Chapter 1

Sets

Contents at a Glance

1.1 INTRODUCTION

The theory of sets was developed by German mathematician Georg Cantor (1845-1918) in the year 1895 while working on *problems on trigonometric series*.

The concept of set serves as a fundamental aspect of present-day mathematics. Sets are used not only in every branch of mathematics but also in other branches of study as well such as computer science. Set are used to define the concepts of relations and functions. The study of other branches of mathematics such as geometry, sequences, probability, etc., also requires sound knowledge of mathematics.

In this chapter, we will be discussing the notations and terminology of set theory, operations on sets and applications of sets..

1.2 SET DEFINED

In our day-to-day life, we have to deal with collections of objects of one kind or the other.

For example

- ❑ Collection of *even natural numbers* less than 12, *i.e.,* of the numbers 2,4,6,8 and 10.
- ❑ Collection of planets, *i.e.,* Sun, Moon, Earth, etc.
- ❑ Collection of *vowels* in the English alphabet, *i.e.,* of the letters *a, e, i, o, u*.
- ❑ Collection of all natural numbers that divide, 12 *i.e.,* the numbers 1,2,3,4,6 and 12.
- ❑ Collection of students who speak a particular language.

In each of the above collections, it is clearly known whether a given object can be included in the collection or not. Each collection is *well-defined*.

However, the collection of all intelligent students in a class of a given school is not a set. Here, it is difficult to decide who is intelligent and who is not. As instance, the same student may be intelligent in the eyes of one teacher and may not be intelligent in the eyes of another. We say that such a collection is *not well defined*.

With this basic notion, we have the following.

> A set is any well-defined (distinguishable) collection of objects.

By *'well-defined collection (distinguishable)'* we mean that given a set and an object, it must be possible to decide whether the object belongs to the set or not.

The objects are called the *members* or the *elements* of the *set*.

Sets are denoted by capital letters such as A, B, C, X, Y, Z, etc, and their corresponding members are denoted by small letters such as a, b, c, x, y, z, etc.

The symbol $\in$ (*epsilon* – a Greek alphabet) is used to indicate, *belong to* (a member or element of).

For example, if x is a member (element) of the set X, then symbolically we write $x \in X$ which is read as x is a member (element) of the set X or x belongs to the set X (or simply as x belongs to X).

The symbol $\notin$ is used to indicate, *does not belong to* (not a member or element of).

Thus, if x is not a member (element) of the set X, then symbolically we write $x \notin X$ which is read as x is not a member (element) of the set X or x does not belongs to the set X (or simply as x does not belongs to X).

Putting together, if x is a member of set X, we write $x \in X$, and if x is not a member the set X, we write $x \notin X$.

Likewise, if a pair of elements, such as x and y, belong to set S, we write $x, y \in S$.

1.3 SOME STANDARD SETS

Following is the list of some standard set which are commonly used in the study of sets.

Letter Used	Description
N	Set of all natural numbers (+ve integer numbers excluding 0).
Z	Set of all integers (–ve as well +ve integer numbers including 0).
Q	Set of rational numbers.
R	Set of real numbers.
Z^+	Set of +ve integers.
Q^+	Set of +ve rational numbers.
R^+	Set of +ve real numbers.

Symbols for these standard sets will be referred to throughout the book.

1.4 REPRESENTATION OF SETS

There are two ways to represent a given set.

(*a*) **Roster or tabular form:** In this form, list all the members of the set, separate these by commas and enclose these within braces (curly brackets).

For example,

- Set S of even natural numbers less than 12 in the tabular form is written as

  $S = \{2, 4, 6, 8, 10\}$.

  Note that $8 \in S$ while $7 \notin S$.

- Set T of prime natural numbers less than 20 in the tabular form is written as

  $T = \{2, 3, 5, 7, 11, 13, 17, 19\}$.

- Set N of natural numbers in the tabular form is written as

  $N = \{1, 2, 3, \ldots\}$

  Three dots (called ellipses) indicating this pattern continues infinitely.

(*b*) **Set builder form:** In this form, write one or more (if necessary) variables (say x, y etc.) representing an arbitrary member(s) of the set, followed by a statement or a property which must be satisfied by each member of the set.

For example,

- Set S of even natural numbers less than 12 in the set builder form is written as

  $S = \{x \mid x \text{ is an even natural number less than } 12\}$.

- Set of prime natural numbers less than 20 in the set builder form is written as

  $S = \{x \mid x \text{ is a prime natural number less than } 20\}$.

  Symbol '|' stands for the words 'such that' or 'where'. Sometimes we use the symbol ';' or ':' in place of the symbol '|'.

- Set N of natural numbers in the set builder form is written as

  $N = \{x \mid x \text{ is a natural number}\}$.

1.5 TYPES OF SETS AND RELATED TERMS

In this section, we will discuss various types of sets.

1.5.1 Empty Set

A set which does not contain any element is called the *empty set*. It is also called the *null set* or *void set*. There is only one such set. It is denoted by ϕ or by {}.

Following are few examples of an empty set.

- Let set $A = \{x : 1 < x < 2, x \text{ is a natural number}\}$. Here the set A is the empty set, because there is no natural number between 1 and 2.
- Let set $B = \{x : x \text{ is an even prime number geater than } 2\}$. Here the set B is the empty set, because 2 is the only even number which is prime.
- Let set $C = \{x : x^2 = 4 \text{ and } x \text{ is odd}\}$. Here the set C is the empty set, because the equation $x^2 = 4$ is not satisfied by an odd value of x.
- Let set $D = \{x : x^2 - 2 = 0 \text{ and } x \text{ is a rational number}\}$. Here the set D is the empty set, because the equation $x^2 - 2 = 0$ is not satisfied by any rational value of x.
- Collection of all girl students in a boy's college is the empty set.
- Collection of all integers whose square is less than 0 is the empty set because square of an integer cannot be less than zero, *i.e.*,, negative.

1.5.2 Finite and Infinite Sets

A set that consists of a definite number of elements is called *finite*; otherwise, the set is called *indefinite*. Note that the *empty* set is always taken as finite with zero cardinality.

Following are few examples of finite sets.

- Let set W be the set of days of week. Here set W is a finite set.
- Let set S = {2, 4, 6, 8}. Here set S is a finite set.
- The set of all students in a given school is a finite set.
- The set of divisors of a given natural number is a finite set.

Following are few examples of infinite sets.

- The set N of all natural numbers is an infinite set.
- The set of all prime number is an infinite set.

1.5.3 Order of a Finite Set

The number of different elements in a finite set S is called *order* or *cardinality* of set S, it is denoted by $O(S)$ or $n(S)$.

The order of the empty set is zero. The set A whose order is 1 is called a *singleton set*. Thus, a singleton set is a set which contains only one distinct element.

Remember, order of an infinite set is not defined.

For example,

- If $A = \{ x \mid x$ is a positive divisor of 20$\}$, then $n(\mathrm{A}) = 6$ as $A = \{1, 2, 4, 5, 10, 20\}$.
- If $B = \{ x : x$ is a positive even prime$\}$, then $n(B) = 1$ as $B = \{2\}$. Note that B is a singleton set.
- If $C = \{ x : x$ is an integer neither positive or negative$\}$, then $n(C) = 1$ as $C = \{0\}$. C is a singleton set.

1.5.4 Equivalent Sets

Two finite sets A and B are said to be equivalent written as $A \sim B$ (or $A \approx B$), if they contain the same number of distinct elements, *i.e.*, $n(A) = n(B)$.

For example,

- Let set $A = \{a, b, c, d, e\}$ and set $B = \{1, 2, 3, 4, 5\}$. Here set A and set B are equivalents sets since $n(A) = n(B)$, *i.e*, each of these sets contains five distinct elements.
- The sets $\{1\}$ and $\{2, 2, 2\}$ are equivalent.
- The sets $\{3, 4\}$ and $\{ x : x^2 = 4 \}$ are equivalent sets.

Remark: Equivalence of infinite sets is beyond the scope of this book.

1.5.5 Equal Sets

Two sets A and B are said to be equal, written as $A = B$, if every member of A is a member of B and every member of B is a member of A.

Remark: Remember that equal sets are always equivalent but equivalent sets may not be equal.

For example,

- The sets $\{-1,1\}$ and $\{x : x^2 = 1\}$ are equal.
- The sets $\{0, 0\}$ and $\{3\}$ are not equal, but they are equivalent.
- The set N of natural numbers and the set $\{x : x \in Z, x > 0\}$ are equal.

ILLUSTRATIVE EXAMPLES 1.1

Example 1.1: Which of the following collection are set? Justify your answer.

(*a*) Collection of handsome boy in a class of a college.
(*b*) Collection of all circles in a given plane.
(*c*) Collection of all natural numbers less than 20.
(*d*) Collection of factors which are divisors of 90.

Solution:

(*a*) This collection is not well defined since for a given object (boy), it is not possible logically to decide which boy is handsome and which boy is not handsome, hence, the given collection is not a set.

(*b*) This collection is a set as it is well defined collection.

(*c*) This collection is well defined since for a given object (number), it is possible to decide with logic whether the given object belong to the set or not. Hence, this collection is a set.

(*d*) This collection is also well defined since for a given object (factor), it is possible logically to decide which it is a divisor of 90 or not, and hence belongs to the set or not. Hence, this collection is a set.

Example 1.2: Following is a list of some sets in the roster form. Write them in the tabular form.

(*a*) $A = \{x : x \text{ is a prime number} < 20\}$

(*b*) $B = \{x : x \text{ is a natural number} < 10\}$

(*c*) $C = \{x : x \text{ is an odd +ve inetegr and } x^2 < 30\}$

(*d*) $D = \{x : x \text{ is letter in the word 'LOVELY'}\}$

Solution:

(a) $A = \{2,3,5,7,11,13,17,19\}$

(b) $B = \{1,2,3,4,5,6,7,8,9\}$

(c) $C = \{1,3,5\}$

(d) $D = \{L,O,V,E,Y\}$

Example 1.3: Following is a list of some sets in the tabular form. Write them in the set builder form.

(a) $A = \{0\}$

(b) $B = \{1,3,9,27\}$

(c) $C = \{1,2,4,16\}$

(d) $D = \{A,E,I,O,U\}$

Solution:

(a) $A = \{x : x \text{ is a whole number but not a natural number}\}$

(b) $B = \{x : x \text{ is a natural number that divided } 27\}$

(c) $C = \{x : x \text{ is a square of a natural number and } x^2 < 16\}$

(d) $D = \{x : x \text{ is a vowel in Elglish alphabet}\}$

Example 1.4: Which of the following sets are empty sets?

(a) $A = \{x : x^2 = 2 \text{ and } x \text{ is a rational number}\}$

(b) $B = \{x : x \text{ is a integer number that is neither positive nor negative}\}$

(c) $C = \{x : x \text{ is a male student in a grils college}\}$

(d) $D = \{x : x \text{ is a real number and } x^2 < 0\}$

Solution:

(a) Since $A = \{x : x^2 = 2 \text{ and } x \text{ is a rational number}\}$

$\Rightarrow \quad A = \{x : x = \sqrt{2} \text{ or } -\sqrt{2} \text{ and } x \text{ is a rational number}\}$

Since $\sqrt{2}$ or $-\sqrt{2}$ are not rational numbers, therefore, set A is empty set.

(b) Since $B = \{x : x \text{ is a integer number that is neither positive nor negative}\}$

$\Rightarrow \quad B = \{0\}$ $\quad [\because$ 0 is an integer which is neithet +ve not -ve $]$

Hence, set A is non-empty set.

(c) $C = \{x : x \text{ is a male student in a grils college}\}$

$\Rightarrow$ set C is an empty set since there can't be a male student in a girls college.

(d) $D = \{x : x \text{ is a real number and } x^2 < 0\}$

$\Rightarrow$ set D is an empty set since square of a real number cannot be negative.

Example 1.5: Which of the following sets are finite or infinite?

(a) $A = \{x : x \in N \text{ and } x \text{ is prime}\}$
(b) $B = \{x : x \in N \text{ and } x \text{ is multiple of } 3\}$
(c) $C = \{x : x \text{ is a real number between 1 and 2}\}$
(d) $D = \{x : x \text{ is a root of quadratic equation } x^2 + 4x + 4 = 0\}$

Solution:

(a) Set A is infinite as there are infinite prime number and no greatest prime number.

(b) $B = \{x : x \in N \text{ and } x \text{ is multiple of } 3\} = \{3, 6, 9, 12, ...\}$ $B = \{x : x = 3n,\ n \in N\}$

Obviously, set B is infinite as there is no greatest natural number.

(c) Set C is infinite as there are infinite real numbers between any two given real numbers.

(d) Set D is finite since a quadratic equation has only two roots.

Example 1.6: In the following pair of sets, find whether $A = B$ or $A \neq B$.

(a) $A = \{a, b, c, d, e\}$ and
$B = \{x : x \text{ is one of the first five letters of the English alphabet}\}$

(b) $A = \{0\}$ and $B = \{x : x \text{ is a natural number number less than } 1\}$

(c) $A = \{1, 5, 25, 125\}$ and $B = \{x : x \text{ is a positive divisor of } 125\}$

(d) $A = \{x : x \text{ is a multiple of } 10\}$ $B = \{10, 15, 20, 25, 30, ...\}$

Solution:

(a) Here $A = B$ as the first five letters of the English alphabet are a, b, c, d, e.

(b) Here $A \neq B$ since $A = \{0\}$ and $B = \{\ \}$. Note that there is no natural number less than 1.

(c) Here $A = B$ since 1, 5, 25, 125 are the only positive divisors of 125.

(d) Here $A \neq B$ since $A = \{x : x \text{ is a multiple of } 10\} = \{10, 20, 30, 40, ...\}$ and 15, 25 $\in B$ but 15, 25 $\notin A$.

Example 1.7: From the following list of sets, select pair of equal and equivalent set:

$$A = \{0, 1\},\ B = \{x : x^2 = 25\},\ C = \{x : x^2 = x\},\ D = \{1, 2, 3, 4\},$$
$$E = \{3, 5, 7, 11\},\ F = \{x : x^2 = 1\},\ G = \{g, h, i, j\},\ H = \{3, 5, 7\}$$

Solution:

Let us rewrite the set as shown below:

$$A = \{0, 1\},\ B = \{-5, 5\},\ C = \{0, 1\},\ D = \{1, 2, 3, 4\},$$
$$E = \{3, 5, 7, 11\},\ F = \{-1, 1\},\ G = \{g, h, i, j\},\ H = \{3, 5, 7\}$$

Since A, B, C and F contain equal number (two) of elements, therefore, they are equivalent sets, i.e.,

$$A \sim B,\ A \sim C,\ A \sim F,\ B \sim C,\ B \sim F,\ C \sim F$$

Since D, E and G contain equal number (four) of elements, therefore, they are equivalent sets, i.e.,

$$D \sim E,\ \ D \sim G,\ \ F \sim G$$

Set H is not equivalent to any set.

Set A and C are equal as they exactly contain the same element, i.e.,

$$A = C$$

EXERCISE 1.1

1. Which of the following are sets? Justify your answer.
   (*a*) Collection of beautiful girls in a college.
   (*b*) Collection of difficult problems in a book.
   (*c*) Collection of all odd numbers.
   (*d*) Collection of month in a year beginning with letter A.
   (*e*) Collection of natural numbers less than 50.
2. Following sets are given in the set builder form. Write them in the tabular form:
   (*a*) $A = \{x : x \text{ is an integer and } -3 \le x < 7\}$
   (*b*) $A = \{x : x = \dfrac{n}{n+1}, n \text{ is a natural number less than } 5\}$
   (*c*) $A = \{x : x \text{ is a letter in the word 'INDIA'}\}$
   (*d*) $A = \{x : x \text{ is a two digit number whose sum of digits is } 9\}$
   (*e*) $A = \{x : x \text{ is an integer and } x^2 \le 4\}$
3. Following sets are given in the tabular form. Write them in the set builder form:
   (*a*) $A = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
   (*b*) $A = \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}\right\}$
   (*c*) $A = \{\text{I,N,D,A}\}$
   (*d*) $A = \{18, 27, 36, 45, 54, 63, 72, 81, 90\}$
   (*e*) $A = \{-2, -1, 0, -, 1, 2\}$
4. Which of the following sets is an empty set?
   (*a*) $A = \{x : x \text{ is a real number which is neither rational not irrational}\}$
   (*b*) $B = \{x : x \text{ is a whole number less than } 1\}$
   (*c*) $C = \{x : x \text{ is a natural number and } x^2 < 1\}$
   (*d*) $D = \{x : x \text{ is a month of a year containing 30 days}\}$

5. Which of the following sets are finite?
   (*a*) $A = \{x : x$ is an odd number$\}$
   (*b*) $B = \{x : x$ is a whole number less than $1\}$
   (*c*) $C = \{x : x$ is a natural number and $x^2 < 1\}$
   (*d*) $D = \{x : x$ is a natural number greater than $7\}$
6. Which of the following sets are equivalent?
   $A = \{x : x$ is a vowel in English alphabet$\}$
   $B = \{x : x$ is a root of the equation $x^2 - 3x = 0\}$
   $C = \{x : x$ is an even positive integer less than $11\}$
   $D = \{x : x$ is a letter in the word 'SET'$\}$
   $E = \{x : x$ is a digit in the number $1000\}$
   $F = \{x : x$ is a positive divisor of $4\}$
7. Which of the following statements are true?
   (*a*) $\phi \subset \{\}$
   (*b*) $0 \in \{\{0\}\}$
   (*c*) $0 \notin \{\}$
   (*d*) $A \subset A$
   (*e*) $0 \notin \{0, \{0\}\}$

ANSWERS

1. (*c*), (*d*), (*e*)
2. (*a*) $A = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
   (*b*) $A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$ (*c*) $A = \{\text{I,N,D,A}\}$
   (*d*) $A = \{18, 27, 36, 45, 54, 63, 72, 81, 90\}$ (*e*) $A = \{-2, -1, 0, -, 1, 2\}$
3. (*a*) $A = \{x : x$ is an integer and $-3 \le x < 7\}$
   (*b*) $A = \{x : x = \dfrac{n}{n+1}$, n is a natural number less than $5\}$
   (*c*) $A = \{x : x$ is a letter in the word 'INDA'$\}$
   (*d*) $A = \{x : x$ is a two digit number whose sum of digits is $9\}$
   (*e*) $A = \{x : x$ is an integer and $x^2 \le 4\}$
4. *A*, *C*
5. *B*, *C*
6. A and *C*, *B* and *E*, *D* and *F*
7. (*a*), (*c*), (*d*)

1.5.6 Subset and Superset

Let A, B be two sets such that every member of A is a member of B, then A is called a *subset* of B, it is written as $A \subset B$.

If A is a subset of B, we say that A is contained in B or B contains A, it is written as $B \supset A$ and we say that B is a *superset* of A.

Thus, $A \subset B$ iff (read as 'if and only if') $x \in A \quad \Rightarrow \quad x \in B$.

If $\exists$ (read as 'there exists') atleast one element in A which is not a member of B, then A is not a subset of B and we write $A \not\subset B$.

For example,

- Let A = {-1,2,5} and B = {3,-1,2,7,5}, then $A \subset B$. note that $B \not\subset A$.
- The set of all even natural numbers is a subset of the set of natural numbers.
- $N \subset W \subset Z \subset Q \subset R$

Remark:

- Two sets A and B are equal iff $A \subset B$ and $B \subset A$.
- Since every element of a set A belongs to A, it follows that every set is a subset of itself.
- Further, note that the empty set is also a subset of every set.

1.5.7 Proper Subset

Let A be a subset of B. We say that A is a *proper subset* of B *if* $A \neq B$, i.e., if there exist atleast one element in B which does not belong to A. Subset A, which is not proper, is called an *improper subset*.

Remark: Every set is an improper subset of itself. If a set A is non-empty, then the null set is a proper subset of A.

1.5.8 Power set

The set formed by all the subsets of a given set A is called the *power set* of A, it is usually denoted by $P(A)$ or by 2^A.

For example,

- Let $A = \{0\}$, then $P(A) = \{\phi, \{0\}\}$. Note that $n(P(A)) = 2 = 2^1$.
- Let $A = \{a, b\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Note that $n(P(A)) = 4 = 2^2$.
- Let $A = \{1, 2, 3\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Note that $n(P(A)) = 8 = 2^3$.

In all these examples, we have observed that $n(P(A)) = 2^{n(A)}$.

Rule to write down the power set of a finite set of *A*:

- First of all write ϕ.
- Next, write down singleton subsets each containing only one element of *A*.
- In the next step write all the subsets which contain two elements from the set *A*.
- Continue this way and in the end write *A* itself as *A* is also a subset of *A*.
- Enclose all these subsets in braces to get the power set of *A*.

1.5.9 Comparable Sets

Two sets *A* and *B* are said to be comparable iff either $A \subset B$ or $B \subset A$.

For example,

- The sets $A = \{1, 2\}$ and $B = \{1, 2, 4, 5\}$ are comparable as $A \subset B$.
- The sets $A = \{0, 1, 3\}$ and $B = \{1, 3\}$ are comparable as $B \subset A$.
- The sets $A = \{-1, 1\}$ and $B = \{x : x^2 = 1\}$ are comparable as $A \subset B$ and also $B \subset A$.

Remark: Equal sets are always comparable. However, comparable sets may not be equal.

1.5.10 Universal Set

In any application of the theory of sets, all sets under investigation are regarded as subsets of a fixed set. We call this set *the universal set,* it is usually denoted by *X or U or* ξ.

For example,

(*a*) When in two dimensional geometry we discuss lines or triangles or circles, then the universal set may be the plane in which the lines or circles lie.

(*b*) When we discuss integers or positive integers or primes numbers, then the universal set may be *Z* (the set of all integers).

1.6 CARTESIAN PRODUCT

Many of the discrete structures that we will be discussing in later chapters are based on the notion of the *cartesian product* of sets.

Cartesian product is a mode by which two or more sets can be combined to obtain another one.

1.6.1 Cartesian Product of two Sets

If A and B are two sets, then for $a \in A, b \in B, (a,b)$ is called an *ordered pair* with a as the first component and b as the second component. Two ordered pairs (a,b) and (c,d) are said to be equal written as $(a,b) = (c,d)$ if and only if $a = c$ and $c = d$.

The set of all the ordered pairs of the type $(a,b), a \in A, b \in B$ is called the *Cartesian product* of A and B (in this very order)) and is written as $A \times B$.

Thus, $A \times B = \{(a,b) : a \in A, b \in B\}$.

For example,

If $A = \{1,2,3\}$ and $B = \{3,4\}$, then

$A \times B = \{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$ and

$B \times A = \{(3,1),(3,2),(3,3),(4,1),(4,2),(4,3)\}$.

Note: Cartesian product $A \times B$ is not always equal to Cartesian product $B \times A$. This can happen only if the both the sets are equal.

1.6.2 Cartesian Product of more than two Sets

The above concept of Cartesian product of two sets can be extended to Cartesian product of any number of sets.

If $A_1, A_2, A_3, ..., A_k$ are k given sets, then their Cartesian product, denoted by $\prod_{i=1}^{k} A_i$, is defined as $\{(a_1, a_2, a_3, ..., a_k);$ where $\forall\ i,\ a_i \in A_i\ \}$.

Further, if $A_1, A_2, A_3, ..., A_k$ are all finite sets, then $A_1 \times A_2 \times A_3 \times ... \times A_k$ is finite, and

$$n(A_1 \times A_2 \times A_3 \times ... \times A_k) = n(A_1) \times n(A_2) \times n(A_3) \times ... \times n(A_k)$$

Also, we observe that $A_1 \times A_2 \times A_3 \times ... \times A_k$ is infinite iff atleast one of $A_1, A_2, A_3, ..., A_k$ is an infinite set.

With this basic idea, A^k, *i.e.*, $A \times A \times ... \times A$ is defined as the set k times

$$\{(a_1, a_2, a_3, ..., a_k) : a_i \in A_i \text{ for each } i = 1,2,3,\ldots,k\}$$

k being any positive integer.

For example,

- If $A = \{1,2\}$, $B = \{3,4\}$ and $C = \{5,6\}$, then

$$A \times B \times C = \{(x, y, z);\ x \in A, y \in B, z \in C\}$$
$$= \{(1,3,5), (1,3,6), (1,4,5), (1,4,6), (2,3,5), (2,3,6), (2,4,5), (2,4,6)\}$$

- If $A = \{a, b\}$, then

$$A \times A \times A = \{(x, y, z) : x, y, z \in A\}$$
$$= \{(a, a, a), (a, a, b), (a, b, a), (a, b, b), (b, a, a), (b, a, b), (b, b, a), (b, b, b)\}$$

1.6.3 Important Results of Cartesian Product

For two non-empty sets A and B, the following results holds (which can be extended to any number of non-empty sets):

- When $B = A$, then $A \times B = A \times A$ is the set of all ordered pairs of the type $(a,b), a \in A, b \in A$, i.e, $A \times A = \{(a,b) : a, b \in A\}$. $A \times A$ is usually written as A^2.
- When $A \neq B$, then $A \times B \neq B \times A$, i.e., $A \times B = B \times A$ only when $A = B$.
- If $A = \phi$ and/or $B = \phi$ then $\mathrm{A} \times \mathrm{B}$ does not contain any ordered pair, i.e., $A \times B$ is empty set.
- If both A and B are finite sets, then $A \times B$ is also a finite set and $n(A \times B) = n(\mathrm{A}) \times n(B)$.
- If A and/or B is an infinite set, then $A \times B$ is also an infinite set.

1.6.4 Properties of Cartesian Product

For two non-empty sets A and B, the following properties hold (which can be extended to any number of non-empty sets):

Property 1: If A, B and C are sets and $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$.

Proof: Let (a, c) be an arbitrary element of $A \times C$, then

$$\begin{aligned}(a,c) \in A \times C \;&\Rightarrow\; a \in A \text{ and } c \in C \\ &\Rightarrow\; a \in B \text{ and } c \in C \qquad [\text{since } A \subseteq B] \\ &\Rightarrow (a,c) \in B \times C\end{aligned}$$

Thus, $A \times C \subseteq B \times C$.

Property 2: If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subseteq (B \times D)$.

Proof: Let (a, c) be an arbitrary element of $A \times C$, then

$$\begin{aligned}(a,c) \in A \times C \;&\Rightarrow\; a \in A \text{ and } c \in C \\ &\Rightarrow\; a \in B \text{ and } c \in D \qquad [\text{since } A \subseteq B \text{ and } \mathrm{C} \subseteq \mathrm{D}] \\ &\Rightarrow (a,c) \in B \times D\end{aligned}$$

Thus, $A \times C \subseteq B \times D$.

Property 3: If A, B and C are sets, then $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof: Let (x, y) be an arbitrary element of $A \times (B \cap C)$, then

$$\begin{aligned}(x,y) \in A \times (B \cap C) \;&\Leftrightarrow\; x \in A \text{ and } y \in (B \cap C) \\ &\Leftrightarrow\; x \in A \text{ and } (y \in B \text{ and } y \in C) \\ &\Leftrightarrow\; (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\ &\Leftrightarrow\; (x,y) \in (A \times B) \text{ and } (x,y) \in (A \times C) \\ &\Leftrightarrow\; (x,y) \in (A \times B) \cap (A \times C)\end{aligned}$$

Thus, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Property 4: If $A \subseteq B$, then $A \times A = (A \times B) \cap (B \times A)$.

Proof: Let (x, y) be an arbitrary element of the set $A \times A$, then

$$
\begin{aligned}
(x, y) \in A \times A &\Leftrightarrow x \in A \text{ and } y \in A. \\
&\Leftrightarrow (x \in A \text{ and } y \in A) \text{ and } (x \in A \text{ and } y \in A) \\
&\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in B \text{ and } y \in A) \qquad [\text{since } A \subseteq B] \\
&\Leftrightarrow (x, y) \in A \times B \text{ and } (x, y) \in B \times A \\
&\Leftrightarrow (x, y) \in (A \times B) \cap (B \times A)
\end{aligned}
$$

Hence, if $A \subseteq B$, $A \times A = (A \times B) \cap (B \times A)$.

Property 5: $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof: Let (x, y) be an arbitrary element of $A \times (B \cup C)$. Then

$$
\begin{aligned}
(x, y) \in A \times (B \cup C) &\Leftrightarrow x \in A \text{ and } y \in (B \cup C) \\
&\Leftrightarrow x \in A \text{ and } (y \in B \text{ or } y \in C) \\
&\Leftrightarrow (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \\
&\Leftrightarrow (x, y) \in A \times B \text{ or } (x, y) \in A \times C \\
&\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)
\end{aligned}
$$

Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Property 6: $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$.

Proof: Let (x, y) be an arbitrary element of $(A \times B) \cap (S \times T)$, then

$$
\begin{aligned}
(x, y) \in (A \times B) \cap (S \times T) &\Leftrightarrow (x, y) \in (A \times B) \text{ and } (x, y) \in (S \times T) \\
&\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in S \text{ and } y \in T) \\
&\Leftrightarrow (x \in A \text{ and } x \in S) \text{ and } (y \in B \text{ and } y \in T) \\
&\Leftrightarrow x \in (A \cap S) \text{ and } y \in (B \cap T) \\
&\Leftrightarrow (x, y) \in (A \cap S) \times (B \cap T)
\end{aligned}
$$

Hence, $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$.

Property 7: For any three set A, B, and C, $A\times(B-C)=A\times B-A\times C$

Proof: Let (x,y) be an arbitrary element of $A\times(B-C)$, then

$$\begin{aligned}
(x,y)\in A\times(B-C) &\Leftrightarrow x\in A \text{ and } y\in(B-C)\\
&\Leftrightarrow x\in A \text{ and } (y\in B \text{ and } y\notin C)\\
&\Leftrightarrow (y\in A \text{ and } y\in B) \text{ and } (x\in A \text{ and } y\notin C)\\
&\Leftrightarrow (x,y)\in A\times B \text{ and } (x,y)\notin A\times C\\
&\Leftrightarrow (x,y)\in (A\times B-A\times C)
\end{aligned}$$

Hence, $A\times(B-C)=A\times B-A\times C$

Property 8: For a non-empty sets A and B, $A\times B=B\times A\Leftrightarrow A=B$.

Proof: Let $A=B$. Then $A\times B=A\times A$ and $B\times A=A\times A$. This shows that $A\times B=B\times A$.

Conversely, let $A\times B=B\times A$. Then, let

$$\begin{aligned}
x\in A &\Rightarrow (x,y)\in A\times B\\
&\Rightarrow (x,y)\in B\times A \qquad [\text{Since } A\times B=B\times A]\\
&\Rightarrow x\in B
\end{aligned}$$

This means, $A\subseteq B$. Similarly $B\subseteq A$. Consequently, $A=B$.

Property 9: If A, B are two non-empty sets having n elements in common, then $A\times B$ and $B\times A$ have n^2 elements in common.

Proof: Let $C=A\cap B$, then assume that $C\times C=(A\times B)\cap(B\times A)$

Also
$$\begin{aligned}
(x,y)\in C\times C &\Leftrightarrow x\in C \text{ and } y\in C\\
&\Leftrightarrow x\in A\cap B \text{ and } y\in A\cap B\\
&\Leftrightarrow (x\in A \text{ and } x\in B) \text{ and } (y\in A \text{ and } y\in B)\\
&\Leftrightarrow (x\in A \text{ and } y\in B) \text{ and } (x\in B \text{ and } y\in A)\\
&\Leftrightarrow (x,y)\in A\times B \text{ and } (x,y)\in B\times A\\
&\Leftrightarrow (x,y)\in (A\times B)\cap(B\times A)
\end{aligned}$$

Hence, $C \times C = (A \times B) \cap (B \times A)$

Since $C \times C$ has n^2 elements, therefore, $(A \times B) \cap (B \times A)$ also has n^2 elements. This implies that both $A \times B$ and $B \times A$ have n^2 elements in common.

ILLUSTRATIVE EXAMPLES 1.2

Example 1.8: If $A = \{1,4\}$, $B = \{2,3\}$, and $C = \{3,5\}$. Prove that $A \times B \neq B \times A$. Also find

$(A \times B) \cap (A \times C)$.

Solution:

$A \times B = \{1,4\} \times \{2,3\} = \{(1,2),(1,3),(4,2),(4,3)\}$

$B \times A = \{2,3\} \times \{1,4\} = \{(2,1),(2,4),(3,1),(3,4)\}$

Clearly, $A \times B \neq B \times A$.

Now $A \times C = \{1,4\} \times \{3,5\} = \{(1,3),(1,5),(4,3),(4,5)\}$

Then $(A \times B) \cap (A \times C) = \{(1,3),(4,3)\}$

Example 1.9: If $A = \{1,2,3\}$, $B = \{3,4\}$, and $C = \{4,5,6\}$, find

(a) $A \times (B \cup C)$ (b) $A \times (B \cap C)$ (c) $(A \times B) \cap (B \times C)$

Solution:

(a) $B \cup C = \{3,4\} \cup \{4,5,6\} = \{3,4,5,6\}$

$A \times (B \cup C) = \{1,2,3\} \times \{3,4,5,6\}$

$= \{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6)\}$

(b) $B \cap C = \{3,4\} \cap \{4,5,6\} = \{4\}$

$A \times (B \cap C) = \{1,2,3\} \times \{4\} = \{(1,4),(2,4),(3,4)\}$

(c) $A \times B = \{1,2,3\} \times \{3,4\} = \{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$

$B \times C = \{3,4\} \times \{4,5,6\} = \{(3,4),(3,5),(3,6),(4,4),(4,5),(4,6)\}$

$(A \times B) \cap (B \times C) = \{(3,4)\}$

EXERCISE 1.2

1. If $A=\{0,1\}$ and $B=\{0,-1\}$ are two sets, find $A\times B$ and $B\times A$. Also show that $A\times B\neq B\times A$.
2. Prove that if A, B and C are three sets and $A\subseteq B$, then $A\times C\subseteq B\times C$.
3. If $A\subseteq B$, then show that $A\times A=(A\times B)\cap(B\times A)$.
4. If A, B and C are any three sets, then prove that $A\times(B\cup C)=(A\times B)\cup(A\times C)$.
5. Find the value of x and y if the following ordered pair are equal.
   (a) $(x+1, y+5)=(0,5)$ (b) $(2x-1,-5)=(x, y+1)$
6. If $A=\{1,2,3\}$, $B=\{2,3,4\}$, $S=\{1,3,4\}$ and $T=\{2,4,5\}$, verify that
   $(A\times B)\cap(S\times T)=(A\cap S)\times(B\cap T)$
7. If R is the set of real numbers, what is represented by R^2 and R^3?
8. Prove that $(A-B)\times C=(A\times C)-(B\times C)$.
9. Prove that
   (a) $(A\cap B)\times C=(A\times C)\cap(B\times C)$ (b) $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$
10. If $A=\{1,4\}$, $B=\{4,5\}$ and $C=\{5,7\}$, find
    (a) $(A\times B)\cup(B\times C)$ (b) $(A\times B)\cap(B\times C)$
11. If A and B are any two having two elements in common. If $n(A)=5$ and $n(A)=3$, find $n\big((A\times B)\cap(B\times A)\big)$.

ANSWERS

1. $A\times B=\{(0,0),(0,-1),(1,0),(1,-1)\}$, $B\times A=\{(0,0),(0,1),(-1,0),(-1,1)\}$
   Clearly, $A\times B\neq B\times A$
5. (a) $x=-1,\ y=0$ (b) $x=1,\ y=-6$
10. (a) $(A\times B)\cup(B\times C)=\{(1,4),(1,5),(4,4),(4,5),(4,7),(5,5),(5,7)\}$
    (b) $(A\times B)\cap(B\times C)=\{(4,5)\}$
11. $n(A\times B)=n(A)\times n(B)=5\times 3=15$. As A and B have 2 elements in common, hence $A\times B$ and $B\times A$ have 2^2 elements in common. Hence, $n\big((A\times B)\cap(B\times A)\big)=4$.

1.7 VENN DIAGRAM

Many of the relations (properties) of sets can be verified with the help of diagrams known as *Venn diagrams* or *Venn-Euler diagrams.*

A Venn diagram is a pictorial representation of set in which set is represented by enclosed areas in the plane. The universal set U is represented by the interior of a rectangle and the other sets by circles lining within the rectangle.

For example,

- Let A and B be two sets such that $A \subset B$. This relation (property) is represented by Venn diagram as shown in Figure 1.1.
- Let A and B be two sets such that $A \cap B = \phi$, i.e., sets are disjoint. This relation (property) is represented by Venn diagram as shown in Figure 1.2.

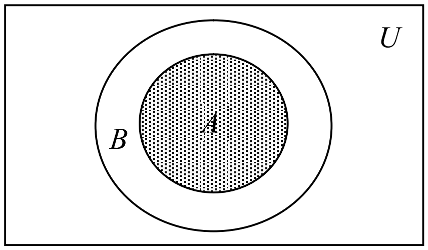


Figure 1.1: $A \subset B$ (shaded area)

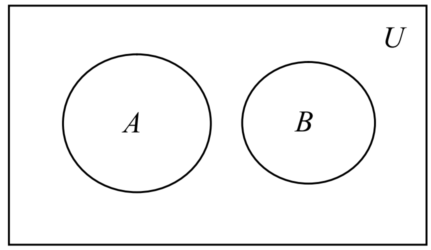


Figure 1.2: $A \cap B = \phi$

1.8 INTERVALS AS SUBSET OF R

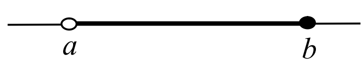
An interval I is a subset of R (the set of real numbers) such that if $x, y \in I$ and z is any real number between x and y then $z \in I$.

Thus, we find that any real number lying between two different elements of an interval must be contained in the interval. If $a, b \in R$ and $a < b$, then we have the following types of intervals.

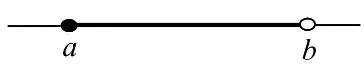
(a) The set $\{x \in R; a < x < b\}$ is called an open interval and is denoted by (a, b). On the number line it is shown as

(b) The set $\{x \in R; a \leq x \leq b\}$ is called a closed interval and is denoted by $[a, b]$. On the number line it is shown as

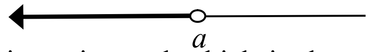
(c) The set $\{x \in R; a < x \leq b\}$ is an interval, open on the left and closed on the right. It is denoted by $(a, b]$. On the number line it is shown as



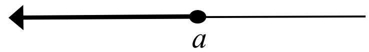
(d) The set $\{x \in R; a \leq x < b\}$ is an interval, closed on the left and open on the right. It is denoted by $[a, b)$. On the number line it is shown as



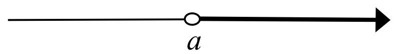
(e) The set $\{x \in R; x < a\}$ is an interval which is denoted by $(-\infty, a)$. It is open on both sides. On the number line it is shown as



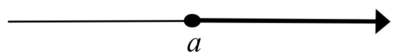
(f) The set $\{x \in R; x \leq a\}$ is an interval which is denoted by $(-\infty, a]$. It is closed on right. On the number line it is shown as



(g) The set $\{x \in R; x > a\}$ is an interval which is denoted by (a, ∞). It is closed on both sides. On the number line it is shown as



(h) The set $\{x \in R; x \geq a\}$ is an interval which is denoted by $[a, \infty)$. It is closed on the left. On the number line it is shown as



The first four intervals, i.e., from (a) – (d), are called *finite intervals*, and the number $a - b$ (which is always positive) is called the *length* (also called *range*) of these four intervals (a,b), $[a,b]$, $(a,b]$ and $[a,b)$.

The last four intervals, i.e., from (e) – (h), are called *infinite intervals*, and the *length* of these four intervals is not defined.

Most often the set R itself is written as an interval $(-\infty, \infty)$.

Note: Remember that each interval (finite or infinite) is an infinite set containing infinitely many rational and infinitely may irrational numbers.

ILLUSTRATIVE EXAMPLES 1.3

Example 1.10: Write all the subsets of the set {}. Also write down its power set.

Solution: Let A = {}, then the only subset of A is A itself *i.e.* {}.

Hence $P(A) = \{A\}$ or {{ }} or $\{\phi\}$.

Example 1.11: Write down the power set for the set $A = \{1, a\}$.

Solution: First, we note that $\phi \subset A$.

Next, we write down those subsets of A which contain only one element. These are {1} and $\{a\}$.

Further, we note that A itself is a subset of A.

Hence $P(A) = \{\phi, \{1\}, \{a\}, A\}$ is the required power set of A.

Example 1.12: Write down all the proper subsets of the set $\{1, 2, 3, 4\}$.

Solution: Let $A = \{1, 2, 3, 4\}$. Note that A contains four distinct elements.

First, we observe that empty set is a subset of every set.

Next, we write down the subsets of A containing only one element; they are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$.

Now, we write down the subsets of A containing exactly two elements: these are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$.

Further, we list the subsets of A containing exactly three elements: these are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$.

Hence, all the proper subsets of the set A are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$.

Though $\{1, 2, 3, 4\}$ is a subset of A, but it is an improper subset.

Example 1.13: Let $A = \{2, 3, 4, 5\}$, $B = \{2, 3, 4\}$ and $C = \{2, 3, 5\}$. Find all the sets X satisfying

(*a*) $X \subset A,\ X \not\subset B$ (*b*) $X \subset B,\ X \neq B,\ X \not\subset C$

Solution:

(*a*) Any one of the sets $\{5\}$, $\{2, 5\}$, $\{3, 5\}$, $\{4, 5\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$ and A can be taken as X as each of these is a subset of A but not a subset of B.

(*b*) Any one of the sets $\{4\}$, $\{2, 4\}$, $\{3, 4\}$ are subsets of B which satisfy all the required properties. Hence any one of these can be taken as X.

Example 1.14: Which of the following statements are true and which false?

(*a*) $0 \subset \{0\}$ (*b*) $\{0\} \subset \{0\}$ (*c*) $\{0\} \subset \{\{0\}\}$ (*d*) $\{0\} \in \{\{0\}\}$
(*e*) $\{0\} \in \{0, \{0\}\}$ (*f*) $\{0\} \subset \{0, \{0\}\}$ (*g*) $\{\} \subset \{\{0\}, 0\}$.

Solution:

(a) As 0 is not a set and '$\subset$' is a relation between two sets, therefore, the given statement is *false*.

(b) Since every set is a subset of itself, therefore, the given statement is *true*.

(c) Let A = {0} and B = {{0}}, then the only element 0 of A is not contained in B. So, $A \not\subset B$. Hence the given statement is *false*.

(d) As {0} is an element of the set {{0}}, therefore, the given statement is *true*.

(e) As {0} is contained in the set $\{0,\{0\}\}$, therefore, the given statement is *true*.

(f) Let A = {0} and B = {0, {0}}. Since the only element 0 of A is contained in B, therefore, $A \subset B$ is true. Hence, the given statement is *true*.

(d) Since empty set is a subset of every set, therefore, the given statement is *true*.

Example 1.15: Write each of the following sets as intervals:

(a) $\{x \in R : -1 < x \leq 2\}$
(b) $\{x \in R : 1 \geq 2x - 3 \geq 0\}$
(c) $\left\{x \in R : \frac{3}{2} < 3x + 2 < 5\right\}$

Solution:

(a) Let $A = \{x \in R : -1 < x \leq 2\} = (-1,\ 2]$

(b) Let $A = \{x \in R : 1 \geq 2x - 3 \geq 0\} = \{x \in R : 4 \geq 2x \geq 3\}$

$$= \left\{x \in R : 2 \geq x \geq \frac{3}{2}\right\} = \left\{x \in R : \frac{3}{2} \leq x \leq 2\right\} = \left[\frac{3}{2}, 2\right]$$

(c) Let $A = \left\{x \in R : \frac{3}{2} < 3x + 2 < 5\right\} = \left\{x \in R : \frac{3}{2} - 2 < 3x < 5 - 2\right\}$

$$= \left\{x \in R : -\frac{1}{2} < 3x < 3\right\} = \left\{x \in R : -\frac{1}{6} < x < 1\right\} = \left(-\frac{1}{6}, 1\right)$$

EXERCISE 1.3

1. If $A = \{\}$, write $P(A)$ and hence find $n(P(A))$.
2. Write $P(A)$ for $A = \{b, c\}$.
3. Write down all the proper subsets of $A = \{q, b, c, d\}$.
4. If $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3\}$ and $C = \{1, 2, 5\}$. Find all the sets X satisfying following:
   (*a*) $X \subset A,\ X \not\subset B$
   (*b*) $X \subset A,\ X \subset B, X \subset C$
5. Which of the following statements are true?
   (*a*) Every not empty set has at least two subsets.
   (*b*) Two sets are equivalent if they have same number of elements.
   (*c*) Let *A* and *B* be two disjoint sets then $A \cap B \neq \phi$.
   (*d*) Empty set is a subset of every set.
   (*e*) Let *A* and *B* be two set such that $n(A) = n(B)$, then $A = B$.
6. Write the following sets in the interval form:
   (*a*) $A = \{x \in R : -8 < x < 3\}$
   (*b*) $A = \{x \in R : 3 \leq 2x < 7\}$
   (*c*) $A = \{x \in R : -1 < 4x - 1 \leq 9\}$

ANSWERS

1. $P(A) = \{\{\}\}$, $n(P(A)) = 1$
2. $P(A) = \{\phi, \{b\}, \{c\}, A\}$
3. $\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$, $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$
4. (*a*) {5}, {1,5}, {4,5}, {3,5},{1,2,5}, {1,3,5}, {2,3,5}, {1,2,3,5},
   (*b*) {}, {1}, {2}, {1,2}
5. (*a*), (*b*), (*d*)
6. (*a*) $(-8,\ 3)$ (*b*) $\left[\dfrac{3}{2},\ \dfrac{7}{2}\right)$ (*c*) $\left(0,\ \dfrac{5}{2}\right]$

1.9 OPERATIONS ON SETS

1.9.1 Union of two Sets

Let *A*, *B* be any two sets, then the set consisting of elements which belong to *A* or to *B* (or to both) is called the union of *A* and *B*. It is written as $A \cup B$ (read as *A* union *B*).

Symbolically, $A \cup B = \{x; x \in A \text{ or } x \in B\}$.

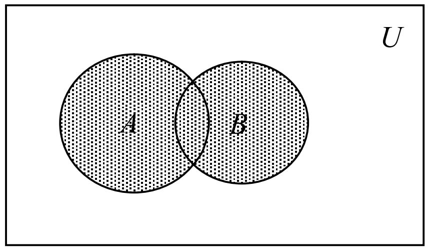


Figure 1.3: $A \cup B$ (shaded area)

It is worthwhile to note that $x \in A$ or $x \in B$ also includes the possibility of x belonging to both A and B. Further, if x belongs to both A and B, it should be taken as the member of the union set only once, since we know that the repetition of the members of a set does not alter it.

For example,

- If $A = \{0, 2, 3, 5\}$ and $B = \{-1, 2, 3, 7, 9\}$ then

  $A \cup B = \{0, 2, 3, 4, -1, 7, 9\}$

- If $A = \{1, 3, 5, 7, ...\}$ and $B = \{2, 4, 6, 8, ...\}$, then

  $A \cup B = \{1, 2, 3, 4, 5, 6, ...\}$

1.9.2 Intersection of two Sets

Let A, B be any two sets, then the set consisting of the elements which belong to both A and B is called the intersection of A and B. It is written as $A \cap B$ (read as A intersection B).

Symbolically, $A \cap B = \{x; x \in A \text{ and } x \in B\}$.

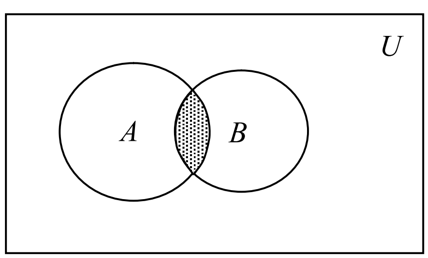


Figure 1.4: $A \cap B$ (shaded area)

For example,

- If $A = \{0, 2, 3, 5\}$ and $B = \{-1, 2, 3, 7, 9\}$ then

  $A \cap B = \{2, 3\}$

- If $A = \{1, 3, 5, 7, ...\}$ and $B = \{2, 4, 6, 8, ...\}$, then

  $A \cap B = \{\} = \phi$

1.9.3 Disjoint Sets

When two sets have no common element, they are called *disjoint sets*.

Thus, if A and B are any two sets such that $A \cap B = \phi$, then we say set A and set B are disjoint.

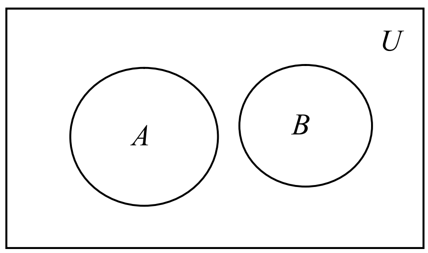


Figure 1.5: $A \cap B = \phi$

For example,

If $A = \{1, 3, 5, ...\}$ and $B = \{2, 4, 6, ...\}$, then

$$A \cap B = \phi$$

Thus, A and B are disjoint sets.

1.9.4 Difference of two Sets

Let A, B be any two sets, then difference $A - B$ is set consisting of the elements which belong to A but do not belong to B.

Symbolically,

$$A - B = \{x; x \in A \text{ and } x \notin B\}$$

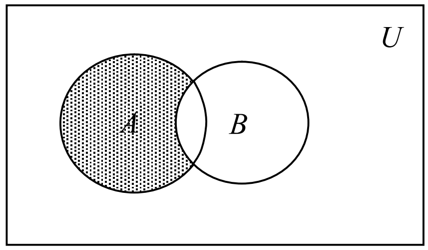


Figure 1.6: $A-B$ (shaded area)

For example,

- If $A=\{a,b,c,d,e\}$ and $B=\{a,e,i,o,u\}$ then

  $A-B=\{b,c,d\}$ and $B-A=\{i,o,u\}$

- If $A=\{1,3,5,7,9\}$ and $B=\{2,3,5,7\}$, then

  $A-B=\{1,9\}$ and $B-A=\{2\}$

1.9.5 Symmetric Difference of two Sets

Let *A*, *B* be any two sets, symmetric difference of *A* and *B*, denoted by $A\oplus B$, is defined as the $(A-B)\cup(B-A)$.

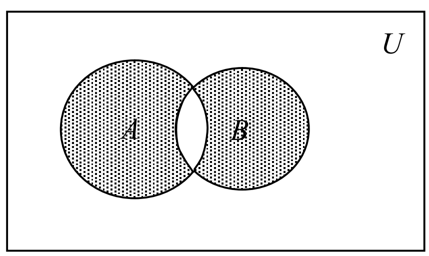


Figure 1.7: $(A-B)\cup(B-A)$ (shaded area)

For example,

- If $A=\{1,2,3,4,5\}$ and $B=\{2,4,5,6,7\}$ then

  $A\oplus B=(A-B)\cup(B-A)=\{1,3\}\cup\{6,7\}=\{1,3,6,7\}$

1.9.6 Complement of a Set

Let A be any given set and U be the universal set, then the set consisting of all the elements of U which do not belong to A is called the compliment of A. It is denoted by A^c or A' or $\overline{A}$.

Symbolically,

$$A^c = \{x; x \in U \text{ and } x \notin A\}$$

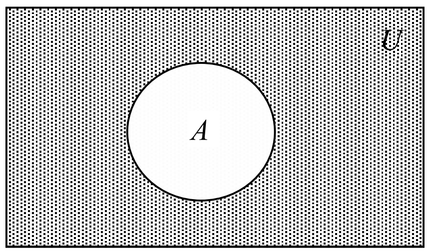


Figure 1.8: A^c (shaded area)

Obviously, $A^c = U - A$.

For example,

- If $U = \{1, 2, 3, ..., 10\}$ and $A = \{1, 3, 5, 7\}$ then

  $A^c = \{2, 4, 6, 8, 9, 10\}$

- If $U = N$, the set of natural numbers, and $A = \{1, 3, 5, 7, ...\}$ then

  $A^c = \{2, 4, 6, 8, ...\}$

1.10 BASIC LAWS OF SET THEORY

Suppose *A, B* and *C* are any three sets and *U*, the universal set. Then, the following table sums up the basic laws of set theory.

Of these laws, *Idempotent laws*, *Identity laws*, *Complement laws*, and *Commutative laws*, are intuitively true and need no proof.

Table: 1.1: Basic laws of Set Theory

Idempotent Laws: $(a)\ A \cup A = A$ $(b)\ A \cap A = A$ **Commutative Laws:** $(a)\ A \cup B = B \cup A$ $(b)\ A \cap B = B \cap A$ **Associative Laws:** $(a)\ (A \cup B) \cup C = A \cup (B \cup C)$ $(b)\ (A \cap B) \cap C = A \cap (B \cap C)$ **Distributive Laws:** $(a)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $(b)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	**Identity Laws:** $(a)\ A \cup U = U$ $(b)\ A \cap U = A$ $(c)\ A \cup \phi = A$ $(d)\ A \cap \phi = \phi$ **Complement Laws:** $(a)\ A \cup A^c = U$ $(b)\ A \cap A^c = \phi$ $(c)\ U^c = \phi$ $(d)\ \phi^c = U$ **Involution Law:** $(A^c)^c = A$ **De Morgan's Laws:** $(a)\ (A \cup B)^c = A^c \cap B^c$ $(b)\ (A \cap B)^c = A^c \cup B^c$

We will proceed to prove *associative laws*, *distributive laws*, and *De-Morgan's laws*.

Associative Laws: If A, B and C are any three sets, then

$(a)\ (A \cup B) \cup C = A \cup (B \cup C)$ $\qquad$ $(b)\ (A \cap B) \cap C = A \cap (B \cap C)$

Proof:

(a) To prove $\left(A \cup B\right) \cup C = A \cup \left(B \cup C\right)$

Let $x \in (A \cup B) \cup C$

$\Rightarrow \quad x \in A \cup B$ or $x \in C$

$\Rightarrow \quad (x \in A \text{ or } x \in B) \text{ or } x \in C$

$\Rightarrow \quad x \in A \text{ or } (x \in B \text{ or } x \in C)$

$\Rightarrow \quad x \in A \text{ or } x \in B \cup C$

$\Rightarrow \quad x \in A \cup (B \cup C)$

Again, let $x \in A \cup (B \cup C)$

$\Rightarrow \quad x \in A \text{ or } x \in B \cup C$

$\Rightarrow \quad x \in A \text{ or } (x \in B \text{ or } x \in C)$

$\Rightarrow \quad (x \in A \text{ or } x \in B) \text{ or } x \in C$

$\Rightarrow \quad x \in A \cup B \text{ or } x \in C$

$\Rightarrow \quad x \in (A \cup B) \cup C$

Thus, $x \in A \cup (B \cup C) \quad \Leftrightarrow \quad x \in (A \cup B) \cup C$

Hence, $\quad (A \cup B) \cup C = A \cup (B \cup C)$

(*b*) To prove $(A \cap B) \cap C = A \cap (B \cap C)$

Let $x \in (A \cap B) \cap C$

$\Rightarrow \quad x \in A \cap B \text{ and } x \in C$

$\Rightarrow \quad (x \in A \text{ and } x \in B) \text{ and } x \in C$

$\Rightarrow \quad x \in A \text{ and } (x \in B \text{ and } x \in C)$

$\Rightarrow \quad x \in A \text{ and } x \in B \cap C$

$\Rightarrow \quad x \in A \cap (B \cap C)$

Again, let $x \in A \cap (B \cap C)$

$\Rightarrow \quad x \in A \text{ and } x \in B \cap C$

$\Rightarrow \quad x \in A \text{ and } (x \in B \text{ and } x \in C)$

$\Rightarrow \quad (x \in A \text{ and } x \in B) \text{ and } x \in C$

$\Rightarrow \quad x \in A \cap B \text{ and } x \in C$

$\Rightarrow \quad x \in (A \cap B) \cap C$

Thus, $x \in (A \cap B) \cap C \quad \Leftrightarrow \quad x \in A \cap (B \cap C)$

Hence, $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Laws: If A, B and C are any three sets, then

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

(a) To prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$

$\Rightarrow$ $x \in A$ and $x \in (B \cup C)$

$\Rightarrow$ ($x \in A$ and $x \in A$) and ($x \in B$ or $x \in C$)

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow$ $x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$

Again, let $x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow$ $x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow$ ($x \in A$ or $x \in A$) and ($x \in B$ or $x \in C$)

$\Rightarrow$ $x \in A$ and ($x \in B$ or $x \in C$)

$\Rightarrow$ $x \in A$ and $x \in B \cup C$

$\Rightarrow$ $x \in A \cap (B \cup C)$

Thus, $x \in (A \cap B) \cup (A \cap C) \quad \Leftrightarrow \quad x \in A \cap (B \cup C)$

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) To prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cap C)$

$\Rightarrow$ ($x \in A$ or $x \in A$) or ($x \in B$ and $x \in C$)

$\Rightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Rightarrow \quad x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C)$

Again, let $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow \quad x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow \quad (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow \quad (x \in A$ and $x \in A)$ or $(x \in B$ and $x \in C)$

$\Rightarrow \quad x \in A$ or $(x \in B \cap C)$

$\Rightarrow \quad x \in A \cup (B \cap C)$

Thus, $x \in (A \cup B) \cap (A \cup C) \quad \Leftrightarrow \quad x \in A \cup (B \cap C)$

Hence, $\quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De-Morgan's Laws: If A and B are any two sets, then

(a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$

Proof:

(a) To prove $(A \cup B)^c = A^c \cap B^c$

Let $x \in (A \cup B)^c$

$\Rightarrow \quad x \notin A \cup B$

$\Rightarrow \quad x \notin A$ and $x \notin B$

$\Rightarrow \quad x \in A^c$ and $x \in B^c$

$\Rightarrow \quad x \in A^c \cap B^c$

Again, let $x \in A^c \cap B^c$

$\Rightarrow \quad x \in A^c$ and $x \in B^c$

$\Rightarrow \quad x \notin A$ and $x \notin B$

$\Rightarrow \quad x \notin A \cup B$

$\Rightarrow \quad x \in (A \cup B)^c$

Thus, $x \in A^c \cap B^c \quad \Leftrightarrow \quad x \in \left(A \cup B\right)^c$

Hence, $\quad (A \cup B)^c = A^c \cap B^c$

(*a*) To prove $(A \cap B)^c = A^c \cup B^c$

Let $x \in (A \cap B)^c$

$\Rightarrow \quad x \notin A \cap B$

$\Rightarrow \quad x \notin A$ or $x \notin B$

$\Rightarrow \quad x \in A^c$ or $x \in B^c$

$\Rightarrow \quad x \in A^c \cup B^c$

Again, let $x \in A^c \cup B^c$

$\Rightarrow \; x \in A^c$ or $x \in B^c$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow \; x \notin A \cap B$

$\Rightarrow \; x \notin \left(A \cap B\right)^c$

Thus, $x \in A^c \cup B^c \quad \Leftrightarrow \quad x \notin \left(A \cap B\right)^c$

Hence, $\quad (A \cap B)^c = A^c \cup B^c$

ILLUSTRATIVE EXAMPLES 1.4

Example 1.16: Given $U = \{1, 2, 3, \ldots, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{2, 4, 8, 10\}$. Verify the following:

(*a*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(*b*) $A - B = A \cap B^c$

(*c*) $(A \cup B)^c = A^c \cap B^c$

Solution:

(*a*) $B \cup C = \{1,3,5,7,9\} \cup \{2,4,8,10\} = \{1,2,3,4,5,7,8,9,10\}$

$\therefore \quad A \cap (B \cap C) = \{1,2,3,4,5\} \cap \{1,2,3,4,5,7,8,9,10\} = \{1,2,3,4,5\}$

$A \cap B = \{1,2,3,4,5\} \cap \{1,3,5,7,9\} = \{1,3,5\}$

$A \cap C = \{1,2,3,4,5\} \cap \{2,4,8,10\} = \{2,4\}$

$\therefore \quad (A \cap B) \cup (A \cap C) = \{1,3,5\} \cup \{2,4\} = \{1,2,3,4,5\}$

$\Rightarrow \quad A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$

(b) $\quad A - B = \{1,2,3,4,5\} - \{1,3,5,7,9\} = \{2,4\}$

$A \cap B^c = \{1,2,3,4,5\} \cap \{1,3,5,7,9\}^c = \{1,2,3,4,5\} \cap \{2,4,6,8,10\} = \{2,4\}$

$\Rightarrow \quad A - B = A \cap B^c$

(c) $\quad A \cup B = \{1,2,3,4,5\} \cup \{1,2,3,4,5,7,9\} = \{1,2,3,4,5,7,9\}$

$(A \cup B)^c = \{1,2,3,4,5,7,9\}^c = \{6,8,10\}$.

$A^c = \{1,2,3,4,5\}^c = \{6,7,8,9,10\}$ $\qquad$ $B^c = \{1,3,5,7,9\}^c = \{2,4,6,8,10\}$

$\therefore \quad A^c \cap B^c = \{6,7,8,9,10\} \cap \{2,4,6,8,10\} = \{6,8,10\}$

$\Rightarrow \quad (A \cup B)^c = A^c \cap B^c$

Example 1.17: In each of the following cases, find $A \cup B$ and $A \cap B$:

(a) $A = \{3, 5, 7, 9, 11\}$, $B = \{1, 5, 9, 13\}$
(b) $A = \{5, 6, 7, 8\}$, $B = \{7, 8, 9, 10\}$
(c) $A = \{a, e, i, o, u\}$, $B = \{a, i, j, k\}$
(d) $A = \{x : x$ is a natural number and $1 < x \le 6\}$,
$B = \{x : x$ is a natural number satisfying $6 < x \le 10\}$
(e) $A = \{x : x$ is a natural number divisible by 3$\}$,
$B = \{x : x$ is a natural number less than 6$\}$.

Solution:

Note that, $A \cup B = \{x : x \in A \text{ or } x \in B\}$ and $A \cap B = \{x : x \in A \text{ and } x \in B\}$

(a) $\quad A \cup B = \{1,3,5,7,9,11,13\}$ and $A \cap B = \{5,9\}$

(b) $\quad A \cup B = \{5,6,7,8,9,10\}$ and $A \cap B = \{7,8\}$

(c) $\quad A \cup B = \{a,e,i,o,u,j,k\}$ and $A \cap B = \{a,i\}$

(d) $\quad$ Here, $A = \{2, 3, 4, 5, 6\}$ and $B = \{7, 8, 9, 10\}$

$\therefore \quad A \cup B = \{2,3,4,5,6,7,8,9,10\}$ and $A \cap B = \{\}$

(e) Here, $A = \{3, 6, 9, 12\ldots\}$ and $B = \{1, 2, 3, 4, 5\}$

$\therefore\ A \cup B = \{1,2,3,4,5,6,9,12,15,...\}$ and $A \cap B = \{3\}$

Example 1.18: In each of the following cases, find $A - B$ and $B - A$:

(a) $A = \{3, 6, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$
(b) $A = \{2, 4, 6, 8, 10, 12, 14, 16\}$, $B = \{3, 6, 12, 15, 18, 21\}$
(c) $A = \{a, b, c, d\}$, $B = \{f, b, g, d\}$
(d) $A = \{x : x$ is an odd integer$\}$, $B = \{x : x$ is a negative integer$\}$
(e) $A = \{x : x$ is an integer divisible by 3$\}$, $B = \{x : x$ is a positive integer$\}$

Solution:

Note that, $A - B = \{x : x \in A \text{ and } x \notin B\}$ and $B - A = \{x : x \in B \text{ and } x \notin A\}$

(a) $A - B = \{3,6,15,18,21\}$ and $B - A = \{4,8,16,20\}$

(b) $A - B = \{2,4,8,10,14,16\}$ and $B - A = \{3,15,18,21\}$

(c) $A - B = \{a,c\}$ and $B - A = \{f,g\}$

(d) Here, $A = \{\ldots, -7, -5, -3, -1, 0, 3, 5, 7, \ldots\}$ and $B = \{\ldots, -3, -2, -1\}$

$\therefore\ A - B = \{1,3,5,7,...\}$ and $B - A = \{..., -6, -4, -2\}$

(e) Here, $A = \{\ldots, -9, -6, -3, 0, 3, 6, 7, \ldots\}$ and $B = \{1, 2, 3, 4, \ldots\}$

$\therefore\ A - B = \{..., -9, -6, -3, 0\}$ and $B - A = \{1,2,4,5,7,8,10,11,...\}$

Example 1.19: If $U = \{1,2,3,...,8,9,10\}$, $A = \{1,2,3,5\}$, $B = \{2,4,6,7\}$ and $C = \{2,3,4,8\}$. Find the following:

(a) $(A - B)^c$
(b) $\left(A^c\right)^c$
(c) $(B \cup C)^c$
(d) $(C \cap A)^c$

Solution:

(a) $A - B = \{1,3,5\} \quad \Rightarrow \quad (A - B)^c = \{2,4,6,7,8,9,10\}$

(b) $A = \{1,2,3,5\} \quad \Rightarrow \quad A^c = \{4,6,7,8,9,10\} \quad \Rightarrow \quad \left(A^c\right)^c = \{1,2,3,5\} = A$

(c) $B \cup C = \{2,3,4,6,7,8\} \quad \Rightarrow \quad (B \cup C)^C = \{1,5,9,10\}$

(d) $C \cap A = \{2,3\} \quad \Rightarrow \quad (C \cap A)^c = \{1,4,5,7,8,9,10\}$

Example 1.20: If A and B are two sets, then $A \subset B$ iff $B^c \subset A^c$.

Solution: Let $A \subset B$, we shall prove that $B^c \subset A^c$.

For all $x \in B^c \quad \Rightarrow \quad x \notin B \quad \Rightarrow \quad x \notin A \quad \Rightarrow \quad x \in A^c$ $\quad [\because \quad A \subset B]$

Hence $B^c \subset A^c$.

Conversely, let $B^c \subset A^c$, we shall prove that $A \subset B$.

For all $x \in A \quad \Rightarrow \quad x \notin A^c \quad \Rightarrow \quad x \notin B^c \quad \Rightarrow \quad x \in B$. $\quad [\because \quad B^c \subset A^c]$

Hence, $A \subset B$.

Therefore,

$A \subset B$ iff $B^c \subset A^c$.

Example 1.21: If A and B are any two sets, then prove that

(a) $A \subset A \cup B$ (b) $A \cap B \subset A$ (c) $A \cap B = A \cup B \quad \Rightarrow \quad A = B$

Solution:

(a) $\forall x \in A \quad \Rightarrow \quad x \in A$ or $x \in B \quad \Rightarrow \quad x \in A \cup B \quad \Rightarrow \quad A \subset A \cup B$

(b) $\forall x \in A \cap B \quad \Rightarrow \quad x \in A$ and $x \in B \quad \Rightarrow \quad x \in A \quad \Rightarrow \quad A \cap B \subset A$

Note. Similarly, we can show that $B \subset A \cup B$ and $A \cup B \subset A$.

(c) Now $A \subset A \cup B$ and $A \cup B = A \cap B$,

$\therefore \quad A \subset A \cap B$ … (i)

Also $A \cap B \subset A$ … (ii)

From Eq. (i) and (ii), we have

$A \cap B = A$ . . . (iii)

Similarly, we can have

$A \cap B = B$ . . . (iv)

From Eq. (*iii*) and (*iv*), we have

$A = B$

Alternatively,

$A \subset A \cup B = A \cap B \subset B \quad \Rightarrow \quad A \subset B$ . . . (*v*)

Also,

$B \subset A \cup B = A \cap B \subset A \quad \Rightarrow \quad B \subset A$ . . . (*vi*)

From Eq. (*v*) and (*vi*), we have $A = B$

Example 1.22: If A and B are any two sets, then prove that

(*a*) $A - B = A \cap B^c$ (*b*) $(A - B) \cap B = \phi$

Solution:

(*a*) $\forall x \in A - B \quad \Leftrightarrow \quad x \in A$ and $x \notin B \quad \Leftrightarrow \quad x \in A$ and $x \in B^c \quad \Leftrightarrow \quad x \in A \cap B^c$

Hence, $A - B = A \cap B^c$

(*b*) LHS = $(A - B) \cap B = (A \cap B^c) \cap B$ [using part (*a*)]

$= A \cap (B^c \cap B)$ [using associative property of sets]

$= A \cap \phi = \phi =$ RHS

Example 1.23: If A and B are any two sets, then prove that

(*a*) $A \cup B = A$ iff $A \supset B$ (*b*) $A \cap B = A$ iff $A \subset B$

Solution:

(*a*) Let $A \cup B = A$

Since, $B \subset A \cup B$, therefore, $B \subset A$ $[\because \; A \cup B = A]$

Hence, $A \cup B = A \quad \Rightarrow \quad B \subset A$, i.e., $A \supset B$

Conversely, let $A \supset B$, then

$x \in A \cup B \quad \Rightarrow \quad x \in A$ or $x \in B \quad \Rightarrow \quad x \in A \quad [\because \; B \subset A, \; \therefore \; x \in B \; \Rightarrow \; x \in A]$

$\therefore \quad A \cup B \subset A$ . . . (*i*)

Also, $A \subset A \cup B$ . . . (*ii*)

From Eqs. (*i*) and (*ii*), we have

$A \cup B = A$

(b) Let $A \cap B = A$

Since, $A \cap B \subset B$ therefore, $A \subset B$ [$\because$ $A \cap B = A$]

Conversely, let $A \subset B$, then $x \in A \Rightarrow x \in A$ and $x \in B$ [$\because$ $A \subset B$]

$x \in A \cup B \Rightarrow x \in A$ or $x \in B \Rightarrow x \in A$

$\Rightarrow x \in A \cap B$

$\therefore$ $A \subset A \cap B$ . . . (iii)

Also, $A \cap B \subset A$ . . . (iv)

From Eqs. (iii) and (iv), we have

$A \cap B = A$

Example 1.24: Using laws of set theory, do as directed

(a) Show that $(A \cup B) \cap (A \cup B^c) = A$

(b) If $A \cap B^c = \phi$, show that $A \subset B$

(c) If $A^c \cup B = U$, show that $A \subset B$

Solution:

(a) $(A \cup B) \cap (A \cup B^c) = A \cup (B \cap B^c)$ [using distributive law]

$= A \cup \phi = A$

(b) Now, $A = A \cap U$, where U is the universal set

$= A \cap (B \cup B^c)$

$= (A \cap B) \cup (A \cap B^c)$ [using distributive law]

$= (A \cap B) \cup \phi$ [given that $A \cap B^c = \phi$]

$= A \cap B$

Hence, $A = A \cap B$ therefore, $A \subset B$ [see example 21]

(c) Now, $B = B \cup \phi$

$= B \cup (A \cap A^c)$

$= (B \cup A) \cap (B \cup A^c)$ [using distributive law]

$= (B \cup A) \cap (A^c \cup B)$ [using commutative law]

$= (B \cup A) \cap U$ [given that $A^c \cup B = U$]

$= B \cup A$

Hence, $B = B \cup A = A \cup B$, therefore, $B \supset A \quad \Rightarrow \quad A \subset B$ [see example 21]

Example 1.25: Show that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

Solution: LHS = $(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)^c$ [$\because \; A - B = A \cap B^c$]

$= (A \cup B) \cap (A^c \cup B^c)$ [De Morgan's law]

$= \{(A \cup B) \cap A^c\} \cup \{(A \cup B) \cap B^c\}$ [Distributive law]

$= \{(A \cap A^c) \cup (B \cap A^c)\} \cup \{(A \cap B^c) \cup (B \cap B^c)\}$

[Distributive law]

$= \{\phi \cup (B \cap A^c)\} \cup \{(A \cap B^c) \cup \phi\}$

$= (B \cap A^c) \cup (A \cap B^c)$ [$\because \; A \cup \phi = A$]

$= (B - A) \cup (A - B)$ [Commutative law]

$= (A - B) \cup (B - A) =$ RHS

EXERCISE 1.4

1. If $U = \{a, b, c, d, e, f, g, h, i\}$, find the complement of each of the following sets:
   (a) $A = \{a, b, d\}$, (b) $B = \{c, f, g, i\}$, (c) $C = \{a, e, i\}$
2. Given $U = \{1, 2, 3, ..., 10\}$, $A = \{1, 2, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{2, 3, 4, 5, 6, 7\}$. Verify the following results:
   (a) $\left(A^c\right)^c = A$, (b) $(A \cup B)^c = A^c \cap B^c$, (c) $A - (B \cup C) = (A - B) \cap (A - C)$
3. Taking N, the set of natural numbers, as the universal set, write down the complement of each of the following sets:
   (a) $A = \{x : x$ is an even natural number$\}$
   (b) $B = \{x : x$ is a prime number$\}$
   (c) $C = \{x : x = n^2, n \in N\}$
   (d) $D = \{x : x$ is a natural number $\leq 7\}$

4. Which of the following pairs of sets are disjoint?
   (*a*) $A = \{1, 2, 3, 4\}$, $B = \{x : x$ is a natural number and $1 \le x \le 4\}$
   (*b*) $A = \{x : x$ is an odd number$\}$, $B = \{x : x$ is an even number$\}$
   (*c*) $A = \{x : x$ is a natural number $\le 7\}$,
   $B = \{x : x$ is a natural number and $7 \le x \le 10\}$
   (*d*) $A = \{x : x$ is a prime number $< 10\}$, $B = \{x : x$ is a prime number and is even$\}$
5. If A, B and C are any three sets, prove that
   (*a*) $(A - B) - C = A - (B \cup C)$
   (*b*) $A - B = A - (A \cap B)$
   (*c*) $A - B = A$ iff $A \cap B = \phi$
   (*d*) $(A \cup B) - C = (A - C) \cup (B \cup C)$
6. If A, B and C are any three sets, prove that
   (*a*) $A^c - B^c = B - A$
   (*b*) $A \cap (B - A) = \phi$
   (*c*) $A \subset B$ and $B \subset C \;\Rightarrow\; A \subset C$
   (*d*) $A \cap (A \cup B)^c = \phi = A - (A \cup B)$

ANSWERS

1. (*a*) $A^c = \{c, e, f, g, h, i\}$, (*b*) $B^c = \{a, b, d, e, h\}$, (*c*) $C^c = \{b, c, d, f, g, h\}$
3. (*a*) $A^c = \{x : x$ is an odd natural number$\}$
   (*b*) $B^c = \{x : x$ is not a prime number$\}$
   (*c*) $C^c = \{x : x$ is a natural number which is not perfect square of a natural number$\}$
   (*d*) $D^c = \{x : x$ is a natural number $> 7\}$
4. (*b*)

1.11 FEW IMPORTANT RELATIONS INVOLVING ORDER OF SET

(*a*) If A and B are finite sets, and $A \cap B \ne \phi$, then number of elements in $A \cup B$ can be obtained by adding the number of elements in the set A and the number of elements in the set B and subtracting from the sum, the number of those elements which occur in both A and B, because in $n(A) + n(B)$, common elements have been counted twice, i.e.,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

The above rule can be applied even when $A \cap B = \phi$ as in that case $n(A \cap B) = 0$ and we shall obtain

$$n(A \cup B) = n(A) + n(B)$$

(b) If A and B are any two sets for which $A \cap B = \phi$, then $A \cup B$ is the union of three mutually disjoint sets $A - B$, $A \cap B$ and $B - A$. Hence, if both A and B are finite sets, then

$$n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$$

(c) If A and B are any two finite sets for which $A \cap B \neq \phi$, then A is union of disjoint sets $A - B$ and $A \cap B$, and B is union of disjoint sets $B - A$ and $A \cap B$.

Hence, $n(A) = n(A - B) + n(A \cap B)$ and $n(B) = n(B - A) + n(A \cap B)$.

From here, we obtain

$$n(A - B) = n(A) - n(A \cap B) \quad \text{and} \quad n(B - A) = n(B) - n(A \cap B).$$

(d) If A, B and C are any three finite sets, then

$$\begin{aligned} n(A \cup B \cup C) &= n((A \cup B) \cup C) = n(A \cup B) + n(C) - n((A \cup B) \cap C) \\ &= \left[n(A) + n(B) - n(A \cap B)\right] + n(C) - n((A \cap C) \cup (B \cap C) \quad \ldots (i) \end{aligned}$$

Again

$$\begin{aligned} n((A \cap C) \cup (B \cap C)) &= n(A \cap C) + n(B \cap C) - n((A \cap C) \cap (B \cap C)) \\ &= n(A \cap C) + n(B \cap C) - n(A \cap B \cap C) \quad \ldots (ii) \end{aligned}$$

From (i) and (ii), we have

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) - n(A \cap B) + n(C) \\ &\quad - \left[n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)\right] \end{aligned}$$

i.e.,

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) \\ &\quad + n(A \cap B \cap C) \end{aligned}$$

ILLUSTRATIVE EXAMPLES 1.5

Example 1.26: If A and B are two sets such that

$$n(A) = 38, \quad n(B) = 42, \quad n(A \cup B) = 60$$

Find $n(A \cap B)$.

Solution: We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

Substituting given values, we get

$$n(A \cap B) = 38 + 42 - 60 = 20$$

Example 1.27: If A and B are two sets such that

$$n(A) = 150, \quad n(B) = 250, \quad n(A \cup B) = 300$$

Find $n(A - B)$ and $n(B - A)$.

Solution: We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

Substituting given values, we get

$$n(A \cap B) = 150 + 250 - 300 = 100$$

Also $\quad n(A) = n(A - B) + n(A \cap B)$

$$\Rightarrow \quad n(A - B) = n(A) - n(A \cap B)$$

Substituting given values, we get

$$n(A - B) = 150 - 100 = 50$$

Similarly $n(B - A) = n(B) - n(A \cap B)$

Substituting given values, we get

$$n(B - A) = 250 - 100 = 150$$

Example 1.28: There are 11 teachers who teach mathematics or physics in a college. Of these, 7 teach mathematics and 3 teach both mathematics and physics. How many teachers teach physics?

Solution: Let A be the set of teachers who teach mathematics and B be the set of teachers who teach physics.

Then $\quad n(A \cup B) = 11, \; n(A) = 7, \; n(A \cap B) = 3$

To find $n(B)$:

We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \qquad n(B) = n(A \cup B) + n(A \cap B) - n(A)$$

Substituting given values, we get

$$n(B) = 11 + 3 - 7 = 7$$

Hence, 7 teachers teach physics.

Example 1.29: In a group of 60 persons, each take either tea or coffee. If 20 of them take tea but not coffee and 35 take tea, find the number of persons who take

(a) both tea and coffee
(b) coffee but not tea

Solution: Let A denote the set of persons who take tea, and B denote the set of persons who take coffee, then we have

$$n(A \cup B) = 60,\ n(A) = 35, \text{and } n(A - B) = 20$$

We know that

$$n(A) = n(A - B) + n(A \cap B)$$

$$\Rightarrow \qquad n(A \cap B) = n(A) - n(A - B)$$

Substituting given values, we get

$$n(A \cap B) = 35 - 20 = 15$$

So, the number of persons who take both tea and coffee = 15

Also

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \qquad n(B) = n(A \cup B) + n(A \cap B) - n(A)$$

Substituting respective values, we get

$$n(B) = 60 + 15 - 35 = 40$$

So, the number of persons who take coffee = 40

Hence, the number of persons who take coffee but not tea $= n(B - A)$

$$= n(B) - n(A \cap B) = 40 - 15 = 25$$

Example 1.30: In a group of 100 persons, 80 take tea, 30 take coffee, and 20 take both tea and coffee. How many persons take neither tea nor coffee?

Solution: Let T denote the set of persons who take tea, and C denote the set of persons who take coffee, then we are given that

$$n(U) = 100,\ n(T) = 80,\ n(C) = 30, \text{and } n(T \cap C) = 20$$

We know that

$$n(T \cup C) = n(T) + n(C) - n(T \cap C)$$

Substituting given values, we get

$$n(T \cup C) = 80 + 30 - 20 = 90$$

So, the number of persons who take both tea and coffee = 90

Now, the number of persons who take neither tea nor coffee $= n\left((T \cup C)^c\right)$

$$= n(U) - n(T \cup C) = 100 - 90 = 10$$

Example 1.31: Let A and B are two sets containing m_1 and m_2 elements, respectively. If $x \le n(A \cup B) \le y$, find x and y.

Solution: We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= m_1 + m_2 - n(A \cap B) \le m_1 + m_2 \qquad \ldots (i)$$

It may be noted that equality holds only when $n(A \cap B) = 0,$ *i.e.*, when A and B are disjoint.

Also, $A \subset A \cup B$ and $B \subset A \cup B \quad \Rightarrow \quad n(A) \le n(A \cup B)$ and $n(B) \le n(A \cup B)$

$\Rightarrow \quad n(A \cup B) \ge n(A)$ and $n(A \cup B) \ge n(B)$

$$\Rightarrow \quad n(A \cup B) \ge \max\left(n(A), n(B)\right) \qquad \ldots (ii)$$

It may be noted that equality holds only when either $A \subset B$ or $B \subset A$ in which case $n(A \cap B)$ is maximum.

From (*i*) and (*ii*), we find that

$$\max(m_1, m_2) \le n(A \cup B) \le m_1 + m_2$$

Hence

$$x = \max(m_1, m_2) \quad \text{and} \quad y = m_1 + m_2$$

EXERCISE 1.5

1. If $n(A) = 33$, $n(B) = 27$ and $n(A \cup B) = 58$, find $n(A \cap B)$.
2. If A and B are two sets such that $n(A) = 27$, $n(B) = 33$ and $n(A \cap B) = 15$, find $n(A \cup B)$.
3. If $n(A \cup B) = 60$, $n(A \cap B) = 10$, and $n(A) = 40$, find $n(B)$.
4. If $n(A) = 300$, $n(B) = 200$ and $n(A \cup B) = 400$, find $n(A - B)$ and $n(B - A)$.
5. If A and B are two sets such that $n(A - B) = 15$, $n(A \cap B) = 10$ and $n(B - A) = 25$, find $n(A \cup B)$, $n(A)$ and $n(B)$.
6. In a group of 500 persons, 350 can speak Hindi, 250 can speak English and 150 can speak both Hindi and English. Find the number of persons who can speak neither Hindi nor English.
7. There are 240 students in a class who study mathematics or physics. Of these, 80 study physics, and 20 study both mathematics and physics. How many student study mathematics?
8. In a group of 100 persons, 80 like bananas, 75 like apples. If x persons like both bananas and apples, show that $35 \le x \le 60$.
9. In a group of 1000 persons, 700 can speak Hindi and 500 can speak English. If every person can speak at least one of the two languages, find the number of persons who can speak
   (*a*) both Hindi and English
   (*b*) only Hindi
   (*c*) only English

ANSWERS

1. 2	2. 45	3. 30	4. 200, 100
5. 50, 25, 35	6. 50	7. 180	

9. (*a*) 200, (*b*) 500 and (*c*) 300

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