$\frac{d\pi}{dA_1} = 0, \quad \frac{d\pi}{dA_2} = 0, \quad \frac{d\pi}{dA_3} = 0 \qquad ...(1.64)$$

Resulting equations in matrix form are

$$\frac{EI}{2}\begin{bmatrix} 8L & 12L^2 & 16L^3 \\ 12L^2 & 24L^3 & 36L^4 \\ 16L^3 & 36L^4 & 288L^5/5 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} qL^3/3 \\ qL^4/4 \\ qL^5/5 \end{Bmatrix} \qquad ...(1.65)$$

Solution gives

$$A_1 = \frac{qL^2}{4EI}, \quad A_2 = \frac{-qL}{6EI}, \quad A_3 = \frac{q}{24EI}$$

Hence, the solution for deflection $w(x)$ is

$$w(x) = \frac{1}{EI}\left[\frac{qL^2x^2}{4} - \frac{qLx^3}{6} + \frac{qx^4}{24}\right] \qquad ...(1.66)$$

This is the exact solution for beam in Fig. 1.10. This was possible because trial function included all terms of the polynomial (e.g. x^2, x^3, x^4) which are present in exact solution. Exact solution will not be obtained if trial function is chosen as $w(x) = A_1x^2 + A_2x^3$.

1.3.3 Review of Stiffness Matrix Method

Before embarking on the study of finite element method, it is advisable to take a hurried review of the stiffness matrix method. The stiffness matrix method is used for analysis of skeletal structures. The method involves formulation and solution of joint equilibrium equations.

An alternate view of the stiffness matrix method is to consider it as a variational approach in which certain functional associated with the structure is sought to be minimised. The functional is the total potential energy π of the structure. It is expressed in terms of displacements of the joints in the structure. The conditions to minimise the total potential energy yield a set of simultaneous equations in which joint displacements are unknowns. It is interesting to note that these equations are same as joint equilibrium equations. This important fact is presented here and then applied to the finite element method for solving continuum structures.

Example 1.11. Analyse simple frame shown in Fig. 1.11 using stiffness matrix method. The axial deformations are neglected and sway is not possible. Each joint is associated with rotation only. For a typical member force-displacement relation is written as

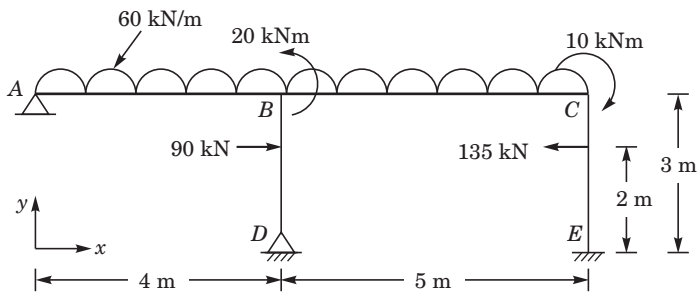


Fig. 1.11. Frame for example 1.11.

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix} + \begin{Bmatrix} M_{FAB} \\ M_{FBA} \end{Bmatrix} \qquad ...(1.67)$$

or $$\{f_e\} = [s_e]\{\delta_e\} + \{q'_e\}$$

in which,

$\{f_e\}$ = Member end forces (moments)
$[s_e]$ = Member stiffness matrix
$[\delta_e]$ = Member displacement vector
$\{q'_e\}$ = Fixed end reactions (moments)

Equation 1.67 is written for all members using appropriate member properties and load acting on them.

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = EI \begin{array}{c} \begin{array}{cc} \theta_A & \theta_B \end{array} \\ \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} \end{array} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix} + \begin{Bmatrix} 80 \\ -80 \end{Bmatrix} \qquad ...(1.68a)$$

$$\begin{Bmatrix} M_{BC} \\ M_{CB} \end{Bmatrix} = EI \begin{array}{c} \begin{array}{cc} \theta_B & \theta_C \end{array} \\ \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} \end{array} \begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} + \begin{Bmatrix} 125 \\ -125 \end{Bmatrix} \qquad ...(1.68b)$$

$$\begin{Bmatrix} M_{DB} \\ M_{BD} \end{Bmatrix} = EI \begin{array}{c} \begin{array}{cc} \theta_D & \theta_B \end{array} \\ \begin{bmatrix} 1.333 & 0.667 \\ 0.667 & 1.333 \end{bmatrix} \end{array} \begin{Bmatrix} \theta_D \\ \theta_B \end{Bmatrix} + \begin{Bmatrix} 20 \\ -40 \end{Bmatrix} \qquad ...(1.68c)$$

$$\begin{Bmatrix} M_{EC} \\ M_{CE} \end{Bmatrix} = EI \begin{array}{c} \begin{array}{cc} \theta_E & \theta_C \end{array} \\ \begin{bmatrix} 1.333 & 0.667 \\ 0.667 & 1.333 \end{bmatrix} \end{array} \begin{Bmatrix} \theta_E \\ \theta_C \end{Bmatrix} + \begin{Bmatrix} -30 \\ 60 \end{Bmatrix} \qquad ...(1.68d)$$

There are four non-zero rotations in the frame namely θ_A, θ_B, θ_C and θ_D. Hence, four joint equilibrium equations are to be developed. These equations, written in terms of end moments, are

$$\left.\begin{aligned} &(i)\ M_{AB} = 0 \\ &(ii)\ M_{BA} + M_{BC} + M_{BD} = M_B \\ &(iii)\ M_{CB} + M_{CE} = M_C \\ &(iv)\ M_{DB} = 0 \end{aligned}\right\} \qquad ...(1.69)$$

M_B(= 20 kNm) and M_C (= – 10 kNm) are external moments acting on joints B and C respectively. It is possible to substitute moments in Eq. 1.69 in terms of joint rotations from Eq. 1.68 and develop four equations. For example, the first equation is written as

$$M_{AB} = EI\,\theta_A + 0.5EI\,\theta_B + 80 = 0$$

Other equations can also be obtained similarly. However, this method of proceeding equation by equation is inconvenient, tedious and not suitable for computer programming. An alternate approach known as assembly procedure is used [15]. The assembly procedure consists of adding elements of member stiffness matrices to the structure stiffness matrix in a systematic way. Also, fixed end forces and moments acting on joints are added to the right hand side of the equations.

We begin with the blank matrices as shown in Eq. 1.70. The unknowns are also entered as column headings. Similarly, the unknowns associated with the members are entered as column headings above respective member stiffness matrices, in Eq. 1.68.

$$\begin{array}{c} \begin{array}{cccc} \theta_A & \theta_B & \theta_C & \theta_D \end{array} \\ \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{bmatrix} \end{array} \begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \end{Bmatrix} = \begin{Bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{Bmatrix} + \begin{Bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{Bmatrix} \qquad \text{...(1.70)}$$

or, $$[S]\ \{D\} = \{Q\} + \{P\}$$

The rows and columns of the structure and member stiffness matrices are identified by the unknowns associated with them.

Assembly procedure proceeds member by member. Consider member AB. The stiffness coefficient $1.0EI$ appears in θ_A row and θ_A column in member stiffness matrix, in Eq. 1.68a. Hence, this coefficient ($1.0\ EI$) is entered at similar position *i.e.* at the intersection of θ_A row and θ_A column in structure stiffness matrix. Same procedure is followed for entering remaining three coefficients. The fixed end moments are entered in corresponding rows on right hand side, but with change of sign. The contribution of member AB to the equilibrium equations is seen as

$$\begin{array}{c} \begin{array}{cccc} \theta_A & \theta_B & \theta_C & \theta_D \end{array} \\ \begin{bmatrix} 1.0 & 0.5 & & \\ 0.5 & 1.0 & & \\ & & & \\ & & & \end{bmatrix} \end{array} \begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \end{Bmatrix} = \begin{Bmatrix} -80 \\ 80 \\ \_\_ \\ \_\_ \end{Bmatrix} + \begin{Bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{Bmatrix} \qquad \text{...(1.71)}$$

Assembly procedure is completed by considering remaining members one by one. The moments acting on joints B and C are added to the right hand side against corresponding unknown rotations. Thus, the cumulative effect is seen as shown below in Eq. 1.72.

$$EI \begin{array}{c} \begin{array}{cccc} \theta_A & \theta_B & \theta_C & \theta_D \end{array} \\ \begin{bmatrix} 1.0 & 0.5 & & \\ 0.5 & \begin{array}{l} 1.0+0.8 \\ +1.333 \end{array} & 0.4 & 0.667 \\ & 0.4 & \begin{array}{l} +0.8 \\ +1.333 \end{array} & \\ & 0.667 & & +1.333 \end{bmatrix} \end{array} \begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \end{Bmatrix} = \begin{Bmatrix} -80 \\ 80-125 \\ +40 \\ 125-60 \\ -20 \end{Bmatrix} + \begin{Bmatrix} \_\_ \\ 20 \\ -10 \\ \_\_ \end{Bmatrix} \qquad \text{...(1.72)}$$

It is noted that the coefficients in θ_E row and θ_E column of member stiffness matrix of member EC (Eq. 1.68d) are not added to structure stiffness matrix because θ_E is not an unknown.

Final form of equilibrium equations is obtained by summing up the quantities in Eq. 1.72.

$$EI\begin{array}{c} \begin{array}{cccc} \theta_A & \theta_B & \theta_C & \theta_D \end{array} \\ \left[\begin{array}{c|c|c|c} 1.0 & 0.5 & 0 & 0 \\ \hline 0.5 & 3.133 & 0.4 & 0.667 \\ \hline 0 & 0.4 & 2.133 & 0 \\ \hline 0 & 0.667 & 0 & 1.333 \end{array}\right] \end{array} \left\{\begin{array}{c} \theta_A \\ \hline \theta_B \\ \hline \theta_C \\ \hline \theta_D \end{array}\right\} = \left\{\begin{array}{c} -80 \\ \hline 15 \\ \hline 55 \\ \hline -20 \end{array}\right\} \qquad ...(1.73)$$

or, $[S]\ \{D\} = \{R\}$

Solution of equations by Gauss elimination method gives

$$\theta_A = -\frac{91.053}{EI}, \qquad \theta_B = \frac{22.105}{EI},$$

$$\theta_C = \frac{21.640}{EI}, \qquad \theta_D = -\frac{26.065}{EI}$$

This completes first stage of solution. Second stage consists of back substitution to calculate member end forces using known joint rotations. Thus,

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} \begin{Bmatrix} -91.053 \\ 22.105 \end{Bmatrix} + \begin{Bmatrix} 80 \\ -80 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -103.422 \end{Bmatrix} \qquad ...(1.74a)$$

$$\begin{Bmatrix} M_{BC} \\ M_{CB} \end{Bmatrix} = \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} \begin{Bmatrix} 22.105 \\ 21.640 \end{Bmatrix} + \begin{Bmatrix} 125 \\ -125 \end{Bmatrix} = \begin{Bmatrix} 151.340 \\ -98.846 \end{Bmatrix} \qquad ...(1.74b)$$

$$\begin{Bmatrix} M_{DB} \\ M_{BD} \end{Bmatrix} = \begin{bmatrix} 1.333 & 0.667 \\ 0.667 & 1.333 \end{bmatrix} \begin{Bmatrix} -26.065 \\ 22.105 \end{Bmatrix} + \begin{Bmatrix} 20 \\ -40 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -27.919 \end{Bmatrix} \qquad ...(1.74c)$$

$$\begin{Bmatrix} M_{EC} \\ M_{CE} \end{Bmatrix} = \begin{bmatrix} 1.333 & 0.667 \\ 0.667 & 1.333 \end{bmatrix} \begin{Bmatrix} 0 \\ 21.640 \end{Bmatrix} + \begin{Bmatrix} -30 \\ 60 \end{Bmatrix} = \begin{Bmatrix} -15.566 \\ 88.846 \end{Bmatrix} \qquad ...(1.74d)$$

It is left to the reader to draw bending moment diagrams for all members. All end moments are in kN m units.

1.3.4 Stiffness Method as a Variational Approach

In Art 1.3.3 the stiffness matrix method was explained as a method to formulate joint equilibrium equations to obtain unknown joint displacements. An alternate view is to consider the stiffness method as a variational approach. A variational approach is one in which certain functional associated with the structure is sought to be minimised. In structural engineering problems the functional is the total potential energy π of the structure. It is interesting to note that the conditions to minimise π with respect to joint displacements yield the equations which are exactly identical to the joint equilibrium equations.

Example 1.11 (Fig. 1.11) is considered again for explanation purpose. If member is not carrying loads, then for a typical member AB :

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix} \qquad ...(1.75)$$

or, $\{f_e\} = [s_e]\{\delta_e\}$

The end forces (moments) will do work on end displacements (rotations). This work is stored in member as strain energy U_e as

$$W = U_e = \frac{1}{2}(\theta_A M_{AB} + \theta_B M_{BA}) \qquad ...(1.76a)$$

$$U_e = \frac{1}{2}[\theta_A \quad \theta_B]\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} \qquad ...(1.76b)$$

Substituting for end moments from Eq. 1.75

$$U_e = \frac{1}{2}[\theta_A \quad \theta_B]\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}\begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix} \qquad ...(1.76c)$$

or,

$$U_e = \frac{1}{2}\{\delta_e\}^T \; [s_e] \; \{\delta_e\} \qquad ...(1.76d)$$

The fixed end forces (moments in this case) are converted to equivalent joint moments. These are equal to negatives of the fixed end moments. Thus,

$$\text{Fixed end moments} = \{q_e'\} = \begin{Bmatrix} M_{FAB} \\ M_{FBA} \end{Bmatrix} \qquad ...(1.77a)$$

$$\text{Equivalent joint moments} = \{q_e\} = -\{q_e'\} = \begin{Bmatrix} -M_{FAB} \\ -M_{FBA} \end{Bmatrix} \qquad ...(1.77b)$$

The potential of equivalent joint loads (moments in this case) is written as

$$V_e = -[\theta_A \quad \theta_B]\begin{Bmatrix} -M_{FAB} \\ -M_{FBA} \end{Bmatrix} = +[\theta_A \quad \theta_B]\begin{Bmatrix} M_{FAB} \\ M_{FBA} \end{Bmatrix} \qquad ...(1.78a)$$

$$V_e = -\{\delta_e\}^T\{q_e\} = +\{\delta_e\}^T\{q_e'\} \qquad ...(1.78b)$$

The total potential energy of the member AB is the sum of strain energy U_e and the potential of loads V. Thus,

$$\left.\begin{aligned} \pi_e &= U_e + V_e \\ \pi_e &= \frac{1}{2}\{\delta_e\}^T\,[s_e]\,\{\delta_e\} + \{\delta_e\}^T\,\{q_e'\} \end{aligned}\right\} \qquad ...(1.79)$$

Substituting from Eqs. 1.76(c) and 1.78 (a) and carrying out multiplications

$$\pi_e = \frac{1}{2}(s_{11}\theta_A^2 + s_{21}\theta_A\theta_B + s_{12}\theta_A\theta_B + s_{22}\theta_B^2) + \theta_A M_{FAB} + \theta_B M_{FBA} \qquad ...(1.80)$$

Taking derivatives with each component of $\{\delta_e\}$ and noting that $s_{12} = s_{21}$:

$$\frac{\partial \pi_e}{\partial \theta_A} = s_{11}\theta_A + s_{12}\theta_B + M_{FAB} \qquad ...(1.81a)$$

$$\frac{\partial \pi_e}{\partial \theta_B} = s_{21}\theta_A + s_{22}\theta_B + M_{FBA} \qquad ...(1.81b)$$

Writing in matrix form

$$\begin{Bmatrix} \dfrac{\partial \pi_e}{\partial \theta_A} \\ \\ \dfrac{\partial \pi_e}{\partial \theta_B} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ \\ s_{21} & s_{22} \end{bmatrix}\begin{Bmatrix} \theta_A \\ \\ \theta_B \end{Bmatrix} + \begin{Bmatrix} M_{FAB} \\ \\ M_{FBA} \end{Bmatrix} \qquad ...(1.81c)$$

or,

$$\begin{Bmatrix} \dfrac{\partial \pi_e}{\partial \delta_e} \end{Bmatrix} = [s_e]\,\{\delta_e\} + \{q'_e\} \qquad ...(1.81d)$$

This relation is derived for an individual member. Later, similar expression will be used at structure level.

A connectivity matrix $[C_i]$ is introduced for i^{th} member such that

$$\left.\begin{aligned} \{\delta_e\}_i &= [C_i]\{D\} \\ \text{and} \quad \{D\} &= [C_i]^T \{\delta_e\}_i \end{aligned}\right\} \qquad \text{...(1.82)}$$

For a typical member AB

$$\begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \\ \theta_C \\ \theta_D \end{Bmatrix}$$

Members AB, BC, DB, EC are numbered 1 to 4 respectively. Total potential energy of member 1 (AB) can be written using Eqs. 1.79 and 1.82 as

$$\pi_1 = \frac{1}{2}\{D\}^T [C_1]^T [s_{e1}] [C_1] \{D\} + \{D\}^T [C_1]^T \{q'_{e1}\} \qquad \text{...(1.83)}$$

writing $[S_1] = [C_1]^T [s_{e1}] [C_1]$ ...(1.84*a*)

and $\{Q'_1\} = [C_1]^T \{q'_{e1}\}$ ...(1.84*b*)

The total potential energy of AB is written as

$$\pi_1 = \frac{1}{2}\{D\}^T [S_1] \{D\} + \{D\}^T \{Q'_1\} \qquad \text{...(1.85)}$$

Matrices $[S_1]$ and $\{Q'_1\}$ for member AB are

$$[S_1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[S_1] = EI \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{Q'_1\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 80 \\ -80 \end{Bmatrix} = \begin{Bmatrix} 80 \\ -80 \\ 0 \\ 0 \end{Bmatrix}$$

Matrix multiplications in Eq. 1.84*a* enlarge member stiffness matrix of 2 × 2 size to a size of structure stiffness matrix of 4 × 4. The coefficients are placed at the locations adopted in assembly procedure. Also, the multiplication in Eq. 1.84*b* enlarges matrix $\{q'_e\}$ of 2 × 1 size to a size of structure load matrix of 4 × 1.

Omitting the details of multiplications, enlarged matrices $[S_i]$ and $\{Q'_1\}$ for remaining members are given below.

For member 2 (BC)

$$[S_2] = EI\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0.4 & 0 \\ 0 & 0.4 & 0.8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad [Q_2'] = \begin{Bmatrix} 0 \\ 125 \\ -125 \\ 0 \end{Bmatrix}$$

For member 3 (DB)

$$[S_3] = EI\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.333 & 0 & 0.667 \\ 0 & 0 & 0 & 0 \\ 0 & 0.667 & 0 & 1.333 \end{bmatrix}; \quad [Q_3'] = \begin{Bmatrix} 0 \\ -40 \\ 0 \\ 20 \end{Bmatrix}$$

For member 4 (EC)

$$[S_4] = EI\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1.333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad [Q_4'] = \begin{Bmatrix} 0 \\ 0 \\ 60 \\ 0 \end{Bmatrix}$$

Total potential energy of entire structure consists of

(i) Strain energies of all members,

(ii) Potentials of loads acting on members and

(iii) Potentials of loads acting on joints proper.

Using Eq. 1.85, total potential energy of the entire structure is

$$\pi = \frac{1}{2}\sum_{i=1}^{4} \{D\}^T [S_i]\{D\} + \sum_{i=1}^{4} \{D\}^T \{Q_i'\} - \{D\}^T \{P\} \qquad ...(1.86)$$

Writing,

$$\left.\begin{aligned} [S] &= \sum [S_i] = [S_1] + [S_2] + [S_3] + [S_4] \\ \{Q'\} &= \sum \{Q'_i\} = \{Q_1'\} + \{Q_2'\} + \{Q_3'\} + \{Q_4'\} \end{aligned}\right\} \qquad ...(1.87)$$

$$\pi = \frac{1}{2}\{D\}^T [S]\{D\} + \{D\}^T \{Q'\} - \{D\}^T \{P\} \qquad ...(1.88)$$

Applying minimum potential energy theorem :

$$\left\{\frac{\partial \pi}{\partial D}\right\} = [S]\{D\} + \{Q'\} - \{P\} = \{0\} \qquad ...(1.89)$$

or, $$[S]\{D\} = -\{Q'\} + \{P\} \qquad ...(1.90)$$

or, $$[S]\{D\} = \{R\} \qquad ...(1.91)$$

Structure stiffness matrix $[S]$ is obtained by adding enlarged matrices $[S_i]$ and $\{Q'\}$ is obtained by adding $\{Q_i'\}$ of all members. Thus,

$$[S] = EI\begin{bmatrix} 1.0 & 0.5 & & \\ 0.5 & 1.0 + 0.8 + 1.333 & 0.4 & 0.667 \\ & 0.4 & 0.8 + 1.333 & \\ & 0.667 & & 1.333 \end{bmatrix} \qquad ...(1.92)$$

$$\{R\} = \begin{Bmatrix} -80 \\ \hline +80-125+40 \\ \hline 125-60 \\ \hline -20 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \hline 20 \\ \hline -10 \\ \hline 0 \end{Bmatrix} \qquad ...(1.93)$$

Matrices of Eqs. 1.92 and 1.93 are exactly same as those obtained in Eq. 1.72. It is, thus, proved that the conditions of minimum total potential energy yield the joint equilibrium equations.

Whether the equations (Eqs. 1.73, 1.90) are treated as joint equilibrium equations or conditions to minimise total potential energy of the structure, the process of obtaining the equations is exactly identical, namely formulation of member stiffness matrices and member load matrices and assembling them to form structure stiffness and structure load matrices. It is important to understand this fact clearly as it will be used while developing formulations in finite element method.

1.3.5 Concept of Finite Element Method

The finite element method is used to analyse continuum structures. The stiffness matrix method, on the other hand, is used to analyse discrete element or skeletal structures. Recent trend, however, is to present unified treatment of both the methods considering discrete member as a special case of continuum elements.

The basic concept of the finite element method is to divide the continuum structure into sub-regions or sub-domains or zones. This process of division into sub-domains is known as dicretisation of the structure. A typical discretisation of a two dimensional plate into triangular regions is shown in Fig. 1.12(*a*). The sub-domains are known as elements. The structure is treated as an assemblage of these elements. An element (in finite element method) has generally more than two nodes. The elements are assumed as connected to each other at nodes. In reality, the elements do not exist as separate entities. One can imagine that the elements are obtained by drawing chalk lines or pencil lines on the structure. Further, nodes are not actual nodes. These are imaginary or notional nodes. Actually, elements are connected together all along the common edge, and not at nodes only. In skeletal structure, the members are actual and they are actually connected together at joints only. Each member of skeletal structure has two nodes-one at each end.

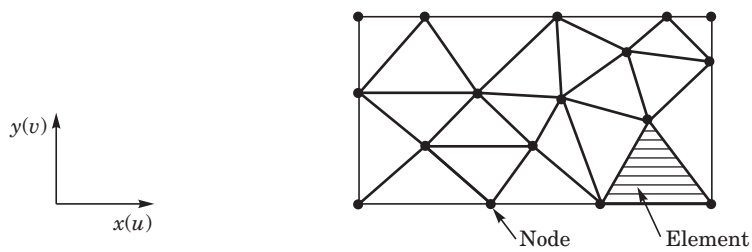


Fig. 1.12 (a) Discretisation.

In finite element method, the displacements at the nodes are treated as primary unknowns. These nodal displacements are obtained by solution of simultaneous equations which arise when conditions of minimum total potential energy of the structure are applied. If nodal displacements are known, then other dependant quantities can be found at any location in the structure.

The simultaneous equations mentioned above are obtained by assembly procedure which consists of adding element stiffness matrices to the structure stiffness matrix. Obviously, the first task is to develop standard stiffness matrix of a typical element. Consider a triangular element shown in Fig. 1.12(*b*). The element has three nodes. In plane stress situation, the nodes undergo displacements u and v in x and y directions respectively. Thus, the element displacement vector for element is

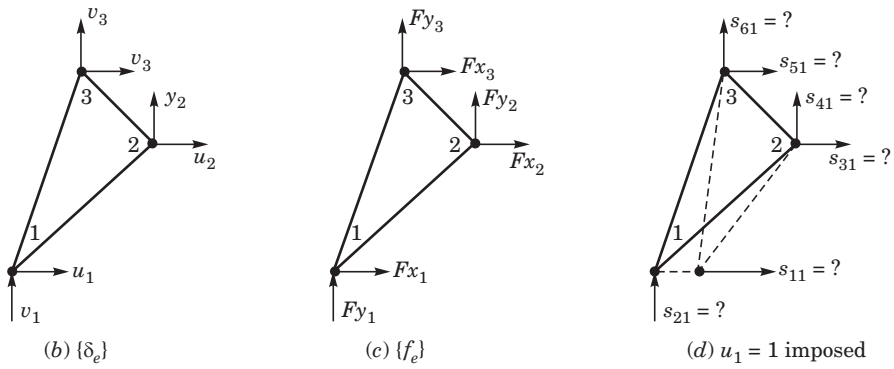


Fig. 1.12. Two dimensional region and triangular element.

$$\{\delta_e\} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3\}^T \qquad ...(1.94)$$

and corresponding matrix of nodal forces is

$$\{f_e\} = \{F_{x1} \quad F_{y1} \quad F_{x2} \quad F_{y2} \quad F_{x3} \quad F_{y3}\}^T \qquad ...(1.95)$$

The element stiffness matrix relates nodal forces to nodal displacements as

$$\{f_e\} = [s_e]\{\delta_e\} \qquad ...(1.96)$$

Stiffness matrix $[s_e]$ is to be developed. The coefficients of stiffness matrix of a discrete element structure are easily found by imposing one displacement of unit value at a time, while restraining other displacements. The forces required to keep the element in that displaced state are the required coefficients. Let us imagine that the triangular element exists as a separate entity and nodal displacement $u_1 = 1$ is imposed as shown in Fig. 1.12*d*. Is it, then, possible to obtain the coefficients s_{11}, s_{21}, ..., s_{61}? The members of the skeletal structure are simple straight members and the standard formulae in structural mechanics are used to calculate stiffness coefficients. But, for a triangular element, there are no such standard formulae available to calculate coefficients. Thus, the idea of obtaining stiffness coefficients using approach used in stiffness method can not succeed.

The stiffness matrix is also seen as a means of expressing strain energy in the element in quadratic form vide Eq. 1.76 (*d*). If it is possible to express strain energy in the element as

$$U_e = \frac{1}{2}\{\delta_e\}^T [X]\{\delta_e\} \qquad ...(1.97)$$

then, the matrix $[X]$ is certainly the stiffness matrix. This matrix may be a single matrix or may be product of two or more matrices. Often, integration of matrices may also be involved. Thus, for example, if energy is expressed as

$$U_e = \frac{1}{2}\{\delta_e\}^T \left[\int [X]^T [Y][X]\, dA\right]\{\delta_e\} \qquad ...(1.98)$$

then, stiffness matrix is

$$[s_e] = \int [X]^T [Y][X]\, dA \qquad ...(1.99)$$

To implement this strategy of obtaining stiffness matrix via strain energy expression, the variation of displacements over element region is expressed in terms of nodal displacements using shape functions. This helps in writing strains and stresses at any point within element. Finally, the strain energy in element is expressed in quadratic form and hence element stiffness matrix is found.

The potential of the loads acting on element is also expressed in terms of nodal displacements of the element. The total potential energy of the entire structure is the sum of the potential energies of all elements. Obviously, this quantity π is the function of the displacements of all nodes of the structure. The total potential energy of the structure is minimised with reference to each component of nodal displacements and a set of equations in unknown displacements is obtained. Actually, these equations are obtained by assembly procedure mentioned earlier. The solution of these equations is obtained using any standard method of solution. Once, the nodal displacements are found, it is easy to calculate strains and stresses at any point in the structure.

In summary, the finite element method aims at determining deformed state of the structure in terms of nodal displacements using energy minimisation principle.

As mentioned earlier, nodes are imaginary and nodal forces do not really exist. Yet, in the earlier stages of the development of finite element method, the nodal forces were considered and were shown as the net effect of the stresses acting on element sides. The concept of nodal forces is useful for a beginner who can study and understand the finite element method on lines of stiffness matrix method, considering force-displacement relations and joint equilibrium conditions. The present day books, however, do not mention nodal forces and present the finite element method in terms of energy minimisation process.

1.3.6 Finite Strip Method

The finite strip method has emerged as a popular, powerful and computationally economic method of structural analysis. General concept of the method is discussed with reference to a rectangular thin plate shown in Fig. 1.13. The plate is discretised into strips and not into elements. The method is a semi-analytical method. For a plate with two opposite edges simply supported, the displacement (w) variation over the strip region is expressed using sine functions in x-direction and beam functions in y-direction. Each strip has two nodal lines and each nodal line is associated with w and $\frac{\partial w}{\partial y}$ per harmonic. The method employs minimum potential energy principle. The resulting $m \times n$ equations are decoupled into m sets of n equations each, where m is the number of harmonics used. Each set of equations is solved independently. The harmonic decoupling of equations results in great computational economy. The method has been used for analysis of annular sector plates, folded plates, box-girders, axi-symmetric shells and solids. Mindlin plates also have been analysed.

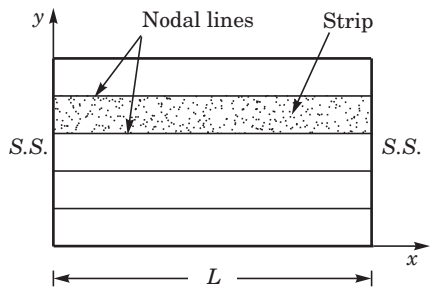


Fig. 1.13. Rectangular plate discretised into strips.

The advantage of harmonic decoupling is not available for other combinations of boundary conditions at $x = 0$ and $x = L$ edges. To overcome this difficulty, a new spline finite strip method has been developed. In this method, nodes are introduced on nodal lines. The coefficients for w and $\frac{\partial w}{\partial y}$ are unknowns at each node. Spline functions are used along nodal lines to describe deflection variation over strip region. Harmonic wise analysis is not required.

1.3.7 Finite Difference Energy Method

This is a variational approach. It is also known as variational finite difference method. It uses minimum potential energy principle. The structure is discretised into elements. An element is associated with nodes some of which lie outside element domain. For beam and plate bending problems, only deflection w is unknown at a node. The derivatives of the displacement appearing in energy expression are related by corresponding finite difference operator. The bending curvatures are treated as constant over element region and hence integration is not required to evaluate stiffness matrix.

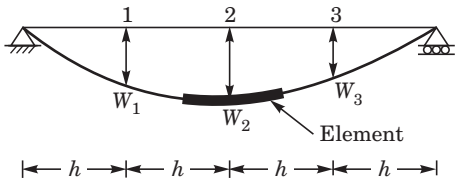


Fig. 1.14. Element and associated node.

For a typical beam element shown in Fig. 1.14, the element displacement vector is composed of w_1, w_2 and w_3. The bending curvature at the centre of element is written as

$$\chi = -\frac{d^2w}{dx^2} = -\frac{1}{h^2}[w_1 - 2w_2 + w_3]$$

This forms the basis for further calculations. The method is computationally economic since integration is not required and the number of unknowns is less than the number of unknowns in finite element method for identical mesh.

1.3.8 Discrete Energy Method

Discrete energy method is a variational approach. It is based on application of minimum total potential energy theorem. For a plate subjected to membrane-bending action, the discretisation is into two types of elements namely type A and type B elements (Fig. 1.15). Bending and extensional energies are calculated for type A elements, while twisting and inplane shear energies are calculated for type B elements. The regions of type A and type B elements are overlapping. Curvatures and strains are treated as constant over element region. Derivatives appearing in strain and curvature expressions are replaced by finite difference operators. The method is computationally economic. The number of unknowns is less compared to the number in finite element method for identical mesh. The method has been applied for analysis of folded plates, box-girders, shallow shells and also for non-linear analysis of plates and shells.

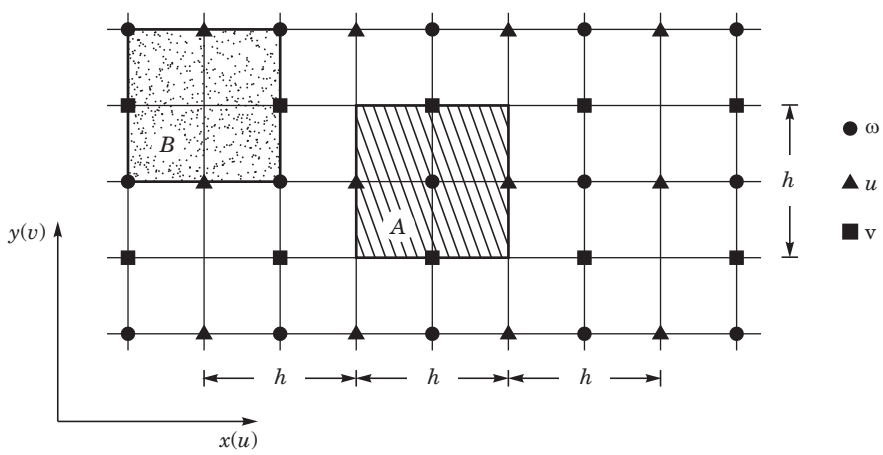


Fig. 1.15. Type *A* and *B* Elements.

1.4 BOUNDARY ELEMENT METHOD

Boundary element method is a numerical approach for analysis of two and three dimensional problems. In this method, the governing differential equation of the problem is transformed into an integral identity [16] which is applicable over boundary or surface. For a two dimensional problem only boundary of the structure is discretised into one dimensional elements. For three dimensional solid the surface is discretised into 2*D* surface elements. The interior of the structure *i.e.* structure domain is not discretised. Thus, the dimensionality of the problem is reduced by one. The integrals on boundary or surface are found using numerical integration process. The method is based on application of Green's theorem and Betti-Maxwell theorem. For applying the method, the knowledge of fundamental solution (Kelvin solution) for unit load applied at a point in infinite domain is necessary. The primary unknowns are displacements and tractions at nodes located on boundary or surface. The coefficient matrix of resulting equations is neither symmetric nor banded. Hence, it is to be stored in full form. Once unknowns at boundary nodes are found, it is possible to calculate quantities of interest at any location within structure.

The method involves complex mathematical expressions and is not as popular as finite difference and finite element methods.

1.5 RAYLEIGH-RITZ METHOD AND FINITE ELEMENT METHOD

Rayleigh-Ritz method is a classical method of structural analysis. It is a variational approach and uses energy minimization principle to predict the deflected shape of structure. The displacement variation over entire structure is described in terms of certain functions. Each function is associated with unknown constant. These constants are primary unknowns. The constants do not have any physical meaning. They are merely mathematical constants. They can be treated as nodeless variables. Each function ϕ_i must satisfy kinematic boundary conditions. The total potential energy of the structure is expressed in terms of unknown constants and then principle of minimum total potential energy is applied. This yields a set of simultaneous equations. The solution gives the values of the constants. The displacements are then available in terms of functions which are continuous over entire structure. Hence, displacements, strains, stresses can be calculated at any location within structure using appropriate coordinate of the location. Force boundary conditions may not be satisfied. The displacement solution can be exact provided all terms appearing in true solution are included in assumed functions. In other cases solution is approximate. The method can be applied only to a structure of regular shape. It is difficult to prescribe displacement variation over arbitrarily shaped structure. The coefficient matrix in simultaneous equations is symmetric, but fully populated.

The finite element method started in real sense since around 1960. The advent of computers helped in accelerating the growth of the method by leaps and bounds.

The finite element method is also a variational method and uses energy minimisation principle to predict the deflected shape of the structure. The structure is divided into zones or elements and the displacement variation is prescribed piecewise over each element of the structure in terms of displacement parameters of the nodes attached to that element. The displacement parameters have physical meaning such as deflection, slope of the tangent, rotation of the normal etc. The nodal displacement parameters are primary unknowns. The total potential energy of the structure is the sum of the potential energies of all elements. Application of principle of minimum potential energy results in a set of simultaneous equations. The solution gives values of nodal parameters. To calculate displacements and other dependant quantities within element, the displacement parameters at nodes of the element are used

along with the shape functions and constitutive relations. It is, thus, seen that finite element method is a particular case of Rayleigh-Ritz method. This realisation came at a later stage.

The coefficient matrix of equations is symmetric and banded, as against fully populated matrix in Rayleigh-Ritz method. The finite element method is appealing to the engineers as the nodal parameters have physical meaning and engineers can draw immediate conclusions about the deflected shape of the structure. The force boundary conditions are not necessarily satisfied. The kinematic boundary conditions are applied by providing suitable data. The shape function chosen to describe displacement variation over element region need not a-priori satisfy kinematic boundary conditions.

The method can be applied to an arbitrarily shaped structure since it is always possible to discretise the structure into finite elements. This is the great advantage of the finite element method over Rayleigh-Ritz method.

1.6 STIFFNESS MATRIX METHOD AND FINITE ELEMENT METHOD

Stiffness matrix method is used for analysis of skeletal or discrete element structures. Finite element method, on the other hand, is used to analyse continuum structures.

The stiffness method can be treated as a special case of finite element method.

Joint displacements in skeletal structure and nodal displacement parameters in continuum structure are treated as primary unknowns.

Joints and members in skeletal structure are actual. Whereas, elements and nodes in continuum structure are notional, imaginary. The elements of continuum structure are attached to each other not at nodes only, but all along the common edge.

The members of skeletal structure have two nodes-one at each end of member. The elements in continuum structure generally have more than two nodes associated with it.

The stiffness matrix of a member of skeletal structure is found easily using standard formulae of structural mechanics. The stiffness matrix of an element of a continuum structure is to be found by expressing strain energy in quadratic form.

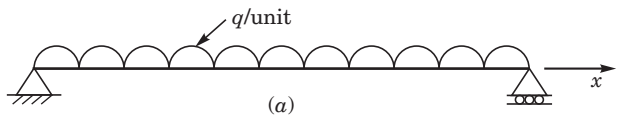
Joint equilibrium conditions are used to formulate equations in unknown joint displacements, in stiffness matrix method. In finite element method, the conditions of minimum total potential energy are used to formulate equations in unknown nodal displacements.

The coefficient matrix is symmetric and banded in both methods.

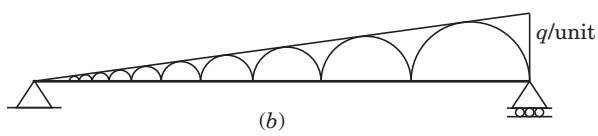
The theory for both methods can be written in concise matrix notations and hence both methods are amenable to computer usage.

EXERCISES I

1. Starting from governing differential equation, obtain exact solution of single span beams shown in Figs. 1.16*a* to 1.16*f*. For all beams span is L and flexural rigidity is EI. Origin is at left end of the beam.



(*a*)



(*b*)

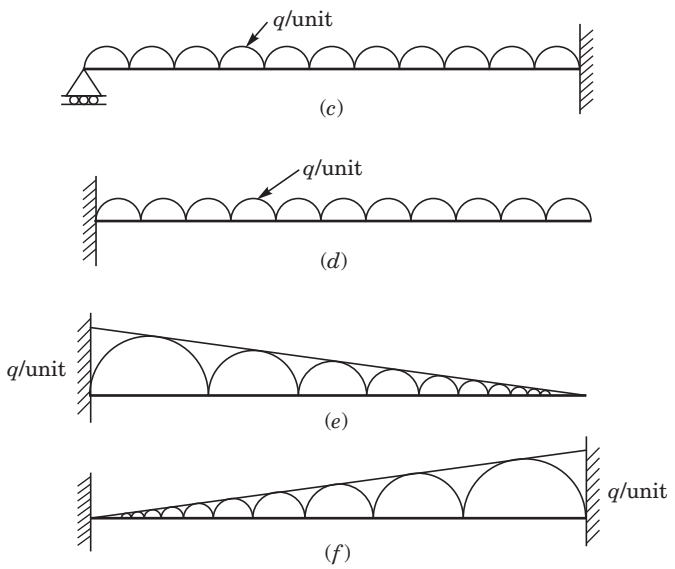


Fig. 1.16. Beams for Exercise I.

2. Obtain trigonometric series solution for beams shown in Fig. 1.17*a* to 1.17*c*. For all beams span is L and flexural rigidity is EI.

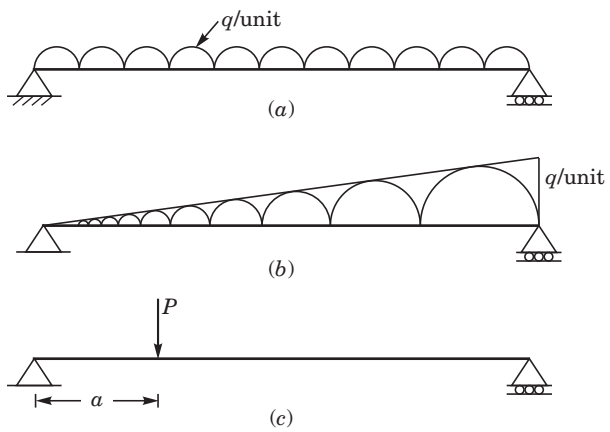


Fig. 1.17. Beams for Exercise 2.

3. Obtain nodal point deflections for beams shown in Fig. 1.18*a* and 1.18*b* by finite difference method. Span of the beam is L and flexural rigidity is EI. Beams are divided in six equal parts.

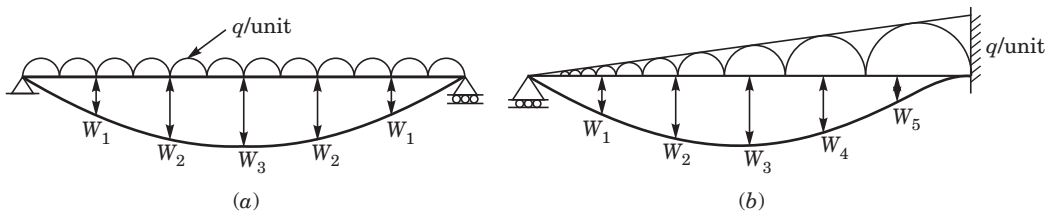


Fig. 1.18. Beams for Exercise 3.

4. For beams in Figs. 1.19*a* to 1.19*c*, assume deflection function $w(x)$ as given below and obtain Rayleigh-Ritz solution.

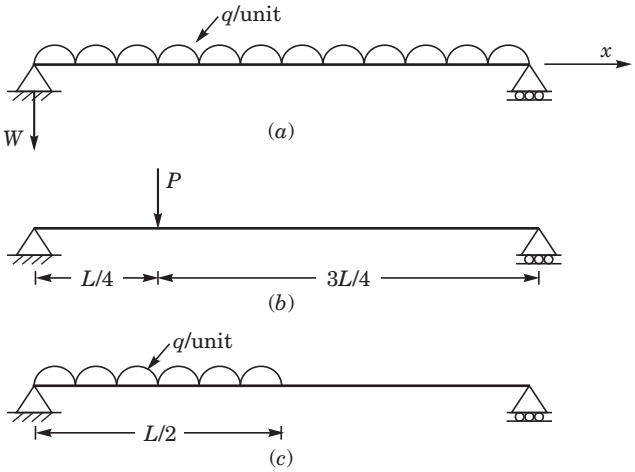


Fig. 1.19. Beams for Exercise 4.

(*a*) $w(x) = A_1 \sin\left(\frac{\pi x}{L}\right) + A_3 \sin\left(\frac{3\pi x}{L}\right)$

(*b*) $w(x) = A_1 \sin\left(\frac{\pi x}{L}\right) + A_2 \sin\left(\frac{2\pi x}{L}\right)$

(*c*) $w(x) = A_1 \sin\left(\frac{\pi x}{L}\right) + A_2 \sin\left(\frac{2\pi x}{L}\right)$

5. Analyse the continuous beam shown in Fig. 1.20 by stiffness matrix method. For all beams, flexure rigidity is *EI*.

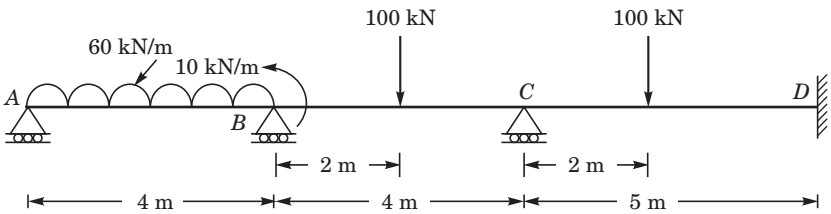


Fig. 1.20. Beam for Exercise 5.

6. Write points of similarity and contrast between :

(*a*) Analytical method and numerical method

(*b*) Rayleigh-Ritz method and finite element method

(*c*) Stiffness matrix method and finite element method

(*d*) Finite difference method and finite difference energy method

(*e*) Finite difference method and finite element method

(*f*) (Standard) finite element method and Galerkin finite element method

7. Starting from governing differential equation, obtain exact solution for beams described below.

$$\left[\textbf{Hint :}\ q(x) = 0 \text{ and } -EI\frac{d^2w}{dx^2} = M_0 \text{ at } x = 0\right]$$

(*a*) A simply supported beam of span L and flexure rigidity EI is subjected to clockwise moment at left end.

(*b*) A propped cantilever AB of span L and flexure rigidity EI has prop support at A and is clamped at B. It carries clockwise moment M_0 at A.

(*c*) A cantilever AB of span L and flexure rigidity EI has free end at A and is clamped at B. It carries clockwise moment M_0 at A.

8. A simply supported beam ABC has supports at A and C. $AB = BC = L/2$. It is subjected to distributed load which varies linearly from zero intensities at A and C and maximum intensity of q/unit length at B.

Obtain solution in the form of infinite trigonometric series.

9. A clamped beam carries concentrated load P at the centre of span. Assume

$$w(x) = A\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]$$

and obtain Rayleigh solution. Compare deflection at centre with the exact value

$$w(x = L/2) = PL^3/(192\,EI)$$

10. A clamped beam of span L carries uniformly distributed load of q/unit length on entire span. Assume

$$w(x) = A\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]$$

and obtain Rayleigh solution. Compare deflection at centre with exact value of

$$w(x = L/2) = qL^4/(384\,EI)$$

11. A cantilever beam ABC is clamped at A and is free at C. $AB = BC = L/2$. It carries a point load P at free end. The flexural rigidity of AB and BC parts are $2EI$ and EI respectively. Assume

$$w(x) = Ax^2$$

and find deflection at free end using Rayleigh approach.

12. A cantilever AB is clamped at A and is free at B. It carries distributed load which varies linearly from q/unit length at A to zero at B. Assume

$$w(x) = Ax^2 + Bx^3 + Cx^4 + Dx^5$$

and obtain Rayleigh-Ritz solution.

13. A simply supported beam ABC has supports at A and C. $AB = BC = L/2$. Beam supports a point load P at B. The flexural rigidity of AB and BC parts are EI and $2EI$ respectively. Assume

$$w(x) = A_1 \sin\left(\frac{\pi x}{L}\right) + A_2 \sin\left(\frac{2\pi x}{L}\right)$$

and obtain Rayleigh-Ritz solution.

14. A clamped beam of 6 m span carries distributed load of 40 kN/m on entire span. Divide the beam in six equal parts and obtain deflections at nodal points using finite difference method. Use symmetry.

15. A clamped beam ABC of 6 m span carries distributed load of 50 kN/m on entire span. $AB = BC =$ 3 m. The flexural rigidities of AB and BC parts are EI and $2EI$ respectively. Divide the beam in six parts and obtain deflections at nodal points using F. D. method. Use flexural rigidity = 1.5 EI at B.

16. A propped cantilever ABC has prop support at A and is clamped at C. $AB = BC = 3$ m. The cantilever carries distributed load of 60 kN/m on entire span. The flexure rigidity of AB and BC are EI and $2EI$ respectively. Divide the beam in six parts and obtain deflections at nodal points using finite difference method.

17. A simply supported beam of 6 m span carries a point load of 100 kN at the centre of span. Divide the beam in six parts and obtain nodal point deflection using F.D. method. Use symmetry. Replace point load by distributed load of 100 kN/m over a length of 1 m around centre of beam.

18. Governing differential equation of thin plate bending phenomenon is written as

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x, y)}{D}$$

Convert g.d.e. to a difference equation using square mesh ($h \times h$).

ANSWERS TO EXERCISES I

1. (*a*) $w = \frac{1}{EI}\left(-\frac{qLx^3}{12} + \frac{qx^4}{24} + \frac{qL^3x}{24}\right)$

(*b*) $w = \frac{1}{EI}\left(-\frac{qLx^3}{36} + \frac{qx^5}{120L} + \frac{7qL^3x}{360}\right)$

(*c*) $w = \frac{1}{EI}\left(-\frac{qLx^3}{16} + \frac{qx^4}{24} + \frac{qL^3x}{48}\right)$

(*d*) $w = \frac{1}{EI}\left(\frac{qL^2x^2}{4} - \frac{qLx^3}{6} + \frac{qx^4}{24}\right)$

(*e*) $w = \frac{1}{EI}\left(\frac{qL^2x^2}{12} - \frac{qLx^3}{12} + \frac{qx^4}{24} - \frac{qx^5}{120L}\right)$

(*f*) $w = \frac{1}{EI}\left(\frac{qL^2x^2}{60} - \frac{qLx^3}{40} + \frac{qx^5}{120L}\right)$

2. (*a*) $w = \frac{4qL^4}{EI\pi^5}\left[\sum_{m=1,3,} \frac{1}{m^5}\sin\left(\frac{m\pi x}{L}\right)\right] m = 1, 3, 5, 7 \ldots$

(*b*) $w = \frac{L^4}{EI\pi^4}\left[\sum_{m=1,2,3} \frac{q_m}{m^4}\sin\left(\frac{m\pi x}{L}\right)\right]$

where, $q_m = \frac{2q}{m\pi}$ $(m = 1, 3, 5, 7 \ldots)$

$q_m = \frac{-2q}{m\pi}$ $(m = 2, 4, 6 \ldots)$

(*c*) $w = \frac{2PL^3}{EI\pi^4}\left[\sum_{m=1,2} \sin\left(\frac{m\pi a}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\right] m = 1, 2, 3, 4, \ldots$

3. (*a*) $w_1 = 0.006752\, qL^4/(EI)$
$w_2 = 0.011574\, qL^4/(EI)$
$w_3 = 0.013310\, qL^4/(EI)$
(*b*) $w_1 = 0.001461\, qL^4/(EI)$
$w_2 = 0.002466\, qL^4/(EI)$
$w_3 = 0.002687\, qL^4/(EI)$
$w_4 = 0.002054\, qL^4/(EI)$
$w_5 = 0.000882\, qL^4/(EI)$

4. (*b*) $w = \dfrac{4qL^4}{EI\pi^5}\left[\sin\left(\dfrac{\pi x}{L}\right) + \dfrac{1}{243}\sin\left(\dfrac{3\pi x}{L}\right)\right]$

(*b*) $w = \dfrac{PL^3}{EI\pi^4}\left[\sqrt{2}\sin\left(\dfrac{\pi x}{L}\right) + \dfrac{1}{8}\sin\left(\dfrac{2\pi x}{L}\right)\right]$

(*c*) $w = \dfrac{qL^4}{EI\pi^5}\left[2\sin\left(\dfrac{\pi x}{L}\right) + \dfrac{1}{8}\sin\left(\dfrac{2\pi x}{L}\right)\right]$

5. (*a*) $\theta_A = -106.725/(EI)$

$\theta_B = 53.449/(EI)$

$\theta_C = -27.071/(EI)$

$M_{AB} = 0$, $M_{BA} = -79.914$

$M_{BC} = 89.914$, $M_{CB} = -50.347$

$M_{CD} = 50.343$, $M_{DC} = -58.828$

(All moments in kN m units)