CHAPTER 1

LINEAR ALGEBRA

1.1 MATRICES AND THEIR TYPES

1.1.1 Definition of a Matrix

A set of *mn* numbers (real or complex) arranged in '*m*' rows and '*n*' columns is called a rectangular matrix or simply a matrix of order '*m* by *n*' (denoted by $m \times n$). Thus,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \quad \ldots(i)$$

is a matrix of order $m \times n$.

The numbers $a_{11}, a_{12}, \ldots a_{mn}$ are called elements of the matrix and a_{ij} is the element of the matrix lying in the i^{th} row and j^{th} column. The matrix given in (*i*) can also be represented by

$$A = [a_{ij}]_{m\times n} \quad \text{or} \quad \text{by } A = (a_{ij})_{m\times n}$$

Examples:

$$A = \begin{pmatrix} 0 & 3 & 5 \\ 1 & 4 & 1 \end{pmatrix} \text{ is a matrix of order } 2 \times 3,$$

$$A = \begin{pmatrix} 4 & 5 \\ 3 & 6 \\ 2 & 1 \end{pmatrix} \text{ is a matrix of order } 3 \times 2.$$

The rows of a matrix are denoted by R_1 (1st row), R_2 (2nd row) and so on. The columns of a matrix are denoted by C_1 (1st column), C_2 (2nd column) and so on.

Remember:

Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ are said to be equal if $m = p$, $n = q$ (*i.e.*, they have same number of rows and columns) and the elements lying in the corresponding places of the two matrices are the same.

Thus for $A = \begin{pmatrix} x & y \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 3 & 5 \\ a & b \end{pmatrix}$;

$$A = B \quad \Leftrightarrow \quad x = 3, y = 5, a = 0, b = 2.$$

1.1.2 Types of Matrices

(*a*) Row matrix

A matrix with only one row is called a row matrix.

Example:

[2 5 1] is a row matrix (order of this matrix is 1×3).

(*b*) Column matrix

A matrix with only one column is called a column matrix.

Example:

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} \text{ is a column matrix}$$

(order of this matrix is 4×1)

(*c*) Square matrix

A matrix which has equal number of rows and columns is called a square matrix.

The elements a_{ij} of the square matrix $A = [a_{ij}]_{n\times n}$ for which $i = j$ are called diagonal elements and the line formed by the diagonal elements is called the principal diagonal of the matrix.

If a matrix *A* has '*n*' rows and '*n*' columns, then we say that *A* is a square matrix of order *n*. Then *A* can be represented by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

where, $a_{11}, a_{22}, \ldots, a_{nn}$ are diagonal elements.

Example:

1, 2, 4 are diagonal elements of the matrix $\begin{pmatrix} 1 & 0 & 5 \\ 3 & 2 & 1 \\ -1 & 9 & 4 \end{pmatrix}$

(*d*) Diagonal matrix

A square matrix whose non-diagonal elements are all zero is called a diagonal matrix.

Example:

$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is a diagonal matrix of order 3.

The above diagonal matrix can also be written as diag (4, 7, – 1) or diag [–1, 7, 4].

(*e*) Scalar matrix

If all the diagonal elements of a diagonal matrix are equal, then the matrix is called a scalar matrix.

Example:

$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a scalar matrix of order 3.

(*f*) Identity matrix

A diagonal matrix whose diagonal elements are all '1', is called an identity matrix. The identity matrix of order n is denoted by I_n or $I_{n\times n}$ or by I.

Example:

$$I_1 = [1]_{1\times 1},\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2},\quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} \text{ etc.}$$

(*g*) Null matrix

If all the elements of a matrix are zero, then it is called a null matrix or zero matrix. The null matrix of order $m\times n$ is denoted by $O_{m\times n}$.

Example:

$$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

etc. are null matrices.

For sake of simplicity sometimes, we denote the null matrix by 'O'.

(*h*) Upper triangular matrix

A square matrix A = $[a_{ij}]_{n\times n}$ is called an upper triangular matrix if $a_{ij} = 0\ \forall\ i > j$.

Thus for any upper triangular matrix, all the elements lying below the principal diagonal are zero.

Example:

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 6 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

etc. are all upper triangular matrices.

(*i*) Lower triangular matrix:

A square matrix $A = [a_{ij}]_{n\times n}$ is called an lower triangular matrix if $a_{ij} = 0\ \forall\ i < j$.

Thus for any lower triangular matrix, all the elements lying above the principal diagonal are zero.

Example:

$$\begin{pmatrix} 3 & 0 \\ 1 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 2 & 6 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 5 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

etc. are all lower triangular matrices.

1.2 ALGEBRA OF MATRICES

1.2.1 Negative, Sum and Differences of Matrices

(*i*) The negative of a matrix $A = [a_{ij}]_{m\times n}$ is the matrix $[-a_{ij}]_{m\times n}$ and is denoted by '$-A$'.

Example:

If $A = \begin{pmatrix} 1 & 2 & 5 \\ 6 & 3 & 4 \end{pmatrix}$, then $-A = \begin{pmatrix} -1 & -2 & -5 \\ -6 & -3 & -4 \end{pmatrix}$

(*ii*) The sum of two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ is denoted by $A + B$ and defined by $A + B = [a_{ij} + b_{ij}]_{m\times n}$.

Thus, the sum of two matrices A and B of same order is the matrix obtained by adding the corresponding elements of the two matrices A and B.

Example:

If $A = \begin{pmatrix} 4 & 5 & 6 \\ 2 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 0 & 1 \\ 2 & 1 & 9 \end{pmatrix}$

Then $A + B = \begin{pmatrix} 11 & 5 & 7 \\ 4 & 0 & 9 \end{pmatrix}$

(*iii*) The difference of two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ is denoted by $A - B$ and is defined to be the sum $A + (-B)$.

Thus, $A - B = A + (-B) = [a_{ij} - b_{ij}]_{m\times n}$.

Example:

For $A = \begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$;

$A - B = \begin{pmatrix} 5 & 5 \\ 2 & -2 \end{pmatrix}$ and $B - A = \begin{pmatrix} -5 & -5 \\ -2 & 2 \end{pmatrix}$

Remember: $A - B \neq B - A$ (in general)

Properties of addition of matrices

If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [c_{ij}]_{m\times n}$, are three matrices of same order, then

(*i*) $A + B = B + A$ (Commutative property)

(*ii*) $A + (B + C) = (A + B) + C$ (Associative property)

(*iii*) $A + O = O + A$, where O is the null matrix of order $m \times n$.

(*iv*) $A + (-A) = (-A) + A = O$

where '$-A$' is called the additive inverse of A w.r.t. the matrix addition.

1.2.2 Multiplication of a Matrix by a Scalar

Let $A = [a_{ij}]_{m\times n}$ be any matrix and k be any scalar quantity. Then the product kA is the matrix of order $m\times n$ defined by $kA = [ka_{ij}]_{m\times n}$.

Example:

If $k = 2, A = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$, then $kA = \begin{pmatrix} 8 & 10 \\ 12 & 14 \end{pmatrix}$.

Properties:

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices and k_1, k_2 be scalars, then

(*i*) $k_1 (A \pm B) = k_1 A \pm k_1 B$

(*ii*) $(k_1 \pm k_2)A = k_1A \pm k_2A$

(*iii*) $k_1(k_2A) = (k_1k_2)A$

(*iv*) $-k_1A = k_1(-A)$

1.2.3 Transpose of a Matrix

The transpose of a matrix A is the matrix obtained from A by changing its rows into columns and columns into rows. The transpose of a matrix A is denoted by A' or A^t or A^T. Thus, if A is matrix of order $m \times n$, then A^T is of order $n \times m$.

Example:

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Properties of transpose of a matrix

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of same order, then

(*i*) $(A^T)^T = A$

(*ii*) $(A \pm B)^T = A^T \pm B^T$

(*iii*) $(\lambda A)^T = \lambda A^T$, where λ is any number (constant).

(*iv*) $(\lambda A \pm \mu B)^T = \lambda A^T \pm \mu B^T$, where λ, μ are scalars.

(*v*) $(AB)^T = B^T A^T$ provided AB is defined.

Remember:

If A is a square matrix, then

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, where $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is skew-symmetric. Thus, every square matrix can be expressed as a sum of a symmetric and skew-symmetric matrix.

1.2.4 Multiplication of Matrices (Product of Matrices)

If the number of columns of a matrix A is equal to the number of rows of another matrix B, then we say that the product AB is defined.

If $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$, then $AB = [c_{ik}]_{m\times p}$

where $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$, where the suffixes i, j, k ranges from 1 to m, 1 to n, 1 to p respectively.

In other words, if A be a matrix of order $m\times n$ and B be a matrix of order $n \times p$, then the product AB is a matrix of order $m\times p$ and the element lying in the i^{th} row and k^{th} column in AB is the sum of the products of the elements of the i^{th} row of A and the corresponding elements of the k^{th} column of B. Sometimes we denote the matrix product by $A.B$ or $A \times B$ also.

Example:

For $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}_{2\times 3}$, $B = \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \\ f_1 & f_2 \end{pmatrix}_{3\times 2}$

$$AB = \begin{pmatrix} a_1c_1 + a_2d_1 + a_3f_1 & a_1c_2 + a_2d_2 + a_3f_2 \\ b_1c_1 + b_2d_1 + b_3f_1 & b_1c_2 + b_2d_2 + b_3f_2 \end{pmatrix}$$

Remember:

(*a*) In the product AB, A is called pre-factor and B is called the post factor. In general, $AB \neq BA$.

(*b*) $A(BC) = (AB)C$, provided the products AB and BC are defined (Associative property)

(*c*) $A(B + C) = AB + AC$, provided the products AB, AC are defined.

(*d*) $A^2 = AA$, $A^3 = AAA$ and so on (provided A is a square matrix)

(*e*) $(A \pm B)^2 \neq A^2 \pm 2AB + B^2$ (in general but equality holds if $AB = BA$)

(*f*) $(A + B)(A - B) \neq A^2 - B^2$ (in general but equality holds if $AB = BA$)

(*g*) $IA = AI = A$, where A is a square matrix of order n and I is the identity matrix of order n.

1.3 DETERMINANT OF A SQUARE MATRIX

1.3.1 Definition of Determinant

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. Then the determinant of A is denoted by $|A|$ or det (A) and is defined by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots\cdots & a_{3n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots & a_{nn} \end{vmatrix}$$

Here $|A|$ is called the determinant of order 'n'. One can use the symbols Δ, Δ_1, Δ_2 etc to denote a determinant. We can find the value of determinant in the following manner:

(*i*) If $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, then $|A| = a_{11}a_{22} - a_{12}a_{21}$

(*ii*) If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

The above process of finding the value of a determinant is called '*expansion of a determinant*'.

1.3.2 Properties of a Determinant

(*i*) A determinant remains unchanged by changing its rows into columns and columns into rows.

Thus, the determinants $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ have the same value.

(*ii*) The interchange of two rows (or columns) of determinant changes the sign of the determinant without changing its numerical value.

Thus if $\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$

(Δ_2 is obtained by interchanging the 1st and 2nd rows), then

$$\Delta_2 = a_2b_1 - a_1b_2 = -(a_1b_2 - a_2b_1) = -\Delta_1.$$

The operation 'interchange of R_1 and R_2' is denoted by $R_1 \leftrightarrow R_2$. Similar notation can be used for interchange of two columns.

(*iii*) If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

For example, in $\Delta = \begin{vmatrix} 1 & 0 & 3 \\ 5 & -1 & 2 \\ 1 & 0 & 3 \end{vmatrix}$,

R_1(1st row) and R_3(3rd row) are identical, and so $\Delta = 0$.

(*iv*) If AB is defined, then $|AB| = |A|\,|B|$.

(*v*) if A be a square matrix of order 'n', then $|kA| = k^n|A|$.

Example:

Let $k = 4$ and $A = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$. So A is a square matrix of order '2' and $|A| = 5$.

Then $|4A| = \begin{vmatrix} 4\times1 & 4\times3 \\ 4\times0 & 4\times5 \end{vmatrix} = \begin{vmatrix} 4 & 12 \\ 0 & 20 \end{vmatrix}$

$= 80 = 4^2 \times 5$

$= 4^{\text{order of A}} \times |A|$.

(*vi*) $|A^n| = |A|^n$, n being a positive integer.

(*vii*) If every element of any row (or column) of a determinant is multiplied by a constant 'k', then the determinant is multiplied by the same constant 'k'.

Example:

If $\Delta = \begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix}$, then

$2\Delta = 2\begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix}$ (by multiplying R_1 by 2)

(*viii*) The value of a determinant corresponding to a lower (or upper) triangular matrix is obtained by taking the product of all the diagonal elements.

Example:

The value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 6 \end{vmatrix}$ is the product of 1, 5 and 6 *i.e*; 30.

(*ix*) The determinant value of the null and identity matrices are respectively 0 and 1.

(*x*) If all elements of any row (or column) are zero, then the determinant value becomes zero.

Example:

The determinant $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{vmatrix}$ has value '0'.

(*xi*) The value of a determinant remains unaltered by adding (or subtracting) 'k' times the elements of any row (or column) to (or from) the corresponding elements of any other row (column), where 'k' is any constant. In this case we use the following notations:

$R_i \to R_i + kR_j$, $R_i \to R_i - kR_j$,

$C_i \to C_i + kC_j$, $C_i \to C_i - kC_j$.

Example:

Let $\Delta = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix}$.

Then $\Delta = (15 - 4) - 0 - 2 = 9$.

Again $\Delta = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1+3\times(-1) & 0 & -1 \\ 2+3\times4 & 3 & 4 \\ 0+3\times5 & 1 & 5 \end{vmatrix}$

(*by* $C_1 \to C_1 + 3C_3$)

$$= \begin{vmatrix} -2 & 0 & -1 \\ 14 & 3 & 4 \\ 15 & 1 & 5 \end{vmatrix} = -2(15-4) - 0 - (14-45) = 9.$$

Also,

$$\Delta = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 2-2\times1 & 3-2\times0 & 4-2\times(-1) \\ 0 & 1 & 5 \end{vmatrix}$$

(*by* $R_2 \to R_2 - 2R_1$)

$$= \begin{vmatrix} 1 & 0 & -1 \\ 0 & 3 & 6 \\ 0 & 1 & 5 \end{vmatrix} = 15 - 6 = 9.$$

Remember:

If the given determinant is of order 4 (or more), then use row (or column) operations to get only one non-zero element in the 1^{st} row (or 1^{st} column). Then the value of the determinant = non-zero element obtained in 1^{st} row (1^{st} column) × it's cofactor.

Example:

If $\Delta = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & 1 & 2 \\ -1 & 5 & 1 & 0 \\ 1 & 0 & -1 & 3 \end{vmatrix}$. Then

$$\Delta = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & 1 & 2 \\ -1 & 5 & 1 & 0 \\ 1 & 0 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 7 & 4 & -1 \\ 0 & -2 & -4 & 4 \end{vmatrix}$$

[by $R_3 \to R_3 + R_1$, $R_4 \to R_4 - R_1$]

= 1 × co-factor of '1'

[since '1' is the only non-zero element in 1^{st} column]

$$= 1 \times (-1)^{1+1} \begin{vmatrix} 2 & 1 & 2 \\ 7 & 4 & -1 \\ -2 & -4 & 4 \end{vmatrix}$$

$= 2(16 - 4) - 1(28 - 2) + 2(-28 + 8) = -42.$

1.3.3 Minors and Cofactors

The minor of an element in a determinant Δ is obtained by eliminations from Δ the row and the column containing the element. Thus if a_{ij} be in i^{th} row and j^{th} column, then the minor of a_{ij} is denoted by M_{ij} and the co-factor of a_{ij} denoted by C_{ij}, is defined by $C_{ij} = (-1)^{i+j} M_{ij}$

Example:

Consider a determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \\ 6 & 9 & 8 \end{vmatrix}$

Then

(*i*) the minor of '1' is $\begin{vmatrix} 7 & 5 \\ 9 & 8 \end{vmatrix}$ (which is obtained by omitting 1^{st} row and 1^{st} column where the element 1 lies),

(*ii*) the minor of '5' is $\begin{vmatrix} 1 & 2 \\ 6 & 9 \end{vmatrix}$ (which is obtained by omitting 2^{nd} row and 3^{rd} column where the element 5 lies) etc.

(*iii*) the co-factor of '1' is

$(-1)^{1+1} \begin{vmatrix} 7 & 5 \\ 9 & 8 \end{vmatrix}$ *i.e.*, $\begin{vmatrix} 7 & 5 \\ 9 & 8 \end{vmatrix}$

(*iv*) the co-factor of '5' is

$(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 6 & 9 \end{vmatrix}$ *i.e.*, $-\begin{vmatrix} 1 & 2 \\ 6 & 9 \end{vmatrix}$

Remember:

The sum of products of all elements of any row (or column) with their respective co-factors give the value of the determinant.

Thus if $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

then considering 1^{st} row, we can write

$\Delta = (a_1 \times \text{cofactor of } a_1) + (a_2 \times \text{cofactor of } a_2) + (a_3 \times \text{cofactor of } a_3)$

Again considering 2^{nd} column, we can write

$\Delta = (a_2 \times \text{cofactor of } a_2) + (b_2 \times \text{cofactor of } b_2) + (c_2 \times \text{cofactor of } c_2)$

1.4 ADJOINT AND INVERSE OF A MATRIX

1.4.1 Adjoint of a Matrix

If $A = [a_{ij}]_{n\times n}$ be a square matrix of order 'n' and C_{ij} be the cofactor of a_{ij} in $|A|$. Then the matrix $[C_{ij}]^T_{n\times n}$ is called the adjoint or adjugate of the matrix A and is denoted by adj(A). Thus adj(A) = $[C_{ij}]^T_{n\times n}$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then

(*i*) co-factor of '1' is $(-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$ *i.e.*, -3

(since '1' lies in 1^{st} row and 1^{st} column)

(*ii*) co-factor of '2' is $(-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$ *i.e.*, $+6$

(since '2' lies in 1^{st} row and 2^{nd} column)

(*iii*) co-factor of '3' is $(-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$ *i.e.*, -7

(since '3' lies in 1^{st} row and 3^{rd} column)

(*iv*) co-factor of '4' is $(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$ *i.e.*, $+6$

(since '4' lies in 2^{nd} row and 1^{st} column)

(*v*) co-factor of '5' is $(-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}$ *i.e.*, -12

(since '5' lies in 2^{nd} row and 2^{nd} column)

(*vi*) co-factor of '6' is $(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$ *i.e.*, $+6$

(since '6' lies in 2^{nd} row and 3^{rd} column)

(*vii*) co-factor of '7' is $(-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$ *i.e.*, -3

(since '7' lies in 3^{rd} row and 1^{st} column)

(*viii*) co-factor of '8' is $(-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$ *i.e.*, $+6$

(since '8' lies in 3rd row and 2nd column)

(*ix*) co-factor of '9' is $(-1)^{3+3}\begin{vmatrix}1 & 2\\4 & 5\end{vmatrix}$ *i.e.*, – 3

(since '9' lies in 3rd row and 3rd column)

Hence adj(A)

$$=\begin{bmatrix}\text{co-factor of 1} & \text{co-factor of 2} & \text{co-factor of 3}\\ \text{co-factor of 4} & \text{co-factor of 5} & \text{co-factor of 6}\\ \text{co-factor of 7} & \text{co-factor of 8} & \text{co-factor of 9}\end{bmatrix}^T$$

$$=\begin{pmatrix}-3 & 6 & -7\\ 6 & -12 & 6\\ -3 & 6 & -3\end{pmatrix}^T=\begin{pmatrix}-3 & 6 & -3\\ 6 & -12 & 6\\ -7 & 6 & -3\end{pmatrix}$$

Properties :

If A be a square matrix of order 'n', then

(*i*) $A \times (\text{adj } A) = (\text{adj } A) \times A = |A| I$

I being the identity matrix of order 'n'.

(*ii*) $|\text{adj } A| = |A|^{n-1}$,

(*iii*) $|\text{adj }(\text{adj } A)| = |\text{adj }(A)|^{n-1} = |A|^{(n-1)^2}$

(*iv*) $\text{adj}(\text{adj } A) = |A|^{n-2} \times A$

(*v*) $\text{adj }(kA) = (\text{adj } A) \times k^{n-1}$, where '$k$' is a real number

(*vi*) $\text{adj }(A^T) = (\text{adj } A)^T$.

1.4.2 Inverse of a Matrix

Suppose A be a square matrix of order 'n' and there exist an another square matrix B of the same order such that $AB = BA = I$, where I is the identity matrix of order 'n', then B is called the inverse of A, denoted by A^{-1} and is defined by

$$A^{-1} = \frac{1}{|A|} adj\ (A) \text{ provided } |A| \neq 0$$

Example:

Let $A = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$, then

$\det(A) = 1(45 - 48) - 2(36 - 42) + 3(32 - 35) = 0$

So A^{-1} does not exist.

Let us consider another matrix, $B = \begin{pmatrix}1 & 0 & 3\\ 4 & 5 & 6\\ 0 & 1 & 1\end{pmatrix}$

Then $\det(B) = 1(5 - 6) - 0 + 3(4 - 0) = 11$.

Therefore B^{-1} exist.

$$\therefore \text{adj}(B) = \begin{pmatrix}-1 & -4 & 4\\ 3 & 1 & -1\\ -15 & 6 & 5\end{pmatrix}^T = \begin{pmatrix}-1 & 3 & -15\\ -4 & 1 & 6\\ 4 & -1 & 5\end{pmatrix}.$$

$$\text{Hence } B^{-1} = \frac{1}{|B|}\text{adj}(B) = \frac{1}{11}\begin{pmatrix}-1 & 3 & -15\\ -4 & 1 & 6\\ 4 & -1 & 5\end{pmatrix}.$$

Properties:

If A be a square matrix of order 'n', then

(*a*) $A \times A^{-1} = A^{-1} \times A = I$

(*b*) $(AB)^{-1} = B^{-1}A^{-1}$

(*c*) $(A^{-1})^{-1} = A$

(*d*) $(A^T)^{-1} = (A^{-1})^T$

(*e*) For $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $A^{-1} = \frac{1}{|A|}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$ provided inverse exist.

(*f*) If $|A| \neq 0$, then A is called non-singular (invertible) matrix; otherwise A is called singular.

1.5 VARIOUS TYPES OF REAL SQUARE MATRICES

1.5.1 Symmetric Matrix

A square matrix A is called symmetric if $A^t = A$.

Example:

$\begin{pmatrix}1 & 3 & 4\\ 3 & 0 & -7\\ 4 & -7 & 4\end{pmatrix}$ is symmetric.

Properties:

(*i*) If A, B are Symmetric matrices of same order, then kA, $aA + bB$, $AB + BA$, A^n ($n \in N$), are symmetric (where 'k' 'a' and 'b' are real numbers).

(*ii*) If A, B are Symmetric matrices of same order, then AB is symmetric if and only if $AB = BA$.

(*iii*) For any square matrix A, the matrix $A + A^T$ is always symmetric.

1.5.2 Skew-Symmetric Matrix

A square matrix A is called skew-symmetric if $A^T = -A$.

Example:

$\begin{pmatrix}0 & -3 & 4\\ 3 & 0 & -7\\ -4 & 7 & 0\end{pmatrix}$ is skew-symmetric.

Properties:

(*i*) If A, B are skew-symmetric matrices of same order, then kA, $aA \pm bB$, $AB - BA$, A^n ($n \in N$) are all skew-symmetric.

(*ii*) If A, B are skew-symmetric matrices of same order, then AB is skew-symmetric if and only if $AB + BA = 0$.

(*iii*) diagonal elements of skew symmetric matrix are all zero.

(*iv*) For any square matrix A, the matrix $A - A^T$ is always skew-symmetric.

1.5.3 Orthogonal Matrix

A square matrix A is said to be orthogonal if and only if $AA^T = A^TA = I$

Example:

$$A=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix} \text{ is orthogonal.}$$

Properties:

(*i*) If A is orthogonal, then $|A| = \pm 1$ and hence A is non-singular.

(*ii*) If A, B are orthogonal matrices of same order, then A^T, AB, BA, A^{-1} are all orthogonal matrices.

(*iii*) If A is orthogonal, then $A^T = A^{-1}$.

1.5.4 Idempotent Matrix

A square matrix 'A' is called idempotent if $A^2 = A$.

Example:

$$\text{The matrix } \begin{pmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{pmatrix} \text{ is idempotent.}$$

Properties:

(*a*) If A, B are idempotent matrices of same order, then

(*i*) AB is idempotent if and only if $AB = BA$

(*ii*) $A + B$ is idempotent if and only if $AB = BA = O$

(*iii*) $I - A$ is idempotent

(*b*) If $AB = A$, $BA = B$, then A and B are both idempotent matrices.

1.5.5 Involutary Matrix

A square matrix 'A' is called involutary if $A^2 = 1$.

Example:

$$\text{The matrix } \begin{pmatrix}-5 & -8 & 0\\ 3 & 5 & 0\\ 1 & 2 & -0\end{pmatrix} \text{ is involutary.}$$

1.5.6 Nilpotent Matrix

A square matrix 'A' is said to be nilpotent of index 'k' if $A^k = O$ for some positive number k 'k' should be the least positive integer).

Example:

(*i*) The matrix $A=\begin{bmatrix}1 & -1 & 1\\ -3 & 3 & -3\\ -4 & 4 & -4\end{bmatrix}$ is nilpotent of index '2' (since $A^2 = O$)

(*ii*) $\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ is a nilpotent matrix of index '2' (since $A^2 = O$).

1.6 COMPLEX MATRICES AND THEIR TYPES

1.6.1 Complex Conjugate of a Matrix

If $A = [a_{ij}]_{m\times n}$ be a complex matrix of order $m \times n$, then the complex conjugate of $A = [a_{ij}]_{m\times n}$, is denoted by $\overline{A}$ and is defined by $\overline{A} = [\overline{a}_{ij}]_{m\times n}$. Thus, the complex conjugate matrix of a given matrix is obtained by replacing all the elements in the given matrix by their respective complex conjugates.

Example:

$$\text{If } A=\begin{pmatrix}5 & -8+i & i\\ 3 & 5 & 0\\ 1-i & 2 & -i\end{pmatrix}, \text{ then}$$

$$\overline{A}=\begin{pmatrix}5 & -8-i & -i\\ 3 & 5 & 0\\ 1+i & 2 & i\end{pmatrix}.$$

Properties:

If A, B are complex matrices, then

(*i*) $\overline{\overline{A}} = A$

(*ii*) $\overline{A+B} = \overline{A} + \overline{B}$

(*iii*) $\overline{kA} = \overline{k} \times \overline{A}$, where '$k$' is any number (real or complex)

(*iv*) $\overline{AB} = \overline{A} \times \overline{B}$, provided AB is defined.

(*v*) $\overline{A^n} = (\overline{A})^n$

(*vi*) $\overline{A} = A$ if and only if A is real.

(*vii*) $\overline{A} = -A$ if and only if A purely imaginary

1.6.2 Transposed Conjugate of a Matrix

The transposed conjugate of a complex matrix A is denoted by A^θ and is defined by $A^\theta = (\overline{A})^T = \overline{A^T}$.

Example:

$$\text{If } A=\begin{pmatrix}5 & -8+i & i\\ 3 & 5 & 0\\ 1-i & 2 & -i\end{pmatrix}, \text{ then}$$

$$A^\theta=\begin{pmatrix}5 & 3 & i+1\\ -8-i & 5 & 2\\ -i & 0 & i\end{pmatrix}.$$

Properties:

If A, B are complex matrices of same order, then

(*i*) $(A^\theta)^\theta = A$

(*ii*) $(A + B)^\theta = A^\theta + B^\theta$

(*iii*) $(kA)^\theta = \overline{k}(A)^\theta$ where 'k' is any number (real or complex)

(*iv*) $(AB)^\theta = B^\theta A^\theta$

(*v*) $(A^n)^\theta = (A^\theta)^n$.

1.6.3 Unitary matrix

A square complex matrix A is said to be unitary matrix if $AA^\theta = A^\theta A = I$.

Example:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} \text{ is unitary.}$$

Properties:

(*i*) If A is unitary then A^T, A^{-1} are both unitary matrix.

(*ii*) If A, B are unitary matrices of same order, then AB and BA are also unitary matrices.

1.6.4 Hermitian Matrix

A square matrix 'A' is called hermitian if $A^\theta = A$.

Example:

$$A = \begin{pmatrix} 1 & 2-i & 3-i \\ 2+i & 2 & 1+2i \\ 3+i & 1-2i & 4 \end{pmatrix} \text{ is hermitian.}$$

Properties:

(*i*) If A, B are hermitian matrices of same order, then

kA, $aA + bB$, AA^θ, $A^\theta A$, $AB + BA$, $\bar{A}$, A^n, $A + A^\theta$ are also hermitian matrices.

(*ii*) If A, B are hermitian matrices of same order, then AB is hermitian if and only if $AB = BA$.

1.6.5 Skew-Hermitian Matrix

A square matrix 'A' is called skew-hermitian if $A^\theta = -A$

Example:

$$\begin{pmatrix} 3i & 3+4i & 4-5i \\ -3+4i & -4i & 5+6i \\ -4-5i & -5+6i & 0 \end{pmatrix} \text{ is skew-hermitian.}$$

Properties:

(*i*) $A = [a_{ij}]_{n\times n}$ is skew-hermitian if and only if $\overline{a_{ij}} = -a_{ji}$

(*ii*) In a skew-hermitian matrix, all the elements lying in the principal diagonal are purely imaginary or zero.

(*iii*) If A, B are skew-hermitian matrices of same order, then kA, $aA + bB$, $AB - BA$, $A - A^\theta$, $\bar{A}$ are also skew-hermitian matrices.

(*iv*) If A is hermitian, then iA is skew-hermitian.

(*v*) If A is skew-hermitian, then iA is hermitian.

(*vi*) For any square matrix A,

$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$. Thus every square matrix can be expressed as a sum of a Hermitian matrix $\frac{1}{2}(A + A^\theta)$ and a skew-hermitian matrix $\frac{1}{2}(A - A^\theta)$.

1.7 RANK OF A MATRIX

1.7.1 Elementary Transformations

The following transformations are known as elementary transformations:

(*i*) **Interchange of two rows (or columns).**

Suppose we interchange 1st row and 2nd row in a matrix, then this operation will be denoted by $R_1 \leftrightarrow R_2$. Again, if we interchange 3rd column and 4th column in a matrix, then this operation will be denoted by $C_3 \leftrightarrow C_4$.

(*ii*) **The multiplication of the elements of a row (or column) by a non-zero number.**

Suppose we multiply each element of 2nd row by 'k', then the operation is denoted by $R_2 \to kR_2$. In a similar way, if we multiply each element of 3rd column by 'k', then the operation is denoted by $C_3 \to kC_3$.

(*iii*) **The addition to the elements of a row (or column), the corresponding elements of a row (or column) multiplied by any number.**

Suppose we first multiply the 2nd row by 'k' and add it with the 1st row, then the corresponding operation is denoted by $R_1 \to R_1 + kR_2$. Similarly if we first multiply the 2nd column by 'k' and add it with the 1st column, then the corresponding operation is denoted by $C_1 \to C_1 + kC_2$.

1.7.2 Equivalent Matrices

Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary operations. The symbol ~ is used for equivalence. Thus if A and B are equivalent matrices, then we write A~B. The relation '~' forms an equivalence relation.

Example:

$$\text{If } A = \begin{pmatrix} 3 & 4 & 5 \\ -3 & -4 & 6 \\ -5 & 6 & 0 \end{pmatrix}, \text{ then}$$

$$A = \begin{pmatrix} 3 & 4 & 5 \\ -3 & -4 & 6 \\ -5 & 6 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 3 & 4 & 5 \\ 0 & 0 & 11 \\ -5 & 6 & 0 \end{pmatrix} \quad (\text{by } R_2 \to R_2 + R_1)$$

$$\sim \begin{pmatrix} 3 & 4 & 11 \\ 0 & 0 & 11 \\ -5 & 6 & -10 \end{pmatrix} \quad (\text{by } C_3 \to C_3 + 2C_1)$$

$$= B \text{ (say) where } B = \begin{pmatrix} 3 & 4 & 11 \\ 0 & 0 & 11 \\ -5 & 6 & -10 \end{pmatrix}$$

Then A and B are equivalent.

1.7.3 Rank of a Matrix

A positive real number 'r' is said to be the rank of a given matrix A of order $m\times n$ if

(i) there exist a square sub-matrix of order 'r' whose determinant is non-zero

(ii) the determinant value of every square sub-matrix of order higher than 'r' is zero.

The symbols $r(A)$, $\rho(A)$, rank(A) are used to denote the rank of a matrix A.

Example:

Consider a matrix $A = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 1 & 2 \\ 6 & 8 & 10 \end{pmatrix}$. Then det ($A$) = 0.

So rank of A can't be '3'.

Let us consider a square sub-matrix $B = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$. Then $\det(B) = -1 \neq 0$. Thus there exist a square sub-matrix B of order '2' whose determinant value is non-zero. Hence rank (A) = 2.

Properties:

(i) If A is a matrix A of order $m\times n$, then rank $(A) \leq \min\{m, n\}$.

(ii) rank (O) = 0

(iii) rank $(I_n) = n$, where I_n is the identity matrix of order 'n'.

(iv) rank (A) = n if A is non-singular square matrix of order 'n' and vice versa.

(v) if A is singular, then rank $(A) < n$ and vice versa, provided A is a square matrix of order 'n'.

(vi) If A and B are matrices of same order, then $r(A + B) \leq r(A) + r(B)$.

(vii) $r(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

($viii$) rank (A) = rank (A^T) and rank (A) = rank (AA^T)

(ix) rank (A) = rank (A^θ) and rank (A) = rank (AA^θ)

(x) Two equivalent matrices have same rank.

(xi) Rank of a matrix remains the same when we multiply any non-singular matrix with it.

(xii) The rank of a skew-symmetric matrix can't be '1'.

($xiii$) Let A be a square matrix of order 'n'. Then

$$\text{rank}(\text{adj}A) = \begin{cases} n, \text{ if rank}(A) = n \\ 1, \text{ if rank}(A) = n-1. \\ 0, \text{ if rank}(A) < n-1 \end{cases}$$

1.7.4 Determination of the Rank of a Matrix

There are many forms namely 'normal form', 'echelon form', 'row reduced echelon form' which are very useful in determining the rank of a matrix. But here we will use a secondary method (quickest method) to find the rank of a matrix discussed below:

"Let A be a matrix of order $m\times n$. Then apply elementary row transformations to get maximum number of zero rows. In this case the number of non-zero rows appearing in the final equivalent matrix will be equal to the rank of the given matrix."

Example:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 5 & 1 \\ 2 & 0 & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 \\ 3 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \left(\text{by } R_2 \to R_2 - 2R_1\right)$$

which has only two non-zero rows. Therefore rank(A) = 2.

1.8 SYSTEM OF LINEAR EQUATIONS AND THEIR SOLUTIONS

1.8.1 Introduction

A linear equation in 'n' unknowns $x_1, x_2, x_3, \ldots x_n$ is an equation of the form

$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$.

If $b = 0$, then the equation is called a homogeneous equation; otherwise it is called a non-homogeneous equation.

A system of 'm' non-homogeneous linear equations in 'n' unknowns is given by:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \qquad \ldots\ldots\ldots(i)$$

If there exist at least one set of values of $x_1, x_2, x_3, \ldots, x_n$ satisfying all the equations in (i), then the system of equations given by (i), is said to be consistent; otherwise inconsistent. In other words, the system of equations given by (i) is consistent if it has a solution; otherwise it is inconsistent.

Let us consider the following system of equations:

$$\left.\begin{aligned} x - y &= 2 \\ 2x + y &= 4 \end{aligned}\right\} \qquad \ldots(ii)$$

$$\left.\begin{aligned} x + y &= 2 \\ 2x + 2y &= 3 \end{aligned}\right\} \qquad \ldots(iii)$$

The system of equations given by (ii) has a solution $x = 2$, $y = 0$ and so it is consistent.

On the other hand, no pair of values of x and y simultaneously satisfy both the equations in (iii). So it is inconsistent.

Remember:

The non-homogeneous system of linear equations given in (*i*) can also be rewritten as $AX = B$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mn} \end{pmatrix},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{bmatrix}$$

Here A is called the coefficient matrix. The augmented matrix, denoted by $[A\colon B]$, is defined by

$$\left[A : B\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

1.8.2 Methods for Solving Non-Homogeneous System of Linear Equations

1.8.2.1 Cramers Rule

Let us consider a system of equations consisting 3 non-homogeneous linear equations in 3 unknowns given by:

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\} \qquad \ldots(i)$$

Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

Then

Case-I:

If $\Delta \neq 0$, then (*i*) has an unique solution and so consistent. In this case the solution is given by $x = \dfrac{\Delta_1}{\Delta},\ y = \dfrac{\Delta_2}{\Delta},\ z = \dfrac{\Delta_3}{\Delta}$.

Example:

Let us consider the system of equations:

$2x - z = 1,\ 2x + 4y - z = 1,\ x - 8y - 3z = -2.$

Then $\Delta = \begin{vmatrix} 2 & 0 & -1 \\ 2 & 4 & -1 \\ 1 & -8 & -3 \end{vmatrix} = -20 \neq 0.$

So the given system of equations has a unique solution.

$$\Delta_1 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 4 & -1 \\ -2 & -8 & -3 \end{vmatrix} = -20,$$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 1 & -2 & -3 \end{vmatrix} = 0,$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & 1 \\ 2 & 4 & 1 \\ 1 & -8 & -2 \end{vmatrix} = -20.$$

Hence, by Cramer's rule, the solution is:

$$x = \frac{\Delta_1}{\Delta} = 1, y = \frac{\Delta_2}{\Delta} = 0, z = \frac{\Delta_3}{\Delta} = 1.$$

Case-II:

If $\Delta = 0$ and at least one of $\Delta_1, \Delta_2, \Delta_3$ be zero, then (*i*) has no solution and so inconsistent. Thus the following sub cases arise:

(*i*) If $\Delta = 0$, $\Delta_1 \neq 0$, then (*i*) has no solution
(*ii*) If $\Delta = 0$, $\Delta_2 \neq 0$, then (*i*) has no solution
(*iii*) If $\Delta = 0$, $\Delta_3 \neq 0$, then (*i*) has no solution

Case-III:

If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then (*i*) has infinite number of solutions and so consistent.

Example:

Let us consider the system of equations:

$x + y + z = 6,\ x + 2y + 3z = 10,\ 2x + 2y + 2z = 12.$

Then

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{vmatrix} = 0,$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ 12 & 2 & 2 \end{vmatrix} = 0,$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 2 & 12 & 2 \end{vmatrix} = 0,$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 2 & 2 & 12 \end{vmatrix} = 0.$$

Since $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, hence the system has infinite number of solutions and so is consistent.

Remark:

If we consider a system of equations consisting two non- homogeneous linear equations in two unknowns given by:

$$\left.\begin{aligned} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{aligned}\right\}$$

.....................(*i*)

Then

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

Then

- If $\Delta \neq 0$, then (*i*) has an unique solution and so consistent. In this case the solution is given by $x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta}$.
- If $\Delta = 0$ and at least one of Δ_1, Δ_2 be zero, then (1) has no solution and so inconsistent. Thus the following sub cases arise:

  (*i*) If $\Delta = 0$, $\Delta_1 \neq 0$, then (*i*) has no solution

  (*ii*) If $\Delta = 0$, $\Delta_2 \neq 0$, then (*i*) has no solution

Example:

Let us consider the system of equations:

$2x + y = 1,\ 4x + 2y = 5$

Then

$$\Delta = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0, \text{ but } \Delta_2 = \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} = 6 \neq 0.$$

So the system has no solution.

- If $\Delta = \Delta_1 = \Delta_2 = 0$, then (*i*) has infinite number of solutions and so consistent.

1.8.2.2 Matrix Method

Let us consider a system of equations consisting 3 non-homogeneous linear equations in 3 unknowns given by:

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\}$$

The above system of equations can be rewritten as: $AX = B$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$

then we say that A^{-1} exist and the solution is given by: $X = A^{-1}B$.

Example:

Let us consider the system of equations:

$x - y + z = 0,\quad 2x + 3y - 5z = 7,\quad 3x - 4y - 2z = -1.$

The above system of equations can be re-written as: $AX = B$, where

Here

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -5 \\ 3 & -4 & -2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 0 \\ 7 \\ -1 \end{pmatrix}.$$

$$\text{Here } |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & -5 \\ 3 & -4 & -2 \end{vmatrix} = -32 \neq 0.$$

so A^{-1} exist and the solution is given by: $X = A^{-1}B$.

Now $X = A^{-1}B$

$$= \frac{1}{|A|} adj(A) \times B$$

$$= \frac{1}{(-32)} \begin{pmatrix} -26 & -6 & 2 \\ -11 & -5 & 7 \\ -17 & 1 & 5 \end{pmatrix} \times \begin{bmatrix} 0 \\ 7 \\ -1 \end{bmatrix}$$

$$= \frac{-1}{32} \begin{bmatrix} 0 - 42 - 2 \\ 0 - 35 - 7 \\ 0 + 7 - 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-44}{-32} \\ \frac{-42}{-32} \\ \frac{2}{-32} \end{bmatrix} = \begin{bmatrix} \frac{11}{8} \\ \frac{21}{16} \\ \frac{-1}{16} \end{bmatrix}$$

$$\therefore x = \frac{11}{8},\ y = \frac{21}{16},\ z = \frac{-1}{16}.$$

1.8.2.3 Rank Method

Let us consider a system of equations consisting ‘m’ non-homogeneous linear equations in ‘n’ unknowns given by:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots + a_{mn}x_n = b_m \end{aligned}\right\}$$

.....................(2)

Then the above system of equations can be re-written as $AX = B$ (discussed earlier).

Case-I:

If rank (A) = rank $([A : B]) = n$, then the system given by (2) has an unique solution and so the system is consistent.

Example:

Let us consider the system of equations:

$2x - z = 1$, $2x + 4y - z = 1$, $x - 8y - 3z = -2$.

$$[A : B] = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 2 & 4 & -1 & 1 \\ 1 & -8 & -3 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -8 & -3 & -2 \\ 2 & 4 & -1 & 1 \\ 2 & 0 & -1 & 1 \end{pmatrix} \quad (\text{by } R_1 \leftrightarrow R_3)$$

$$\sim \begin{pmatrix} 1 & -8 & -3 & -2 \\ 0 & 20 & 5 & 5 \\ 0 & 16 & 5 & 5 \end{pmatrix}$$

$(\text{by } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1)$

$$\sim \begin{pmatrix} 1 & -8 & -3 & -2 \\ 0 & 20 & 5 & 5 \\ 0 & -4 & 0 & 0 \end{pmatrix}$$

$(by\ R_3 \to R_3 - R_2)$

which has 3 non-zero rows.

So rank $([A{:}B]) = 3$.

Ignoring the last column of the final equivalent matrix of $[A{:}B]$, we see that there are three non-zero rows. So rank$(A) = 3$.

Thus rank(A) = rank $([A{:}B]) = 3$ (=number of unknowns). Hence the system is consistent and has the unique solution.

Case-II:

If rank (A) = rank $([A{:}B]) < n$, then the system given by (2) has an infinite number of solutions and so the system is consistent.

Example:

Let us consider the system of equations:

$x + y + z = 6$, $x + 2y + 3z = 1$

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, $[\text{A} : \text{B}] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$.

Then $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \ (\text{by } R_2 \to R_2 - R_1)$$

which have 2 non-zero rows.

Hence rank$(A) = 2$.

$$[\text{A} : \text{B}] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -5 \end{bmatrix} (\text{by } R_2 \to R_2 - R_1)$$

which has two non-zero rows.

So rank $([A{:}B]) = 2$.

Thus rank(A) = rank $([A{:}B]) = 2 < 3$ (= number of unknowns). Hence the system has an infinite number of solutions and so the system is consistent.

Case-III:

If rank (A) ≠ rank $([A{:}B])$, then the system given by (2) has no solution and so the system is inconsistent.

Example:

Let us consider the system of equations:

$2x + y - z = 12$, $x - y - 2z = -3$, $3y + 3z = 10$.

$$[A : B] = \begin{pmatrix} 2 & 1 & -1 & 12 \\ 1 & -1 & -2 & -3 \\ 0 & 3 & 3 & 10 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 3 & 3 & 18 \\ 1 & -1 & -2 & -3 \\ 0 & 0 & 3 & 10 \end{pmatrix}$$

$(by\ R_1 \to R_1 - 2\text{R}_2)$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 8 \\ 1 & -1 & -2 & -3 \\ 0 & 3 & 3 & 10 \end{pmatrix}$$

$(by\ R_1 \to R_1 - \text{R}_3)$

$$\sim \begin{pmatrix} 0 & 3 & 3 & 10 \\ 1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

$(by\ R_1 \leftrightarrow \text{R}_3)$

which has 3 non-zero rows.

So rank $([A : B]) = 3$.

Ignoring the last column of the final equivalent matrix of $[A{:}\ B]$, we see that there are two non-zero rows. So rank $(A) = 2$.

Thus rank (A) ≠ rank $([A{:}B])$. Hence the system is inconsistent.

1.8.3 Homogeneous System of Linear Equations

A homogeneous system of m linear equations in n unknowns is given by:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots + a_{mn}x_n &= 0 \end{aligned}\right\} \quad \ldots(i)$$

The above homogeneous system can also be rewritten as $AX = O$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}.$$

Here A is called the coefficient matrix. It is easy to verify that $x_1 = x_2 = x_3 = \dots = x_n = 0$ is a solution of (i) [called the trivial solution]. Hence the system (i) is consistent.

Further if rank (A) < n(= number of unknowns), then the system (i) has an infinite number of solutions.

Remember:

If a system has 'n' homogeneous equations in 'n' unknowns, then

System has infinite number of solutions ⇔ determinant of the co-efficient matrix is zero.

Example:

Let us consider the system of equations:

$x + 2y + 3z = 0$, $2x + 3y + 4z = 0$, $3x + 4y + 5z = 0$.

$$\text{Here } A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix}$$

$\left(\text{using } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1\right)$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \left(\text{using } R_3 \to R_3 - 2R_2\right)$$

which has '2' non-zero rows.

Hence rank (A) = 2 < 3 (= number of unknowns). Thus the given system of equations has an infinite number of solutions.

Alternative method:

Here $\det(A) = (15 - 16) - 2(10 - 12) + 3(8 - 9) = 0$.

Hence the system has infinite number of solutions.

1.9 EIGEN VALUES AND EIGEN VECTORS

1.9.1 Characteristic Roots (Eigen Values) of a Matrix

If A be a square matrix of order n and I be an identity matrix of order n, then $A - \lambda I$ is called the characteristic matrix of A, λ being a scalar.

The determinant $|A - \lambda I|$ is called the characteristic polynomial of A, which is basically a polynomial in λ of n th degree.

The equation $|A - \lambda I| = 0$ *i.e.*, $\det(A - \pi I) = 0$ is called characteristic equation of the matrix A and the roots of the characteristic equation are called the eigen values or characteristic roots of the matrix A.

Example:

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}.$$

$$\text{Then } A - \lambda I = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{bmatrix}.$$

Then the Characteristic equation is given by

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0$

or, $(1 - \lambda)(-1 - \lambda) - 3 = 0$

or, $-(1 - \lambda^2) = 3$

or, $\lambda^2 = 4$

or, $\lambda = \pm 2$

Therefore eigen values of A are 2 and – 2.

Properties:

(i) If A be a square matrix, then A and A^T both have the same eigen values.

(ii) If A be a square matrix of order n, then $\det(A)$ = product of all eigen values of A.

(iii) If the matrix A is invertible and λ is an eigen values of A, then λ^{-1} will be an eigen value of A^{-1}

(iv) If a square matrix A have eigen values $\lambda_1, \lambda_2, \dots, \lambda_m$ then the matrix kA have eigen values $k\lambda_1, k\lambda_2, \dots, k\lambda_m$.

(v) If $\lambda_1, \lambda_2, \dots, \lambda_k$ are eigen values of a square matrix A, then $\lambda_1^m, \lambda_2^m, \dots, \lambda_k^m$ will be the eigen values of A^m, m being a positive integer.

(vi) If both A and P are square matrices and P is non-singular, then, both the matrices A and $P^{-1}AP$ have the same eigen values.

(vii) If both A and B are square invertible matrices of same order, then AB and BA will have the same eigen values.

($viii$) '0' is an eigen value of a square matrix A if and only if A is singular.

(ix) All the eigen values of an identity matrix is '1'.

(x) The eigen values of a lower (or upper) triangular matrix are the diagonal elements of the matrix.

(xi) The eigen values of a diagonal matrix are the diagonal elements of the matrix.

(xii) The eigen values of a unitary matrix are of unit modulus i.e., if λ be an eigen value of an unitary matrix A, then $|\lambda| = 1$.

($xiii$) The eigen values of a Hermitian matrix are all real.

(xiv) The eigen values of a real symmetric matrix are real.

(*xv*) The eigen values of a skew-symmetric matrix are either purely imaginary or zero.

(*xvi*) The eigen values of an orthogonal matrix are of unit modulus.

(*xvii*)The eigen values of an idempotent matrix are 0 and 1.

(*xviii*)(*Cayley-Hamilton theorem*). Every square matrix satisfies its own characteristic equation.

Application:

If $\lambda^2 + \lambda + 3 = 0$ be the characteristic equation of a square matrix A of order 3, then we can write $A^2 + A + 3I = O$.

1.9.2. Trace of a Matrix

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. Then the trace of the matrix A is denoted by tr(A) and is defined by

$$tr(A) = \sum_{r=1}^{n} a_{rr}$$

Thus trace of a given square matrix is equal to the sum of the diagonal elements of the matrix.

Example:

Let $A = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 1 \\ 0 & -1 & 4 \end{pmatrix}$

Then tr(A)

= Sum of the diagonal elements = 2 + 7 + 4 =13.

Properties:

If A and B be two square matrices of same order. Then

(*i*) tr (kA) = k tr(A), k being a scalar

(*ii*) tr ($A + B$) = tr (A) + tr (B)

(*iii*) tr (AB) = tr (BA)

(*iv*) tr (A) = sum of all eigen values of A.

Remember:

(*i*) If $\lambda^2 + a\lambda + b$ be the characteristic polynomial of a 2 × 2 matrix A, then tr(A) = $-a$ and det(A) = b.

(*ii*) If $\lambda^3 + a\lambda^2 + b\lambda + c$ be the characteristic polynomial of a 3 × 3 matrix A, then tr(A) = $-a$ and det(A) = $-c$.

1.9.3. Eigen vectors or Characteristic Vectors

Let A be a square matrix of order n and be λ an eigen value of A. If there exists a column matrix X ($\neq 0$) of order $n \times 1$ such that $AX = \lambda$ X holds, then X is called an eigen vector or characteristic vector corresponding to the eigen value λ.

Remember:

If A is a square matrix of order 2, then we assume X in the form $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and if A is a square matrix of order 3, then we assume X in the form $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Example:

Let $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$.

Then as discussed earlier, the Eigen value of A are 2 and – 2 (see section 1.9.1).

For λ = 2:

$$AX = \lambda X \;\Rightarrow\; \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \left[\text{where } X = \begin{pmatrix} x \\ y \end{pmatrix}\right]$$

$$\Rightarrow \begin{pmatrix} x+3y \\ x-y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

Equating the corresponding elements we get, $x + 3y = 2x$ and $x - y = 2y$ *i.e.*, $x - 3y = 0$

$$\therefore \quad X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ y \end{pmatrix} = y\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Which is the Eigen vector for λ = 2

For λ = –2.

$$AX = \lambda X \Rightarrow \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \left[\text{where } X = \begin{pmatrix} x \\ y \end{pmatrix}\right]$$

$$\Rightarrow \begin{pmatrix} x+3y \\ x-y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

Equating the corresponding elements we get, $x + 3y = -2x$ *i.e.*, $x + y = 0$ *i.e.*, $y = -x$

and $x - y = -2y$ *i.e.*, $x + y = 0$ *i.e.*, $y = -x$.

$$\therefore \quad X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which is the eigen vector for λ = –2.

1.10 VECTORS

1.10.1 Introduction

An ordered n-tuple $X = (x_1, x_2, x_3, ..., x_n)$ is called a n-vector. $x_1, x_2, x_3, ..., x_n$ are called components of the vector X.

Remember:

(*i*) $(x_1, x_2, x_3, ..., x_n)$ is also termed as n-tuple vector.

(*ii*) Every row matrix as well as every column matrix is also termed as a vector. Thus [2 5 1] and $\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ are vectors.

(*iii*) A vector whose components belong to a field F is said to be a vector over the field F. The set of all n-tuple vectors over a field F (denoted by $V_n(F)$ is called a vector space over F. The elements of F are called scalars.

(*iv*) A subset S of a vector space $V_n(F)$ is called a vector subspace if $aX + bY \in S \quad \forall\, X, Y \in S$ and $a, b \in F$.

Example:

Let $S = \left\{ X = (x, y, z) \in \mathbb{R}^3 : x + y + z = 0 \right\}$

(where $\mathbb{R}^3$ is the set of all points in three dimensional space)

Then for $X = (x, y, z)$, $Y = (x', y', z') \in S$ and $a, b \in F$, we have

$$aX + bY = a(x, y, z) + b(x', y', z')$$
$$= (ax, ay, az) + (bx', by', bz')$$
$$= (ax + bx', ay + bx', az + bz')$$

Now $(ax + bx') + (ay + bx') + (az + bz')$

$$= a(x + y + z) + b(x' + y' + z')$$
$$= a \times 0 + b \times 0$$
$$= 0$$
$$\left(\begin{array}{c} \because X, Y \in S \\ \Rightarrow x + y + z = 0,\ x' + y' + z' = 0 \end{array} \right)$$

So $aX + bY \in S$.

Hence S is a vector sub-space.

1.10.2 Linear Dependence and Linear Independence

(*i*) The vectors $X_1, X_2, X_3, ..., X_k$ are said to be linearly dependent if there exist scalars $a_1, a_2, a_3, ..., a_k$ (not all zero) such that

$$a_1X_1 + a_2X_2 + a_3X_3 + ... + a_kX_k = O$$

Example:

Let $X_1 = (1, 0, -2)$ and $X_2 = (-2, 0, 4)$. Then

$$2X_1 + X_2 = 2(1, 0, -2) + (-2, 0, 4)$$
$$= (2 - 2, 0 + 0, -4 + 4)$$
$$= (0, 0, 0) = O$$

Therefore the vectors X_1 and X_2 are linearly dependent.

(*ii*) The vectors $X_1, X_2, X_3, ..., X_k$ are said to be linearly independent if there exist scalars $a_1, a_2, a_3, ..., a_k$ such that

$$a_1X_1 + a_2X_2 + a_3X_3 + ... + a_kX_k = O$$
$$\Rightarrow a_1 = a_2 = a_3 = ... = a_k = 0$$

Example:

Let $X_1 = (1, 0 - 2)$ and $X_2 = (0, 3, 0)$.

Then

$$a_1X_1 + a_2X_2 = O$$
$$\Rightarrow a_1(1, 0, -2) + a_2(0, 3, 0) = O$$
$$\Rightarrow (a_1, 0, -2a_1) + (0, 3a_2, 0) = O$$
$$\Rightarrow (a_1 + 0, 0 + 3a_2, -2a_1 + 0) = (0, 0, 0)$$
$$\Rightarrow a_1 = 0, 3a_2 = 0, -2a_1 = 0$$
$$\Rightarrow a_1 = a_2 = 0$$

Therefore the vectors X_1 and X_2 are linearly independent.

Remember:

(*i*) Rank of a matrix = number of linearly independent eigen vectors of the matrix.

(*ii*) Rank of a matrix = number of linearly independent columns (or rows) of the matrix.

(*iii*) If A be a matrix of order $n \times n$ such that A has 'n' distinct eigen values, then A has 'n' linearly independent eigen vectors.

(*iv*) Two functions $f(x)$ and $g(x)$ are said to be linearly independent if $\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} \neq 0$; otherwise they called linearly dependent.

(*v*) Three functions $f(x)$, $g(x)$ and $h(x)$ are said to be linearly independent if $\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} \neq 0$; otherwise they are called linearly dependent.

(*vi*) Three vectors $X_1 = (x_1, x_2, x_3)$, $X_2 = (y_1, y_2, y_3)$ and $X_3 = (z_1, z_2, z_3)$ are said to be linearly independent if $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \neq 0$; otherwise they are said to be linearly dependent.

Example:

$$\because \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 5 \\ 2 & -2 & 4 \end{vmatrix} = 0,$$

the vectors $(1, -1, 2), (3, 0, 5)$ and $(2, -2, 4)$ are linearly dependent.

1.10.3 Inner Product and Norm of Vectors

(*i*) The inner product of two vectors $X = (x_1, x_2, x_3, ..., x_n)$ and $Y = (y_1, y_2, y_3, ..., y_n)$ over the field R of real numbers is denoted by $\langle X, Y \rangle$ and is defined by

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + ... + x_ny_n$$

Example:

Let $X_1 = (1, 0, -2)$ and $X_2 = (3, 3, 4)$.

Then $\langle X, Y \rangle = 1 \times 3 + 0 \times 3 + (-2) \times 4 = -5$.

(*ii*) The norm of a vector $X = (x_1, x_2, x_3, ... x_n)$ over the field R of real numbers is denoted by $\|X\|$ and is defined by $\|X\| = \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2 + ... + {x_n}^2}$

Example:

Let $X = (1, 3, -2)$. Then $\|X\| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}.$

Remember:

If θ be the angle between the two vectors X and Y, then $\cos\theta = \dfrac{\langle X, Y \rangle}{\|X\|\|Y\|}$.

Example:

Let $X_1 = (2, 1, -2)$ and $X_2 = (3, 0, 4)$.

Then $\langle X,Y\rangle = 2\times 3+0\times 1+(-2)\times 4 = -2.$ Also

$$\|X\| = \sqrt{2^2+1^2+(-2)^2} = 3,$$

$$\|Y\| = \sqrt{3^2+0^2+4^2} = 5.$$

$$\therefore \cos\theta = \frac{\langle X,Y\rangle}{\|X\|\|Y\|} = \frac{-2}{3\times 5} = -\frac{2}{15}.$$

Now, So $\theta = \cos^{-1}\left(-\frac{2}{15}\right).$

Remember : $\|x\|^2 = \langle x, x\rangle$.

1.10.4 Orthogonal and Orthonormal Vectors

(*i*) Two vectors X and Y are said to be orthogonal if $\langle X, Y\rangle = 0$.

Example:

Let $X_1 = (2, 1, -2)$ and $X_2 = (-3, 0, -3)$.

Then $\langle X,Y\rangle = 2\times(-3)+1\times 0+(-2)\times(-3) = 0.$

Therefore the vectors X and Y are orthogonal.

(*ii*) A set of n-tuple vectors $X_1, X_2, X_3, ..., X_k$ is called an orthonormal set of vectors if

(*i*) $\|X_i\| = 1$ for $i = 1,2,3,...,k$

(*ii*) $\langle X_i, Y_j\rangle = 0$ for $i \neq j;\ i,j = 1,2,3,...,k$

Example:

Let $X_1 = \frac{1}{\sqrt{3}}(1,1,1)$ and $X_2 = \frac{1}{\sqrt{2}}(-1,0,1)$.

Then

$$X_1 = \frac{1}{\sqrt{3}}(1,1,1) = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \text{and}$$

$$X_2 = \frac{1}{\sqrt{2}}(-1,0,1) = \left(\frac{-1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right).$$

$$\therefore \langle X_1,X_2\rangle = \frac{1}{\sqrt{3}}\times\left(\frac{-1}{\sqrt{2}}\right)+0+\frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{2}}$$
$$= 0$$

Therefore the vectors X_1 and X_2 are orthogonal.

Also

$$\|X_1\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2} = 1,$$

$$\|X_2\| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2+0^2+\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

Hence the vectors X_1 and X_2 form an orthonormal set of vectors.

1.10.5 Basis and Dimension

A collection of n-tuple vectors $X_1, X_2, X_3, ... X_k$ is said to form a basis of the vector space $V_n(F)$ if

(*i*) $X_1, X_2, X_3, ..., X_k$ are linearly independent

(*ii*) any arbitrary n-tuple vector X can be expressed as:

$$X = b_1X_1 + b_2X_2 + b_3X_3 + ... + b_kX_k$$

where $b_1, b_2, b_3, ..., b_k \in F$.

If the n-tuple vectors $X_1, X_2, X_3, ..., X_k$ forms a basis of $V_n(F)$, then we say that dimension of the vector space $V_n(F)$ is 'k'.

Example:

Let $X_1 = (1,0,0), X_2 = (0,1,0)$ and $X_3 = (0,0,1)$.

Then $\begin{vmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{vmatrix} = 1 \neq 0.$

Therefore the vectors X_1, X_2 and X_3 are linearly independent.

Now let $X = (a,b,c)$ be any arbitrary 3-tuple vector.

Then we can write

$$X = (a,b,c) = a(1,0,0)+b(0,1,0)+c(0,0,1)$$
$$= aX_1 + bX_2 + cX_3$$

Hence the vectors X_1, X_2 and X_3 form a basis of $V_3(R)$ and dimension of $V_3(R)$ is '3'.

Fully Solved MCQs (Level-I)

1. For the matrices $A_{3\times1}$, $B_{1\times3}$, $C_{3\times5}$, $D_{5\times3}$ which of the followings is possible?

 (a) AB only (*b*) CD only
 (c) AB and CD only
 (*d*) AB, BA, BC, CD, DA, DC only.

2. A square matrix 'A' of order 2 which commutes with every 2 × 2 matrix is of the form

 (*a*) $\begin{pmatrix}0 & a\\a & 0\end{pmatrix}$ (*b*) $\begin{pmatrix}a & 0\\0 & a\end{pmatrix}$
 (*c*) $\begin{pmatrix}a & 0\\0 & b\end{pmatrix}$ (*d*) $\begin{pmatrix}0 & a\\b & 0\end{pmatrix}$

3. If $A = \begin{pmatrix}0 & \alpha\\\beta & 0\end{pmatrix}$, then $A^3 + A = 0$, whenever

 (*a*) $\alpha\beta = 2$ (*b*) $\alpha\beta = 1$
 (*c*) $\alpha\beta \neq 0$ (*d*) $\alpha\beta = -1$

4. A square matrix becomes a diagonal matrix if and only if

 (*a*) it is upper triangular
 (*b*) it is lower triangular

(c) both lower and upper triangular
(*d*) none of these

5. If a matrix A is symmetric as well as skew-symmetric, then A is
(*a*) Diagonal matrix (*b*) Null matrix
(*c*) Unit matrix (*d*) None of these

6. Let $A = \begin{pmatrix} 6 & x \\ y & 0 \end{pmatrix}$ and $A = A'$. Then
(*a*) $x = 0, y = 6$ (*b*) $x + y = 6$
(*c*) $x = y$ (*d*) None of these

7. If $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 0 \end{pmatrix}$, then
(*a*) AB, BA exist but not equal
(*b*) AB exists but BA does not exist
(*c*) AB does not exist but BA exist
(*d*) AB, BA exist and both are equal

8. If A and B are square matrices of order 3 such that det $(A) = -1$, det $(B) = 3$; then det $(3AB)$ = ?
(*a*) 9 (*b*) 81
(*c*) 27 (*d*) –81

9. If $A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$,, then A. adj A = ?
(*a*) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (*b*) $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$
(*c*) $\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$ (*d*) $\begin{pmatrix} 1 & 3 \\ -3 & 0 \end{pmatrix}$

10. If A, B, C are square matrix of the same order, then $AB = AC \Rightarrow B = C$ if
(*a*) $|A| \neq 0$ (*b*) $|A| = 0$
(*c*) $A = I$ (*d*) $A = 0$

11. If ω is a complex cube root of unity, then the matrix $A = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}$ is
(*a*) Singular (*b*) Non-singular
(*c*) Symmetric (*d*) None of these

12. If $A = \begin{pmatrix} 2x & 0 \\ x & x \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, then x = ?
(*a*) 1 (*b*) 2
(*c*) 1/2 (*d*) – 1

13. If A is a square matrix, then adj$(A') - (\text{adj } A)'$ is equal to
(*a*) $2|A|$ (*b*) $2|A|I$
(*c*) null matrix (*d*) I

14. If A is an idempotent matrix and $A + B = I$, then B is
(*a*) idempotent (*b*) involutary
(*c*) null matrix (*d*) none of these

15. If A is a real skew-symmetric matrix such that $A^2 + I = 0$, then A is
(*a*) idempotent (*b*) involutary
(*c*) null matrix (*d*) orthogonal

16. The matrix $A = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$ is
(*a*) Idempotent (*b*) Orthogonal
(*c*) Nilpotent (*d*) none of these

17. The matrix $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is
(*a*) Orthogonal (*b*) Idempotent
(*c*) Symmetric (*d*) None of these.

18. The matrix $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$ is
(*a*) Unitary (*b*) Idempotent
(*c*) Symmetric (*d*) None of these

19. For the matrix $M = \begin{bmatrix} 7 & 1+i & 2 \\ 1-i & 4 & 3i \\ 2 & -3i & 0 \end{bmatrix}$ which of the following is correct?
(*a*) M is skew-Hermitian and iM is Hermitian
(*b*) M is Hermitian and iM is Skew-Hermitian
(*c*) M and iM both are Hermitian
(*d*) M and iM both are Skew-Hermitian

20. If the rank of the matrix $A = \begin{pmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{pmatrix}$ is '3', then the value of x is?
(*a*) $\neq \frac{3}{5}$ (*b*) $\neq \frac{4}{5}$
(*c*) $\neq \frac{2}{5}$ (*d*) none of these

21. The rank of the unit matrix I of order n is
(*a*) $n - 1$ (*b*) n
(*c*) n^2 (*d*) $n+1$

22. If A is a non-singular matrix of order n, then the rank of A is
(*a*) $n - 1$ (*b*) n
(*c*) 2 (*d*) $n+1$

23. The rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{pmatrix}$ is

(*a*) 2 (*b*) 0
(*c*) 1 (*d*) 3

24. If $A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then rank (A) =?
(*a*) 2 (*b*) 0
(*c*) 1 (*d*) 3

25. The rank of the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 5 & 0 \end{pmatrix}$ is
(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 4

26. The system of given equations $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$ has
(*a*) unique solution (*b*) two solutions
(*c*) infinite number of solutions
(*d*) no solution

27. A system of '*m*' homogeneous linear equations $AX = 0$ in '*n*' unknowns has only trivial solution if
(*a*) $m \neq n$ (*b*) $m = n$
(*c*) rank $(A) = m$ (*d*) rank $(A) = n$

28. The system of equations $4x + 6y = 5$, $6x + 9y = 7$ has
(*a*) a unique solution
(*b*) an infinite number of solutions
(*c*) no solution
(*d*) finite number of solutions

29. The equations $2x - 3y + 6z = 4$, $5x + 7y - 14z = 1$, $3x + 2y - 4z = 0$ has
(a) a unique solution (*b*) no solution
(*c*) infinitely many solutions
(*d*) none of these

30. The values of '*a*' for which the system of equations $ax + y + z = 0$, $x + ay + z = 0$, $x + y + z = 0$ posses non-zero solutions are given by
(*a*) 1, 2 (*b*) 1, 1
(*c*) 1, – 1 (*d*) – 1, – 2

31. The system of equations $x + 2y + 3z = 1$, $2x + y + 3z = 2$, $5x + 5y + 9z = 4$ has
(*a*) a unique solution
(*b*) no solution
(*c*) infinitely many solutions
(*d*) none of these

32. The system of equations $x + y + z = 0$, $2x + y - z = 0$, $3x + 2y = 0$ has
(*a*) a unique solution
(*b*) no solution
(*c*) infinitely many solutions
(*d*) none of these

33. The system of equations $x + 2y + 3z = 0$, $2x + 3y + 4z = 0$, $3x + 4y + 5z = 0$ has
(*a*) unique solution
(*b*) infinite number of solutions
(*c*) trivial solution
(*d*) none of these

34. The system of equations $x - 2y + z = 0$, $x - 2y - z = 0$, $2x - 4y - 5z = 0$ has
(*a*) unique solution
(*b*) infinite number of solutions
(*c*) trivial solution
(*d*) a none of these

35. Consider the system of equations $5x + 2y - z = 1$, $2x + 3y + 4z = 7$, $4x - 5y + \lambda z = \lambda - 5$
It will have a unique solution if
(*a*) $\lambda = 14$ (*b*) $\lambda \neq 14$
(*c*) $\lambda \neq -14$ (*d*) $\lambda = -14$

36. The system of equations
$x_1 + x_2 + x_3 + x_4 = 0$,
$x_1 + 3x_2 + 2x_3 + 4x_4 = 0$,
$2x_1 + x_3 - x_4 = 0$.
has
(*a*) unique solution
(*b*) infinite number of solutions
(*c*) trivial solution
(*d*) none of these

37. The system of equations $x + 2y + 3z = 0$, $2x + 3y + z = 0$, $3x + y + 2z = 0$ has
(*a*) two solutions
(*b*) infinite number of solutions
(*c*) trivial solution
(*d*) none of these

38. If $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, , then A^4 = ?
(*a*) $24A - 55I$ (*b*) $-4A + 55I$
(*c*) $A + I$ (*d*) $A - I$

39. The Eigen values of $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ are
(*a*) 1, 2, 3 (*b*) 1, 3, 3
(*c*) 1, 1, 3 (*d*) 1, 1, 1

40. The eigen value of $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ are
(*a*) 1, 3, 1 (*b*) 1, 2, 3
(*c*) 0, 2, 3 (*d*) 1, 1, 1

41. The Eigen value of $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$ are

(*a*) 1, 2, 4 (*b*) 1, –2, 4
(*c*) 1, –2, –4 (*d*) –1, 2, 4

42. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then $A^{-1} = ?$

(*a*) $A^2 + A - I$ (*b*) $A^2 - A - I$
(*c*) $A^2 + A + I$ (*d*) None of these

43. The eigen value and eigen vector of $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are respectively

(*a*) 0, $\begin{pmatrix} k \\ 0 \end{pmatrix}$ (*b*) 1, $\begin{pmatrix} 0 \\ k \end{pmatrix}$
(*c*) 0, $\begin{pmatrix} k \\ k \end{pmatrix}$ (*d*) 1, $\begin{pmatrix} k \\ k \end{pmatrix}$

44. If $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of $\begin{bmatrix} 1 & -n \\ -3 & 2n \end{bmatrix}$, then '$n$' = ?

(*a*) 2 (*b*) –2
(*c*) 3 (*d*) 1

45. Let P be a 3 × 3 matrix with real entries such that det (P) = 6 and trace (P) = 0. If det ($P + I$) = 0, where I is the identity matrix, then the eigen values of P are?

(*a*) –1, –2, 3 (*b*) 4, –4, 0
(*c*) 1, 2, 3 (*d*) 1, 1, 6

46. If the vectors (0, 1, a), (1, a, 1) and (a, 1, 0) are linearly dependent, then the value of 'a' will be

(*a*) 0, 1 (*b*) 1, –1
(*c*) $0, \pm\sqrt{2}$ (*d*) 0

47. If the vectors (a, b) and (c, d) are linearly dependent, then which of the following is true?

(*a*) $ab = ac$ (*b*) $a + c = b + d$
(*c*) $a = d = 0$ (*d*) $ad - bc = 0$

48. If the vectors (1, 0, 1), (1, 1, 0) and (p, q, r) are linearly dependent, then which of the following is true?

(*a*) $p + q + r = 0$ (*b*) $p = q = r$
(*c*) $p = q + r$ (*d*) $r = p + q$

Answer key

1. (*d*)	2. (*b*)	3. (*d*)	4. (*c*)	5. (*b*)
6. (*c*)	7. (*a*)	8. (*d*)	9. (*a*)	10. (*a*)
11. (*a*)	12. (*c*)	13. (*c*)	14. (*a*)	15. (*d*)
16. (*c*)	17. (*a*)	18. (*a*)	19. (*b*)	20. (*a*)
21. (*b*)	22. (*b*)	23. (*a*)	24. (*d*)	25. (*c*)
26. (*a*)	27. (*b*)	28. (*c*)	29. (*b*)	30. (*b*)
31. (*a*)	32. (*c*)	33. (*b*)	34. (*b*)	35. (*c*)
36. (*b*)	37. (*c*)	38. (*a*)	39. (*c*)	40. (*a*)
41. (*b*)	42. (*d*)	43. (*a*)	44. (*b*)	45. (*a*)
46. (*c*)	47. (*d*)	48. (*c*)		

Explanation

1. (*d*) The number of column of A is 1 and the number of rows of B is 1. So AB is possible.

The number of column of B is 3 and the number of rows of both A and C are 3. So BA and BC both are defined.

The number of column of C is 5 and the number of rows of D is 5. So CD is possible.

The number of column of D is 3 and the number of rows of both A and C are 3. So DA and DC are possible.

2. (*b*) Let, $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ be any arbitrary 2 × 2 matrix and $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then

$$AB = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ax & ay \\ az & at \end{pmatrix},$$

$$BA = \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} ax & ay \\ az & at \end{pmatrix}$$

Hence, $AB = BA$ which means A commutes with B.

3. (*d*) $A^2 = AA = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta \end{pmatrix}$

$$A^3 = A^2A = \begin{pmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta \end{pmatrix}\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha^2\beta \\ \alpha\beta^2 & 0 \end{pmatrix}$$

Then,

$$A^3 + A = 0 \Rightarrow \begin{pmatrix} 0 & \alpha^2\beta \\ \alpha\beta^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & \alpha^2\beta + \alpha \\ \alpha\beta^2 + \beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \alpha^2\beta + \alpha = 0,\ \alpha\beta^2 + \beta = 0$
$\Rightarrow \alpha(\alpha\beta + 1) = 0,\ \beta(\alpha\beta + 1) = 0$
$\Rightarrow \alpha\beta + 1 = 0$ (assuming $\alpha \neq 0, \beta \neq 0$)
$\Rightarrow \alpha\beta = -1$

4. (*c*) Consider $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$

If A is upper as well as lower triangular, then the elements lying below and above of the principal diagonal will be all zero. So $b = c = d = f = g = h = 0$.

Hence A becomes $\begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & k \end{pmatrix}$, which is clearly a diagonal matrix.

5. (*b*) A is symmetric $\Rightarrow A^T = A$ ...(*i*)

A is skew-symmetric $\Rightarrow A^T = -A$ ...(*ii*)

Adding (*i*) and (*ii*) we get $2A^T = O$

$\Rightarrow (A^T)^T = O^T \Rightarrow A = O$.

6. (*c*) $A = A' \Rightarrow \begin{pmatrix} 6 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 6 & y \\ x & 0 \end{pmatrix}$

Then equating the corresponding elements we get $x = y$.

7. (*a*) A is a matrix of order 2 × 3 and B is a matrix of order 3 × 2. Therefore AB and BA are defined.

Also AB is a matrix of order 2 × 2 and BA is a matrix of order 3 × 3. So obviously they are not equal.

8. (*d*) Det $(3AB)$

$$= 3^3 \det(A)\det(B) = 27 \times (-1) \times 3$$
$$= -81$$

(since AB is a square matrix of order '3').

9. (*a*) $|A| = \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 4 \times 2 - 5 \times 1 = 3$

Then

$A.\ \text{adj}\ A = |A| I = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

10. (*a*) $|A| \neq 0 \Rightarrow A^{-1}$ exist.

$\therefore\ AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow\ (A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow\ IB = IC$

$\Rightarrow\ B = C$

11. (*a*)

$$|A| = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{vmatrix} = \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega & 1 \\ 1+\omega+\omega^2 & 1 & \omega^2 \end{vmatrix}$$

$$(\text{by}\ C_1 \to C_1 + C_2 + C_3)$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega & 1 \\ 0 & 1 & \omega^2 \end{vmatrix} = 0$$

$$(\because 1 + \omega + \omega^2 = 0)$$

Hence A is singular.

12. (*c*)

$$|A| = \begin{vmatrix} 2x & 0 \\ x & x \end{vmatrix} = 2x^2$$

$$\therefore A^{-1} = \frac{1}{|A|} adj(A)$$

$$= \frac{1}{2x^2}\begin{pmatrix} x & -x \\ 0 & 2x \end{pmatrix}^T$$

$$= \frac{1}{2x^2}\begin{pmatrix} x & 0 \\ -x & 2x \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2x} & 0 \\ -\frac{1}{2x} & \frac{1}{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \text{(given)}$$

$$\Rightarrow \frac{1}{2x} = 1 \Rightarrow x = \frac{1}{2}.$$

13. (*c*) $adj(A') - (adj\ A') = adj\ (A') - adj(A')$ $= O$ = null matrix.

14. (*a*) $A + B = I \quad \Rightarrow \quad B = I - A$

$\Rightarrow\ B^2 = (I - A)^2 = I - 2IA + A^2$

$\Rightarrow\ B^2 = I - 2A + A$ (since A is idempotent, so $A^2 = A$)

$\Rightarrow\ B^2 = I - A = B$

Hence B is idempotent.

15. (*d*) A is skew-symmetric $\Rightarrow A^T = -A \Rightarrow AA^T = -A\,A = -A^2 = I$ (using $A^2 + I = 0$).

Hence A is orthogonal.

16. (*c*) $A^2 = A \times A = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} \times \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$

$$= \begin{pmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^2b + ab & -a^2b^2 + a^2b^2 \end{pmatrix} = O$$

Hence A is a nilpotent matrix of index 2.

17. (*a*)

$$AA^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}+\frac{1}{6}+\frac{1}{2} & \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{1}{6}-\frac{1}{2} \\ \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{4}{6}+0 & \frac{1}{3}-\frac{2}{6}+0 \\ \frac{1}{3}+\frac{1}{6}-\frac{1}{2} & \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{1}{6}+\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

So A is orthogonal.

18. (*a*)

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

$$\Rightarrow A^{\theta} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

$$\therefore AA^{\theta} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1-i^2 & i-i \\ -i+i & -i^2+1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Similarly $AA^{\theta} = I$. Hence $A^{\theta}A = AA^{\theta} = I$. So A is unitary.

19. (*b*)

$$\bar{M} = \begin{bmatrix} 7 & 1-i & 2 \\ 1+i & 4 & -3i \\ 2 & 3i & 0 \end{bmatrix} \text{and so}$$

$$M^{\theta} = \left(\bar{M}\right)^T = \begin{bmatrix} 7 & 1+i & 2 \\ 1-i & 4 & 3i \\ 2 & -3i & 0 \end{bmatrix}^T = \begin{bmatrix} 7 & 1-i & 2 \\ 1+i & 4 & -3i \\ 2 & 3i & 0 \end{bmatrix} = M$$

Hence M is Hermitian matrix and so iM is skew-Hermitian.

20. (*a*) Rank $(A) = 3$

$\Rightarrow$ A has non-zero determinant

$\Rightarrow$ $\det(A) \neq 0$.

$$\Rightarrow \begin{vmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} \neq 0$$

$$\Rightarrow 2(x-0) - 4(3x-2) + 2(0-1) \neq 0$$

$$\Rightarrow -10x + 6 \neq 0$$

$$\Rightarrow x \neq \frac{3}{5}.$$

21. (*b*) Since I is an unit matrix of order 'n' and det (I) = $1 \neq 0$. So rank (I) = order $(I) = n$.

22. (*b*) Since A is non-singular, so det $(A) \neq 0$.

Hence, rank (A) = order of $A = n$.

23. (*a*)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix} (by\ R_3 \to R_3 - 2R_2)$$

which has two non-zero rows.

Hence rank = 2.

24. (*d*) A has only three non-zero rows. Hence rank $(A) = 3$.

25. (*c*)

$$\text{Let, } A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 5 & 0 \end{pmatrix}$$

Then A is a matrix of order 3×4. Therefore rank $(A) \leq \min\{3, 4\} = 3$.

Now consider a sub matrix B of order 3, where

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}. \text{ Then } \det(B) = 1 \times 4 \times 5 = 20 \neq 0$$

Therefore rank (A) = order of the square sub-matrix $B = 3$.

Alternative method:

The given matrix has three non-zero rows. Hence it's rank = 3.

26. (*a*)

$$\text{Here } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 6 \neq 0.$$

Therefore by Cramer's rule, the given system of equations has a unique solution.

27. (*b*) Any system of 'n' homogeneous linear equations in 'n' unknowns has a trivial solution.

28. (*c*)

$$\Delta = \begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 36 - 36 = 0$$

$$\Delta_1 = \begin{vmatrix} 5 & 6 \\ 7 & 9 \end{vmatrix} = 45 - 42 = 3 \neq 0$$

Therefore by Cramer's rule, the system has no solution.

29. (*b*)

$$\text{Here } \Delta = \begin{vmatrix} 2 & -3 & 6 \\ 5 & 7 & -14 \\ 3 & 2 & -4 \end{vmatrix} = 0,$$

$$\Delta_1 = \begin{vmatrix} 4 & -3 & 6 \\ 1 & 7 & -14 \\ 0 & 2 & -4 \end{vmatrix} = 0,$$

$$\Delta_2 = \begin{vmatrix} 2 & 4 & 6 \\ 5 & 1 & -14 \\ 3 & 0 & -4 \end{vmatrix} \neq 0$$

Therefore by Cramer's rule, the system has no solution.

30. (*b*) The homogeneous system has a non-zero solution

$\Rightarrow$ rank of the co-efficient matrix < 3

(here number of unknowns = 3)

$\Rightarrow |A| = 0$

$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

$\Rightarrow a^2 - 2a + 1 = 0$

$\Rightarrow a = 1,1$

31. (*a*)

Here $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 5 & 5 & 9 \end{vmatrix} = 3 \neq 0.$

Therefore by Cramer's rule, the system has a unique solution.

32. (*c*)

Here $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 0 \end{vmatrix} = 0.$

Therefore rank of the coefficient matrix < 3 (here number of variables = 3).

Hence the system has an infinite number of solutions.

33. (*b*)

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix}$

(by $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$)

$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

(by $R_3 \to R_3 - 2R_2$)

which has two non-zero rows.

Therefore rank (A) = 2 < 3 (= number of unknowns)

Hence the system has infinite number of solutions.

Alternative method:

Here co-efficient determinant

$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$= (15 - 16) - 2(10 - 12) + 3(8 - 9)$

$= 0.$

Therefore rank of the coefficient matrix < 3 (here number of variables = 3).

Hence the system has infinite number of solutions.

34. (*b*)

$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & -1 \\ 2 & -4 & -5 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 7 \end{pmatrix}$

(by $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$)

$\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

(by $R_3 \to R_3 - \frac{7}{2}R_2$)

which has two non-zero rows.

Therefore rank (A) = 2 < 3 (= number of unknowns)

Hence the system has infinite number of solutions.

Alternative method:

Here co-efficient determinant

$= \begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & -1 \\ 2 & -4 & -5 \end{vmatrix}$

$= (10 - 4) + 2(-5 + 2) + 1(-4 + 4)$

$= 0.$

Therefore rank of the coefficient matrix < 3 (here number of variables=3).

Hence the system has infinite number of solutions.

35. (*c*)

$|A| = \begin{vmatrix} 5 & 2 & -1 \\ 2 & 3 & 4 \\ 4 & -5 & \lambda \end{vmatrix}$

$= 5(3\lambda + 20) - 2(2\lambda - 16) - 1(-10 - 12)$

$= 11\lambda + 154$

System has a unique solution

$\Rightarrow \quad |A| \neq 0 \Rightarrow 11\lambda + 154 \neq 0 \Rightarrow \lambda \neq -14.$

36. (*b*)

$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & -2 & -1 & -3 \end{pmatrix}$

(by $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$)

$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(by $R_3 \to R_3 + R_2$)

which has two non zero rows

Therefore rank $(A) = 2 < 4$.

(here number of unknowns = 4)

Hence the system has infinite number of solutions.

37. (*c*)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{pmatrix}$$

(by $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$)

which has three non zero rows

Therefore rank $(A) = 3$ = number of unknowns

Hence the system has a unique solution. So the only solution is $x = y = z = 0$ (trivial solution).

Alternative method:

Here co-efficient determinat

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= (6-1) - 2(4-3) + 3(2-9)$$

$$= -18 \neq 0.$$

Hence the system has a unique solution which is the trivial solution.

38. (*a*)

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1 - \lambda)(3 - \lambda) + 2 = 0$

$\Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$

Therefore by Cayley Hamilton theorem,

$$A^2 - 4A + 5I = O$$

$$\Rightarrow A^2 = 4A - 5I$$

$$\Rightarrow \left(A^2\right)^2 = \left(4A - 5I\right)^2 = 16A^2 - 40A + 25I$$

$$\Rightarrow A^4 = 16\left(4A - 5I\right) - 40A + 25I = 24A - 55I.$$

39. (*c*)

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2(2 - 0) + 1(0 - 1) = 3 = 1 \times 1 \times 3$$

= product of the eigen values.

40. (*a*)

$$|A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2(2 - 0) + 1(0 - 1) = 3 = 1 \times 1 \times 3$$

= product of the eigen values.

41. (*b*) Trace of the matrix = 2 + 1 + 0 = 3 = sum of eigen values = 1 + 4 + (–2).

42. (*d*)

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1 - \lambda) \times (\lambda^2 - 1) = 0$

$\Rightarrow \quad \lambda^2 - 1 - \lambda^3 + \lambda = 0$

$\Rightarrow \quad \lambda^3 - \lambda^2 - \lambda + 1 = 0$

Therefore by Cayley Hamilton theorem,

$A^3 - A^2 - A + I = 0$

$\Rightarrow \quad A^{-1}(A^3 - A^2 - A + I) = A^{-1}0$

$\Rightarrow \quad A^{-1}A^3 - A^{-1}A^2 - A^{-1}A + A^{-1}I = 0$

$\Rightarrow \quad A^2 - A - I + A^{-1} = 0$

$\Rightarrow \quad A^{-1} = A + I - A^2$

43. (*a*)

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

$$\therefore \quad |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda^2 = 0 \Rightarrow \lambda = 0, 0$

Therefore 0 is the only eigen value.

$\therefore AX = \lambda X$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \pi \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow y = 0$

$$\text{Hence } X = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix}$$

(by replacing x by k)

which is the eigen vector for eigen value ‘0’.

44. (*b*)

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & -n \\ -3 & 2n \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+n \\ -3-2n \end{bmatrix} = \begin{bmatrix} \lambda \\ -\lambda \end{bmatrix}$$

$\Rightarrow 1 + n = \lambda, -3 - 2n = -\lambda$

$\Rightarrow -3 - 2n = -(1 + n)$

$\Rightarrow n = -2.$

45. (*a*) Det(P) = 6 = product of the eigen values

= (–1) × (–2) × 3

Trace(P) = 0 = sum of the eigen values = (–1) + (–2) + 3.

46. (c) The vectors $(0, 1, a), (1, a, 1)$ and $(a, 1, 0)$ are linearly dependent

$$\Rightarrow \begin{vmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow 0-(0-a)+a(1-a^2)=0$$

$$\Rightarrow 2a-a^3=0$$

$$\Rightarrow a(2-a^2)=0$$

$$\Rightarrow a=0,\pm\sqrt{2}$$

47. (d) The vectors (a, b) and (c, d) are linearly dependent

$$\Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

$$\Rightarrow ad-bc=0$$

48. (c) The vectors (1, 0, 1), (1, 1, 0) and (p, q, r) are linearly dependent

$$\Rightarrow \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ p & q & r \end{vmatrix} = 0$$

$$\Rightarrow (r-0)-0+(q-p)=0$$

$$\Rightarrow p=r+q$$

Fully Solved MCQs (Level-II)

1. If A and B are square matrices of the same order such that $(A + B)^2 = A^2 + B^2 + 2AB$, then
   (a) $AB = BA$ (b) $A = B$
   (c) $A + B = 0$ (d) $A = -B^T$

2. Let A be a 3 × 5 matrix and B be a matrix such that A^TB and BA^T are both defined. Then B is of the type:
   (a) 3 × 5 (b) 5 × 3
   (c) 3 × 3 (d) 5 × 5

3. If $A_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, then
   (a) $A_\alpha{}^n = A_\alpha$ (b) $A_\alpha{}^n = nA_\alpha$
   (c) $A_\alpha{}^n = A_{n\alpha}$ (d) $A_\alpha{}^n = 0$

4. If A and B are square matrices of same order so that $AB = A$, $BA = B$, then
   (a) Both A and B are singular
   (b) Both A and B are non-singular.
   (c) Both A and B are unit matrix.
   (d) $A^2 = A$, $B^2 = B$

5. If the matrices A and B commute, then
   (a) $(AB)^n = A^nB^n$ (b) $(AB)^n = AB$
   (c) $(AB)^n = B^n$ (d) none of these.

6. If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, then A^n = ?
   (a) $\begin{pmatrix} n & 0 \\ -n & n \end{pmatrix}$ (b) $\begin{pmatrix} n & 0 \\ -1 & n \end{pmatrix}$
   (c) $\begin{pmatrix} 1 & 0 \\ -n & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$

7. If $A = \begin{pmatrix} i & -i \\ -i & i \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and then A^8 = ?
   (a) 128B (b) 130B
   (c) 116B (d) 8B

8. Two matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ commute under multiplication. Then
   (a) $a = b$ or $\theta = n\pi$ (n is an integer)
   (b) always
   (c) $a = 4b$ (d) $\theta = \pi/3$

9. Find P^{50} if P is given by

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

   (a) $\begin{bmatrix} 1 & 100 & 500 \\ 0 & 1 & 100 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 50 & 100 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 50 & 100 & 150 \\ 0 & 50 & 100 \\ 0 & 0 & 50 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 50 & 1275 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix}$

10. Let $P = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}$, where ω is the complex cube root of unity . Then P^{24} is equal to
   (a) P^2 (b) P
   (c) Identity matrix (d) 0

11. Let $A = (a_{ij})_{3\times3}$ be a matrix with $a_{ij} \in R$. Let B be a matrix obtained by interchanging two columns of A. Then det ($A + B$) = ?
   (a) 2det (A)+ det (B) (b) 0
   (c) 2 det (A) (d) det (A) – det (B)

12. What is the value of the following determinant of order 'n'?

$$\begin{vmatrix} 1 & 0 & 0 & \ldots & \ldots & 0 \\ 1 & \frac{1}{2} & 0 & \ldots & \ldots & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} & \ldots & \ldots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 1 & \frac{1}{2} & \frac{1}{3} & .. & .. & \frac{1}{n} \end{vmatrix}$$

(*a*) $\frac{1}{n!}$ (*b*) $\frac{1}{n}$

(*c*) 0 (*d*) $\frac{1}{n(n+1)}$

13. If $A = \begin{pmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix} = f(x)$, then $A^{-1} = ?$

(*a*) $f(x)$ (*b*) $f(-x)$

(*c*) $-f(x)$ (*d*) $-f(-x)$

14. If $A^k = O$, then $I + A + A^2 + ... + A^{k-1}$ is equal to

(*a*) Null matrix (*b*) $(I + A)^k$

(*c*) I (*d*) $(I - A)^{-1}$

15. If the matrix A, B, $A + B$ are non-singular, then $[A(A + B)^{-1}B]^{-1}$?

(*a*) $A + B$ (*b*) $A^{-1} + B^{-1}$

(*c*) $(A + B)^{-1}$ (*d*) AB

16. Let A, B be two 3×3 invertible matrices with $A + B = AB$. Then

(*a*) $A^{-1} + B^{-1} = 0$ (*b*) $A^{-1} + B^{-1} = B^{-1} A^{-1}$

(*c*) $I - A^{-1}$ is invertible (*d*) $I + B^{-1}$ is invertible

17. Let A and B be two non-zero 2×2 matrices such that $AB = 0$. Then

(*a*) both A and B are non-singular

(*b*) exactly one of A and B is singular

(*c*) both A and B are singular

(*d*) $A + B$ is singular

18. If $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & y \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$, then

(*a*) $x = 1, y = -1$ (*b*) $x = -1, y = 1$

(*c*) $x = y = \frac{1}{2}$ (*d*) $x = y = -\frac{1}{2}$

19. Let A and B be symmetric and skew-symmetric matrices respectively of same order. Then $A^m B^n A^m$ is

(*a*) Skew-symmetric for all m, n

(*b*) Symmetric for all m, n

(*c*) Skew-symmetric if n is odd and symmetric if n is even.

(*d*) Symmetric if m is even and skew-symmetric if m is odd.

20. If A and B are symmetric and commutes, then

(*a*) $A^{-1}B$ is symmetric only

(*b*) AB^{-1} is symmetric only

(*c*) $A^{-1} B^{-1}$ is symmetric only

(*d*) $A^{-1}B$, AB^{-1}, $A^{-1}B^{-1}$ are all symmetric

21. Let A (*a* square matrix of order n) be a nilpotent matrix of index 'p'. Then

(*a*) $I_n - A$ is invertible (*b*) $I_n + A$ is *a* zero matrix

(*c*) $I_n + A$ is nilpotent (*d*) $I_n - A$ is *a* zero matrix

22. If A, a square matrix of order 'n', be nilpotent, then

(*a*) $I_n - A$ is singular and $I_n + A$ is non-singular

(*b*) $I_n + A$ is singular and $I_n - A$ is non-singular

(*c*) Both $I_n - A$ and $I_n + A$ are singular.

(*d*) Both $I_n + A$ and $I_n - A$ are non-singular.

23. The matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -1 \end{pmatrix}$ is

(*a*) idempotent (*b*) Nilpotent of index 3

(*c*) skew-symmetric (*d*) none of these

24. The values of α, β and γ so that the matrix

$A = \begin{pmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{pmatrix}$ is orthogonal are:

(*a*) $\alpha = \pm\frac{1}{\sqrt{2}}, \beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}$

(*b*) $\alpha = \pm\frac{1}{\sqrt{2}}, \beta = \pm\frac{1}{\sqrt{2}}, \gamma = \pm\frac{1}{\sqrt{3}}$

(*c*) $\alpha = \pm\frac{1}{\sqrt{2}}, \beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{2}}$

(*d*) none of these

25. If A is a square matrix and $A - \frac{I}{2}$ and $A + \frac{I}{2}$ are both orthogonal, then

(*i*) A is

(*a*) idempotent (*b*) involutary

(*c*) skew-symmetric (*d*) symmetric

(*ii*) $A^2 =$

(*a*) $-\frac{3}{4}I$ (*b*) $-\frac{3}{2}I$

(*c*) $-\frac{3}{4}I$ (*d*) I

26. Let A be a $n \times n$ matrix with integral entries and $B = A + (1/2)I$, where I is the $n \times n$ identity matrix. Then B is

(*a*) idempotent (*b*) nilpotent

(*c*) invertible (*d*) none of these

27. Suppose A and B are two orthogonal matrices such that det (A) + det (B) = 0, then

(*a*) $A + B = -I$ (*b*) $A + B = I$

(*c*) det $(A + B) = 0$ (*d*) $A + B$ = null matrix

28. If A and B are 3×3 matrices such that rank (AB) = 1, then rank (BA) cannot be

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

29. The rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & 2 \end{pmatrix}$ is:

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

30. The rank of the matrix $A = \begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$ is:

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

31. The rank of the matrix $A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{pmatrix}$ is:

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

32. The rank of the matrix $A = \begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$ is:

(*a*) 4 (*b*) 1
(*c*) 2 (*d*) 3

33. The rank of the matrix $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$ is:

(*a*) 4 (*b*) 1
(*c*) 2 (*d*) 3

34. The rank of the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$ is:

(*a*) 4 (*b*) 1
(*c*) 2 (*d*) 3

35. The rank of the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$ is:

(*a*) 4 (*b*) 1
(*c*) 2 (*d*) 3

36. The rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 5 \\ -1 & 0 & -3 \end{pmatrix}$ is:

(*a*) 4 (*b*) 1
(*c*) 2 (*d*) 3

37. The rank of the matrix $A = \begin{pmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$ is:

(*a*) 4 (*b*) 1
(*c*) 2 (*d*) 3

38. The rank of the matrix $A = \begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ is:

(*a*) 4 (*b*) 1
(*c*) 2 (*d*) 3

39. The rank of the matrix $\begin{pmatrix} 1 & 3 & 5 & 9 \\ 0 & 7 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is

(*a*) 3 (*b*) 4
(*c*) 1 (*d*) 2

40. The rank $A = \begin{pmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -6 & 11 & -6 & 1 \end{pmatrix}$ is 3. Then the value of λ is

(*a*) 1 (*b*) – 1
(*c*) 0 (*d*) 4

41. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, then rank ($A$) = ?

(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 4

42. If a, b, c are in A.P. with common difference 'd' and the rank of the matrix $\begin{bmatrix} 4 & 5 & a \\ 5 & 6 & b \\ 6 & \lambda & c \end{bmatrix}$ is '2'. Then the value of d and λ are given by:

(*a*) $d = 0$, λ = any arbitrary number
(*b*) d = any arbitrary number, $\lambda = 7$
(*c*) $d = \lambda = 7$
(*d*) none of these

43. The system of given equations $x + 2y + z = 8$, $2x + y + 3z = 13$, $3x + 4y - \lambda z = \mu$

(*i*) has an unique solution if

(*a*) $\lambda \neq -\dfrac{11}{3}$ (*b*) $\lambda = -\dfrac{11}{3}$
(*c*) $\lambda \neq -1$ (*d*) $\lambda = 0$

(*ii*) No solution for

(*a*) $\lambda = -\frac{11}{3}, \mu = 22$ (*b*) $\lambda = -\frac{11}{3}, \mu \neq 22$

(*c*) $\lambda \neq -\frac{11}{3}, \mu \neq 22$ (*d*) $\lambda \neq -\frac{11}{3}, \mu \neq 22$

(*iii*) Infinite Solution

(*a*) $\lambda = -\frac{11}{3}, \mu = 22$ (*b*) $\lambda \neq -\frac{11}{3}, \mu \neq 22$

(*c*) $\lambda \neq -\frac{11}{3}, \mu = 22$ (*d*) $\lambda \neq -\frac{11}{3}, \mu \neq 22$

44. Consider the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$

Then the system has

(*i*) an unique solution if

(*a*) $\lambda = 3$ (*b*) $\lambda \neq 3$

(*c*) $\lambda \neq 1$ (*d*) $\lambda = 1$

(*ii*) No solution for

(*a*) $\lambda = 3, \mu \neq 10$ (*b*) $\lambda \neq 3, \mu \neq 10$

(*c*) $\lambda = 3, \mu = 10$ (*d*) $\lambda \neq 3, \mu = 10$

(*iii*) Infinite Solution

(*a*) $\lambda = 3, \mu \neq 10$ (*b*) $\lambda \neq 3, \mu \neq 10$

(*c*) $\lambda = 3, \mu = 10$ (*d*) $\lambda \neq 3, \mu = 10$

45. Consider the system of equations

$x + 4y + 2z = 1,$

$2x + 7y + 5z = 2\mu,$ It has

$4x + \lambda y + 10z = 2\mu + 1.$

(*i*) no solution for

(*a*) $\lambda = 14, \mu \neq \frac{1}{2}$ (*b*) $\lambda = 14, \mu = \frac{1}{2}$

(*c*) $\mu \neq 14$ (*d*) None of these

(*ii*) a unique solution if

(*a*) $\lambda = 14$ (*b*) $\lambda \neq 14$

(*c*) $\lambda \neq \frac{1}{2}$ (*d*) $\lambda = \frac{1}{2}$

(*iii*) infinite number of solutions for

(*a*) $\lambda = 14, \mu = \frac{1}{2}$ (*b*) $\lambda \neq 14, \mu = \frac{1}{2}$

(*c*) $\lambda = \mu = 14$ (*d*) None of these

46. Consider the system of equations

$x_1 + x_2 + x_3 = 1,$

$x_1 + 2x_2 - x_3 = \mu,$ It has

$5x_1 + 7x_2 + \lambda x_3 = \mu^2$

(*i*) no solution for

(*a*) $\lambda = 1, \mu \neq -1, 3$

(*b*) $\lambda \neq 1, \mu = -1, 3$

(*c*) $\lambda \neq 1, \mu = -1, 3$

(*d*) none of these

(*ii*) a unique solution if

(*a*) $\lambda \neq -1$ (*b*) $\lambda \neq 3$

(*c*) $\lambda \neq 1$ (*d*) $\lambda = 1$

(*iii*) infinite number of solutions

(*a*) $\lambda = 1, \mu = -1$ or 3

(*b*) $\lambda \neq 1, \mu \neq -1$ or 3

(*c*) $\lambda = 1, \mu \neq -1$ or 3

(*d*) None of these

47. The value of λ for which the system

$x + 2y + 3z = \lambda x,$

$3x + y + 2z = \lambda y,$

$2x + 3y + z = \lambda z;$

has a non-trivial soluation, is

(*a*) $\lambda = 4$ (*b*) $\lambda = 5$

(*c*) $\lambda = 6$ (*d*) $\lambda = 1$

48. The system of equations given below is

$x_1 + 2x_2 + 3x_3 + x_4 = 1,$

$x_1 - x_2 + 2x_3 - x_4 = -3,$

$3x_1 + 3x_2 + 8x_3 + x_4 = -3$

(*a*) consistent

(*b*) inconsistent

(*c*) non-trivial solution

(*d*) none of these

49. The system of equations given below is

$2x_1 + x_2 + 4x_3 = 4,$

$x_1 - 3x_2 - x_3 = -5,$

$-3x_1 + 2x_2 - 2x_3 = 1,$

$8x_1 - 3x_2 + 8x_3 = 2$

(*a*) consistent

(*b*) inconsistent

(*c*) non-trivial solution

(*d*) none of these

50. The characteristic values of $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ are

(*a*) 1, 1, 4 (*b*) 1, 4, 4

(*c*) 4, 4, 4 (*d*) 1, 1, 1

51. The eigen vectors of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ are

(*a*) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$ (*b*) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

(*c*) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ (*d*) None of these

52. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then $A^9 = ?$

(a) $12A^2 - 22A - 9I$ (b) $-12A^2 + 22A - 9I$
(c) $12A^2 + 22A + 9I$ (d) None of these

53. Which of the following can be an Eigen vector of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$?

(a) $\begin{pmatrix} 2 \\ 2 \\ 9 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

54. The possible set of eigen values of a 4 × 4 skew symmetric orthogonal real matrix is

(a) $\pm i$ (b) $\pm 1, \pm i$
(c) ± 1 (d) $0, \pm i$

55. Let A be a 2 × 2 complex matrix such that tr(A) = 1 and det(A) = – 6. Then tr($A^4 - A^3$) = ?

(a) 55 (b) 78
(c) 70 (d) 88

56. Let $\alpha = e^{\frac{2\pi i}{5}}$ and $M = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$

then trace of $I + M + M^2$ is

(a) 5 (b) – 5
(c) 1 (d) 0

57. If a 3 × 3 real skew symmetric matrix has an eigen value $3i$, then one of the remaining eigen values is

(a) 0 (b) $1/3i$
(c) $-1/3i$ (d) 1

58. Let H be a 3 × 3 complex Hermitian matrix which is unitary. Then the distinct eigen values of H are:

(a) ± 1 (b) $\pm i, \pm i$
(c) $1 \pm i$ (d) $(1 \pm i)/2$

59. If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$, then tr($A^{102}$) = ?

(a) 0 (b) 1
(c) 2 (d) 3

60. Suppose that the matrix $A = \begin{bmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{bmatrix}$ has a certain complex number $\lambda \neq 0$ as an eigen value. Then which of the following must also be an eigen value of A?

(a) $\lambda + 20$ (b) $\lambda - 20$
(c) $20 - \lambda$ (d) None of these

61. Consider the matrix $M = \begin{bmatrix} 0 & 3 & 2 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$. Then

(a) M has no real eigen values
(b) all eigen values of M are positive
(c) all eigen values of M are negative
(d) M has both are positive and negative real eigen values

62. The linear operation $L(X)$ is defined by the cross product $L(X) = b \times X$, where $b = [0\ 1\ 0]^T$ and $X = [x_1\ x_2\ x_3]^T$ are three dimensional vectors. Also given that $L(X) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then the eigen values of A are:

(a) $0, \pm 1$ (b) $0, \pm i$
(c) $1, \pm i$ (d) $-1, \pm i$

63. Let A be a 3 × 3 matrix with eigen values 0, 1 and –1. Then det $(I + A^{100})$ = ?

(a) 3 (b) 4
(c) 5 (d) 6

64. If ω is a non real cube roots of unity, then the eigen values of the following matrix are:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{\omega} & \frac{1}{\omega^2} \\ 1 & \frac{1}{\omega^2} & \frac{1}{\omega^4} \end{bmatrix}$$

(a) 1, 1, –1 (b) 1/3, 1/3, 0
(c) $1, -\omega, -\omega^2$ (d) 3, –3, 0

65. Let P, M, N be square matrices of order 'n' such that the matrices M and N are non-singular. If X be an eigen vector of P corresponding to the eigen value λ, then an eigen vector of $N^{-1}MPM^{-1}N$ corresponding to the eigen value λ is

(a) $MN^{-1}X$ (b) $M^{-1}NX$
(c) $M^{-1}N^{-1}X$ (d) $N^{-1}MX$

66. Let $A = \begin{bmatrix} a & -1 & 4 \\ 0 & b & 7 \\ 0 & 0 & 3 \end{bmatrix}$ be a matrix with real entries. If the sum and product of all the eigen values of A are 10 and 30 respectively then $a^2 + b^2$ equals

(*a*) 29 (*b*) 45
(*c*) 58 (*d*) 60

67. If A be a 3 × 3 non – zero matrix such that $A^2 = O$, then the number of non- zero eigen values of A is

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 4

68. Let A be a 3 × 3 matrix with eigen values 1, –1 and 3. Then

(*a*) $A^2 + A$ is non – singular
(*b*) $A^2 - A$ is non – singular
(*c*) $A^2 + 3A$ is non – singular
(*d*) $A^2 - 3A$ is non – singular

69. If $a_{ij} = 1$ for $1 \le i, j \le m$, then the characteristic equation of the matrix $A = \{a_{ij}\}_{m \times m}$ is

(*a*) $\lambda^m - 2\lambda^{m-1} + 2 = 0$
(*b*) $\lambda^m - m = 0$
(*c*) $\lambda^m - m\lambda^{m-1} = 0$
(*d*) $\lambda^m + m = 0$

70. Consider the 2 × 2 matrix $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. Then which of the following vectors is not a valid eigen vector:

(*a*) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
(*c*) $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ (*d*) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

71. One of the eigen vectors of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ is $V = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Then the corresponding eigen value is?

(*a*) –1 (*b*) 2
(*c*) 0 (*d*) 4

72. Let $f(x) = x - 1$, $g(x) = x + 1$, $h(x) = x^2 - 1$ and $q(x) = x^2 + 1$.

Then these functions are

(*a*) linearly independent
(*b*) linearly dependent because $f(x) \times g(x) = h(x)$
(*c*) linearly dependent because $f(x) - g(x) - h(x) = q(x)$
(*d*) none of the above

73. If $V_n(R)$ be the vector space over the field of real numbers and $W = \left\{(a_1, a_2, ...) : \lim_{n\to\infty} a_n = \beta\right\}$ be a subspace of $V_n(R)$. Then

(*a*) β is real (*b*) $0 < \beta < 1$
(*c*) $\beta = 0$ (*d*) $\beta = 1$

74. The set $\{1, x, x(1 - x)\}$ is

(*a*) linearly independent for all x
(*b*) linearly independent for $x = 0$ only
(*c*) linearly independent for $x = 1$ only
(*d*) linearly dependent

Answer key

1. (*a*)	2. (*a*)	3. (*c*)	4. (*d*)	5. (*a*)
6. (*d*)	7. (*a*)	8. (*a*)	9. (*d*)	10. (*c*)
11. (*b*)	12. (*a*)	13. (*b*)	14. (*d*)	15. (*b*)
16. (*c*)	17. (*c*)	18. (*a*)	19. (*c*)	20. (*d*)
21. (*a*)	22. (*d*)	23. (*b*)	24. (*a*)	
25. (*i*)-(c), (*ii*)-(*a*)		26. (*c*)	27. (*c*)	28. (*d*)
29. (*c*)	30. (*c*)	31. (*c*)	32. (*d*)	33. (*c*)
34. (*d*)	35. (*a*)	36. (*d*)	37. (*c*)	38. (*a*)
39. (*b*)	40. (*a*)	41. (*b*)	42. (*b*)	
43. (*i*) (*a*), (*ii*) (*b*), (*iii*) (*a*)				
44. (*i*) (*b*), (*ii*) (*a*), (*iii*) (*c*)				
45. (*i*) (*a*), (*ii*) (*b*), (*iii*) (*a*)				
46. (*i*) (*a*), (*ii*) (*c*), (*iii*) (*a*)				
47. (*c*)	48. (*b*)	49. (*a*)	50. (*a*)	51. (*a*)
52. (*b*)	53. (*c*)	54. (*a*)	55. (*b*)	56. (*a*)
57. (*a*)	58. (*a*)	59. (*d*)	60. (*c*)	61. (*d*)
62. (*b*)	63. (*b*)	64. (*d*)	65. (*d*)	66. (*a*)
67. (*a*)	68. (*c*)	69. (*c*)	70. (*d*)	71. (*c*)
72. (*c*)	73. (*c*)	74. (*a*)		

Explanation

1. (*a*) $(A + B)^2 = A^2 + 2AB + B^2$

$\Rightarrow (A + B)(A + B) = A^2 + 2AB + B^2$

$\Rightarrow A^2 + AB + BA + B^2 = A^2 + AB + AB + B^2$

$\Rightarrow AB = BA$

2. (*a*) Order of A is 3 × 5 $\Rightarrow$ Order of A^T is 5 × 3

A^TB is defined if number of columns of A^T = Number of rows of B.

Therefore number of rows of B = 3.

BA^T is defined if number of columns of B = number of rows of A^T.

Therefore number of columns of B = 5.

Hence B is a matrix of order 3 × 5.

3. (*c*) A^2_α

$= A_\alpha \times A_\alpha$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \times \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\alpha - \sin^2\alpha & \sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha - \sin\alpha\cos\alpha & -\sin^2\alpha + \cos^2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix}$$

$\therefore$ $A^n_\alpha = A_{n\alpha}$ is satisfied for $n = 2$.

4. (*d*) $AB = A$

$\Rightarrow$ $A(BA) = A$ [$\because$ $BA = B$]

$\Rightarrow$ $(AB)A = A$ [by associative law]

$\Rightarrow$ $AA = A$ [$\because$ $AB = A$]

$\Rightarrow$ $A^2 = A$

Similarly considering $BA = B$ it can be shown that $B^2 = B$.

5. (*a*) $(AB)^2 = (AB)\,(AB)$

$= A(BA)B$

$= A(AB)B$

[$\because$ A and B commutative, so $AB = BA$]

$= A^2B^2$,

which satisfies $(AB)^n = A^nB^n$ for $n = 2$.

6. (*d*) $A^2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1-0 & 0 \\ -1-1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$\therefore$ $A^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$ is true for $n = 2$.

7. (*a*)

$$A^2 = \begin{pmatrix} i & -i \\ -i & i \end{pmatrix} \times \begin{pmatrix} i & -i \\ -i & i \end{pmatrix}$$

$$= \begin{pmatrix} i^2 + i^2 & -i^2 - i^2 \\ -i^2 - i^2 & i^2 + i^2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$= -2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = -2B$$

$A^8 = (-2B)^4 = 16B^4$ ...(*i*)

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow B^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 2B$$

$\Rightarrow B^4 = (2B)^2 = 4B^2 = 4 \times 2B = 8B$

$\therefore (i) \Rightarrow A^8 = 16 \times 8B = 128B.$

8. (*a*) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

Then $BA = AB$

$$\Rightarrow \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a\cos\theta & -a\sin\theta \\ b\sin\theta & b\cos\theta \end{bmatrix} = \begin{bmatrix} a\cos\theta & -b\sin\theta \\ a\sin\theta & b\cos\theta \end{bmatrix}$$

$\Rightarrow -a\sin\theta = -b\sin\theta$

$\Rightarrow (a-b)\sin\theta = 0$

$\Rightarrow a = b$ or $\sin\theta = 0$

$\Rightarrow a = b$ or $\theta = n\pi$

$[\because \sin\theta = 0 = \sin 0 \Rightarrow \theta = n\pi + (-1)^n 0\,]$

9. (*d*) $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$P^2 = P \times P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1+2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^3 = P^2 \times P = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 1+2+3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^4 = P^3 \times P = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 1+2+3+4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Continuing like this we get,

$$= \begin{bmatrix} 1 & 50 & 1+2+3+\text{.........}+50 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{50} = \begin{bmatrix} 1 & 50 & \frac{50(50+1)}{2} \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 50 & 1275 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

10. (*c*)

$$P = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}$$

$$P^2 = P \times P = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}\begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

$$P^4 = P^2 \times P^2 = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}\begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

$$\begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix}$$

$$P^6 = P^4 \times P^2 = \begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix}\begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} = \begin{bmatrix} \omega^6 & 0 \\ 0 & \omega^6 \end{bmatrix}$$

Continuing like this we get, P^{24}

$$= \begin{bmatrix} \omega^{24} & 0 \\ 0 & \omega^{24} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad (\text{since } \omega^{24} = (\omega^3)^8 = 1).$$

11. (*b*) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 6 & 7 & 5 \end{pmatrix}$

Let B be a matrix obtained by interchanging two columns C_2 and C_3 of A.

Then $B = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 6 & 5 & 7 \end{pmatrix}$

and $A + B = \begin{pmatrix} 2 & 5 & 5 \\ 8 & 3 & 3 \\ 12 & 12 & 12 \end{pmatrix}$

$\therefore$ det $(A + B)$

$= 2(36 - 36) - 5(96 - 36) + 5(96 - 36) = 0$

12. (*a*) Since the associated matrix is lower triangular, so value of the determinant = product of the elements lying in the principal diagonal $= 1 \times \frac{1}{2} \times \frac{1}{3} \times \text{.....} \times \frac{1}{n} = \frac{1}{n!}$.

13. (*b*)

$$|A| = \begin{vmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

$$= \begin{pmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-x) & \sin(-x) & 0 \\ -\sin(-x) & \cos(-x) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= f(-x).$$

14. (*d*)

$I - A^k = (I - A)(I + A + A^2 + \dots + A^{k-1})$

$\Rightarrow I = (I - A)(I + A + A^2 + \dots + A^{k-1})$

$[\because A^k = O]$..........(1)

$\Rightarrow |(I - A)(I + A + A^2 + \dots + A^{k-1})| = |I|$

$\Rightarrow |I - A||I + A + A^2 + \dots + A^{k-1}| = 1$

(by $|AB| = |A||B|$)

$\Rightarrow |I - A| \neq 0$

$\Rightarrow I - A$ is invertible

$\Rightarrow (I - A)^{-1}$ exist

Then multiplying both sides of (1) by $(I - A)^{-1}$, we get

$(I-A)^{-1} I = (I-A)^{-1} (I-A)(I+A+A^2+...+A^{k-1})$

$\Rightarrow \quad (I-A)^{-1} = I(I+A+A^2+...+A^{k-1})$

$= I+A+A^2+...+A^{k-1}$

15. (*b*)

$$\left[A(A+B)^{-1}B\right]^{-1}$$

$$= B^{-1}\left[(A+B)^{-1}\right]^{-1}A^{-1}$$

$$= B^{-1}(A+B)A^{-1}$$

$$= B^{-1}\left(AA^{-1}+BA^{-1}\right)$$

$$= B^{-1}\left(I+BA^{-1}\right)$$

$$= B^{-1}I+\left(B^{-1}B\right)A^{-1}$$

$$= B^{-1}+IA^{-1}$$

$$= B^{-1}+A^{-1}$$

16. (*c*)

$$A+B = AB \Rightarrow A^{-1}(A+B) = A^{-1}(AB)$$

$$\Rightarrow A^{-1}A + A^{-1}B = (A^{-1}A)B$$

$$\Rightarrow A^{-1}A + A^{-1}B = (A^{-1}A)B$$

$$\Rightarrow I + A^{-1}B = IB = B$$

$$\Rightarrow I = B - A^{-1}B = (I-A^{-1})B$$

$$\Rightarrow |I| = |(I-A^{-1})B|$$

$$\Rightarrow |I-A^{-1}||B| = 1$$

$$\Rightarrow |I-A^{-1}| \neq 0$$

$\Rightarrow I-A^{-1}$ is invertible.

17. (*c*)

Let $A = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, B = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$.

Then $AB = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$.

Here $|A| = 0, |B| = 0$ and so both are singular.

Now $A+B = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$

$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

Hence $|A+B| = \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} = 1 \neq 0$.

So $A+B$ is non-singular.

Here $|A| = 0$, $|B| = 0$ and so both are singular

Note:

If A and B be two non-zero $n \times n$ matrices such that $AB = 0$. Then both of them are singular.

18. (*a*)

$$|A| = \begin{vmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & x & 1\end{vmatrix} = 0-(1-9)+2(x-6)$$

$= 2x-4$.

$$\text{adj}(A) = \begin{bmatrix} \begin{vmatrix}2 & 3\\x & 1\end{vmatrix} & -\begin{vmatrix}1 & 3\\3 & 1\end{vmatrix} & \begin{vmatrix}1 & 2\\3 & x\end{vmatrix} \\ -\begin{vmatrix}1 & 2\\x & 1\end{vmatrix} & \begin{vmatrix}0 & 2\\3 & 1\end{vmatrix} & -\begin{vmatrix}0 & 1\\3 & x\end{vmatrix} \\ \begin{vmatrix}1 & 2\\2 & 3\end{vmatrix} & -\begin{vmatrix}0 & 2\\1 & 3\end{vmatrix} & \begin{vmatrix}0 & 1\\1 & 2\end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix}2-3x & 8 & x-6\\2x-1 & -6 & 3\\-1 & 2 & -1\end{bmatrix}^T$$

$$= \begin{bmatrix}2-3x & 2x-1 & -1\\8 & -6 & 2\\x-6 & 3 & -1\end{bmatrix}$$

Now $\quad A^{-1} = \dfrac{1}{|A|}\text{adj}(A)$

$$\Rightarrow \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\-4 & 3 & y\\\frac{5}{2} & -\frac{3}{2} & \frac{1}{2}\end{pmatrix} = \frac{1}{2x-4}\begin{pmatrix}2-3x & 2x-1 & -1\\8 & -6 & 2\\x-6 & 3 & -1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\-4 & 3 & y\\\frac{5}{2} & -\frac{3}{2} & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{2-3x}{2x-4} & \frac{2x-1}{2x-4} & \frac{-1}{2x-4}\\\frac{8}{2x-4} & \frac{-6}{2x-4} & \frac{2}{2x-4}\\\frac{x-6}{2x-4} & \frac{3}{2x-4} & \frac{-1}{2x-4}\end{pmatrix}$$

$$\Rightarrow \frac{1}{2} = \frac{2-3x}{2x-4},\ y = \frac{2}{2x-4}$$

$$\Rightarrow 2x-4 = 4-6x,\ y = \frac{2}{2x-4}$$

$$\Rightarrow 8x = 8,\ y = \frac{2}{2x-4}$$

$$\Rightarrow x = 1,\ y = \frac{2}{2-4} = -1.$$

19. (*c*) A is symmetric $\Rightarrow A^T = A$. Also B is skew-symmetric $\Rightarrow B^T = -B$.

$$(A^m B^n A^m)^T$$

$$= (A^m)^T (B^n)^T (A^m)^T$$

$$= (A^T)^m (B^T)^n (A^T)^m$$

$$= A^m (-B)^n A^m$$

$= (-1)^n A^m B^n A^m$

$= \begin{cases} A^m B^n A^m & \text{if } n \text{ is even} \\ -A^m B^n A^m & \text{if } n \text{ is odd} \end{cases}$

Thus $(A^m B^n A^m)^T = A^m B^n A^m$ if n is even

$\Rightarrow A^m B^n A^m$ is symmetric if n is even

Also $(A^m B^n A^m)^T = -A^m B^n A^m$ if n is odd

$\Rightarrow A^m B^n A^m$ is skew-symmetric if n is odd

20. (*d*) As A and B are symmetric, so $A^T = A, B^T = B$. Also A and B commutes, so $AB = BA$. Then

$$\left(A^{-1}B\right)^T = B^T\left(A^{-1}\right)^T = B\left(A^T\right)^{-1} = BA^{-1} \quad \text{...(1)}$$

Now AB = BA $\Rightarrow A^{-1}AB = A^{-1}BA$

$\Rightarrow IB = \left(A^{-1}B\right)A$

$\Rightarrow BA^{-1} = \left(A^{-1}B\right)AA^{-1}$

$\Rightarrow BA^{-1} = \left(A^{-1}B\right)I = A^{-1}B$

$\therefore (1) \Rightarrow \left(A^{-1}B\right)^T = A^{-1}B$

So $A^{-1}B$ is symmetric.

Similarly it can be shown that AB^{-1} is symmetric.

Now $\left(A^{-1}B^{-1}\right)^T = \left(B^{-1}\right)^T\left(A^{-1}\right)^T$

$= \left(B^T\right)^{-1}\left(A^T\right)^{-1}$

$= B^{-1}A^{-1}$

$= (AB)^{-1}$

$= (BA)^{-1} = A^{-1}B^{-1}$

Hence $A^{-1}B^{-1}$ is symmetric.

21. (*a*) Since A is a nilpotent matrix of index p, so $A^P = 0$. Then

$I_n - A^p = (I_n - A)(I_n + A + A^2 + ... + A^{p-1})$

$\Rightarrow I_n = (I_n - A)(I_n + A + A^2 + ... + A^{p-1})$

$\Rightarrow |I_n| = |(I_n - A)(I_n + A + A^2 + ... + A^{p-1})|$

$\Rightarrow 1 = |I_n - A||I_n + A + A^2 + ... + A^{p-1}|$

$(\because |AB| = |A||B|)$

$\Rightarrow |I_n - A| \neq 0$

$\Rightarrow I_n - A$ is invertible

22. (*d*) Let A be a nilpotent matrix of index p. So $A^P = 0$. Then

$I_n + A^n = (I_n + A)(I_n - A + A^2 - ... - A^{p-1})$

$\Rightarrow I_n = (I_n + A)(I_n - A + A^2 - ... - A^{p-1})$

$\Rightarrow |I_n| = |(I_n + A)(I_n - A + A^2 - ... - A^{p-1})|$

$\Rightarrow 1 = |I_n + A||I_n - A + A^2 - ... - A^{p-1}|$

(by $|AB| = |A||B|$)

$\Rightarrow |I_n + A| \neq 0$

$\Rightarrow I_n + A$ is non-singular

Again

$I_n - A^n = (I_n - A)(I_n + A + A^2 + ... + A^{p-1})$

$\Rightarrow I_n = (I_n - A)(I_n + A + A^2 + ... + A^{p-1})$

$\Rightarrow |I_n| = |(I_n - A)(I_n + A + A^2 + ... + A^{p-1})|$

$\Rightarrow 1 = |I_n - A||I_n + A + A^2 + ... + A^{p-1}|$

(*by* $|AB| = |A||B|$)

$\Rightarrow |I_n - A| \neq 0$

$\Rightarrow I_n - A$ is non-singular.

23. (*b*) $A^2 = A \times A$

$$= \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix},$$

$A^3 = A^2 \times A$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -3-5+6 & -1-2+3 & -3-6+9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Hence A is a nilpotent matrix of index 3.

24. (*a*) A is orthogonal

$\Rightarrow \quad A \times A^T = I$

$$\Rightarrow \begin{pmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{pmatrix} \times \begin{pmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{pmatrix} = I$$

$$\Rightarrow \begin{pmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow 4\beta^2+\gamma^2 = 1$ ...(*i*)

$2\beta^2-\gamma^2 = 0$ ...(*ii*)

$\alpha^2+\beta^2+\gamma^2 = 1$ ...(*iii*)

Solving the above three equations, we get

$$\alpha = \pm\frac{1}{\sqrt{2}}, \beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}.$$

25. (*i*)-(*c*); (*ii*)-(*a*)

$A - \frac{I}{2}$ is orthogonal

$$\Rightarrow \left(A-\frac{I}{2}\right)\left(A-\frac{I}{2}\right)^T = I \Rightarrow \left(A-\frac{I}{2}\right)\left(A^T-\frac{I}{2}\right) = I$$

$$\Rightarrow AA^T - \frac{AI}{2} - \frac{IA^T}{2} + \frac{I}{4} = I$$

$$\Rightarrow AA^T - \frac{A}{2} - \frac{A^T}{2} + \frac{I}{4} = I \quad ...(i)$$

Similarly $A + \frac{I}{2}$ is orthogonal

$$\Rightarrow AA^T + \frac{A}{2} + \frac{A^T}{2} + \frac{I}{4} = I \quad ...(ii)$$

Subtracting (*i*) from (*ii*), we get, $A + A^T = O$ which gives $A^T = -A$. Thus A is skew-symmetric.

Adding (*i*) and (*ii*), we get, $2AA^T + \frac{I}{2} = 2I$ which implies $AA^T = \frac{3I}{4}$ and so $A(-A) = \frac{3I}{4}$ *i.e.*,

$$A^2 = -\frac{3}{4}I.$$

26. (*c*)

Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Then $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -1 \\ 2 & \frac{7}{2} \end{pmatrix}$$

$$\therefore \det(B) = \frac{21}{4} + 2 \neq 0.$$

So B is invertible.

27. (*c*) Since A and B are two orthogonal matrices, so $A^TA = A\,A^T = I$ and $B^TB = B\,B^T = I$. Then

$$\left|A^T(A+B)B^T\right| = \left|\left(A^TA + A^TB\right)B^T\right|$$

$$\Rightarrow \left|A^T\right|\left|A+B\right|\left|B^T\right| = \left|\left(I + A^TB\right)B^T\right|$$

$$\Rightarrow |A||A+B||B| = \left|B^T + A^TBB^T\right|$$

$$\Rightarrow |A||A+B||B| = \left|B^T + A^TI\right| = \left|B^T + A^T\right|$$

$$\Rightarrow |A||A+B||B| = \left|(B+A)^T\right| = |A+B|$$

$$\Rightarrow |A+B|\left(|A|\,|B| - 1\right) = 0$$

$$\Rightarrow |A+B| = 0$$

$$\begin{pmatrix} \because |A|\,|B| - 1 = 0 \\ \Rightarrow |A| = |B| = -1 \text{ or } |A| = |B| = -1 \\ \Rightarrow |A| + |B| \neq 0, \text{ a contradiction} \end{pmatrix}$$

28. (*d*)

$\text{rank}(AB) = 1 \neq 3 = \text{order of } AB$

$\Rightarrow |AB| = 0 \Rightarrow |A||B| = 0.$

So, at least either $|A|$ or $|B|$ should be zero

Hence $|BA| = |B|\,|A| = 0$. Therefore BA is singular.

Hence, rank (BA) cannot be 3.

29. (*c*)

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 2 \end{pmatrix} \quad (\text{by } R_2 \to R_2 - 2R_3)$$

which has two non-zero rows.

Hence rank (A) = 2.

30. (*c*)

$$A = \begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$$

$$\sim \begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad [\text{by } R_4 \to R_4 - (R_1 + R_3)]$$

$$\sim \begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad [\text{by } R_3 \to R_3 - (R_1 + R_2)]$$

which has two non-zero rows.

Hence rank $(A) = 2$.

31. (c)

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\left(\text{by } R_3 \to R_3 - (2R_1 + R_2)\right)$

Which has two non-zero rows.

Hence rank $(A) = 2$.

32. (d)

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\left(\text{by } R_4 \to R_4 - (R_1 + R_2 + R_3)\right)$

which has three non-zero rows.

Hence rank $(A) = 3$.

33. (c)

$$A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

(by $R_1 \leftrightarrow R_2$)

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{pmatrix}$$

(by $R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$)

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(by $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$)

which has two non-zero rows.

Hence rank $(A) = 2$.

34. (d) Since order of A is 3×4, so rank $(A) \le \min\{3, 4\} = 3$. Consider a sub-matrix B, where

$$B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}.$$

Then $|B| = 0 - (1-9) + 2(1-6)$

$= -2 \neq 0.$

Thus there exist a square sub-matrix B of order '3' whose determinant value is non zero.

Hence rank $(A) = 3$.

35. (a)

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \left[\text{by } R_1 \to R_1 - (R_2 + R_3)\right]$$

$$\sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \left[\text{by } R_1 \to -\frac{1}{2}R_1\right]$$

$$\sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \left(\text{by } R_5 \to R_5 - R_1\right)$$

$$\sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\left(\text{by } R_2 \to R_2 - R_1, R_4 \to R_4 - R_5\right)$

$$\sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \left(\text{by } R_3 \to R_3 - R_1\right)$$

$$\sim\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix}\left(\text{by } R_4 \to R_4 - R_3\right)$$

which has four non-zero rows.

Hence rank $(A) = 4$.

36. (*d*)

$$A=\begin{pmatrix}1&1&1\\-1&1&-1\\1&1&5\\-1&0&-3\end{pmatrix}$$

$$\sim\begin{pmatrix}1&1&1\\0&2&0\\1&1&5\\-1&0&-3\end{pmatrix}(\text{by } R_2 \to R_2 + R_1)$$

$$\sim\begin{pmatrix}1&1&1\\0&2&0\\0&0&4\\-1&0&-3\end{pmatrix}(\text{by } R_3 \to R_3 - R_1)$$

$$\sim\begin{pmatrix}0&1&-2\\0&1&0\\0&0&1\\-1&0&-3\end{pmatrix}$$

$$\left[\begin{array}{l}\text{by } R_1 \to R_1 + R_4,\ R_2 \to \frac{1}{2}R_2,\\ \quad R_3 \to \frac{1}{4}R_3\end{array}\right]$$

$$\sim\begin{pmatrix}0&1&0\\0&1&0\\0&0&1\\-1&0&-3\end{pmatrix}(\text{by } R_1 \to R_1 + 2R_3)$$

$$\sim\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\\-1&0&-3\end{pmatrix}(\text{by } R_1 \to R_1 - R_2)$$

which has three non-zero rows.

Hence rank $(A) = 3$.

Alternative method:

Rank$(A) \le$ min {number of rows, number of colums} = min {4, 3}. So rank $(A) \le 3$.

Let us consider a 3rd order square sub-matrix

$$B=\begin{pmatrix}1&1&1\\-1&1&-1\\1&1&5\end{pmatrix}$$

Then $\det(B) = 1(5 + 1) - (- 5 + 1) + 1(-1-1) \ne 0$.

Thus there exist a square sub-matrix B of order 3 whose determinant value is non-zero. Hence rank (A)=3.

37. (*c*)

$$A=\begin{pmatrix}-2&-1&-3&-1\\1&2&3&-1\\1&0&1&1\\0&1&1&-1\end{pmatrix}$$

$$\sim\begin{pmatrix}0&-1&-1&1\\0&2&2&-2\\1&0&1&1\\0&1&1&-1\end{pmatrix}$$

$(\text{by } R_1 \to R_1 + 2R_3,\ R_2 \to R_2 - R_3)$

$$\sim\begin{pmatrix}0&-1&-1&1\\0&0&0&0\\1&0&1&1\\0&0&0&0\end{pmatrix}$$

$(\text{by } R_2 \to R_2 + 2R_1,\ R_4 \to R_4 + R_1)$

which has two non-zero rows.

Hence rank $(A) = 2$.

38. (*a*) rank$(A) \le$ min {number of rows of A, number of columns of A} = min {4, 5}. So rank$(A) \le 4$.

Let us consider a 4th order square sub-matrix

$$B=\begin{pmatrix}0&2&0&2\\1&0&1&0\\1&0&2&1\\1&0&0&0\end{pmatrix}$$

$$\det(B)=\begin{vmatrix}0&2&0&2\\1&0&1&0\\1&0&2&1\\1&0&0&0\end{vmatrix}$$

$$=\begin{vmatrix}1&0&0&0\\1&0&1&0\\1&0&2&1\\0&2&0&2\end{vmatrix}\quad (by\ R_1 \leftrightarrow R_4)$$

$= 1 \times$ co-factor of 1

(since 1 is the only non-zero element in 1st row)

$$=\begin{vmatrix}0&1&0\\0&2&1\\2&0&2\end{vmatrix}=2. \ne 0$$

Thus there exist a square sub-matrix B of order 4 whose determinant value is non-zero. Hence rank $(A) = 4$.

39. (*b*) Let A be the given matrix. Since A is upper triangular, so $\det(A) = 1 \times 7 \times 1 \times 2 = 14 \neq 0$.

So rank (A) = order of the square matrix $A = 4$.

40. (*a*)

$$\text{rank}(A) = 3$$

$$\Rightarrow \text{rank}(A) < 4 \text{ (= order of the square matrix)}$$

$$\Rightarrow |A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -6 & 11 & -6 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ -6 & 11 & \lambda-6 & 0 \\ -6 & 11 & -6 & 1 \end{vmatrix} = 0$$

(by $R_3 \to R_3 + R_4$)

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -6 & 11 & \lambda-6 \end{vmatrix} = 0$$

$$\Rightarrow \lambda[\lambda(\lambda-6)+11]-(-1)(0-6) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 5\lambda(\lambda-1) + 6(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

So $\lambda = 1, 2, 3$.

41. (*b*)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\left(\begin{matrix} \text{by } R_2 \to R_2 + R_1, R_3 \to R_3 - 2R_1, \\ R_4 \to R_4 + R_1 \end{matrix}\right)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \left(\text{by } R_2 \to R_2 - 2R_4\right)$$

which has two non-zero rows.

Therefore rank (A) = 2.

42. (*b*)

$$\begin{bmatrix} 4 & 5 & a \\ 5 & 6 & b \\ 6 & \lambda & c \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 5 & a \\ 10 & 12 & 2b \\ 6 & \lambda & c \end{bmatrix} \quad (\text{by } R_2 \to 2R_2)$$

$$\sim \begin{bmatrix} 4 & 5 & a \\ 0 & 7-\lambda & 2b-(a+c) \\ 6 & \lambda & c \end{bmatrix}$$

[by $R_2 \to R_2 - (R_1 + R_3)$]

$$\sim \begin{bmatrix} 4 & 5 & a \\ 0 & 7-\lambda & 0 \\ 6 & \lambda & c \end{bmatrix}$$

[$\because$ a, b, c are in A.P, so $2b = a + c$]

$$\sim \begin{bmatrix} 4 & 5 & a \\ 0 & 0 & 0 \\ 6 & \lambda & c \end{bmatrix} \quad (\text{if } 7-\lambda = 0)$$

Thus for $\lambda = 7$, we get two non-zero rows in the final equivalent matrix and so rank becomes '2'.

But since the rank is independent of 'd', so we have d = any arbitrary number.

43. (i) – (a), (ii) – (b), (iii) – (a)

(i) By Cramer's rule system has a unique solution $\Leftrightarrow \Delta \neq 0$

$$\Leftrightarrow \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & -\lambda \end{vmatrix} \neq 0$$

$$\Leftrightarrow 3\lambda + 11 \neq 0$$

$$\Leftrightarrow \lambda \neq -\frac{11}{3}.$$

(ii) Now if the system has no solution, then $\Delta = 0$.

$$\Delta = 0 \Rightarrow = -\frac{11}{3}$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 8 \\ 2 & 1 & 13 \\ 3 & 4 & \mu \end{vmatrix} = -3\mu + 66$$

If $\Delta_3 \neq 0$, then $-3\mu + 66 \neq 0$ *i.e*; $\mu \neq 22$ and vice versa.

But $\Delta = 0$, $\Delta_3 \neq 0 \Rightarrow$ system has no solutions (by Cramer's rule).

Hence the system has no solution for $\lambda = -\dfrac{11}{3}$ and $\mu \neq 22$

(*iii*) System has infinite number of solution

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ (by Cramer's rule)

$\Rightarrow \Delta = \Delta_3 = 0$

$\Rightarrow \lambda = -\frac{11}{3}, \mu = 22$

Alternative method:

$[A:B]$

$$= \begin{bmatrix} 1 & 2 & 1 & 8 \\ 2 & 1 & 3 & 13 \\ 3 & 4 & -\lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 1 & -3 \\ 0 & -2 & -\lambda-3 & \mu-24 \end{bmatrix}$$

(by $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$)

$$\sim \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & -\lambda-\frac{11}{3} & \mu-22 \end{bmatrix}$$

(by $R_3 \to R_3 - \frac{2}{3}R_2$)

Case-I: $-\lambda - \frac{11}{3} \neq 0$ i.e; $\lambda \neq -\frac{11}{3}$.

In this case rank (A) = 3 = number of variables. So the system has a unique solution.

Case-II:

$$-\lambda - \frac{11}{3} = 0 \text{ and } \mu - 22 \neq 0 \text{ i.e; } \lambda = -\frac{11}{3} \text{ and } \mu \neq 22.$$

In this case rank (A) = 2 ≠ 3 = rank ([A:B]). So system has no solution.

Case-III:

$$-\lambda - \frac{11}{3} = 0 \text{ and } \mu - 22 = 0 \text{ i.e; } \lambda = -\frac{11}{3} \text{ and } \mu = 22.$$

In this case rank (A) = rank([A:B]) = 2 < 3 = number of variables. Hence the system has infinite number of solutions.

44. (*i*) – (*b*), (*ii*) – (*a*), (*iii*) – (*c*)

(*i*) System has a unique solution

$\Rightarrow \Delta \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} \neq 0$$

$\Rightarrow \lambda - 3 \neq 0$

$\Rightarrow \lambda \neq 3.$

(*ii*) Now if the system has no Solution, then Δ = 0.

But $\Delta = 0 \Rightarrow \lambda = 3$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = \mu - 10$$

If $\Delta_3 \neq 0$, then $\mu \neq 10$ and vice versa.

But $\Delta = 0$, $\Delta_3 \neq 0 \Rightarrow$ System has no solution.

Hence the system has no solution for $\lambda = 3$ and $\mu \neq 10$

(*iii*) System has infinite number of solution

$\Leftrightarrow \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0 \Leftrightarrow \Delta = \Delta_3 = 0 \Leftrightarrow \pi = 3, \mu = 10$ (by Cramer's rule).

Alternative method

$[A:B]$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

(by $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$)

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

(by $R_3 \to R_3 - R_2$)

Case-I: $\lambda - 3 \neq 0$ *i.e.*, $\lambda \neq 3$.

In this case rank (A) = 3 = number of variables. So the system has a unique solution.

Case-II: $\lambda - 3 = 0$ and $\mu - 10 \neq 0$ *i.e*; $\lambda = 3$ and $\mu \neq 10$.

In this case rank (A) = 2 ≠ 3 = rank([A:B]). So system has no solution.

Case-III:

$\lambda - 3 = 0$ and $\mu - 10 = 0$ *i.e.*, $\lambda = 3$ and $\mu = 10$.

In this case rank (A) = rank([A:B]) = 2 < 3 = number of variables. Hence the system has infinite number of solutions.

45. (*i*) – (*a*), (*ii*) – (*b*), (*iii*) – (*a*)

(*i*) $$\Delta = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & \lambda & 10 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & \lambda-16 & 2 \end{vmatrix}$$

(by $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$)

$= 1(-2 - \lambda + 16)$

$= -\lambda + 14$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2\mu & 5 \\ 4 & 2\mu+1 & 10 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2\mu-2 & 1 \\ 0 & 2\mu-3 & 2 \end{vmatrix}$$

(by $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$)

$$= 1(4\mu - 4 - 2\mu + 3)$$

$$= 2\mu - 1$$

Then $\Delta = 0 \Rightarrow -\lambda + 14 = 0$ *i.e.*, $\lambda = 14$ and,

$\Delta_2 \neq 0 \Rightarrow 2\mu - 1 \neq 0$ *i.e*; $\mu \neq \frac{1}{2}$.

But $\Delta = 0$, $\Delta_2 \neq 0 \Rightarrow$ System has no solution (by Cramer's rule).

Hence the system has no solution for $\lambda = 14$ and $\mu \neq \frac{1}{2}$

(*ii*) Now if the system has unique solution, then $\Delta \neq 0$. $\Delta \neq 0 \Leftrightarrow \lambda \neq 14$. Thus system has a unique solution for $\pi \neq 14$

(*iii*) System has infinite number of solution

$\Leftrightarrow \Delta = \Delta_1 = \Delta_2 = \Delta_3$

$\Leftrightarrow \Delta = \Delta_2 = 0 \Leftrightarrow \Delta = 0, \Delta_2 = 0 \Leftrightarrow \pi = 14, \mu = \frac{1}{2}$

Alternative method:

$[A:B]$

$$= \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & 2\mu \\ 4 & \lambda & 10 & 2\mu+1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 2\mu-2 \\ 0 & \lambda-16 & 2 & 2\mu-3 \end{bmatrix}$$

(by $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$)

$$\sim \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 2\mu-2 \\ 0 & \lambda-14 & 0 & 1-2\mu \end{bmatrix}$$

(by $R_3 \to R_3 - 2R_2$)

Case-I: $\lambda - 14 \neq 0$ *i.e.*, $\lambda \neq 14$

In this case rank (A) = 3 = number of variables. So the system has a unique solution.

Case-II:

$\lambda - 14 = 0$ and $1 - 2\mu \neq 0$ *i.e.*, $\lambda = 14$ and $\mu \neq \frac{1}{2}$.

In this case rank (A) = 2 ≠ 3 = rank([A:B]). So system has no solution.

Case-III:

$\lambda - 14 = 0$ and $1 - 2\mu = 0$ *i.e.*, $\lambda = 14$ and $\mu = \frac{1}{2}$.

In this case,

rank (A) = rank ([A:B]) = 2 < 3 = number of variables. Hence the system has infinite number of solutions.

46. (*i*) – (*a*), (*ii*) – (*c*), (*iii*) – (*a*)

(*i*) $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & \lambda-5 \end{vmatrix} = 0$$

(by $R_2 \to R_2 - R_1, R_3 \to R_3 - 5R_1$)

$$\Rightarrow 1(\lambda - 5 + 4) = 0$$

$$\Rightarrow \lambda = 1$$

$\Delta_3 \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & \mu \\ 5 & 7 & \mu^2 \end{vmatrix} \neq 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & \mu-1 \\ 0 & 2 & \mu^2-5 \end{vmatrix} \neq 0$$

(by $R_2 \to R_2 - R_1, R_3 \to R_3 - 5R_1$)

$$\Rightarrow 1(\mu^2 - 5 - 2\mu + 2) \neq 0$$

$$\Rightarrow (\mu + 1)(\mu - 3) \neq 0$$

$$\Rightarrow \mu \neq -1 \text{ and } \mu \neq 3.$$

But $\Delta = 0$, $\Delta_3 \neq 0 \Rightarrow$ System has no solution (by Cramer's rule)

Hence the system has no solution for $\lambda = 1$ and $\mu \neq -1$ or 3.

(*ii*) If the system has a unique solution, then $\Delta \neq 0$ (by Cramer's rule) and vice versa.

But $\Delta \neq 0 \Rightarrow \lambda \neq 1$. Hence the system has a unique solution for $\lambda \neq 1$.

(*iii*) System has infinite number of solutions

$\Leftrightarrow \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ (by Cramer's rule)

$\Leftrightarrow \Delta = \Delta_3 = 0$

$\Leftrightarrow \lambda = 1, \mu = -1$ or 3

Alternative method:

$[A:B]$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \mu \\ 5 & 7 & \lambda & \mu^2 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & \lambda & 10 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & \lambda-16 & 2 \end{vmatrix}$$

(by $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$)

$= 1(-2-\lambda+16)$

$= -\lambda + 14$

Case-I: $\lambda - 1 \neq 0$ *i.e.*, $\lambda \neq 1$.

In this case rank (A) = 3 = number of variables. So the system has a unique solution.

Case-II:

$\lambda - 1 = 0$ and $\mu^2 - 2\mu - 3 \neq 0$ *i.e.*, $\lambda = 1$ and $\mu \neq -1, 3$

In this case rank (A)=2≠3=rank([A:B]). So system has no solution.

Case-III:

$\lambda - 1 = 0$ and $\mu^2 - 2\mu - 3 = 0$ *i.e.*, $\lambda = 1$ and $\mu = -1$ or 3.

In this case rank (A) = rank ([A:B]) = 2 < 3 = number of variables. Hence, the system has infinite number of solutions.

47. (*c*) The given system of equations can be written as:

$(1-\lambda)x + 2y + 3z = 0$

$3x + (1-\lambda)y + 2z = 0$

$2x + 3y + (1-\lambda)z = 0$

The system of equation has a non-trivial solution

⇒ Rank of the coefficient matrix < 3 (here Number of variables = 3)

⇒ co-efficient determinant = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & 2 & 3 \\ 6-\lambda & 1-\lambda & 2 \\ 6-\lambda & 3 & 1-\lambda \end{vmatrix} = 0 \text{ (by } C_1 \to C_1 + C_2 + C_3)$$

$$\Rightarrow (6-\lambda)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1-\lambda & 2 \\ 1 & 3 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow 6 - \lambda = 0$

$\Rightarrow \lambda = 6.$

48. (*b*)

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & -1 & 2 & -1 & -3 \\ 3 & 3 & 8 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & -3 & -1 & -2 & -4 \\ 0 & -3 & -1 & -2 & -6 \end{bmatrix}$$

$(by\ R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1)$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & -3 & -1 & -2 & -4 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$(by\ R_3 \to R_3 - R_2)$

which has three non-zero rows.

Therefore rank ([A: B]) = 3. Now ignoring the last column in the final equivalent matrix, we see that there are only two non-zero rows. Hence rank (A) = 2.

Thus rank (A) ≠ rank ([A: B]) and so the system is inconsistent. (*i.e.*, no solution exist)

49. (*a*)

$$[A:B] = \begin{bmatrix} 2 & 1 & 4 & 4 \\ 1 & -3 & -1 & -5 \\ -3 & 2 & -2 & 1 \\ 8 & -3 & 8 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -1 & -5 \\ 2 & 1 & 4 & 4 \\ -3 & 2 & -2 & 1 \\ 8 & -3 & 8 & 2 \end{bmatrix}$$

$(by\ R_1 \leftrightarrow R_2)$

$$\sim \begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 7 & 6 & 14 \\ 0 & -7 & -5 & -14 \\ 0 & 21 & 16 & 42 \end{bmatrix}$$

[by $R_2 \to R_2 - 2R_1$,
$R_3 \to R_3 + 3R_1$,
$R_4 \to R_4 - 8R_1$]

$$\sim \begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 7 & 6 & 14 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

[by $R_3 \to R_3 + R_2$
$R_4 \to R_4 - 3R_2$]

$$\sim \begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 7 & 6 & 14 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ [by } R_4 \to R_4 + 2R_3]$$

which has three non-zero rows.

Therefore rank ([A: B]) = 3. Now ignoring the last column in the final equivalent matrix, we see that there are only three non-zero rows. Hence rank(A) = 3.

Thus rank (A) = rank $([A: B])$ and so the system is consistent.

50. (*a*)

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2(4-1)+(-2+1)+(1-2) = 4$$

Using $|A|$ = product of eigen value, we get $\lambda = 1, 1, 4$.

51. (*a*)

$$A - \lambda I$$

$$= \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}$$

$$\therefore |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\{(3-\lambda)^2 - (-1)\times(-1)\}$$
$$-(-2)\{-2(3-\lambda) - (-1)\times 2\}$$
$$+2\{(-1)\times(-2) - 2\times(3-\lambda)\} = 0$$

$$\Rightarrow (6-\lambda)\{9 - 6\lambda + \lambda^2 - 1\} - (12 - 4\lambda - 4)$$
$$+(-8+4\lambda) = 0$$

$$\Rightarrow (6-\lambda)(\lambda^2 - 6\lambda + 8) + 8\lambda - 16 = 0$$

$$\Rightarrow 6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 8\lambda - 16 = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\Rightarrow \lambda^2(\lambda-2) - 10\lambda(\lambda-2) + 16(\lambda-2) = 0$$

$$\Rightarrow (\lambda-2)(\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0$$

$$\Rightarrow \lambda = 2, 2, 8$$

For $\lambda = 2$:

$$AX = \lambda X$$

$$\Rightarrow \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6x - 2y + 2z \\ -2x + 3y - z \\ 2x - y + 3z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\Rightarrow 6x - 2y + 2z = 2x, -2x + 3y - z = 2y,$$
$$2x - y + 3z = 2z$$

$$\Rightarrow 4x - 2y + 2z = 0, -2x + y - z = 0,$$
$$2x - y + z = 0$$

$$\Rightarrow 2x - y + z = 0, 2x - y + z = 0,$$
$$2x - y + z = 0$$

$$\Rightarrow y = 2x + z$$

$$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x+z \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Therefore $x = 1, z = 0$ gives eigen vector = $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and

$x = 0, z = 1$ gives eigen vector = $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

For $\lambda = 8$:

$$AX = \lambda X$$

$$\Rightarrow \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6x - 2y + 2z \\ -2x + 3y - z \\ 2x - y + 3z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix}$$

$$\Rightarrow 6x - 2y + 2z = 8x, -2x + 3y - z = 8y,$$
$$2x - y + 3z = 8z$$

$$\Rightarrow -2x - 2y + 2z = 0, -2x - 5y - z = 0,$$
$$2x - y - 5z = 0$$

$\Rightarrow x + y - z = 0$ *i.e.*, $z = x + y$............(1)

$2x + 5y + z = 0$..........(2)

$2x - y - 5z = 0$..........(3)

(3) & (1) $\Rightarrow 2x - y - 5(x + y) = 0$

$\Rightarrow x = -2y$........(4)

(4) & (1) $\Rightarrow z = -2y + y = -y$

$$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y \\ y \\ -y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{(for } y = -1)$$

Therefore y = –1 gives eigen vector = $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

52. (*b*)

$$\therefore |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(-\lambda)(-1-\lambda) - 1\} - 0 + 0 = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 + \lambda - 1) = 0$$

$$\Rightarrow \lambda^2 + \lambda - 1 - \lambda^3 - \lambda^2 + \lambda = 0$$

$\Rightarrow \lambda^3 - 2\lambda + 1 = 0$

Therefore by Cayley Hamilton theorem,

$A^3 - 2A + I = 0$

$or, A^3 = 2A - I$..........(1)

$\therefore A^9 = (A^3)^3 = (2A - I)^3$

$= 8A^3 - 12A^2 + 6A - I$

$= 8(2A - I) - 12A^2 + 6A - I$ [Using (1)]

$= -12A^2 + 22A - 9I$

53. (*c*)

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\times\{(1-\lambda)^2 - 1\} - 1\times(1-\lambda-1)$
$+ 1\times(1-1+\lambda) = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 2\lambda) + 2\lambda = 0$

$\Rightarrow \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + 2\lambda = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 = 0$

$\Rightarrow \lambda^2(\lambda - 3) = 0$

$\Rightarrow \lambda = 0, 0, 3$

Therefore the eigen values are 0, 0, 3

For $\lambda = 0$

$AX = \lambda X$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x + y + z = 0$

$\Rightarrow x = -y - z$

$$\therefore X = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore $y = 1, z = 0$ gives eigen vector $= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

and $y = 0, z = 1$ gives eigen vector $= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For λ=3: $AX = \lambda X$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$\Rightarrow x + y + z = 3x, x + y + z = 3y, x + y + z = 3y$

$$\Rightarrow \begin{cases} -2x + y + z = 0 \text{ ....(1)} \\ x - 2y + z = 0 \text{.............(2)} \\ x + y - 2z = 0 \text{.............(3)} \end{cases}$$

$(1) - (2) \Rightarrow -3x + 3y = 0 \Rightarrow x = y$

$\therefore (3) \Rightarrow x + x - 2z = 0 \Rightarrow z = x$

$$\therefore X = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore $x = 1$ gives eigen vector $= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

54. (*a*) We know that the eigen values of a skew symmetric matrix are either imaginary number or zero. But since the eigen values of an orthogonal matrix are of unit modulus, so the possible eigen values are $\pm$ i .

55. (*b*)

Let $A = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$

Then $\text{tr}(A) = 0 + 1 = 1; \det(A) = 0 - 6 = -6$

Now $A^2 = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 2 & 7 \end{pmatrix}$

$\therefore A^3 = A^2 \times A = \begin{pmatrix} 6 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 6 & 21 \\ 14 & 13 \end{pmatrix}$

$A^4 = A^3 \times A = \begin{pmatrix} 6 & 21 \\ 14 & 13 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 42 & 39 \\ 26 & 55 \end{pmatrix}$

Hence

$A^4 - A^3 = \begin{pmatrix} 42 & 39 \\ 26 & 55 \end{pmatrix} - \begin{pmatrix} 6 & 21 \\ 14 & 13 \end{pmatrix}$

$= \begin{pmatrix} 36 & 18 \\ 12 & 42 \end{pmatrix}$

So $\text{tr}(A^4 - A^3) = 36 + 42 = 78.$

56. (*a*)

$$\alpha = e^{\frac{2\pi i}{5}} = \left(e^{\pi i}\right)^{\frac{2}{5}}$$

$$= (\cos\pi + i\sin\pi)^{\frac{2}{5}}$$

$$= (-1)^{\frac{2}{5}} = \left[(-1)^2\right]^{\frac{1}{5}} = 1^{\frac{1}{5}}$$

$\therefore \alpha$ is a fifth root of 1.

So $1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$ and

$1.\alpha.\alpha^2.\alpha^3.\alpha^4 = 1$ *i.e.*, $\alpha^{10} = 1$

Here $I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore tr(\text{I}) = 1+1+1+1+1 = 5$

Also $tr(\text{M}) = 1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$

Now $\text{M}^2 = MM$

$$= \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & - & - & - & - \\ - & \alpha^2 & - & - & - \\ - & - & \alpha^4 & - & - \\ - & - & - & \alpha^6 & - \\ - & - & - & - & \alpha^8 \end{bmatrix}$$

$$tr(M^2) = 1+\alpha^2+\alpha^4+\alpha^6+\alpha^8$$

$$= \frac{1\times\left[1-\left(\alpha^2\right)^5\right]}{1-\alpha^2}$$

$$= \frac{1-\alpha^{10}}{1-\alpha^2} = \frac{1-1}{1-\alpha^2} = 0$$

Hence $tr(I+M+M^2) = tr(I) + tr(\text{M}) + tr(\text{M}^2)$

$= 5+0+0 = 5.$

57. (*a*) We know that the eigen value of a 3×3 real skew symmetric matrix are either purely imaginary number or zero and since complex eigen values occurs in conjugate pairs, so the other eigen values are 0 and $-3i$.

58. (*a*) Since the eigen value of Hermitian matrix are all real and the eigen values of an unitary matrix are of unit modulus. So ± 1 are the only possible eigen values for H.

59. (*d*) We know that 1, ω and ω^2 are cube root of 1 where $\omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2}$

Also $\omega^3 = 1$ and $1+\omega+\omega^2 = 0$

$A = \begin{bmatrix} 1 & 0 & 0 \\ i & \omega & 0 \\ 0 & 1+2i & \omega^2 \end{bmatrix}$ which is the lower triangular matrix with $1, \omega, \omega^2$ as the diagonal elements.

Therefore eigen value of A are $1, \omega, \omega^2$.

So tr(A) = Sum of eigen values $= 1+\omega+\omega^2 = 0$

Now $1, \omega, \omega^2$ are eigen values of $A \Rightarrow 1^{102}, \omega^{102}, (\omega^2)^{102}$ are eigen values of A^{102}.

Therefore $\text{tr}(A^{102}) = 1^{102} + \omega^{102} + (\omega^2)^{102} = 1 + (\omega^3)^{34} + (\omega^3)^{68} = 1 + (1)^{34} + (1)^{68} = 1+1+1 = 3.$

60. (*c*)

$$\det(A) = \begin{vmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{vmatrix} = \begin{vmatrix} 0 & -29 & -11 \\ 0 & 30 & -12 \\ 0 & 24 & -50 \end{vmatrix}$$

$[By\ C_1 \to C_1 + C_2 + C_3]$

$= 0$

Therefore A is singular and so '0' is an eigen value of A. Let λ_1 and λ_2 be other two eigen value.

Now sum of eigen values = trace of the matrix

$\Rightarrow \quad 0 + \lambda_1 + \lambda_2 = 40 + 30 - 50 \Rightarrow \lambda_2 = 20 - \lambda_1$.

Thus if $\lambda(\neq 0)$ be eigen value of A, then $20 - \lambda$ will be another eigen value.

61. (*d*) Since $M^T = M$, so M is a real symmetric matrix with real eigen values.

But since tr(M) = 0, so M has some positive and some negative eigen values.

62. (*b*) $L(X) = b \times X$ = cross product of b and X

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix} = x_3\hat{i} - x_1\hat{k} = [x_3 \;\; 0 \;\; -x_1]^{\text{T}}$$

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Then $L(X) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1x_1 + b_1x_2 + c_1x_3 \\ a_2x_1 + b_2x_2 + c_2x_3 \\ a_3x_1 + b_3x_2 + c_3x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

Comparing both sides we get,

$a_1x_1 + b_1x_2 + c_1x_3 = x_3 = 0x_1 + 0x_2 + 1x_3$

$a_2x_1 + b_2x_2 + c_2x_3 = 0 = 0x_1 + 0x_2 + 0x_3$,

$a_3x_1 + b_3x_2 + c_3x_3 = -x_1 = (-1)x_1 + 0x_2 + 0x_3$.

Therefore we have,

$a_1 = b_1 = 0,\ c_1 = 1, a_2 = b_2 = c_2 = 0,$

$a_3 = -1, b_3 = c_3 = 0.$

$$\text{Therefore } A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Then $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 0-\lambda & 0 & 1 \\ 0 & 0-\lambda & 0 \\ -1 & 0 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 + \lambda = 0$

$\Rightarrow \lambda(\lambda^2 + 1) = 0$

$\Rightarrow \lambda = 0, \pm i \quad (\because \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 = i^2)$

63. (*b*)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then A has eigen values 0, 1 and –1 (since A is a diagonal matrix). Then

$A^2 = A \times A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$A^4 = A^2 \times A^2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and so on.}$$

$$\therefore A^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and so}$$

$$I + A^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $\det(I + A^{100}) = 2 \times 2 \times 1 = 4.$

64. (*d*)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{\omega} & \frac{1}{\omega^2} \\ 1 & \frac{1}{\omega^2} & \frac{1}{\omega^4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & -\omega & 0 \\ 1 & -\omega^2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \left[\begin{array}{l} \because \frac{1}{\omega} = \frac{\omega^3}{\omega} = \omega^2, \\ \frac{1}{\omega^2} = \frac{\omega^3}{\omega^2} = \omega \\ \frac{1}{\omega^4} = \frac{\omega^6}{\omega^4} = \omega^2 \end{array}\right]$$

$$= \begin{bmatrix} 1-1 & 1-\omega^2 & 1-\omega \\ 1-\omega & 1-\omega^3 & 1-\omega^2 \\ 1-\omega^2 & 1-\omega^4 & 1-\omega^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1-\omega^2 & 1-\omega \\ 1-\omega & 0 & 1-\omega^2 \\ 1-\omega^2 & 1-\omega & 0 \end{bmatrix}$$

$[\because \omega^3 = 1 \text{ and } \omega^4 = (\omega^3)\omega = \omega]$

Therefore tr(A) = 0 = sum of eigen values = 3 + (–3) + 0.

65. (*d*) Since X is the eigen vector of P corresponding to the eigen value λ, so $PX = \lambda X$.

Now $(N^{-1}MPM^{-1}N)(N^{-1}MX)$

$= N^{-1}MPM^{-1}(NN^{-1})MX$

$= N^{-1}MP(M^{-1}M)X$

$= (N^{-1}M)PX$

$= (N^{-1}M)\lambda X$

$= \lambda(N^{-1}MX)$ [since $NN^{-1} = I = M^{-1}M$]

This shows that $N^{-1}MX$ is an eigen vector of $N^{-1}MPM^{-1}N$ corresponding to the eigen value λ.

66. (*a*) Trace of a matrix = sum of eigen values

$a + b + 3 = 10 \Rightarrow a + b = 7$ ...(*i*)

Determinant of a matrix = product of eigen values

$3ab = 30 \Rightarrow ab = 10$ ...(*ii*)

By (*i*) and (*ii*)

$a^2 + b^2 = (a + b)^2 - 2ab = 49 - 20 = 29$

67. (*a*) The eigen values of A^2 are square of eigen values of A. Since A^2 is a zero matrix. So, all its eigen values will be zero. Hence all the eigen values of A will also be zero.

68. (*c*) We knot that the characteristic equation for A is
$|A - \lambda I| = 0$ ...(*i*)
Since '1' is an eigen value, so (*i*) gives
$|A - I| = 0 \Rightarrow |A^2 - A| = 0.$
Hence $A^2 - A$ is singular.
'–1' is eigen value
$\Rightarrow \quad |A + I| = 0.$
$\Rightarrow \quad |A^2 + A| = 0$. Hence $A^2 + A$ is singular
Since '3' is an eigen value, so $|A - 3I| = 0$(by (*i*))
$\therefore \quad |A^2 - 3A| = 0$. Hence $A^2 - 3A$ is singular
Since '0' and '–3' are not eigen values,
so, $|A| \neq 0$ and $|A + 3I| \neq 0$.
Hence $\left|A^2 + 3A\right| = \left|A\right| \times \left|A + 3I\right| \neq 0.$
Therefore, $A^2 + 3A$ is non–singular.

69. (*c*)
Let $m = 2$. Then $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
The characteristic equation of A is given by
$\det(A - \lambda I) = 0$
or, $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 0$
or, $\lambda^2 - 2\lambda = 0$
This satisfies the equation $\lambda^m - m\lambda^{m-1} = 0$ for $m = 2$.

70. (*d*) Let A be the given matrix. Then $A = 4I$ and the eigen values of A are 4 and 4.
Then $AX = \lambda X \Rightarrow 4IX = 4I$, which is true for any $X \neq O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

71. (*c*)
$$AV = \lambda V \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 1-1 \\ 1-1 \\ -1+1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix}$$
$$\Rightarrow \lambda = 0$$

72. (*c*) $f(x) - g(x) - h(x) = (x - 1) - (x + 1) - (x^2 - 1) = x^2 + 1 = q(x)$
$\Rightarrow \quad f(x) - g(x) - h(x) - q(x) = 0$
$\Rightarrow$ the functions are linearly dependent.

73. (*c*) Let $X = (a_1, a_2, ...)$ and $Y = (b_1, b_2, ...) \in W$.
Then $\lim_{n\to\infty} a_n = \beta$ and $\lim_{n\to\infty} b_n = \beta$.
Now
$aX + bY$
$= a(a_1, a_2, ...) + b(b_1, b_2, ...)$
$= (aa_1 + bb_1, aa_2 + bb_2, ..., aa_n + bb_n, ...)$
$\therefore \lim_{n\to\infty}(aa_n + bb_n) = a \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
$= a\beta + b\beta$
$= \beta \;$ (if $\beta = 0$)
Thus W is a subspace of $V_n(R)$ if $\beta = 0$.

74. (*a*)
Let $f(x) = 1$, $g(x) = x$, $h(x) = x(1 - x) = x - x^2$.
Then
$$\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$
$$= \begin{vmatrix} 1 & x & x - x^2 \\ 0 & 1 & 1 - 2x \\ 0 & 0 & -2 \end{vmatrix}$$
$= -2(1 - 0) = -2 \neq 0$
Therefore the given set of vectors is linearly independent for all x.

PREVIOUS YEARS SOLVED PAPERS (2000-2018)

1. The eigen values of the matrix $\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ are
(*a*) 2, –2, 1, –1 (*b*) 2,3,–2,4
(*c*) 1, 2, 3, 4 (*d*) none of these
(EC GATE 2000)

2. An $n \times n$ array V is defined as follows:
$V[i, j] = i - j$ for all i, j where $1 \leq i, j \leq n$. Then the sum of the elements of the array V is
(*a*) 0 (*b*) n–1
(*c*) $n^2 - 3n + 2$ (*d*) $n(n + 1)$
(CS GATE 2000)

3. The determinant of the matrix $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{bmatrix}$ is
(*a*) 4 (*b*) 0
(*c*) 15 (*d*) 20
(CS GATE 2000)

4. If A, B, C are square matrices of the same order, then $(ABC)^{-1}$ is equal to
(*a*) $C^{-1}A^{-1}B^{-1}$ (*b*) $C^{-1}B^{-1}A^{-1}$
(*c*) $A^{-1}B^{-1}C^{-1}$ (*d*) $A^{-1}C^{-1}B^{-1}$
(CE GATE 2000)

5. The rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$ is

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

(IN GATE 2000)

6. The product $[P][Q]^T$ of the following two matrices $[P]$ and $[Q]$ is

$$[P] = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, [Q] = \begin{bmatrix} 4 & 8 \\ 9 & 2 \end{bmatrix}$$

(*a*) $\begin{bmatrix} 32 & 24 \\ 56 & 46 \end{bmatrix}$ (*b*) $\begin{bmatrix} 46 & 56 \\ 24 & 32 \end{bmatrix}$

(*c*) $\begin{bmatrix} 35 & 22 \\ 61 & 42 \end{bmatrix}$ (*d*) $\begin{bmatrix} 32 & 56 \\ 24 & 46 \end{bmatrix}$

(CE GATE 2001)

7. The rank of a 3 × 3 matrix $C = BA$, found by multiplying a non-zero column matrix A of size 3 × 1 and a non-zero row matrix B of size 1 × 3 is

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

(GATE 2001)

8. The rank of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is

(*a*) 4 (*b*) 1
(*c*) 2 (*d*) 0

(CE GATE 2001)

9. The eigen values of the matrix $\begin{pmatrix} 5 & 3 \\ 2 & 9 \end{pmatrix}$ are

(*a*) (5.13, 9.42) (*b*) (3.85, 2.93)
(*c*) (9, 5) (*d*) (10.16, 3.84)

(CE GATE 2001)

10. Consider the system of equation gives below:

$x + y = 2$

$2x + 2y = 5$

This system has

(*a*) one solution (*b*) no solution
(*c*) infinite solution (*d*) four solutions

(GATE 2001)

11. The necessary condition to diagonalize a matrix is that

(*a*) its all eigen values should be distinct
(*b*) its eigen values should be independent
(*c*) its eigen values should be real
(*d*) the matrix is non-singular

(IN GATE 2001)

12. The following set of equations has

$3x + 2y + z = x - y + z = 2, -2x + 2z = 5$

(*a*) no solution (*b*) a unique solution
(*c*) multiple solutions (*d*) an inconsistency

(GATE 2001)

13. Consider the following statements:

S_1 : The sum of two singular matrices may be singular

S_2 : The sum of two non-singular matrices may be non-singular

Which of the following statements is true?

(*a*) S_1 and S_2 are both true
(*b*) S_1 and S_2 are both false
(*c*) S_1 is true and S_2 is false
(*d*) S_1 is false and S_2 is true

(CS GATE 2001)

14. Obtain the eigen values of the matrix $\begin{bmatrix} 1 & 2 & 34 & 49 \\ 0 & 2 & 43 & 94 \\ 0 & 0 & -2 & 104 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

(*a*) 1, 2, –2, –1 (*b*) –1, –2, –1, –2
(*c*) 1, 2, 2, 1 (*d*) none

(CS GATE 2002)

15. The determinant of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{bmatrix}$ is

(*a*) 100 (*b*) 200
(*c*) 1 (*d*) 300

(EE GATE 2002)

16. If $X = \begin{bmatrix} a & 1 \\ -a^2 + a - 1 & 1 - a \end{bmatrix}$ and $X^2 - X + I = 0$, then the inverse of X =?

(*a*) $\begin{bmatrix} 1-a & -1 \\ a^2 & a \end{bmatrix}$ (*b*) $\begin{bmatrix} 1-a & -1 \\ a^2 - a + 1 & a \end{bmatrix}$

(*c*) $\begin{bmatrix} -a & 1 \\ -a^2 + a - 1 & a - 1 \end{bmatrix}$

(*d*) $\begin{bmatrix} a^2 - a + 1 & a \\ 1 & 1 - a \end{bmatrix}$

(GATE 2002)

17. Consider the system of simultaneous equations $x + 2y + z = 6$, $2x + y + 2z = 6$, $x + y + z = 5$

This system has

(*a*) a unique solution
(*b*) infinite number of solutions
(*c*) no solution
(*d*) exactly two solution

(ME GATE 2003)

18. Consider the following system of linear equations:

$$\begin{pmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and third columns of the coefficient matrix are linearly dependent. For how many value of α, does this system of equations have infinitely many solutions?

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) infinitely many

(CS GATE 2003)

19. A system of equation represented by $AX = 0$, when X is a column vector of unknowns and A is a matrix containing coefficients has non trivial solution when A is

(*a*) non-singular (*b*) singular
(*c*) symmetric (*d*) Hermitian

(GATE 2003)

20. Given matrix $A = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{pmatrix}$. The rank of the matrix is

(*a*) 4 (*b*) 3
(*c*) 2 (*d*) 1

(CE GATE 2003)

21. For the matrix $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$, the eigen values are

(*a*) 3 and –3 (*b*) –3 and –5
(*c*) 3 and 5 (*d*) 5 and 0

(ME GATE 2003)

22. The eigen values of the matrix $A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ are

(*a*) 1, 4 (*b*) –1, 2
(*c*) 0, 5 (*d*) can't be determined

(CE GATE 2004)

23. The sum of the eigen values of the matrix given below is

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

(*a*) 5 (*b*) 7
(*c*) 9 (*d*) 18

(ME GATE 2004)

24. The number of $n \times n$ symmetric matrices with each elements being either 0 or 1 is

(*a*) 2^n (*b*) $2n^2$
(*c*) $2^{\frac{n^2+n}{2}}$ (*d*) $2^{\frac{n^2-n}{2}}$

(CS GATE 2004)

25. For which value of 'x' will the matrix $\begin{pmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{pmatrix}$ become singular?

(*a*) 4 (*b*) 6
(*c*) 12 (*d*) 8

(ME GATE 2004)

26. Let A, B, C and D be $n \times n$ matrices, each with non-zero determinant. If $ABCD = I$, then B^{-1} = ?

(*a*) $D^{-1}C^{-1}A^{-1}$ (*b*) CDA
(*c*) ADC
(*d*) does not necessarily exist.

(CS GATE 2004)

27. Real matrices $A_{3\times1}$, $B_{3\times3}$, $C_{3\times5}$, $D_{5\times3}$, $E_{5\times5}$, $F_{5\times1}$ are given, where the matrices B and E are symmetric.

Following statements are made with respect to these matrices:

(*i*) The matrix product F^TC^TBCF is a scalar
(*ii*) The matrix product D^TFD is always symmetric

With reference to above statements, which of the following is true?

(*a*) Statement (*i*) is true but (*ii*) is false.
(*b*) Statement (*i*) false but (*ii*) is true.
(*c*) Both the statements are true.
(*d*) Both the statements are false.

(CE GATE 2004)

28. How many solutions does the following system of linear equations have?

$-x + 5y = -1$, $x - y = 2$, $x + 3y = 3$

(*a*) infinitely many
(*b*) two distinct solutions
(*c*) unique
(*d*) none

(CS GATE 2004)

29. What values of x, y, z satisfy the following system of linear equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

(*a*) $x = 6, y = 3, z = 2$
(*b*) $x = 12, y = 3, z = -4$
(*c*) $x = 6, y = 6, z = -4$
(*d*) $x = 12, y = -3, z = 4$

(GATE 2005)

30. Which of the following is an eigen vector of the matrix

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}?$$

(*a*) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

(ME GATE 2005)

31. The eigen values of the matrix $A = \begin{pmatrix} 1 & 4 \\ a & 2 \end{pmatrix}$ are real and non-negative for the condition

(*a*) $-1/16 \le a \le 1/16$ (*b*) $-1/2 \le a \le 1/2$
(*c*) $-1/2 \le a \le 1/16$ (*d*) $-1/16 \le a \le 1/2$

(CS GATE 2005)

32. For the matrix $P = \begin{pmatrix} 3 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, one of the eigen value is equal to –2. Then which of the following is an eigen vector?

(*a*) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

(EE GATE 2005)

33. What are the eigen values of the following 2 × 2 matrix:

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

(*a*) –1 and 1 (*b*) 1 and 6
(*c*) 2 and 5 (*d*) 4 and –1

(CS GATE 2005)

34. Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$. The eigen vector is

(*a*) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (*b*) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(*c*) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(EC GATE 2005)

35. Consider the system of equations $A_{n\times n} X_{n\times 1} = \lambda X_{n\times 1}$, where λ is a scalar. Let (λ_i, X_i) be an eigen pair of an eigen value and its corresponding eigen vector for real matrix *A*. Let *I* be a $n \times n$ unit matrix. Then which of the following statements is not correct ?

(*a*) For a homogeneous $n \times n$ system of linear equations $(A - \lambda I)X = 0$, having a non-trivial solution, the rank of $A - \lambda I$ is less than *n*.
(*b*) For matrix A^m, *m* be a positive integer, (λ_i^m, X_i^m) will be eigen pair for all *i*.
(*c*) If $A^T = A^{-1}$, then $|\lambda_i| = 1$ for all *i*.
(*d*) If $A^T = A$, then λ_i is real for all *i*.

(CE GATE 2005)

36. The eigen value of the matrix *M* given below are 15, 3, 0.

$$M = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Then the value of the determinant of the matrix is

(*a*) 20 (*b*) 10
(*c*) 0 (*d*) –10

(PI GATE 2005)

37. Identify which one of the followings is are eigen vector of the matrix $A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$

(*a*) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (*d*) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(ME GATE 2005)

38. *A* is a 3 × 4 real matrix and $AX = B$ is an inconsistent system of equations. The highest possible rank of A is

(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 4

(ME GATE 2005)

39. Consider the following system of equations in three real variables x_1, x_2 and x_3:

$$2x_1 - x_2 + 3x_3 = 1,$$
$$3x_1 - 2x_2 + 5x_3 = 2,$$
$$-x_1 - 4x_2 + x_3 = 3.$$

This system of equations has
(*a*) no solution
(*b*) a unique solution
(*c*) more than one but finite number of solution
(*d*) an infinite number of solutions

(CS GATE 2005)

40. Consider a non-homogeneous system of linear equations representing mathematically an over determined system. Such a system will be
(*a*) consistent having a unique solution
(*b*) consistent having many solutions
(*c*) inconsistent having a unique solution
(*d*) inconsistent having no solution

(CE GATE 2005)

41. Let A be a 3×3 matrix with rank 2. Then $AX = 0$ has
   (*a*) only the trivial solution $X = 0$
   (*b*) one independent solution
   (*c*) two independent solutions
   (*d*) three independent solutions

(GATE 2005)

42. In the matrix equation $PX = Q$, which of the following is a necessary condition for the existence of at least one solution for the unknown vector X:
   (*a*) The augmented matrix $[P:Q]$ must have the same rank as matrix P
   (*b*) Vector Q must have only one-zero elements
   (*c*) Matrix P must be singular
   (*d*) Matrix P must be square

(EE GATE 2005)

43. The determinant of the matrix given below is

$$\begin{vmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{vmatrix}$$

(*a*) – 1 (*b*) 0
(*c*) 1 (*d*) 2

(GATE 2005)

44. For an orthogonal matrix A of order 4×4, $(AA')^{-1}$ is

(*a*) $\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$ (*b*) $\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$

(*c*) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (*d*) $\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$

(EC GATE 2005)

45. Let $A = \begin{pmatrix} 2 & -0.1 \\ 0 & 3 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} \frac{1}{2} & a \\ 0 & b \end{pmatrix}$, then $a + b = ?$

(*a*) 7/20 (*b*) 3/20
(*c*) 19/60 (*d*) 11/20

(EC GATE 2005)

46. Consider the matrices $X_{(4\times3)}$, $Y_{(4\times3)}$ and $P_{(2\times3)}$. Then the order of $[P(X^TY)^{-1}P^T]^T$ will be

(*a*) 2×2 (*b*) 3×3
(*c*) 4×3 (*d*) 3×4

(CE GATE 2005)

47. If $R = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$ then top row R^{-1} is

(*a*) [5 6 4] (*b*) [5 –3 1]
(*c*) [2 0 –1] (*d*) $\begin{bmatrix} 2 & -1 & \frac{1}{2} \end{bmatrix}$

(EE GATE 2005)

48. Multiplication of matrices E and F is G. Matrices E and G are as follows

$$E = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What is the matrix F?

(*a*) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(*b*) $\begin{pmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(*c*) $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(*d*) $\begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ME GATE 2006)

49. A system of linear simultaneous equations is given as $AX = B$, where

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then the rank of the matrix A is

(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 4

(IN GATE 2006)

50. The rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

(EC GATE 2006)

51. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is ?

(*a*) 2 (*b*) 4
(*c*) 6 (*d*) 8

(EC GATE 2006)

52. Match the items in columns I and II

Column-I	Column-II
P. Singular matrix	1. Determinant is not defined
Q. Non square matrix	2. Determinant is always one.
R. Real symmetric	3. Determinant is zero
S. Orthogonal matrix	4. Eigen values are always real
	5. Eigen values are not defined

(*a*) $P \to 3, Q \to 1, R \to 4, S \to 2$
(b) $P \to 2, Q \to 3, R \to 4, S \to 1$
(*c*) $P \to 3, Q \to 2, R \to 5, S \to 4$
(*d*) $P \to 3, Q \to 4, R \to 2, S \to 1$

(ME GATE 2006)

53. For a given matrix, $A = \begin{pmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{pmatrix}$ one of the eigen value is 3. The other eigen values are:

(*a*) 2, –5 (*b*) 3, –5
(*c*) 2, 5 (*d*) 3, 5

(CE GATE 2006)

54. Eigen values of a matrix $S = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ are 5 and 1. What are the eigen values of S^2 ?

(*a*) 1, 25 (*b*) 6, 4
(*c*) 1, 5 (*d*) 2, 10

(ME GATE 2006)

55. The eigen values and the corresponding eigen vectors of a 2×2 matrix are given by:

Eigen value	Eigen vector
$\lambda_1 = 8$	$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = 4$	$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then the matrix is:

(*a*) $\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$ (*b*) $\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$

(*c*) $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ (*d*) $\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$

(EC GATE 2006)

56. For a given 2×2 matrix A, it is observed that

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}, A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Then the matrix A is

(*a*) $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

(*b*) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

(*d*) $\begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}$

(IN GATE 2006)

57. If a square matrix A is real and symmetric, then the eigen values

(*a*) are always real
(*b*) are always real and positive
(*c*) are always real and non-positive
(*d*) occur in complex conjugate pairs

(ME GATE 2007)

58. The minimum and maximum eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are –2 and 6 respectively. Then what is the other eigen value?

(*a*) 5 (*b*) 3
(*c*) 1 (*d*) –1

(CE GATE 2007)

59. Consider the set of column vectors defined by $X = \left\{x \in R^3 : x_1 + x_2 + x_3 = 0, \text{where } x^T = [x_1, x_2, x_3]^{\text{T}}\right\}$. Then which of the following is true?

(*a*) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is the basis for the subspace X
(*b*) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set, but it doesn't span X and therefore not a basis of X
(*c*) X is not a subspace of R^3
(*d*) none of the above

(CS GATE 2007)

60. (common data linked question)

Cayley Hamilton Theorem states that a square matrix satisfies its own characteristic equation. Consider a matrix $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$

(*i*) A satisfies the relation
(*a*) $A + 3I + 2A^{-1} = O$
(*b*) $A^2 + 2A + 2I = O$

(*c*) $(A + I)(A + 2I) = I$
(*d*) $\exp(A) = O$
(*ii*) A^9 Equal to
(*a*) $511A + 510I$ (*b*) $309A + 104I$
(*c*) $154A + 155I$ (*d*) $\exp(9A)$

(EE GATE 2007)

61. The determinant $\begin{vmatrix} 1+b & b & 1 \\ b & b+1 & 1 \\ 1 & 2b & 1 \end{vmatrix}$ equals to
(*a*) 0 (*b*) $2b(b-1)$
(*c*) $2(1-b)(2b+1)$ (*d*) $3b(1+b)$

(PI GATE 2007)

62. $X = \begin{bmatrix} x_1 & x_2 & . & . & . & . & x_n \end{bmatrix}^T$ is a n-tuple non-zero vector. Then the $n \times n$ matrix $V = XX'$
(*a*) has rank zero (*b*) has rank 1
(*c*) is orthogonal (*d*) has rank n.

(EE GATE 2007)

63. The inverse of the 2 × 2 matrix $A = \begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix}$ is
(a) $\frac{1}{3}\begin{pmatrix} -7 & 2 \\ 5 & -1 \end{pmatrix}$ (b) $\frac{1}{3}\begin{pmatrix} 7 & 2 \\ 5 & 1 \end{pmatrix}$
(c) $\frac{1}{3}\begin{pmatrix} 7 & -2 \\ -5 & 1 \end{pmatrix}$ (d) $\frac{1}{3}\begin{pmatrix} -7 & 2 \\ -5 & 1 \end{pmatrix}$

(CE GATE 2007)

64. Solution for the system defined by the set of equations:
$4y + 3z = 8$, $2x - z = 2$ and $3x + 2y = 5$ is
(*a*) $x = 0, y = 1, z = \frac{4}{3}$, (*b*) $x = 0, y = \frac{1}{2}, z = 2$
(*c*) $x = 1, y = \frac{1}{2}, z = 2$ (*d*) non- existent

(GATE 2007)

65. For what value of α and β, the following simultaneous equations have an infinite number of solutions?
$x + y + z = 5$, $x + 3y + 3z = 9$, $x + 2y + \alpha z = \beta$
(*a*) 2, 7 (*b*) 3, 8
(*c*) 8, 3 (*d*) 7, 2

(CE GATE 2007)

66. Let A be an $n \times n$ real matrix such that $A^2 = I$ and Y be an n-dimensional vector. Then the linear system of equations $AX = Y$ has
(*a*) no solution
(*b*) unique solution
(*c*) more than one but finitely many dependent solutions
(*d*) infinitely many dependent solution.

(IN GATE 2007)

67. The number of linearly independent eigen vectors of $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is
(*a*) 0 (*b*) 1
(*c*) 2 (*d*) infinite

(ME GATE 2007)

68. It is given that $X_1, X_2, ..., X_M$ are M non-zero orthogonal vectors. Then the dimension of the vector space spanned by the $2M$ vectors $X_1, X_2, ..., X_M, -X_1, -X_2, ... -X_M$ is
(*a*) $2M$ (*b*) $M + 1$
(*c*) M
(d) dependent on the choices of $X_1, X_2, ..., X_M$

(EC GATE 2007)

69. The following simultaneous equations $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + kz = 6$ will not have a unique solution for k is equal to
(*a*) 0 (*b*) 5
(*c*) 6 (*d*) 7

(CE GATE 2008)

70. For what value of '*a*' if any, the following system of equations is *x*, *y*, *z* have a solution?
$2x + 3y = 4$, $x + y + z = 4$, $x + 2y - z = a$
(*a*) 0 (*b*) any real number
(*c*) 1 (*d*) there is no such value

(ME GATE 2008)

71. The following system of equations
$x_1 + x_2 + 2x_3 = 1,$
$x_1 + 2x_2 + 3x_3 = 2,$
$x_1 + 4x_2 + \alpha x_3 = 4.$
has a unique solution. Then the only possible values of 'a' is / are ?
(*a*) 0
(*b*) either 0 or 1
(*c*) one of 0, 1 and – 1
(*d*) any real number other than 5

(CS GATE 2008)

72. The system of linear equations $4x + 2y = 7$, $2x + y = 6$ has
(*a*) a unique solution
(*b*) an infinite number of solutions
(*c*) no solution
(*d*) exactly two distinct solutions

(EC GATE 2008)

73. The product of matrices $(PQ)^{-1}P$ is
(*a*) P^{-1} (*b*) Q^{-1}
(*c*) $P^{-1}Q^{-1}P$ (*d*) PQP^{-1}

(CE GATE 2008)

74. A is $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let $A^1 = (A^TA)^{-1}A^T$. Then which of the following statement is true?

(a) $AA^1A = A$ (b) $(AA^1)^2 = A$
(c) $AA^1A = I$ (d) $AA^1A = A^1$

(EE GATE 2008)

75. All the four entries of the 2×2 matrix $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ are non-zero and one of its eigen values is zero. Then which of the following is true?

(a) $P_{11}P_{22} - P_{12}P_{21} = 1$
(b) $P_{11}P_{22} - P_{12}P_{21} = -1$
(c) $P_{11}P_{22} - P_{12}P_{21} = 0$
(d) $P_{11}P_{22} + P_{12}P_{21} = 0$

(EC GATE 2008)

76. The matrix $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & P \end{pmatrix}$ has eigen value equal to 3. Then the sum of other two eigen values is

(a) p (b) p–1
(c) $p - 2$ (d) $p - 3$

(ME GATE 2008)

77. How many of the following matrices have eigen value '1'?

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(a) one (b) two
(c) three (d) four

(CS GATE 2008)

78. The eigen values of the matrix $P = \begin{pmatrix} 4 & 5 \\ 2 & -5 \end{pmatrix}$ are:

(a) –7, 8 (b) –6, 5
(c) 3, 4 (d) 1, 2

(CE GATE 2008)

79. The eigen vectors of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. Then $a + b$ = ?

(a) 0 (b) 1/2
(c) 1 (d) 2

(ME GATE 2008)

80. The characteristic equation of a 3 × 3 matrix P is defined as :

$P(\lambda) = |P - \lambda I| = p^3 + p^2 + 2p + 1 = 0.$

If I denotes identify matrix, then the inverse of the matrix P will be

(a) $P^2 + P + 2I$ (b) $P^2 + P + I$
(c) $-(P^2 - P + 2I)$ (d) $-(P^2 + P + 2I)$

(EE GATE 2008)

81. The trace and determinant of a 2 × 2 a matrix are known to be –2 and –35 respectively. Its eigen values are?

(a) –30, –5 (b) –37, –1
(c) –7, 5 (d) 17.5, –2

(EE GATE 2009)

82. The eigen values of the following matrix $\begin{pmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ are

(a) 3, 3 + 5j, 6 – j
(b) –6 + 5j, 3 + j, 3 – j
(c) 3 + j, 3 – j, 5 + j
(d) 3, – 1 + 3j, –1 –3j

(EC GATE 2009)

83. The eigen values of a 2 × 2 matrix X are – 2 and – 3. Then the eigen values of the matrix $(X + I)^{-1} (X + 5I)$ are

(a) – 3, – 4 (b) – 1, – 2
(c) – 1, – 3 (d) – 2, – 4

(IN GATE 2009)

84. For a matrix $M = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{pmatrix}$,, the transpose of the matrix is equal to the inverse of the matrix. Then the value of x = ?

(a) – 4/5 (b) – 3/5
(c) 3/5 (d) 4/5

(ME GATE 2009)

85. A square matrix B is skew-symmetric if

(a) $B^T = -B$ (b) $B^T = B$
(c) $B^{-1} = B$ (d) $B^{-1} = B^T$

(CE GATE 2009)

86. The value of x_3 obtained by solving the followings systems of linear equation is

$x_1 + 2x_2 - 2x_3 = 4$
$2x_1 + x_2 + x_3 = -2,$
$-x_1 + x_2 - x_3 = 2$

(a) – 12 (b) –2
(c) 0 (d) 12

(GATE 2009)

87. The value of 'q' for which the following set of linear equations $2x + 3y = 0$, $6x + qy = 0$ can have non- trivial solution is

(a) 2 (b) 7
(c) 9 (d) 11

(GATE 2009)

88. The inverse of the matrix $\begin{pmatrix} 3+2i & i \\ -i & 3-2i \end{pmatrix}$ is

(a) $\frac{1}{12}\begin{pmatrix} 3+2i & -i \\ i & 3-2i \end{pmatrix}$ (b) $\frac{1}{12}\begin{pmatrix} \frac{3-2i \quad -i}{i \quad 3+2i} \end{pmatrix}$

(c) $\frac{1}{14}\begin{pmatrix} 3+2i & -i \\ i & 3-2i \end{pmatrix}$ (d) $\frac{1}{14}\begin{pmatrix} 3+2i & -i \\ i & 3+2i \end{pmatrix}$

(CE GATE 2010)

89. For the set of equations $x_1 + 2x_2 + x_3 + 4x_4 = 2$, $3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$ which of the following statement is true?

(a) Only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$ exist
(b) There are no solutions
(c) A unique non-trivial solution exists
(d) Multiple non-trivial solution exists

(EE GATE 2010)

90. A real $n \times n$ matrix $A = [a_{ij}]_{n \times n}$ is defined as follows:

$$a_{ij} = \begin{cases} i, & \text{for } i = j \\ 0, & \text{otherwise} \end{cases}$$

Then the sum of all n eigen values of A is

(a) $\frac{n(n+1)}{2}$ (b) $n(n-1)$

(c) $\frac{n(n+1)(2n+1)}{2}$ (d) n^2

(IN GATE 2010)

91. If $(1, 0, -1)^T$ is an eigen vector of the following matrix $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, then the corresponding eigen value is

(a) 1 (b) 2
(c) 3 (d) 5

(PI GATE 2010)

92. Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ x & y \end{pmatrix}$. If the eigen values are 4 and 8, then

(a) $x = 4, y = 10$ (b) $x = 5, y = 8$
(c) $x = -3, y = 9$ (d) $x = -4, y = 10$

(CS GATE 2010)

93. One of the eigen vector of the matrix $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ is

(a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(ME GATE 2010)

94. An eigen vector of $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

(a) $[-1\ 1\ 1]^T$ (b) $[1\ 2\ 1]^T$
(c) $[1\ -1\ 2]^T$ (d) $[2\ 1\ -1]^T$

(EE GATE 2010)

95. The matrix $M = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}$ has eigen values 3, –3 and 5. An eigen vector corresponding to the eigen value 5 is $[1\ 2\ -1]^T$. One of the eigen vector of the matrix M^3 is

(a) $[1\ 8\ -1]^T$ (b) $[1\ 2\ -1]^T$

(c) $\begin{bmatrix} 1 & \sqrt[3]{2} & -1 \end{bmatrix}^T$ (d) $[1\ 1\ -1]^T$

(IN GATE 2011)

96. Consider the matrix as given below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$$

Which one of the following options provides the correct values of the eigen values of the matrix?

(a) 1,4, 3 (b) 3, 7, 3
(c) 7, 3, 2 (d) 1, 2, 3

(CS GATE 2011)

97. The system of equations $x + y + z = 6$, $x + 4y + 6z = 20$, $x + 4y + \lambda z = \mu$ has no solution for values of λ and μ and given by

(a) $\lambda = 6, \mu = 20$ (b) $\lambda = 6, \mu \neq 20$
(c) $\lambda \neq 6, \mu = 20$ (d) $\lambda = 6, \mu \neq 20$

(EC GATE 2011)

98. [A] is a square matrix which is neither symmetric nor skew symmetric and $[A]^T$ is its transpose. The sum and difference of these matrices and defined as

$[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$ respectively. Then which of the following statement is true?

(a) both [S] and [D] are symmetric.
(b) both [S] and [D] are skew-symmetric.
(c) [S] is skew-symmetric and [D] is symmetric
(d) [S] is symmetric and [D] is skew-symmetric

(CS GATE 2011)

99. The matrix $[A] = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decompose into a product of lower triangular matrix [L] and an upper triangular matrix [U]. The properly decomposed [L] and [U] matrices respectively are

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

(EE GATE 2011)

100. If $A = \begin{pmatrix} -5 & -3 \\ 2 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $A^3 = ?$

(*a*) 15A + 12I (*b*) 19A + 30I

(*c*) 17A + 15I (*d*) 17A + 21I

(EC GATE 2012)

101. Consider the following system of equations:

$2x_1 + x_2 + x_3 = 0,\ x_2 - x_3 = 0,\ x_1 + x_2 = 0$

This system has

(a) a unique solution

(*b*) no solution

(*c*) infinite number of solutions

(*d*) five solutions

(GATE 2012)

102. Consider

$x + 2y + z = 4,\ 2x + y + 2z = 5,\ x - y + z = 1.$

Then the system of algebraic equations given above has

(*a*) a unique solution: $x = 1, y = 1, z = 1$

(*b*) only two solutions: $x = 1, y = 1, z = 1$ and $x = 2, y = 1, z = 0$

(*c*) infinite number of solutions

(*d*) no feasible solution

(ME GATE 2012)

103. The eigen values of the matrix $\begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$ are

(*a*) –2.42 and 6.86 (*b*) 3.48 and 13.52

(*c*) 4.70 and 6.86 (*d*) 6.86 and 9.50

(CE GATE 2012)

104. Let A be the 2 × 2 matrix with elements $a_{11} = a_{12} = a_{21} = 1$ and $a_{22} = -1$. Then the eigen values of the matrix A^{19} are:

(*a*) 1024, –1024

(*b*) $1024\sqrt{2}, -1024\sqrt{2}$

(*c*) $4\sqrt{2}, -4\sqrt{2}$

(*d*) $512\sqrt{2}, -512\sqrt{2}$

(CS GATE 2012)

105. One pair of eigen vectors corresponding to the eigen values of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is

(*a*) $\begin{bmatrix} 1 \\ -j \end{bmatrix}, \begin{bmatrix} j \\ -1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix}$

(IN GATE 2013)

106. A matrix has eigen values –1 and –2. The corresponding eigen vectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. The matrix is

(*a*) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

(*c*) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ (*d*) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

(EE GATE 2013)

107. The minimum eigen value of the following matrix $\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

(EC GATE 2013)

108. Choose the correct set of functions which are linearly dependent

(*a*) $\sin x, \sin^2 x$ and $\cos^2 x$

(*b*) $\cos x, \sin x$ and $\tan x$

(*c*) $\cos 2x, \sin^2 x$ and $\cos^2 x$

(*d*) $\cos 2x, \sin x$ and $\cos x$

(ME GATE 2013)

109. The equation $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

(*a*) no solution

(*b*) only one solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(*c*) non zero unique solution

(*d*) multiple solutions

(GATE 2013)

110. There are three matrices $P_{4\times2}$, $Q_{2\times4}$ and $R_{4\times1}$. Then the minimum number of multiplication required to compute the matrix PQR is____________________

(CE GATE 2013)

111. Which of the followings doesn't equal to

$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$?

(*a*) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$ (*b*) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

(d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$

(CS GATE 2013)

112. Let A be a $m \times n$ matrix and B be a $n \times m$ matrix. It is given that $\det(I_m + AB) = \det(I_n + BA)$, where I_k denote the identity matrix of order k. Use the above property, the determinant of the matrix given below is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(a) 2 (b) 5
(c) 8 (d) 16

(CE GATE 2013)

113. Consider the matrix $J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

which is obtained by reversing the order of the columns of the identity matrix I_6. Let $P = I_6 + \alpha J_6$, where a is non-negative real number. Then the value of α for which $\det(P) = 0$ is____________

(ECE GATE 2014)

114. Which one of the following equations is a correct identity for arbitrary 3 × 3 real matrices P, Q and R?

(a) $P(Q + R) = PQ + RP$
(b) $(P - Q)^2 = P^2 - 2PQ + Q^2$
(c) $\det(P + Q) = \det(P) + \det(Q)$
(d) $(P + Q)^2 = P^2 + PQ + QP + Q^2$

(ME GATE 2014)

115. If the matrix A is such that $A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 \end{bmatrix}$, then the det($A$) =?

(CS GATE 2014)

116. The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is ________________.

(CE GATE 2014)

117. Two matrices A and B are given below:

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, B = \begin{pmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{pmatrix}.$$

If the rank of A is 'N', then the rank of matrix B is

(a) $N/2$ (b) $N - 1$
(c) N (d) $2N$

(EE GATE 2014)

118. With reference to the conventional Cartesian (x, y) co-ordinate system, the vertices of a triangle have the following co-ordinates:

$(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 2)$ and $(x_3, y_3) = (4, 3)$.

The area of the triangle is equal to

(a) 3/2 (b) 3/4
(c) 4/5 (d) 5/2

(CE GATE 2014)

119. The matrix form of the linear system

$$\frac{dx}{dt} = 3x - 5y \text{ and } \frac{dy}{dt} = 4x + 8y \text{ is}$$

(a) $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(b) $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(c) $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(d) $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(GATE 2014)

120. Consider the following system of equations:

$3x + 2y = 1$, $4x + 7z = 1$, $x + y + z = 3$, $x - 2y + 7z = 0$.

The number of solutions for this system is________________.

(CS GATE 2014)

121. The system of equations given below has

$x + 2y + 4z = 2$, $4x + 3y + z = 5$, $3x + 2y + 3z = 1$

(a) a unique solution (b) two solutions
(c) no solution
(d) more than two solutions

(GATE 2014)

122. The system of linear equations has

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 14 \end{pmatrix}$$

(a) a unique solution
(b) infinitely many solutions
(c) no solution
(d) exactly two solutions

(EC GATE 2014)

123. Given a system of equations: $x + 2y + 2z = b_1$, $5x + y + 3z = b_2$

which of the following is true regarding it's solution?

(*a*) The system has a unique solution for any given b_1 and b_2

(*b*) The system will have infinitely many solutions for any given b_1 and b_2

(*c*) Whether or not a solution exists depends on the given b_1 and b_2

(*d*) The system would have no solution for any values of b_1 and b_2

(EE GATE 2014)

124. The determinant of matrix A is 5 and the determinant of matrix B is 40. Then determinant of matrix AB is____?

(EC GATE 2014)

125. For the matrix A satisfying the equation given below, the eigen values are

$$A\begin{bmatrix}1 & 2 & 3\\ 7 & 8 & 9\\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \text{ is}$$

(*a*) (1, -1, 1) (*b*) (1, 1, 0)

(*c*) (1, 1, –1) (*d*) (1, 0, 0)

(IN GATE 2014)

126. Which of the following statements is true for all symmetric matrices?

(*a*) all eigen values are real

(*b*) all eigen values are positive

(*c*) all eigen values are distinct

(*d*) sum of the eigen values is zero

(EE GATE 2014)

127. Which of the following statements is not true for a square matrix A?

(*a*) If A is upper triangular, the eigen values of A are the diagonal elements of it

(*b*) If A is real symmetric, the eigen values of A are always real and positive

(*c*) If A is real, the eigen values of A and A^T are always the same

(*d*) If all the principal minors of A are positive, all the eigen values are all positive.

(EC GATE 2014)

128. The sum of eigen values of matrix M is, where $M =$

$$\begin{bmatrix}215 & 650 & 795\\ 655 & 150 & 835\\ 485 & 355 & 550\end{bmatrix}$$

(*a*) 915 (*b*) 1355

(*c*) 1640 (*d*) 2180

(CE GATE 2014)

129. The maximum value of the determinant among all 2 × 2 real symmetric matrices with trace 14 is______________

(EC GATE 2014)

130. One of the eigen vectors of the matrix $\begin{bmatrix}-5 & 2\\ -9 & 6\end{bmatrix}$ is

(*a*) $\begin{bmatrix}-1\\ 1\end{bmatrix}$ (*b*) $\begin{bmatrix}-2\\ 9\end{bmatrix}$

(*c*) $\begin{bmatrix}2\\ -1\end{bmatrix}$ (*d*) $\begin{bmatrix}1\\ 1\end{bmatrix}$

(ME GATE 2014)

131. Which one of the following statements is true about every $n \times n$ matrix with only real eigen values?

(*a*) If the trace of the matrix is positive and the determinant of the matrix is negative, at least one of its eigen values is negative

(*b*) If the trace of the matrix is positive, all its eigen values are positive

(c) If the determinant of the matrix is positive, all its eigen values are positive

(*d*) If the product of the trace and determinant of the matrix is positive, all its eigen values are positive

(CS GATE 2014)

132. Consider a 3 × 3 real symmetric matrix A such that two of its eigen values are $a \neq 0$, $b \neq 0$ with respective eigen vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ and $\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$. If $a \neq b$, then $x_1y_1 + x_2y_2 + x_3y_3$ equals

(*a*) a (*b*) b

(*c*) ab (*d*) 0

(ME GATE 2014)

133. The product of the non-zero eigen values of the matrix $\begin{bmatrix}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1 & 0\\ 1 & 0 & 0 & 0 & 1\end{bmatrix}$ is______________.

(CS GATE 2014)

134. A real 4 × 4 matrix A satisfies the equation $A^2 = I$, where I is the identity matrix of order 4. Then the positive eigen value of A is___________.

(EC GATE 2014)

135. A system matrix is given as follows:

$$A = \begin{pmatrix}0 & 1 & -1\\ -6 & -11 & 6\\ -6 & -11 & 5\end{pmatrix}$$

The absolute value of the ratio of the maximum eigen value to the minimum eigen value is______________.

(EE GATE 2014)

136. The smallest and the largest eigen values of the following matrix are

$$\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}:$$

(*a*) 1.5 and 2.5 (*b*) 0.5 and 2.5
(*c*) 1 and 3 (*d*) 1 and 2

(CE GATE 2015)

137. The value of 'p' such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigen vector of the matrix

$$\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}$$ is ________

(EC GATE 2015)

138. In the given matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, one of the eigen values is '1'. The eigen vectors corresponding to the eigen value '1' are?

(*a*) $\{\alpha(4, 2, 1): \alpha \neq 0, \alpha \in R\}$
(*b*) $\{\alpha(-4, 2, 1): \alpha \neq 0, \alpha \in R\}$
(*c*) $\{\alpha(\sqrt{2}, 0, 1): \alpha \neq 0, \alpha \in R\}$
(*d*) $\{\alpha(-\sqrt{2}, 0, 1): \alpha \neq 0, \alpha \in R\}$

(CS GATE 2015)

139. The two eigen values of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & p \end{pmatrix}$ have a ratio 3 : 1 for $p = 2$. What is the another value of 'p' for which the eigen values have the same ratio of 3:1 ?

(*a*) –2 (*b*) 1
(*c*) 7/ 3 (*d*) 14/3

(CE GATE 2015)

140. At least one eigen value of a singular matrix is

(*a*) positive (*b*) zero
(*c*) negative (*d*) imaginary

(ME GATE 2015)

141. The maximum value of 'a' such that the matrix $\begin{bmatrix} -3 & 0 & -2 \\ 1 & -1 & 0 \\ 0 & a & -2 \end{bmatrix}$ has three linearly independent real eigen vectors is

(*a*) $\frac{2}{3\sqrt{3}}$ (*b*) $\frac{1}{3\sqrt{3}}$
(*c*) $\frac{1+2\sqrt{3}}{3\sqrt{3}}$ (*d*) $\frac{1+\sqrt{3}}{3\sqrt{3}}$

(EE GATE 2015)

142. Consider the following 2 × 2 matrix A where two elements are unknown and are marked by 'a' and 'b'. The eigen values of this matrix are –1 and 7. What are the values of 'a' and 'b'?

$$\begin{bmatrix} 1 & 4 \\ b & a \end{bmatrix}$$

(*a*) $a = 6, b = 4$ (*b*) $a = 4, b = 6$
(*c*) $a = 3, b = 5$ (*d*) $a = 5$, b = 3

(CS GATE 2015)

143. The value of 'x' for which all the eigen values of the matrix given below are real is

$$\begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

(*a*) $5 + j$ (*b*) $5 - j$
(*c*) $1 - 5j$ (*d*) $1 + 5j$

(EC GATE 2015)

144. For $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, the determinant of $A^T A^{-1}$ is

(*a*) $\sec^2 x$ (*b*) $\cos^4 x$
(*c*) 1 (*d*) 0

(EC GATE 2015)

145. If any two columns of a determinant $\begin{bmatrix} 4 & 7 & 8 \\ 3 & 1 & 5 \\ 9 & 6 & 2 \end{bmatrix}$ are interchanged, which one of the statement is correct?

(*a*) absolute value remains unchanged but sign will change
(*b*) both value and sign will change
(*c*) absolute value will change but sign will not change.
(*d*) both absolute value and sign will remain unchanged.

(ME GATE 2015)

146. Let $A = [a_{ij}]$, $1 \leq i, j \leq n$ with $n \geq 3$ and $a_{ij} = i \times j$. Then the rank of A is

(*a*) 0 (*b*) 1
(*c*) n–1 (*d*) n

(CE GATE 2015)

147. Consider a system of linear equations:

$x - 2y + 3z = -1$, $x - 3y + 4z = 1$, $-2x + 4y - 6z = k$

The value of 'k' for which the system has infinitely many solutions is____________

(EC GATE 2015)

148. For what value of 'p', the following set of equations will have no solution?

$2x + 3y = 5$, $3x + py = 10$

(CE GATE 2015)

149. If the following system has non-trivial solution:
$px + qy + rz = 0, \quad qx + ry + pz = 0, \quad rx + py + qz = 0;$
then which of the following options is true?
(*a*) $p - q + r = 0$ or $p = q = -r$
(*b*) $p + q - r = 0$ or $p = -q = r$
(*c*) $p + q + r = 0$ or $p = q = r$
(*d*) $p - q + r = 0$ or $p = -q = -r$

(CS GATE 2015)

150. The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has det(A) = 100 and trace(A) = 14. The value of $|a-b|$ is______________

(EC GATE 2016)

151. Let $P = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Consider the set S of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $a^2 + b^2 = 1$ where $\begin{bmatrix} a \\ b \end{bmatrix} = P\begin{bmatrix} x \\ y \end{bmatrix}$. Then S is
(*a*) a circle of radius $\sqrt{10}$
(*b*) a circle of radius $\dfrac{1}{\sqrt{10}}$
(*c*) an ellipse with major axis along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
(*d*) an ellipse with minor axis along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

[EE GATE 2016]

152. If the entries in each column of a square matrix M add up to '1', then an eigen value of M is
(*a*) 4 (*b*) 3
(*c*) 2 (*d*) 1

[CE GATE 2016]

153. Consider a 3 × 3 matrix with every element being equal to '1'. It's only non-zero eigen value is _____________.

(EE GATE 2016]

154. Let $M^4 = I$ (where I denote the identity matrix) and $M \neq I$, $M^2 \neq I$, $M^3 \neq I$. Then for any natural number k, M^{-1} equals
(*a*) M^{4k+1} (*b*) M^{4k+2}
(*c*) M^{4k+3} (*d*) M^{4k}

(EC GATE 2016)

155. The condition for which the eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ are positive is
(*a*) $k > 1/2$ (*b*) $k > -2$
(*c*) $k > 0$ (*d*) $k < -1/2$

(ME GATE 2016)

156. Two eigen values of a 3 × 3 matrix real matrix P are $2 + \sqrt{-1}$ and 3. Then the determinant of P is_________

(CS GATE 2016)

157. Consider the following linear system:
$x + 2y - 3z = a,\ 2x + 3y + 3z = b,\ 5x + 9y - 6z = c.$
The system is consistent if a, b and c satisfy the equation
(*a*) $7a - b - c = 0$ (*b*) $3a + b - c = 0$
(*c*) $3a - b + c = 0$ (*d*) $7a - b + c = 0$

(CE GATE 2016)

158. The solution to the system of equations $\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$ is
(*a*) 6, 2 (*b*) –6, 2
(*c*) –6, –2 (*d*) 6, –2

(ME GATE 2016)

159. The number of solutions of the simultaneous algebraic equations $y = 3x + 3$ and $y = 3x + 5$ is?
(*a*) zero (*b*) 1
(*c*) 2 (*d*) infinite

(PI GATE 2016)

160. Consider the systems, each consisting of 'm' linear equations in 'n' variables.
I. If $m < n$, then all such systems have a solution
II. If $m > n$, then none of the system has a solution
III. If $m = n$, there exist a system which has a solution.
Then which of the following is correct?
(*a*) I, II and III are true
(*b*) only II and III are true
(*c*) only III is true
(*d*) none of them is true

(CSE GATE 2016)

161. Consider a 2 × 2 square matrix $P = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$ where 'x' is unknown. If the eigen values of the matrix A are ($\sigma + j\omega$) and ($\sigma - j\omega$), then 'x' is equal to
(*a*) $j\omega$ (*b*) $-j\omega$
(*c*) w (*d*) $-\omega$

(EC GATE 2016)

162. The eigen value of the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ are
(*a*) $i, -i$ (*b*) 1, – 1
(*c*) 0, 1 (*d*) 0, – 1

(PI GATE 2016)

163. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$ whose eigen values are 1, – 1 and 3. Then the trace of $A^3 - A^2$ is_______

(IN GATE 2016)

164. The value of 'x' for which the matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{bmatrix}$ has zero as an eigen value, is _____

(EC GATE 2016)

165. Suppose that the eigen values of a matrix A are 1,2,4. Then the determinant of $(A^{-1})^T$ is ________________

(CS GATE 2016)

166. A real squire matrix A is called skew-symmetric if

(*a*) $A^T = A$ (*b*) $A^T = A^{-1}$

(*c*) $A^T = -A$ (*d*) $A^T = A + A^{-1}$

(ME GATE 2016)

167. A 3 × 3 Matrix P is such that $P^3 = P$. Then the eigen values of P are

(*a*) 1, 1, –1

(*b*) 1, 0.5 + 0.866*j*, 0.5–0.866*j*

(*c*) 1, –0.5 + 0.866*j*, –0.5 – 0.866*j*

(*d*) 0, 1, –1

(EE GATE 2016)

168. A sequence $x[n]$ is defined as:

$$\begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ for } n \geq 2.$$

The initial conditions are : $x[0] = 1$, $x[1] = 1$ and $x[n] = 0$ for $n < 0$. The the value of $x[12]$ is___________

(EC GATE 2016)

169. Let A be a 4 × 3 real matrix with rank 2. Then which of the following statements is true?

(*a*) Rank of A^TA is less than 2

(*b*) Rank of A^TA is equal to 2

(*c*) Rank of A^TA is greater than 2

(*d*) Rank of A^TA can be any number between 1 and 3

(EE GATE 2016)

170. If the vectors $e_1 = (1, 0, 2)$, $e_2 = (0, 1, 0)$ and $e_3 = (-2, 0, 1)$ form an orthogonal basis of the three dimensional real space R^3, then the vector u = (4, 3, – 3) ∈ R^3 can be expressed as

(*a*) $u = -\frac{2}{5}e_1 - 3e_2 - \frac{11}{5}e_3$

(*b*) $u = -\frac{2}{5}e_1 - 3e_2 + \frac{11}{5}e_3$

(*c*) $u = -\frac{2}{5}e_1 + 3e_2 + \frac{11}{5}e_3$

(*d*) $u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$

(EC GATE 2016)

171. The rank of the matrix $M = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

(EC GATE 2017)

172. If the determinant of a 2 × 2 matrix is 50 and one eigen value is 10, then the other eigen value is_____________________

(ME GATE 2017)

173. The product of the eigen values of the matrix P is

$$\begin{pmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & -1 \end{pmatrix}$$

(*a*) –6 (*b*) 2

(*c*) 6 (*d*) –2

(ME GATE 2017)

174. Let A be a $n \times n$ real valued square symmetric matrix of rank 2 with $\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2 = 50$. Consider the following statements:

I. one eigen value must be in [–5, 5]

II. The eigen value with the largest magnitude must be strictly greater than '5'.

Which of the above statements about eigen values of A is/are necessarily correct?

(*a*) both *I* and *II*

(*b*) *I* only

(*c*) *II* only

(*d*) neither *I* nor *II*

(CS GATE 2017)

175. Consider the matrix $A = \begin{pmatrix} 50 & 70 \\ 70 & 80 \end{pmatrix}$ whose eigen vectors corresponding to eigen values λ_1 and λ_2 are $X_1 = \begin{pmatrix} 70 \\ \lambda_1 - 50 \end{pmatrix}$ and $X_2 = \begin{pmatrix} \lambda_2 - 80 \\ 70 \end{pmatrix}$ respectively.

The value of $X_1^T X_2$ is____________

(ME GATE 2017)

176. The eigen values of the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & 6 \\ 0 & -6 & 5 \end{bmatrix}$ are

(*a*) 1, 5, 6 (*b*) 1, –5 ± 6*j*

(*c*) 1, 5 ± 6*j* (*d*) 1, 5, 5

(IN GATE 2017)

177. Consider the matrix $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$. Then which one of the following statements is true for the eigen values and eigen vectors of this matrix?

(*a*) Eigen value 3 has a multiplicity of 2 and only one independent eigen vector exists (*b*) Eigen value 3 has a multiplicity of 2 and two independent eigen vector exist

(*c*) Eigen value 3 has a multiplicity of 2 and no independent eigen vector exists

(*d*) Eigen values are 3 and -3 and two independent eigen vector exist

(CE GATE 2017)

178. Consider the 5 × 5 matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$

It is given that A has only one real eigen value. Then the real eigen value of A is

(*a*) –25 (*b*) 0

(*c*) 15 (*d*) 25

(EC GATE 2017)

179. Consider the matrix $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$.

Which one of the following statements about P is incorrect?

(*a*) Det(P) = 1 (*b*) P is orthogonal

(*c*) $P^{-1} = P^T$

(*d*) All eigen values of P are real numbers

(ME GATE 2017)

180. Consider the following simultaneous equations (c_1 and c_2 being constants):

$$3x_1 + 2x_2 = c_1$$

$$4x_1 + x_2 = c_2$$

The characteristic equation for these simultaneous equations is

(*a*) $\lambda^2 - 4\lambda - 5 = 0$ (*b*) $\lambda^2 - 4\lambda + 5 = 0$

(*c*) $\lambda^2 + 4\lambda - 5 = 0$ (*d*) $\lambda^2 + 4\lambda + 5 = 0$

(CE GATE 2017)

181. The matrix $A = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$ has three distinct eigen values and one of the eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Which of the following can be another eigen vector of A?

(*a*) $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ (*b*) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(EE GATE 2017)

182. If the characteristic polynomial of a 3 × 3 matrix M over R (where R is the set of real numbers) is $\lambda^3 - 4\lambda^2 + a\lambda + 30$ for $a \in R$ and one eigen value of M is 2; then the largest among the absolute values of the eigen value of M is ____________.

(CS GATE 2017)

183. The rank of the matrix $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$ is________________

(EC GATE 2017)

184. Let $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ *and* $Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two martices, then the rank of $P + Q$ is_______

(CS GATE 2017)

185. The matrix P is the inverse of a matrix Q. If I denotes the identity matrix, which one of the following option is correct?

(*a*) $PQ = I$ but $QP \neq I$

(*b*) $QP = I$ but $PQ \neq I$

(*c*) $PQ = I$ and $QP = I$

(*d*) $PQ - QP = I$

(CE GATE 2017)

186. If V is a non-zero vector of dimension 3 × 1, then the matrix $A = VV^T$ has a rank _____________.

(IN GATE 2017)

187. If $A = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$, then AB^T is equal to

(*a*) $\begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$ (*b*) $\begin{bmatrix} 3 & 40 \\ 42 & 8 \end{bmatrix}$

(*c*) $\begin{bmatrix} 43 & 27 \\ 34 & 50 \end{bmatrix}$ (*d*) $\begin{bmatrix} 38 & 32 \\ 28 & 56 \end{bmatrix}$

(CE GATE 2017)

188. Which of the following matrix is singular?

(*a*) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ (*b*) $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

(*c*) $\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$ (*d*) $\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

(CE GATE 2018)

189. For the given orthogonal matrix Q,

$$Q = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$$

The inverse is

(*a*) $Q = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ \frac{-6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & \frac{-3}{7} \end{pmatrix}$ (*b*) $Q = \begin{pmatrix} \frac{-3}{7} & \frac{-2}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{-3}{7} & \frac{-2}{7} \\ \frac{-2}{7} & \frac{-6}{7} & \frac{3}{7} \end{pmatrix}$

(*c*) $Q = \begin{pmatrix} \frac{3}{7} & \frac{-6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & \frac{-3}{7} \end{pmatrix}$ (*d*) $Q = \begin{pmatrix} \frac{-3}{7} & \frac{6}{7} & \frac{-2}{7} \\ \frac{-2}{7} & \frac{-3}{7} & \frac{-6}{7} \\ \frac{-6}{7} & \frac{-2}{7} & \frac{3}{7} \end{pmatrix}$

(CE GATE 2018)

190. The rank of the following matrix is

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{pmatrix}$$

(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 4

(CE GATE 2018)

191. The rank of the matrix $\begin{pmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{pmatrix}$ is

(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 4

[ME GATE 2018]

192. The matrix $\begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix}$ has

(*a*) Real eigen values and eigen vectors.
(*b*) Real eigen values but complex eigen vectors.
(*c*) complex eigen values but real eigen vectors.
(*d*) complex eigen values and eigen vectors.

(CS/IT-GATE-2018)

193. Let M be a 4×4 matrix. Consider the following statements:

$S1$: M has 4 linearly independent eigenvectors.
$S2$: M has distinct eigen values.
$S3$: M is non-singular (invertible).

Which one among the following is TRUE?

(*a*) S1 implies S2 (*b*) S1 implies S3
(*c*) S2 implies S1 (*d*) S3 implies S2

[EC GATE 2018]

194. Let $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ and $B = A^3 - A^2 - 4A + 5I$, where I is the 3×3 identity matrix. The determinant of B is _________? (Up to 1 decimal places).

[EC GATE 2018]

195. Consider the following system of linear equation

$3x + 2ky = -2, \; kx + 6y = 2$

Here x and y are the unknowns and k is a real constant. The value of k for which there are infinite number of solutions is

(*a*) 3 (*b*) 1
(*c*) – 3 (*d*) – 6

[IN GATE 2018]

196. Consider matrix $A = \begin{pmatrix} k & 2k \\ k^2 - k & k^2 \end{pmatrix}$ and vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The number of distinct real values of k for which the equation $AX = O$ has infinitely many solution is _________? **[EC GATE 2018]**

197. The diagonal elements of a 3 × 3 matrix are – 10, 5 and 0 respectively. If two of its eigen values are – 15 each, the third eigen values is _______?

[PI GATE 2018]

198. Consider a matrix $A = UV^T$ where $U = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Note that V^T denotes the transpose of V. The largest eigen value of A is _____? **[CS/IT GATE 2018]**

199. Consider a non-singular 2×2 square matrix A. If trace(A) = 4 and trace (A^2) = 5, the determinant of the matrix A is __________ ?

[EC GATE 2018]

200. Let N be a 3×3 matrix with real numbers entries. The matrix N is such that $N^2 = 0$. The eigen values of N are

(*a*) 0, 0, 0 (*b*) 0, 0, 1
(*c*) 0, 1, 1 (*d*) 1, 1, 1

Answer key

1. (*b*)	2. (*a*)	3. (*a*)	4. (*b*)	5. (*c*)
6. (*a*)	7. (*b*)	8. (*b*)	9. (*d*)	10. (*b*)
11. (*d*)	12. (*b*)	13. (*a*)	14. (*a*)	15. (*c*)
16. (*b*)	17. (*c*)	18. (*b*)	19. (*b*)	20. (*c*)
21. (*c*)	22. (*c*)	24. (*c*)	25. (*a*)	26. (*b*)

27. (*d*)	28. (*c*)	29. (*c*)	30. (*a*)	31. (*d*)
32. (*d*)	33. (*b*)	34. (*c*)	35. (*b*)	36. (*c*)
37. (*b*)	38. (*b*)	39. (*b*)	40. (*a*), (*b*), (*d*)	
41. (*b*)	42. (*a*)	43. (*a*)	44. (*c*)	45. (*a*)
46. (*a*)	47. (*b*)	48. (*a*)	49. (*d*)	50. (*c*)
51. (*c*)	52. (*a*)	53. (*b*)	54. (*a*)	55. (*a*)
56. (*c*)	57. (*a*)	58. (*b*)	59. (*a*)	
60. (*i*) – (*a*); (*ii*) – (*a*)				
61. (*a*)	62. (*b*)	63. (*a*)	64. (*d*)	65. (*a*)
66. (*b*)	67. (*b*)	68. (*c*)	69. (*d*)	70. (*a*)
71. (*d*)	72. (*c*)	73. (*b*)	74. (*a*)	75. (*c*)
76. (*c*)	77. (*a*)	78. (*b*)	79. (*b*)	80. (*d*)
81. (*c*)	82. (*d*)	83. (*c*)	84. (*a*)	85. (*a*)
86. (*b*)	87. (*c*)	88. (*b*)	89. (*d*)	90. (*a*)
91. (*a*)	92. (*d*)	93. (*a*)	94. (*b*)	95. (*b*)
96. (*a*)	97. (*b*)	98. (*d*)	99. (*b*)	100. (*b*)
101. (*c*)	102. (*c*)	103. (*b*)	104. (*d*)	105. (*a*),(*d*)
106. (*d*)	107. (*a*)	108. (*c*)	109. (*d*)	110. 16
111. (*a*)	112. (*b*)	113. 1, –1	114. (*d*)	115. 0.
116. 2.	117. (*c*)	118. (*a*)	119. (*a*)	120. one
121. (*a*)	122. (*b*)	123. (*b*)	124. 200	125. (*c*)
126. (*a*)	127. (*b*)	128. (*a*)	129. 49	130. (*d*)
131. (*a*)	132. (*d*)	133. 6.	134. 1.	135. 3.
136. (*d*)	137. 17	138. (*b*)	139. (*d*)	140. (*b*)
141. (*b*)	142. (*d*)	143. (*a*)	144. (*c*)	145. (*a*)
146.(*b*)	147. 2	148. 4.5	149. (*c*)	150. 3
151. (*d*)	152. (*d*)	153. 3	154. (*c*)	155. (*a*)
156. 15.	157. (*b*)	158. (*d*)	159. (*a*)	160. (*c*)
161. (*d*)	162. (*a*)	163. –6.	164. 1.	165. 1/8.
166. (*c*)	167. (*d*)	168. 233	169. (*b*)	170. (*d*)
171. (*c*)	172. 5.	173. (*b*)	174. (*b*)	175. [0]
176. (*c*)	177. (*a*)	178. (*c*)	179. (*d*)	180. (*a*)
181. (*c*)	182. 5.	183. 4	184. 2.	185. (*c*)
186. 1	187. (*a*)	188. (*c*)	189. (*c*)	190. (*b*)
191. (*b*)	192. (*d*)	193. (*c*)	194. 1.	195. (*c*)
196. 2.	197. 25.	198. 3.	199. $\frac{11}{2}$.	200. (*a*)

Explanation

1. (*b*) Trace of the matrix = 2 + 3 + (–2) + 4 = 7 = sum of eigen values = 2 + 3 + (–2) + 4.

2. (*a*) Here $V[i, j]$ denotes the element lying in i^{th} row and j^{th} column.

$$\text{For } n = 3,\ V = \begin{pmatrix} V[1,1] & V[1,2] & V[1,3] \\ V[2,1] & V[2,2] & V[2,3] \\ V[3,1] & V[3,2] & V[3,3] \end{pmatrix}$$

$$= \begin{pmatrix} 1-1 & 1-2 & 1-3 \\ 2-1 & 2-2 & 2-3 \\ 3-1 & 3-2 & 3-3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \quad (\because V[i, j] = i - j)$$

Therefore the sum of the elements in the array V is zero.

3. (*a*)

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{vmatrix}$$

$= 2 \times$ co-factor of '2'

(since '2' is the only non-zero element in 1st row)

$$= 2\begin{vmatrix} 1 & 7 & 2 \\ 0 & 2 & 0 \\ 0 & 6 & 1 \end{vmatrix}$$

$= 2 \times 2$

$= 4.$

4. (*b*) $(ABC)^{-1} = [(A(BC)]^{-1} = (BC)^{-1} A^{-1} = C^{-1}B^{-1}A^{-1}$.

5. (*c*)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} [by\ R_3 \to R_3 - (R_1 + R_2)]$$

which has 2 non-zero rows

Therefore rank (A) = 2.

6. (*a*)

$$[P][Q]^T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 4 & 9 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 2\times4+3\times8 & 2\times9+3\times2 \\ 4\times4+5\times8 & 4\times9+5\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 24 \\ 56 & 46 \end{bmatrix}.$$

7. (*b*) $C = BA = B_{1\times3}\ A_{3\times1}$. Then C is a matrix of order 1×1. Therefore rank(A) ≤ min {1, 1} = 1.

So the possible cases are:

rank(C) = 0 or rank(C) = 1.

But since both A and B are non-zero matrices, so the product AB (=C) contains at least one non-zero element. Hence rank(C) = 1.

8. (*b*) Since the given matrix has only one non-zero row, so rank = 1.

9. (*d*) Trace of the matrix = 5 + 9 = 14 = sum of eigen values = 10.16 + 3.84.

10. (*b*)

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0, \qquad \Delta_1 = \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} \neq 0$$

Therefore the system has no solution (by Cramer's rule).

11. (*d*) The necessary condition to diagonalize a matrix is that the matrix is non-singular; whereas the sufficient condition is that the matrix has n linearly independent eigen vectors (where '*n*' is the order of the square matrix).

12. (*b*)

$$\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 2 \end{vmatrix}$$

$$= 3(-2-0) - 2(2+2) + 1(0-2) \neq 0$$

Thus system has a unique solution.

13. (*a*)

Case-I: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ .Then det(A) = 0, det(B) = 0 and det($A + B$) = $\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0.$

In this case each of A and B is a singular matrix and A+B is also singular. Thus the statement S_1 is true.

Case-II: Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Then det($A$) ≠ 0, det($B$) ≠ 0 and det($A + B$) = $\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} \neq 0.$

In this case each of A and B is non-singular matrix and $A + B$ is also non-singular. Thus the statement S_2 is true.

14. (*a*) Trace of the matrix = 1 + 2 + (–2) + (–1) = 0 = sum of eigen values = 1 + 2 + (–2) + (–1).

15. (*c*) Since the determinant value of an upper triangular matrix is the product of the diagonal elements, so the determinant value of the given matrix =1 × 1 × 1 × 1 = 1.

16. (*b*)

$$X^2 - X + I = O$$

$$\Rightarrow I = X - X^2$$

$$\Rightarrow IX^{-1} = \left(X - X^2\right)X^{-1}$$

$$\Rightarrow X^{-1} = XX^{-1} - X^2X^{-1} = I - X$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 1 \\ -a^2 + a - 1 & 1 - a \end{bmatrix}$$

$$= \begin{bmatrix} 1 - a & -1 \\ a^2 - a + 1 & a \end{bmatrix}$$

17. (*c*)

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Delta_1 = \begin{vmatrix} 6 & 2 & 1 \\ 6 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix} = 3 \neq 0$$

∵ Δ = 0, $\Delta_1 \neq 0$, so the system has no solutions (by Cramer's rule).

Alternative method:

$$[A : B] = \begin{bmatrix} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

(by $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$)

$$\sim \begin{bmatrix} 1 & 2 & 1 & 6 \\ 0 & 0 & 0 & -3 \\ 0 & -1 & 0 & -1 \end{bmatrix} \text{ by } (R_2 \to R_2 - 3R_3)$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 6 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

(by $R_2 \leftrightarrow R_3$)

Hence rank ([A:B]) = 3 and rank(A) = 2. Since they are un-equal, so system has no solution.

18. (*b*) The given system of equations can be written as: $2x + y - 4z = \alpha$, $4x + 3y - 12z = 5$, $x + 2y - 8z = 7$.
The system has infinitely many solutions

$$\Rightarrow \Delta_3 = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 1 & \alpha \\ 4 & 3 & 5 \\ 1 & 2 & 7 \end{vmatrix} = 0$$

$$\Rightarrow 2(21 - 10) - (28 - 5) + \alpha(8 - 3) = 0$$

$$\Rightarrow \alpha = \frac{1}{5}.$$

So α has only one value .

19. (*b*) A homogeneous system of equations has a non-trivial solution if the coefficient matrix is singular.

20. (*c*)

$$A = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

(by $R_2 \to R_2 - \frac{3}{2}R_1$)

$$\sim \begin{pmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \text{(by } R_2 \to \frac{2}{5}R_2\text{)}$$

$$\sim \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \text{(by } R_1 \to R_1 - 2R_3\text{)}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \text{(by } R_1 \to R_1 - R_2\text{)}$$

Which has two non-zero rows

Therefore rank (A) = 2.

21. (c) Trace of the matrix = 4 + 4 = sum of eigen values = 3 + 5.

22. (c)

$$|A - \lambda I| = O \Rightarrow \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4 - \lambda)\ (1 - l) - 4 = 0$

$\Rightarrow \lambda^2 - 5\lambda = 0$

$\Rightarrow \lambda = 0, 5$

Alternative Method

If we take the eigen values 0 and 5, then product of the eigen values = 0, which is equal to det(A).

23. (b) Sum of eigen values = trace of the matrix = 1 + 5 + 1 = 7.

24. (c) For n = 2, the symmetric matrices with each element equal to either 0 or 1 are:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus for n = 2, there exist eight such symmetric matrices.

Now $8 = 2^{\frac{2^2+2}{2}}$

Hence option (c) is correct.

25. (a)

$$A = \begin{pmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{pmatrix}$$

Then A is Singular $\Rightarrow$ det(A) = 0 $\Rightarrow x = 4$.

26. (b) A, B, C, D has non-zero determinants A^{-1}, B^{-1}, C^{-1}, D^{-1} exist

Then

$ABCD = I \Rightarrow A^{-1}(ABCD) = A^{-1}I = A^{-1}$

$\Rightarrow\ (A^{-1}A)(BCD)D^{-1} = A^{-1}D^{-1}$

$\Rightarrow\ IBC(DD^{-1}) = A^{-1}D^{-1}$

$\Rightarrow\ BCI = A^{-1}D^{-1}$

$\Rightarrow\ BC = A^{-1}D^{-1}$

$\Rightarrow\ (BC)C^{-1} = A^{-1}D^{-1}C^{-1}$

$\Rightarrow\ B(CC^{-1}) = A^{-1}D^{-1}C^{-1}$

$\Rightarrow\ BI = A^{-1}D^{-1}C^{-1}$

$\Rightarrow\ B = A^{-1}D^{-1}C^{-1} = (CDA)^{-1}$

$\Rightarrow\ B^{-1} = ((CDA)^{-1})^{-1} = CDA$

27. (d) Since product of matrices (if defined) is again matrix. Hence the product F^TC^TBCF can't be a scalar. Hence the statement (i) is false.

Again F is a matrix of order 5×1 and D is a matrix of order 5×3 $\Rightarrow FD$ is not defined

(since number of columns of $F \neq$ number of rows of D and so D^TFD is not defined.

Therefore statement (ii) is false.

28. (c) Solving the 1st two equations we get, x = 9/4 and y = 1/4; which satisfies the 3rd equation. So the system has a unique solution given by x = 9/4 and y = 1/4.

29. (c) Using Crammer's rule we have,

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{vmatrix} = -1$$

$$\Delta_1 = \begin{vmatrix} 6 & 2 & 3 \\ 8 & 3 & 4 \\ 12 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 6 & 2 & 3 \\ 2 & 1 & 1 \\ 6 & 0 & 0 \end{vmatrix} = -6$$

$(by\, R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 3 \\ 1 & 8 & 4 \\ 2 & 12 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{vmatrix}$$

$(by\, R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1)$

= 1(–6 – 0) = –6

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 1 & 3 & 8 \\ 2 & 2 & 12 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & -2 & 0 \end{vmatrix}$$

$(by\, R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1)$

= 4

Thus,

$$x = \frac{\Delta_1}{\Delta} = 6, y = \frac{\Delta_2}{\Delta} = 6, z = \frac{\Delta_3}{\Delta} = -4.$$

Alternative Method

The given system of equations can be written as:
$x + 2y + 3z = 6$, $x + 3y + 4z = 8$, $2x + 2y + 3z = 12$.
These equations are satisfied for $x = y = 6$ and $z = -4$.

30. (*a*)

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 5 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)\begin{vmatrix} 5-\lambda & 5 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 5$ is an eigen value.

Then $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 5\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5x \\ 5y+5z \\ 2z+t \\ 3z+t \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \\ 5z \\ 5t \end{bmatrix}$$

$\Rightarrow 5x = 5x, 5y + 5z = 5y, 2z + t = 5z, 3z + t = 5t$

$\Rightarrow z = 0, t = 0$

Then eigen vector,

$$X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} \quad (\text{taking } x = 1,\ y = -2)$$

31. (*d*)

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ a & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda) - 4a = 0$

$\Rightarrow \lambda^2 - 3\lambda + (2 - 4a) = 0$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(2-4a)}}{2}$$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{16a+1}}{2},$$

which is real if

$16a + 1 \geq 0$ i.e; $a \geq -\frac{1}{16}$

Also, $\lambda \geq 0$

$$\Rightarrow \frac{3 \pm \sqrt{16a+1}}{2} \geq 0$$

$$\Rightarrow \frac{3}{2} \geq \mp\frac{\sqrt{16a+1}}{2}$$

$$\Rightarrow \frac{9}{4} \geq \frac{16a+1}{4}$$

$$\Rightarrow a \leq \frac{1}{2}$$

$$\therefore -\frac{1}{16} \leq a \leq \frac{1}{2}.$$

32. (*d*)

$PX = \lambda X$

$$\Rightarrow \begin{pmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x-2y+2z \\ -2y+z \\ z \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix}$$

$\Rightarrow 3x - 2y + 2z = -2x, -2y + z = -2y, z = -2z$

$\Rightarrow 5x - 2y + 2z = 0, z = 0 \Rightarrow x = \frac{2}{5}y, z = 0$

$$\therefore \text{ eigen vector}, X = \begin{bmatrix} \frac{2}{5}y \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \quad (\text{for } y = 5)$$

33. (*b*) Determinant of the matrix = 10 – 4 = 6 = product of eigen values = 1 × 6.

34. (*c*)

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda - 3)(4 + \lambda) - 8 = 0$

$\Rightarrow \lambda^2 + \lambda - 20 = 0$

$\Rightarrow \lambda = -5, 4$

For $\lambda = -5$,

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -5\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4x+2y \\ 4x+3y \end{bmatrix} = \begin{bmatrix} -5x \\ -5y \end{bmatrix}$$

$\Rightarrow -4x + 2y = -5x,\ 4x + 3y = -5y$

$\Rightarrow x = -2y$

$$\therefore \text{Eigen vector}, X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(taking $y = -1$)

35. (*b*) System has a non-trivial solution

$\Rightarrow$ rank $(A - \lambda I) < n$.

Thus (*a*) is correct.

$A^T = A^{-1} \Rightarrow A$ is orthogonal

$\Rightarrow |\lambda_i| = 1$, (*c*) is correct

$A^T = A \Rightarrow A$ is symmetric

$\Rightarrow$ eigen values λ_i are real

Thus (*d*) is correct.

Therefore (*b*) is not a correct statement.

36. (*c*) Value of the determinant

= product of eigen values

= 15 × 3 × 0 = 0

37. (*b*) Since A is a lower triangular, so the eigen values are the diagonal elements *i.e.* 1 and –2.

For $\lambda = 1$

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ -x-2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow x = x, -x - 2y = y$$

$$\Rightarrow x = -3y$$

Thus eigen vector,

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3y \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(for $y = -1$)

38. (*b*) A has order 3 × 4

$\Rightarrow$ rank (A) $\leq$ min{3, 4}

$\Rightarrow$ rank (A) $\leq$ 3

Also rank ([A : B]) $\leq$ min {3, 5} = 3

If rank (A) = rank ([A : B]) = 3, then the system will be consistent. Hence for the system to be inconsistent, the highest possible rank of A will be 2.

39. (*b*)

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 3 & -2 & 5 \\ -1 & -4 & 1 \end{vmatrix} = 2 \neq 0$$

Therefore system has a unique solution (by Cramer's rule).

40. (*a*), (*b*), (*d*) In case of non-homogeneous system of equations, a consistent system will have either unique solution or infinitely many solutions; where as an inconsistent system will have no solution.

41. (*b*) No. of independent solution

= order of the square matrix – rank of the coefficient matrix

= 3 – 2 = 1.

42. (*a*) Rank (P) = rank ([P : Q]) $\Leftrightarrow$ at least one solution exist.

43. (*a*)

$$\begin{vmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{vmatrix} \quad [\text{by } R_2 \to R_2 + R_4]$$

$= 1 \times$ cofactor of '1' [$\because$ '1' lies in 1st column]

$$= \begin{vmatrix} 1 & 0 & 2 \\ -1 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = -1$$

44. (*c*) A is orthogonal $\Rightarrow AA' = I$

$$\Rightarrow (AA')^{-1} = I^{-1} = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

45. (*a*)

$$|A| = \begin{vmatrix} 2 & -0.1 \\ 0 & 3 \end{vmatrix} = 6$$

$$A^{-1} = \frac{1}{|A|} adj(A)$$

$$= \frac{1}{6}\begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{0.1}{6} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix} \text{(given)}$$

$$\therefore a = \frac{0.1}{6}, b = \frac{1}{3}. \text{ So } a + b = \frac{1}{60} + \frac{1}{3} = \frac{7}{20}.$$

46. (*a*)

$$[P(X^TY)^{-1}P^T]^T$$

$$= (P^T)^T[(X^TY)^{-1}]^T P^T = P[(X^TY)^{-1}]^T P^T$$

$$= P_{2\times3}[\{(X^T)_{3\times4} Y_{4\times3}\}^{-1}]^T (P^T)_{3\times2}$$

$$= P_{2\times3}[\{(X^TY)_{3\times3}\}^{-1}]^T (P^T)_{3\times2}$$

$$= P_{2\times3}[\{(X^TY)^{-1}\}_{3\times3}]^T (P^T)_{3\times2}$$

$= P_{2\times3}[\{(X^TY)^{-1}\}^T]_{3\times3}(P^T)_{3\times2}$

$= P_{2\times3}[\{(X^TY)^{-1}\}^TP^T]_{3\times2}$

$= [P\{(X^TY)^{-1}\}^TP^T]_{2\times2}$

47. (*b*)

$$|R| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix} = 1(2+3) - 0 - 1(6-2) = 5 - 4 = 1$$

$$\text{adj}(R) = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & -6 & 4 \\ -3 & 4 & -3 \\ 1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$$

$$R^{-1} = \frac{1}{|R|}adj(R) = \frac{1}{1}\begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\Rightarrow R^{-1} = \begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$$

Therefore top row of R^{-1} = [5 –3 1]

48. (*a*)

$$|E| = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \cos^2\theta + \sin^2\theta = 1 \neq 0.$

$\therefore E^{-1}$ exist.

adj(E)

$$= \begin{bmatrix} \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} & -\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix} & \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E^{-1} = \frac{1}{|R|}adj(E) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $EF = G$

$\Rightarrow E^{-1}(EF) = E^{-1}G$

$\Rightarrow (E^{-1}E)F = E^{-1}I$ ($\because$ G = I, the identity matrix)

$\Rightarrow IF = E^{-1}$

$$\Rightarrow F = E^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

49. (*d*)

$$|A| = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (by\ R_3 \to R_3 - R_1)$$

= 1 × co-factor of '1'

(since '1' is the only non-zero element of 1st column)

$$= \begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$

Thus A is a non-singular matrix of order 4. Therefore rank (A) = 4

50. (*c*)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{(by } R_3 \to R_3 - R_1)$$

which has 2 non-zero rows

Therefore rank = 2.

51. (*c*)

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix} = \lambda\begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\times101 + 2\times101 \\ 2\times101 + 4\times101 \end{bmatrix} = \begin{bmatrix} 101\lambda \\ 101\lambda \end{bmatrix}$$

$\Rightarrow 101\lambda = 4\times101 + 2\times101$

$\Rightarrow \lambda = 4 + 2 = 6.$

52. (*a*)

53. (*b*) Let λ_1 and λ_2 be the other two eigen values.
Then sum of eigen values = trace of the matrix
$\Rightarrow 3 + \lambda_1 + \lambda_2 = 2 - 1 + 0 = 1$
$\Rightarrow \lambda_2 = -2 - \lambda_1$
If $\lambda_1 = 3$, then $\lambda_2 = -2 - 3 = -5$.
If $\lambda_1 = 2$, then $\lambda_2 = -2 - 2 = -4$.

54. (*a*) Eigen values of S^2 = square of the eigen values of $S = 1, 25$.

55. (*a*)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then

$Av_1 = 8v_1$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$

$\Rightarrow \begin{cases} a+b = 8 \text{............(1)} \\ c+d = 8 \text{............(2)} \end{cases}$

Also

$Av_2 = 4v_2$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

$\Rightarrow \begin{cases} a-b = 4 \text{............(3)} \\ c-d = -4 \text{............(4)} \end{cases}$

Solving (1) and (3) we get, $a = 6$, $b = 2$.
Solving (2) and (4) we get, $c = 2$, $d = 6$.

Hence $A = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$.

56. (*c*)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then

$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow a - b = -1 \text{......(1)}, \ c - d = 1 \text{.....(2)}$

Again $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-2b \\ c-2d \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$\Rightarrow a - 2b = -2 \text{......(3)}, \ c - 2d = 4 \text{.....(4)}$

Solving (1) and (3) we get $b = 1$, $a = 0$
Solving (2) and (4) we get $d = -3$, $c = -2$.

Thus $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. Now

$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \times \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2+2 & -1+2 \\ 2-4 & 1-4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = A$

57. (*a*) A is symmetric $\Rightarrow$ eigen values are all real.

58. (*b*) Let λ be the other eigen value. Then
Sum of eigen values = trace of the matrix $\Rightarrow \lambda - 2 + 6 = 5 + 1 + 1 \Rightarrow \lambda = 3$.

59. (*a*) Clearly the set X forms a subspace of R^3.

$a[1,-1,0]^T + b[1,0,-1]^T = O$

$\Rightarrow [a,-a,0]^T + [b,0,-b]^T = O$

$\Rightarrow [a+b,-a+0,0-b]^T = [0,0,0]^T$

$\Rightarrow a + b = 0, -a = 0, -b = 0$

$\Rightarrow a = b = 0$

Hence the vectors $[1,-1,0]^T$ and $[1,0,-1]^T$ are linearly independent.

Now $x^T = [x_1, x_2, x_3]^T = a[1,-1,0]^T + b[1,0,-1]^T$

$\Rightarrow [x_1, x_2, x_3]^T = [a,-a,0]^T + [b,0,-b]^T$

$\Rightarrow [x_1, x_2, x_3]^T = [a+b,-a+0,0-b]^T$

$\Rightarrow x_1 = a + b, x_2 = -a = 0, x_3 = -b$

$\therefore x_1 + x_2 + x_3 = a + b - a - b = 0, a = x_2, b = -x_3.$

so $x^T = [x_1, x_2, x_3]^T$
$= (-x_2)[1, -1, 0]^T + (-x_3)\ [1, 0, -1]^T$

Thus the vector 'x' can be expressed as a linear combination of the vectors $[1,-1,0]^T$ and $[1,0,-1]^T$ Consequently $\{[1,-1,0]^T, [1,0,-1]^T\}$ is the basis for the subspace X.

60. (*i*) – (*a*) ; (*ii*) – (*a*)

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda(\lambda+3) + 2 = 0$

$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$ ...(*)

$\Rightarrow \lambda = -1, -2$

Using Cayley Hamilton theorem, we get from (*),

$A^2 + 3A + 2I = O$........(**)

$\Rightarrow A^{-1}(A^2 + 3A + 2I) = A^{-1}O$

$\Rightarrow A + 3A^{-1}A + 2A^{-1} = O$

(since, $A^{-1}A^2 = (A^{-1}A)A = IA = A$)

$\Rightarrow A + 3I + 2A^{-1} = O$

Again (**) $\Rightarrow A^2 = -3A - 2I$

$\therefore \quad A^4 = A^2.A^2 = (3A + 2I)^2$

$= 9A^2 + 12A + 4I$

$= 9(-3A - 2I) + 12A + 4I$

$= -15A - 14I$

$A^8 = A^4.A^4 = (15A + 14I)^2$

$= 225A^2 + 420A + 196I$

$= 225(-3A - 2I) + 420A + 196I$

$= -255A - 254I$

Hence $A^9 = A^8 \times A = (-255A - 254I)A$

$= -255A^2 - 254A$

$= -255(-3A - 2I) - 254A$

$= 511A + 510I.$

61. (*a*)

$$\begin{vmatrix} 1+b & b & 1 \\ b & b+1 & 1 \\ 1 & 2b & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+b & b & 1 \\ -1 & 1 & 0 \\ -b & b & 0 \end{vmatrix}$$

[by $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$]

$= -b + b = 0$

62. (*b*)

$$V = XX' = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix} \times \begin{bmatrix} x_1 & x_2 & . & . & . & . & x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1^2 & x_1x_2 & .. & .. & .. & .. & x_1x_n \\ x_2x_1 & x_2^2 & .. & .. & .. & .. & x_2x_n \\ .. & .. & .. & .. & .. & .. & .. \\ x_nx_1 & x_nx_2 & .. & .. & .. & .. & x_n^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} x_1 & x_2 & .. & .. & .. & .. & x_n \\ x_1 & x_2 & .. & .. & .. & .. & x_n \\ .. & .. & .. & .. & .. & .. & .. \\ x_1 & x_2 & .. & .. & .. & .. & x_n \end{bmatrix}$$

$$\left(by\, R_1 \to \frac{1}{x_1} R_1,\, R_2 \to \frac{1}{x_2} R_2, ..., R_n \to \frac{1}{x_n} R_n \right)$$

$$\sim \begin{bmatrix} x_1 & x_2 & .. & .. & .. & .. & x_n \\ 0 & 0 & .. & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & .. & .. & .. & .. & 0 \end{bmatrix}$$

$\left(by\ R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, ..., R_n \to R_n - R_1 \right)$

which has one non-zero row

Therefore rank (A) = 1.

63. (*a*)

Let $A = \begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix}$. Then det ($A$) = 7 – 10 = – 3

adj (A) = $\begin{pmatrix} 7 & -5 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 7 & -2 \\ -5 & 1 \end{pmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{(-3)} \begin{pmatrix} 7 & -2 \\ -5 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -7 & 2 \\ 5 & -1 \end{pmatrix}$$

64. (*d*)

Here

$$\Delta = \begin{vmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{vmatrix} = 0 - 4(0+3) + 3(4-0) = 0,$$

$$\Delta_1 = \begin{vmatrix} 8 & 4 & 3 \\ 2 & 0 & -1 \\ 5 & 2 & 0 \end{vmatrix} = 8(0+2) - 4(0+5) + 3(4-0)$$

$\neq 0$

Since $\Delta = 0$ and $\Delta_1 \neq 0$, so the system is inconsistent (by Cramer's rule) and hence solution in non-existent.

65. (*a*) By Cramer's rule, system has an infinite number of solutions

$\Leftrightarrow \Delta = 0, \Delta_3 = 0$

$$\Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 2 & \alpha \end{vmatrix} = 0, \begin{vmatrix} 1 & 1 & 5 \\ 1 & 3 & 9 \\ 1 & 2 & \beta \end{vmatrix} = 0$$

$\Leftrightarrow 2\alpha - 4 = 0, 2\beta - 14 = 0$

$\Rightarrow \alpha = 2, \beta = 7$

Alternative Method:

[A : B]

$$= \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha-1 & \beta-5 \end{bmatrix}$$

(by $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$)

$$\sim \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha-2 & \beta-7 \end{bmatrix}$$

(by $R_3 \to R_3 - \frac{1}{2}R_2$)

Hence if $\alpha - 2 = 0$ and $\beta - 7 = 0$ *i.e*; if $\alpha = 2$ and $\beta = 7$, then rank([A:B]) = rank(A) = 2<no. of variables. This means the system has infinite number of solutions for $\alpha = 2$ and $\beta = 7$.

66. (*b*) $A^2 = I \Rightarrow |A^2| = |I| = 1$

$\Rightarrow \quad |A|^2 = 1$

$\Rightarrow \quad |A| = 0 \pm 1 \Rightarrow |A| \neq 0$

$\Rightarrow$ The system has a unique solution (by Cramer's rule).

67. (*b*) Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Since A is upper triangular, so its eigen values are diagonal elements *i.e*; 2, 2.

Then $AX = \lambda X$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+y \\ 2y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$\Rightarrow 2x + y = 2x,\ 2y = 2y$

$\Rightarrow y = 0$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence A has only one linearly independent eigen vector, which is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

68. (*c*) $X_1, X_2, ..., X_M$ are M non-zero orthogonal vectors $\Rightarrow$ the vectors are linearly independent.

Let X be any arbitrary vector. Then

$$X = a_1X_1 + a_2X_2 + a_3X_3 + ... + a_MX_M$$
$$+ b_1(-X_1) + b_2(-X_2) + b_3(-X_3) + ... + b_M(-X_M)$$
$$= (a_1 - b_1)X_1 + (a_2 - b_2)X_2 + ... + (a_M - b_M)X_M$$

Thus X is expressed as a linear combination of M vectors $X_1, X_2, ..., X_M$ instead of $2M$ vectors

$X_1, X_2, ..., X_M, -X_1, -X_2, ..., -X_M$.

Hence dimension of the vector space spanned by the $2M$ vectors $X_1, X_2, ..., X_M, -X_1, -X_2, ..., -X_M$ is M.

69. (*d*) The system will not have a unique solution

$\Rightarrow$ system have an infinite number of solutions (assuming that the system is consistent) $\Rightarrow \Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} = 0$$

$\Rightarrow (2k - 12) - (k - 3) + (4 - 2) = 0$

$\Rightarrow k = 7.$

70. (*a*)

$$\Delta = \begin{vmatrix} 2 & 3 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 2(-1-2) - 3(-1-1) = 0$$

Therefore the system cannot have a unique solution and so it must have infinite number of solutions (since system is consistent) .

Now the systems has infinite number of solutions

$\Rightarrow \Delta_1 = 0$

$$\Rightarrow \begin{vmatrix} 4 & 3 & 0 \\ 4 & 1 & 1 \\ a & 2 & -1 \end{vmatrix} = 0$$

$\Rightarrow 4(-1-2) - 3(-4-a) = 0$

$\Rightarrow a = 0.$

Alternative Method

[A : B]

$$= \begin{bmatrix} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 0 & 4 \\ 1 & 2 & -1 & a \end{bmatrix} \quad \text{(by } R_2 \leftrightarrow R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -2 & a-4 \end{bmatrix}$$

(by $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$)

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & a \end{bmatrix} \quad \text{(by } R_3 \to R_3 - R_2\text{)}$$

Hence if $a = 0$, then rank([A:B]) = rank(A) = 2 < no. of variables. This means the system has infinite number of solutions for $a = 0$.

71. (*d*) System has a unique solution

$\Rightarrow \Delta \neq 0$ (by Cramer's Rule)

$$\Rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & a \end{vmatrix} \neq 0$$

$\Rightarrow (2a - 12) - (a - 3) + 2(4 - 2) \neq 0$

$\Rightarrow a - 5 \neq 0$

$\Rightarrow a \neq 5.$

72. (*c*)

$$\Delta = \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 0$$

$$\Delta_1 = \begin{vmatrix} 7 & 2 \\ 6 & 1 \end{vmatrix} = 7 - 12 = 5 \neq 0$$

$\because$ $\Delta = 0$, $\Delta_1 \neq 0$, so the system has no solutions (by Cramer's rule).

73. (*b*) $(PQ)^{-1}P = Q^{-1}P^{-1}P = Q^{-1}I = Q^{-1}$

74. (*a*) Here A^1 doesn't mean A^T.

$$AA'A = A\left\{\left(A^T A\right)^{-1} A^T\right\}A$$

$$= A\left\{A^{-1}\left(A^T\right)^{-1}\right\}A^T A$$

$$= AA^{-1}\left\{\left(A^T\right)^{-1} A^T\right\}A$$

$$= I(IA)$$

$$= A.$$

75. (*c*) One of the eigen value is zero

$\Rightarrow \det(P) = 0$

$$\Rightarrow \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} = 0$$

$\Rightarrow P_{11}P_{22} - P_{12}P_{21} = 0$

76. (*c*) Let λ_1 and λ_2 be the other two eigen values.
Then sum of eigen values = trace of the matrix
$\Rightarrow 3 + \lambda_1 + \lambda_2 = 1 + 0 + p$
$\Rightarrow \lambda_1 + \lambda_2 = p - 2.$

77. (*a*) If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then the eigen values of A are 0 and 1 (since A is upper triangular).

If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then the eigen values of A are 0 and 0 (since A is upper triangular).

If $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, then

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda)^2 + 1 = 0$

$\Rightarrow \lambda \neq 1.$

If $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, the eigen values of A are -1 and -1 (since A is diagonal).

Thus only one matrix is there with eigen value 1.

78. (*b*) Trace of the matrix = 4 + (–5) = –1 = sum of eigen values = (–6) + 5.

79. (*b*) Let A be the square matrix. Then the eigen values of A are 1 and 2 (since A is upper triangular).

For $\lambda = 1$:

$AX = \lambda X$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 2y \\ 2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow x + 2y = x,\ 2y = y$

$\Rightarrow y = 0$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{for } x = 1)$$

Comparing $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with $\begin{bmatrix} 1 \\ a \end{bmatrix}$ we get $a = 0$.

For $\lambda = 2$:

$AX = \lambda X$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 2y \\ 2y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$\Rightarrow x + 2y = 2x,\ 2y = 2y$

$\Rightarrow 2y = x$

$\Rightarrow y = \dfrac{x}{2}$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{x}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \quad (\text{for } x = 1)$$

Comparing $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ with $\begin{bmatrix} 1 \\ b \end{bmatrix}$ we get $b = \dfrac{1}{2}$.

Hence a + b = 1/2.

80. (*d*) The characteristic equation is
$p^3 + p^2 + 2p + 1 = 0$
By Cayley Hamiloton theorem,

$$P^3 + P^2 + 2P + I = O$$
$$\Rightarrow P^{-1}(P^3 + P^2 + 2P + I) = P^{-1}O$$
$$\Rightarrow P^2 + P + 2I + P^{-1} = O$$
$$\Rightarrow P^{-1} = -(P^2 + P + 2I)$$

81. (*c*) Let λ_1 and λ_2 be the two eigen values.

Then sum of eigen values = trace of the matrix

$\Rightarrow \lambda_1 + \lambda_2 = -2$

$\Rightarrow \lambda_2 = -2 - \lambda_1$ ...(1)

Product of the eigen values

= determinant of the matrix

$\Rightarrow \lambda_1\lambda_2 = -35$

$\Rightarrow \lambda_1(-2 - \lambda_1) = -35$ [using (1)]

$\Rightarrow -2\lambda_1 + \lambda_1^2 + 35 = 0$

$\Rightarrow (\lambda_1 + 7)(\lambda_1 - 5) = 0$

$\Rightarrow \lambda_1 = -7, 5.$

82. (*d*) Trace of the matrix = (– 1) + (– 1) + 3 = 1 = sum of eigen values = 3 + (– 1 + 3*i*) + (–1 –3*i*).

83. (*c*) If λ be an Eigen value of *X*, then $(\lambda + 1)^{-1}(\lambda + 5)$ will be an eigen value of $(X + I)^{-1}(X + 5I)$

Now

$\lambda = -2 \Rightarrow (\lambda+1)^{-1}(\lambda+5) = -3$

$\lambda = -3 \Rightarrow (\lambda+1)^{-1}(\lambda+5) = -1$

84. (*a*) $M^T = M^{-1} \Rightarrow M$ is orthogonal

$\Rightarrow \det(M) = \pm 1$

$$\Rightarrow \begin{vmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{vmatrix} = \pm 1$$

$$\Rightarrow \frac{9}{25} - \frac{4x}{5} = \pm 1$$

$\Rightarrow 9 - 20x = \pm 25$

$\Rightarrow 20x = 9 \mp 25 = -16, 34$

$$\Rightarrow x = -\frac{4}{5}, \frac{17}{10}$$

85. (*a*) By definition of skew-symmetric matrix.

86. (*b*)

$$\Delta = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = -6,$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & -2 \\ -1 & 1 & 2 \end{vmatrix} = 12$$

Therefore by Cramer's rule, $x_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{12}{-6} = -2.$

87. (*c*) The homogeneous system of equations have a non-trivial solution

$\Rightarrow$ co-efficient determinant = 0

$$\Rightarrow \begin{vmatrix} 2 & 3 \\ 6 & q \end{vmatrix} = 0$$

$\Rightarrow 2q - 18 = 0$

$\Rightarrow q = 9.$

88. (*b*)

$$\text{Let } A = \begin{pmatrix} 3+2i & i \\ -i & 3-2i \end{pmatrix}$$

$$\text{Then } |A| = (3+2i)(3-2i) - (-i)i$$
$$= 3^2 + 2^2 + i^2 = 9 + 4 - 1 = 12$$

$$adj(A) = \begin{pmatrix} 3-2i & i \\ -i & 3+2i \end{pmatrix}^T$$
$$= \begin{pmatrix} 3-2i & -i \\ i & 3+2i \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{12}\begin{pmatrix} 3-2i & -i \\ i & 3+2i \end{pmatrix}.$$

89. (*d*)

$$[A : B]$$
$$= \begin{pmatrix} 1 & 2 & 1 & 4 & 2 \\ 3 & 6 & 3 & 12 & 6 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} (by\, R_2 \to R_2 - 3R_1)$$

Which has one non-zero row.

Therefore rank ([*A*:*B*]) = 1. Now ignoring last column in the final equivalent matrix, we see that there are only one non-zero row and so rank (*A*) = 1. Thus rank (*A*) = rank ([*A*:*B*]) = 1 < 4 = number of unknowns.

Hence the system has infinite number of solutions and so multiple non-trivial solutions exist.

90. (*a*)

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ a_{31} & a_{32} & .... & a_{3n} \\ ...... & ...... & .... & ...... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} 1 & 0 & 0 & .... & 0 \\ 0 & 2 & 0 & ...... & 0 \\ 0 & 0 & 3 & .... & 0 \\ ..... & .... & ...... & ...... & ..... \\ 0 & 0 & .... & .... & n \end{bmatrix}_{n\times n}$$

Thus sum of *n* eigen values

= trace of A

= sum of the diagonal elements

$= 1+2+3+\dots\dots\dots+n = \frac{n(n+1)}{2}.$

91. (*a*)

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+0+0 \\ -1+0+1 \\ 0+0-1 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ -\lambda \end{bmatrix}$$

$\Rightarrow \lambda = 1$

92. (*d*) trace of A = sum of eigen values of A

$\Rightarrow 2 + y = 4 + 8$

$\Rightarrow y = 10$

Product of eigen values of A = det(A)

$$\Rightarrow 4\times 8 = \begin{vmatrix} 2 & 3 \\ x & y \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ x & 10 \end{vmatrix}$$

$\Rightarrow 32 = 20 - 3x$

$\Rightarrow x = -4$

93. (*a*)

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow \lambda = 1, 4$

Then $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix} \quad (\text{for } \lambda = 1)$$

$$\Rightarrow \begin{bmatrix} 2x+2y \\ x+3y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow 2x + 2y = x,\; x + 3y = y$

$\Rightarrow x = -2y$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} (\text{for } y = -1)$$

94. (*b*) Since P is upper triangular, so eigen values of P are 1, 2, 3 (since 1,2 and 3 are the diagonal elements of the upper triangular matrix P).

Let us consider $\lambda = 3$. Then

$PX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+y \\ 2y+2z \\ 3z \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \\ 3z \end{bmatrix}$$

$\Rightarrow x + y = 3x,\; 2y + 2z = 3y,\; 3z = 3z$

$\Rightarrow y = 2x,\; y = 2z$

$\Rightarrow x = z,\; y = 2z$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 2z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$= \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$ (for $z = 1$)

95. (*b*) If λ and X are respectively the eigen value and eigen vector (corresponding to λ) of the matrix M, then λ^K and X will be the eigen value and eigen vector (corresponding to λ^K) of the matrix M^K.

96. (*a*) The given matrix is upper triangular, so its eigen values are the diagonal elements *i.e.* 1, 4, 3.

97. (*b*)

$\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 6 \\ 1 & 4 & \lambda \end{vmatrix} = 0$$

$\Rightarrow 1(4\lambda - 24) - 1(\lambda - 6) + 6(4 - 4) = 0$

$\Rightarrow \lambda = 6.$

$\Delta_3 \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 6 \\ 1 & 4 & 20 \\ 1 & 4 & \mu \end{vmatrix} \neq 0$$

$\Rightarrow 1(4\mu - 80) - 1(\mu - 20) + 6(4 - 4) \neq 0$

$\Rightarrow \mu \neq 20.$

$\because$ $\Delta = 0$, $\Delta_3 \neq 0 \Leftrightarrow$ the system has no solution, so we must have $\lambda = 6$, $\mu \neq 20$.

Alternative Method:

[A : B]

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{bmatrix}$$

(by $R_2 \to R_2 - R_1; R_3 \to R_3 - R_1$)

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\0 & 3 & 5 & 14\\0 & 0 & \lambda-6 & \mu-20\end{bmatrix}$$
(by $R_3 \to R_3 - R_2$)

Hence if $\lambda = 6$ and $\mu \neq 20$, then rank ([A:B]) = 3 ≠ 2 = rank(A). This means the system has no solution for $\lambda = 6$ and $\mu \neq 20$.

98. (*d*) We know that for any square matrix A, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

99. (*b*)

Let $[A] = \begin{bmatrix}2 & 1\\4 & -1\end{bmatrix} = [\text{L}][\text{U}] = \begin{bmatrix}a & 0\\b & c\end{bmatrix}\begin{bmatrix}d & g\\0 & f\end{bmatrix}$

Then $\begin{bmatrix}2 & 1\\4 & -1\end{bmatrix} = \begin{bmatrix}ad & ag\\bd & bg+cf\end{bmatrix}$

Equating the corresponding elements we get

$ad = 2$ ...(*i*)

$ag = 1$ ...(*ii*)

$bd = 4$ ...(*iii*)

$bg + cf = -1$ ...(*iv*)

All these equations are satisfied if

$[L] = \begin{bmatrix}a & 0\\b & c\end{bmatrix} = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$ and

$[U] = \begin{bmatrix}d & g\\0 & f\end{bmatrix} = \begin{bmatrix}2 & 1\\0 & -3\end{bmatrix}$

100. (*b*)

$$A^2 = \begin{pmatrix}-5 & -3\\2 & 0\end{pmatrix}\times\begin{pmatrix}-5 & -3\\2 & 0\end{pmatrix}$$
$$= \begin{pmatrix}25-6 & 15-0\\-10+0 & -6+0\end{pmatrix} = \begin{pmatrix}19 & 15\\-10 & -6\end{pmatrix}$$
$$\therefore A^3 = A^2 \times A$$
$$= \begin{pmatrix}19 & 15\\-10 & -6\end{pmatrix}\times\begin{pmatrix}-5 & -3\\2 & 0\end{pmatrix}$$
$$= \begin{pmatrix}-95+30 & -57+0\\50-12 & 30+0\end{pmatrix}$$
$$= \begin{pmatrix}19\times(-5)+30\times 1 & 19\times(-3)+30\times 0\\19\times 2+30\times 0 & 19\times 0+30\times 1\end{pmatrix}$$
$$= 19A + 30I.$$

101. (*c*)

$$|A| = \begin{vmatrix}2 & 1 & 1\\0 & 1 & -1\\1 & 1 & 0\end{vmatrix}$$
$$= 2(0+1) - 1(0+1) + 1(0-1) = 0.$$

So rank (A) < 3 = number of variables.

Therefore system has infinite number of solutions.

102. (*c*)

Here $\Delta = \begin{vmatrix}1 & 2 & 1\\2 & 1 & 2\\1 & -1 & 1\end{vmatrix} = 0,$

$$\Delta_1 = \begin{vmatrix}4 & 2 & 1\\5 & 1 & 2\\1 & -1 & 1\end{vmatrix} = 0,$$
$$\Delta_2 = \begin{vmatrix}1 & 4 & 1\\2 & 5 & 2\\1 & 1 & 1\end{vmatrix} = 0,$$
$$\Delta_3 = \begin{vmatrix}1 & 2 & 4\\2 & 1 & 5\\1 & -1 & 1\end{vmatrix} = 0.$$

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, so the system has infinite number of solutions (by Cramer's rule).

Alternative Method:

[A : B]

$$= \begin{bmatrix}1 & 2 & 1 & 4\\2 & 1 & 2 & 5\\1 & -1 & 1 & 1\end{bmatrix}$$
$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\0 & -3 & 0 & -3\\0 & -3 & 0 & -3\end{bmatrix}$$
(by $R_2 \to R_2 - 2R_1; R_3 \to R_3 - R_1$)
$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\0 & -3 & 0 & -3\\0 & 0 & 0 & 0\end{bmatrix}$$
(by $R_3 \to R_3 - R_2$)

Hence rank ([A:B]) = 2 = rank(A) < 3 (=number of variables). This means the system has an infinite number of solutions.

103. (*b*) Sum of eigen values = trace of the matrix = 9 + 8 = 17, which is satisfied for eigen values 3.48 and 13.52 only.

104. (*d*) Here $A = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$. Then

$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix}1-\lambda & 1\\1 & -1-\lambda\end{vmatrix} = 0$$
$$\Rightarrow -(1-\lambda)(1+\lambda) - 1 = 0$$
$$\Rightarrow \lambda^2 - 1 - 1 = 0$$
$$\Rightarrow \lambda = \pm\sqrt{2}$$

A has an eigen value $\pi \Rightarrow A^{19}$ has an eigen value λ^{19}. Hence the eigen values of A^{19} are:

$(\pm\sqrt{2})^{19}$ *i.e*; $512\sqrt{2}$, $-512\sqrt{2}$.

105. (*a*), (*d*)

$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda^2 = -1 = j^2$

$\Rightarrow \lambda = \pm j \quad (j = \sqrt{-1})$

For $\lambda = j$

$$AX = \lambda X \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = j\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} jx \\ jy \end{bmatrix}$$

$\Rightarrow -y = jx, \quad x = jy$.........(*)

$\Rightarrow -jy = j^2x, \quad x = jy$

$\Rightarrow x = jy \quad$ (since $j^2 = -1$)

Thus $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} jy \\ y \end{bmatrix} = \begin{bmatrix} j \\ 1 \end{bmatrix}$ (for $y = 1$)

Again from (*) we can write,

$y = -jx$, $x = jy$

$\Rightarrow y = -j x$, $xj = j^2y$

$\Rightarrow y = -jx$ (since $j^2 = -1$)

Thus $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -jx \end{bmatrix} = \begin{bmatrix} 1 \\ -j \end{bmatrix}$ (for $x = 1$)

For $\lambda = -j$

$$AX = \lambda X \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -j\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} -jx \\ -jy \end{bmatrix}$$

$\Rightarrow -y = -jx, \quad x = -jy$

$\Rightarrow y = jx, \quad x = -jy \quad$ ....(**)

$\Rightarrow y = jx$, $jx = -j^2y = y$ (since $j^2 = -1$)

$\Rightarrow y = jx$

Thus $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ jx \end{bmatrix} = \begin{bmatrix} 1 \\ j \end{bmatrix}$ (for $x = 1$)

Again (**) $\Rightarrow x = -jy$,

Thus $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -jy \\ y \end{bmatrix} = \begin{bmatrix} j \\ -1 \end{bmatrix}$ (for $y = -1$)

Therefore two possible pair of eigen vectors are: $\begin{bmatrix} 1 \\ -j \end{bmatrix}$ and $\begin{bmatrix} j \\ -1 \end{bmatrix}$; $\begin{bmatrix} 1 \\ j \end{bmatrix}$ and $\begin{bmatrix} j \\ 1 \end{bmatrix}$.

106. (*d*)

Let $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

For $\lambda = -1$,

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x-y \\ z-t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x-y = -1 \text{........(1)} \\ z-t = 1 \text{............(2)} \end{cases}$$

For $\lambda = -2$,

$$AX = \lambda X \Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x-2y \\ z-2t \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x-2y = -2 \text{........(3)} \\ z-2t = 4 \text{...........(4)} \end{cases}$$

Solving (1) and (3) we get, $x = 0$, $y = 1$ and solving (2) and (4) we get, $z = -2$, $t = -3$

Therefore $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

107. (*a*)

$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 5 & 2 \\ 5 & 12-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & \lambda & -\lambda \\ 5 & 12-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{vmatrix} = 0$$

[*by* $R_1 \rightarrow R_1 + (R_3 - R_2)$]

$$\Rightarrow (-\lambda)\begin{vmatrix} 1 & -1 & 1 \\ 5 & 12-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda\begin{vmatrix} 1 & 0 & 0 \\ 5 & 17-\lambda & 2 \\ 2 & 9 & 3-\lambda \end{vmatrix} = 0$$

[*by* $C_2 \rightarrow C_2 + C_1$, $C_3 \rightarrow C_3 - C_1$]

$\Rightarrow \lambda\{(17-\lambda)(3-\lambda)-18\}=0$

$\Rightarrow \lambda\{\lambda^2-20\lambda+33\}=0$

$\Rightarrow \lambda=0, \dfrac{20\pm\sqrt{400-4\times 33}}{2}$

$\Rightarrow \lambda=0, \dfrac{20\pm\sqrt{268}}{2}$

Therefore minimum eigen value = 0.

108. (*c*)

$\cos 2x=\cos^2 x-\sin^2 x$

$\Rightarrow \cos 2x+(-1)\cos^2 x+\sin^2 x=0$

$\Rightarrow 1\times\cos 2x+(-1)\cos^2 x+1\times\sin^2 x=0$

Therefore the functions $\cos 2x, \sin^2 x$ and $\cos^2 x$ are linearly dependent.

109. (*d*) The given system represents a homogeneous system and the coefficient determinant

$= \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = -2+2=0.$

Therefore the system has infinite number of solutions (multiple solutions).

110. 16

We know that number of multiplication required to multiply $A_{m\times n}$ with $B_{n\times k}$ is mnk.

Now $PQR=(PQ)R=(P_{4\times 2}\ Q_{2\times 4})\ R_{4\times 1}=(PQ)_{4\times 4}\ R_{4\times 1}$

So the number of multiplication required to compute $(PQ)_{4\times 4}\ R_{4\times 1}$ is (4 × 2 × 4) + (4 × 4 × 1) *i.e.*, 48

Again $PQR=P(QR)=P_{4\times 2}\ (Q_{2\times 4}\ \ R_{4\times 1})=P_{4\times 2}\ (QR)_{2\times 1}$

So the number of multiplication required to compute $P_{4\times 2}\ (QR)_{2\times 1}$ is (2 × 4 × 1) + (4 × 2 × 1) *i.e.*, 16

Hence minimum number of multiplication required to compute PQR is 16.

111. (*a*)

$$\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x(x+1) & x \\ 1 & y(y+1) & y \\ 1 & z(z+1) & z \end{vmatrix} \quad (\text{by } C_3 \to C_3 - C_1)$$

$$= \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} \quad (\text{by } C_2 \to C_2 - C_3)$$

$$= -\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (\text{by } C_2 \Leftrightarrow C_3)$$

Therefore (a) is correct.

$$\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$(\text{by } C_3 \to C_3 - C_1,\ C_2 \to C_2 - C_1)$

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (\text{by } R_1 \to R_1 - R_2)$$

$$= \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix} \quad (\text{by } R_2 \to R_2 - R_3)$$

∴ (*c*) is incorrect.

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 2 & x+y & x^2+y^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (\text{by } R_1 \to R_1 + R_2)$$

$$= \begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix} \quad (\text{by } R_2 \to R_2 + R_3)$$

∴ (*d*) is incorrect.

112. (*b*)

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 & -1 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$[by\ R_1 \to R_1 - 2R_4, R_2 \to R_2 - R_4,$
$R_3 \to R_3 - R_4]$

$= 1\times$ cofactor of '1'

$$= (-1)^{4+1} \begin{vmatrix} -1 & -1 & -3 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$= -\{-1(0+1)+1(-1+0)-3(1-0)\} = 5.$

113. 1.

$P = I_6 + \alpha J_6$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \det(P) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-\alpha^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$(by\ R_1 \to R_1 - \alpha R_6)$

$= (1-\alpha^2) \times \text{cofactor of } (1-\alpha^2)$

$$= (1-\alpha^2) \begin{vmatrix} 1 & 0 & 0 & \alpha & 0 \\ 0 & 1 & \alpha & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 \\ \alpha & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1-\alpha^2) \begin{vmatrix} 1-\alpha^2 & 0 & 0 & \alpha & 0 \\ 0 & 1 & \alpha & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$(by\ C_1 \to C_1 - \alpha C_4)$

$= (1-\alpha^2) \times \left\{(1-\alpha^2) \times \text{cofactor of } (1-\alpha^2)\right\}$

$$= (1-\alpha^2)^2 \begin{vmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1-\alpha^2)^2 \begin{vmatrix} 1-\alpha^2 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$(by\ R_1 \to R_1 - \alpha R_2)$

$$= (1-\alpha^2)^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

(by taking the element $(1-\alpha^2)$ and it's co-factor)

$= (1-\alpha^2)^3.$

Now det(P) = 0 gives $(1-\alpha^2)^3 = 0$, which implies $\alpha = -1, 1$.

114. (d) Since matrix product is not commutative, so options (a) and (b) are not correct.

Let $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} -2 & 0 \\ 6 & 4 \end{bmatrix}$. So $P + Q = \begin{pmatrix} 0 & 3 \\ 7 & 5 \end{pmatrix}$.

Then det(P) = –1, det (Q) = –8 and so

det(P) + det(Q) = –9. But det(P + Q) = –21. Hence option (c) is not correct.

115. 0.

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}_{3\times1} \begin{bmatrix} 1 & 9 & 5 \end{bmatrix}_{1\times3} = \begin{bmatrix} 2 & 18 & 10 \\ -4 & -36 & -20 \\ 7 & 63 & 35 \end{bmatrix}$$

$$\therefore \det(A) = \begin{vmatrix} 2 & 18 & 10 \\ -4 & -36 & -20 \\ 7 & 63 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 18 & 10 \\ 0 & 0 & 0 \\ 7 & 63 & 35 \end{vmatrix} \quad [by\ R_2 \to R_2 + 2R_1]$$

$= 0.$

116. 2.

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix} \sim \begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$[by\ R_3 \to R_3 - (2R_1 - R_2)]$

which has two non-zero rows

Therefore rank = 2.

117. (c)

$$|A| = \begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr$$

$|B|$

$$= \begin{vmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{vmatrix}$$

$$= (p^2+q^2)(r^2+s^2)-(pr+qs)^2$$

$$= p^2r^2+p^2s^2+q^2r^2+q^2s^2-p^2r^2$$

$$-q^2s^2-2pqrs$$

$$= p^2s^2-2ps\times qr+q^2r^2$$

$$= (ps-qs)^2 = |A|^2$$

Thus the matrices A and B become singular or non-singular together. So, rank(A) = Rank(B) = N.

118. (a) The area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{1(2-3)+1(6-8)\} = -\frac{3}{2}.$$

Therefore magnitude of the area = 3/2

119. (a)

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 3x-5y \\ 4x+8y \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

120. one

$[A : B]$

$$= \begin{pmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 7 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -2 & 7 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{pmatrix} \quad [\text{by } R_1 \leftrightarrow R_3]$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

[by $R_2 \to R_2 - 4R_1, R_3 \to R_3 - 3R_1$,
$R_4 \to R_4 - R_1$]

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[by $R_4 \to R_4 - (R_2 - R_3)$]

which has three non-zero rows.

So $r([A{:}B]) = 3$. Now ignoring the last column in the final equivalent matrix, we see that there are only three non-zero rows. Therefore $r(A) = 3$. Thus $r([A{:}B]) = r(A) = 3$ = number of unknowns. Hence the system has a unique solution. Thus number of solutions is one.

121. (a)

$$\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 3 & 2 & 3 \end{vmatrix}$$

$$= 1(9-2)-2(12-3)+4(8-9)$$

$$\neq 0$$

So the system has a unique solution.

122. (b)

$$\Delta = \begin{vmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= 2(0-2)-1(15-1)+3(6-0) = 0,$$

$$\Delta_1 = \begin{vmatrix} 5 & 1 & 3 \\ -4 & 0 & 1 \\ 14 & 2 & 5 \end{vmatrix}$$

$$= 5(0-2)-1(-20-14)+3(-8-0) = 0,$$

$$\Delta_2 = \begin{vmatrix} 2 & 5 & 3 \\ 3 & -4 & 1 \\ 1 & 14 & 5 \end{vmatrix}$$

$$= 2(-20-14)-5(15-1)+3(42+4) = 0,$$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 5 \\ 3 & 0 & -4 \\ 1 & 2 & 14 \end{vmatrix}$$

$$= 2(0+8)-1(42+4)+5(6-0) = 0.$$

$\because \ \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ so the system has infinite number of solution (by Cramer's rule).

Alternative Method

[A : B]

$$= \begin{bmatrix} 2 & 1 & 3 & 5 \\ 3 & 0 & 1 & -4 \\ 1 & 2 & 5 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 & 14 \\ 3 & 0 & 1 & -4 \\ 2 & 1 & 3 & 5 \end{bmatrix}$$

(by $R_3 \leftrightarrow R_1$)

$$\sim \begin{bmatrix} 1 & 2 & 5 & 14 \\ 0 & -6 & -14 & -46 \\ 0 & -3 & -7 & -23 \end{bmatrix}$$

(by $R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$)

$$\sim\begin{bmatrix}1 & 2 & 5 & 14\\ 0 & -6 & -14 & -46\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\left(\text{by } R_3 \to R_3 - \frac{1}{2}R_2\right)$$

Hence rank ([A:B]) = rank(A) = 2 < 3 (=number of variables). This means the system has an infinite number of solutions.

123. (b)

$$A = \begin{pmatrix}1 & 2 & 2\\ 5 & 1 & 3\end{pmatrix}$$

$$\sim\begin{pmatrix}1 & 2 & 2\\ 0 & -9 & -7\end{pmatrix} (by\ R_2 \to R_2 - 5R_1)$$

Which has two non-zero rows.

Therefore rank (A) = 2

$$[A:B] = \begin{pmatrix}1 & 2 & 2 & b_1\\ 5 & 1 & 3 & b_2\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 2 & b_1\\ 0 & -9 & -7 & b_2 - 5b_1\end{pmatrix}$$

$$(by\ R_2 \to R_2 - 5R_1)$$

Which has two non-zero rows.

Therefore rank ([A:B]) = 2

So rank (A) = rank ([A:B]) = 2 < 3 (= number of unknowns)

Hence the system has infinitely many solutions for any given b_1 and b_2.

124. 200

$$|AB| = |A||B| = 5 \times 40 = 200.$$

125. (c)

$$\text{Let } B = \begin{bmatrix}1 & 2 & 3\\ 7 & 8 & 9\\ 4 & 5 & 6\end{bmatrix} \text{ and } C = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}.$$

Then we have $AB = C$.

Then $AB = C$

$$\Rightarrow |AB| = |C| \Rightarrow |A||B| = -|B|$$

($\because C$ is obtained by interchanging R_1 and R_2 in B)

$$\Rightarrow |B|(|A| + 1) = 0$$

$$\Rightarrow |A| = -1 \text{ (since } |B| \neq 0)$$

Let us take the eigen values 1, 1 and –1. Then product of these eigen values = –1 = det(A).

126. (a)

127. (b) Let $A = \begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix}$. Then A is a symmetric matrix with one negative eigen value '–1' and one positive eigen value '2'.

128. (a) Sum of eigen values = trace of the matrix = 215 + 150 + 550 = 915.

129. 49 Let A = $\begin{bmatrix}x & a\\ a & 14 - x\end{bmatrix}$. Then A is a real symmetric matrix with trace $x + (14 - x)$ $i.e$; 14.

Then det(A) = $x(14 - x) - a^2$, which is maximum when $a = 0$. Let $f(x) = x(14 - x)$.

Now det(A) is maximum $\Rightarrow f(x)$ is maximum and vice versa.

$f'(x) = 0 \Rightarrow 14 - 2x = 0 \Rightarrow x = 7$.

Also $f''(x)|_{x=7} = -2 < 0$. Thus $x = 7$ is a point of maxima and so $f(x)$ is maximum when $x = 7$. Consequently det(A) is maximum when $x = 7$ and maximum value of det(A) is 7(14 – 7) $i.e$; 49.

130. (d)

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix}-5-\lambda & 2\\ -9 & 6-\lambda\end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}-3-\lambda & 2\\ -3-\lambda & 6-\lambda\end{vmatrix} = 0$$

$$[by\ C_1 \to C_1 + C_2]$$

$$\Rightarrow (-3-\lambda)\begin{vmatrix}1 & 2\\ 1 & 6-\lambda\end{vmatrix} = 0$$

$$\Rightarrow -3 - \lambda = 0$$

Therefore $\lambda = -3$ is an eigen value. Then

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix}-5 & 2\\ -9 & 6\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = -3\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-5x + 2y\\ -9x + 6y\end{bmatrix} = \begin{bmatrix}-3x\\ -3y\end{bmatrix}$$

$$\Rightarrow -5x + 2y = -3x,\ -9x + 6y = -3y$$

$$\Rightarrow x = y$$

$$\therefore \text{Eigen vector, } X = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ x\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}$$

$$(\text{putting } x = 1)$$

131. (a) *Let* $A = \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$. Then trace($A$) = 2 + (–1) > 0, but the eigen values are –1 and 2 (since A is diagonal matrix). So option (b) is incorrect.

Let $B = \begin{bmatrix}-1 & 2\\ 0 & -1\end{bmatrix}$. Then det($B$) = 1 > 0, but the eigen values are –1 and –1 (since A is upper triangular). So option (c) is incorrect.

Let $C = \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$. Then trace($C$) = 1 + (–2) = –1 and det(C) = –2. Therefore trace(C) × det(C) = 2 > 0, but the eigen values are 1 and –2 (since A is diagonal matrix). So option (d) is incorrect.

132. (d) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then clearly A is symmetric with two non-zero eigen values 1 and 2 with respective eigen vectors X and Y.

For $\lambda = 1$:

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ x_2 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow x_1 = 0, x_3 = 0$

For $\lambda = 2$:

$AY = \lambda Y$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 2\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ y_2 \\ 2y_3 \end{bmatrix} = \begin{bmatrix} 2y_1 \\ 2y_2 \\ 2y_3 \end{bmatrix}$$

$\Rightarrow y_1 = 0, y_2 = 0.$

Hence

$x_1y_1 + x_2y_2 + x_3y_3$

$= 0 \times 0 + x_2 \times 0 + 0 \times y_3$

$= 0.$

133. 6.

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1-(1-\lambda)^2 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$[by\ C_1 \to C_1 - (1-\lambda)C_5]$

$$\Rightarrow \begin{vmatrix} 0 & 1-\lambda & 1 & 1 \\ 0 & 1 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \\ 1-(1-\lambda)^2 & 0 & 0 & 0 \end{vmatrix} = 0$$

(by expanding using R_1)

$$\Rightarrow \{1-(1-\lambda)^2\}\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

(by expanding using C_1)

$$\Rightarrow \lambda(2-\lambda)\begin{vmatrix} 3-\lambda & 1 & 1 \\ 3-\lambda & 1-\lambda & 1 \\ 3-\lambda & 1 & 1-\lambda \end{vmatrix} = 0$$

$[by\ C_1 \to C_1 + C_2 + C_3]$

$$\Rightarrow \lambda(2-\lambda)(3-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(2-\lambda)(3-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$[by\ C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1]$

$\Rightarrow \lambda^3(2-\lambda)(3-\lambda) = 0$

$\Rightarrow \lambda = 0, 0, 0, 2, 3.$

Hence product of the non-zero eigen values $= 2 \times 3 = 6$.

134. 1.

$A^2 = I \Rightarrow \lambda^2 = 1$

(by Cayley Hamilton Theorem)

$\Rightarrow \lambda = \pm 1$

Hence positive eigen value of A is '1'.

135. 3.

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 0-\lambda & 1 & -1 \\ -6 & -11-\lambda & 6 \\ -6 & -11 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 1 & -1 \\ 0 & -11-\lambda & 6 \\ -1-\lambda & -11 & 5-\lambda \end{vmatrix} = 0$$

$[by\ C_1 \to C_1 + C_3]$

$$\Rightarrow (-1-\lambda)\begin{vmatrix} 1 & 1 & -1 \\ 0 & -11-\lambda & 6 \\ 1 & -11 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda)\begin{vmatrix} 1 & 1 & -1 \\ 0 & -11-\lambda & 6 \\ 0 & -12 & 6-\lambda \end{vmatrix} = 0$$

[by $R_3 \to R_3 - R_1$]

$$\Rightarrow (\lambda+1)\{(\lambda-6)(\lambda+11)+72\} = 0$$

$$\Rightarrow (\lambda+1)(\lambda+2)(\lambda+3) = 0$$

$$\Rightarrow \lambda = -1, -2, -3.$$

Hence required absolute ratio =. $\dfrac{|-3|}{|-1|} = 3$.

136. (*d*)

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 2 \\ 4 & -4-\lambda & 6 \\ 2 & -3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1+\lambda & 1-\lambda \\ 4 & -4-\lambda & 6 \\ 2 & -3 & 5-\lambda \end{vmatrix} = 0$$

(by $R_1 \to R_1 + (R_3 - R_2)$)

$$\Rightarrow (1-\lambda)\begin{vmatrix} 1 & -1 & 1 \\ 4 & -4-\lambda & 6 \\ 2 & -3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 4 & -\lambda & 2 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

(by $C_2 \to C_2 + C_1, C_3 \to C_3 - C_1$)

$$\Rightarrow (1-\lambda)\{\lambda(\lambda-3)+2\} = 0$$

$$\Rightarrow \lambda = 1, 1, 2.$$

137. 17

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12 \\ p+7 \\ 36 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \\ 3\lambda \end{bmatrix}$$

$$\Rightarrow \lambda = 12, \ p+7 = 2\lambda$$

$$\Rightarrow \lambda = 12, \ p = 17$$

138. (*b*)

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x-y+2z \\ y \\ x+2y+z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow x-y+2z = x, \ y = y, \ x+2y+z = z$$

$$\Rightarrow -y+2z = 0, \ x+2y = 0$$

$$\Rightarrow x = -2y, \ z = \frac{y}{2}$$

Therefore eigen vector,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ y/2 \end{bmatrix} = \alpha \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$$

(replacing y by 2α and then taking α as common)

$$= \alpha(-4, 2, 1)$$

139. (*d*)

Let λ_1 and λ_2 be two eigen values.

Then ATQ, $\dfrac{\lambda_1}{\lambda_2} = \dfrac{3}{1}$ and so $\lambda_1 = 3\lambda_2$.

Now $\lambda_1 + \lambda_2$ = trace of the matrix

=2+p

$$\Rightarrow 3\lambda_2 + \lambda_2 = 2+p \Rightarrow \lambda_2 = \frac{2+p}{4}$$

Again $\lambda_1 \lambda_2$ = determinant of the matrix

$$= \begin{vmatrix} 2 & 1 \\ 1 & p \end{vmatrix} = 2p - 1$$

$$\Rightarrow 3\lambda_2^2 = 2p - 1$$

$$\Rightarrow 3\left(\frac{2+p}{4}\right)^2 = 2p - 1$$

$$\Rightarrow 3p^2 - 20p + 28 = 0$$

$$\Rightarrow p = 2, \frac{14}{3}$$

140. (*b*) A square matrix is singular

$\Leftrightarrow$ '0' is a eigen value of the matrix.

141. (*b*)

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -3-\lambda & 0 & -2 \\ 1 & -1-\lambda & 0 \\ 0 & a & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(3+\lambda)(1+\lambda)(2+\lambda) - 2a = 0$$

$$\Rightarrow a = -\frac{1}{2}(3+\lambda)(1+\lambda)(2+\lambda)$$

$$= f(\lambda) \text{ (say)}$$

Then $f'(\lambda)$

$$= -\frac{1}{2}\begin{bmatrix}(1+\lambda)(2+\lambda)+(1+\lambda)(3+\lambda) \\ +(2+\lambda)(3+\lambda)\end{bmatrix} = 0$$

$$\Rightarrow -\frac{1}{2}\left[3\lambda^2 + 3\lambda + 4\lambda + 5\lambda + 11\right] = 0$$

$$\Rightarrow 3\lambda^2 + 12\lambda + 11 = 0$$

$$\Rightarrow \lambda = \frac{-12 \pm \sqrt{144 - 12\times 11}}{2\times 3}$$

$$\Rightarrow \lambda = \frac{-12 \pm \sqrt{12}}{6} = \frac{-12 \pm 2\sqrt{3}}{6}$$

$$\Rightarrow \lambda = -2 \pm \frac{1}{\sqrt{3}}$$

Now $f''(\lambda)$

$$= -\frac{1}{2}\left[6\lambda + 12\right]$$

$$= -\frac{1}{2}\left[6\left(-2+\frac{1}{\sqrt{3}}\right)+12\right]\left(\text{at } \lambda = -2+\frac{1}{\sqrt{3}}\right)$$

$$= -\frac{3}{\sqrt{3}} < 0$$

$\therefore$ 'a' *i.e.*, $f(\lambda)$ is maximum for

$$\lambda = -2 + \frac{1}{\sqrt{3}}$$

$\therefore a_{\max}$

$$= f(\lambda)\Big|_{\lambda=-2+\frac{1}{\sqrt{3}}}$$

$$= -\frac{1}{2}\left(3+\left(-2+\frac{1}{\sqrt{3}}\right)\right)\times\left(1+\left(-2+\frac{1}{\sqrt{3}}\right)\right) \times\left(2+\left(-2+\frac{1}{\sqrt{3}}\right)\right)$$

$$= -\frac{1}{2}\left(-1+\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{3}}\right)$$

$$= -\frac{1}{2\sqrt{3}}\left\{\left(\frac{1}{\sqrt{3}}\right)^2 - 1^2\right\}$$

$$= \frac{1}{3\sqrt{3}}.$$

142. (*d*)

$trace(A)$ = sum of eigen value

$$\Rightarrow 1 + a = (-1) + 7 = 6 \Rightarrow a = 5$$

$\det(A)$ = product of eigen value

$$\Rightarrow a - 4b = (-1)\times 7$$

$$\Rightarrow 5 - 4b = -7 \Rightarrow b = 3.$$

143. (*a*) Eigen value will be real if the matrix is symmetric *i.e.*, if $5 + j = x$.

144. (*c*)

$$\left|A^T A^{-1}\right| = \left|A^T\right|\left|A^{-1}\right| = |A| \times \frac{1}{|A|} = 1$$

$$\left(\because AA^{-1} = I \Rightarrow \left|AA^{-1}\right| = |I| \Rightarrow |A|\left|A^{-1}\right| = 1\right)$$

145. (*a*)

$$\Delta = \begin{vmatrix}4 & 7 & 8\\ 3 & 1 & 5\\ 9 & 6 & 2\end{vmatrix} = 4(2-30) - 7(6-45) + 8(18-9)$$

$$= -112 + 273 + 72 = 233$$

Now interchanging C_1 and C_2 in Δ, we get

$$\begin{vmatrix}7 & 4 & 8\\ 1 & 3 & 5\\ 6 & 9 & 2\end{vmatrix} = 7(6-45) - 4(2-30) + 8(9-18)$$

$$= -273 + 112 - 72 = -233 = -\Delta$$

Therefore (*a*) is correct.

146. (*b*) For $n = 3$, we have

$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$= \begin{bmatrix}1\times 1 & 1\times 2 & 1\times 3\\ 2\times 1 & 2\times 2 & 2\times 3\\ 3\times 1 & 3\times 2 & 3\times 3\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

[*by* $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$]

which has only one non-zero row

Therefore rank = 1.

147. 2 The system has infinitely many solutions

$\Leftrightarrow \Delta_3 = 0$ (by Cramer's rule)

$$\Leftrightarrow \begin{vmatrix}1 & -2 & -1\\ 1 & -3 & 1\\ -2 & 4 & k\end{vmatrix} = 0$$

$$\Rightarrow (-3k-4) + 2(k+2) - (4-6) = 0$$

$$\Rightarrow k = 2.$$

148. 4.5 The system has no solution

$\Rightarrow \Delta = 0$ (by Cramer's rule)

$$\Rightarrow \begin{vmatrix}2 & 3\\ 3 & p\end{vmatrix} = 0 \Rightarrow 2p - 9 = 0$$

$$\Rightarrow p = 4.5$$

149. (*c*) The homogeneous system has a non-trivial solution $\Rightarrow$ coefficient determinant = 0

$$\Rightarrow \begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} p+q+r & q & r \\ p+q+r & r & p \\ p+q+r & p & q \end{vmatrix} = 0$$

(by $C_1 \to C_1 + C_2 + C_3$)

$$\Rightarrow (p+q+r)\begin{vmatrix} 1 & q & r \\ 1 & r & p \\ 1 & p & q \end{vmatrix} = 0$$

(taking $p+q+r$ common from 1st column)

$\Rightarrow (p+q+r)(p^2+q^2+r^2-pq-pr-qr)=0$

$\Rightarrow (p+q+r)(2p^2+2q^2+2r^2-2pq-2pr-2qr)=0$

$\Rightarrow (p+q+r)\{(p-q)^2+(q-r)^2+(r-p)^2\}=0$

$\Rightarrow p+q+r=0$ or $p=q=r$

($\because$ $a^2+b^2+c^2=0 \Leftrightarrow a=b=c=0$)

150. 3

$trace(A)=14$

$\Rightarrow a+5+2+b=14$

$\Rightarrow a+b=7 \Rightarrow b=7-a$

$\det(A)=100$

$$\Rightarrow \begin{vmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{vmatrix} = 100$$

$$\Rightarrow b\begin{vmatrix} a & 0 & 3 \\ 2 & 5 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 100$$

$$\Rightarrow 2b\begin{vmatrix} a & 0 \\ 2 & 5 \end{vmatrix} = 100$$

$\Rightarrow 2b \times 5a = 100$

$\Rightarrow ab=10$

$\Rightarrow a(7-a)=10$

$\Rightarrow a^2-7a+10=0$

$\Rightarrow a=2,5$

For $a=2, b=5$ and for $a=5, b=2$.

So $|a-b|=3$.

151. (d)

$$P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x+y \\ x+3y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{cases} 3x+y=a\text{............}(1) \\ x+3y=b\text{............}(2) \end{cases}$$

Then $(1)^2+(2)^2$

$\Rightarrow (3x+y)^2+(x+3y)^2=a^2+b^2=1$

$\Rightarrow 10x^2+10y^2+12xy=1$............(3)

Comparing (3) with $Ax^2+By^2+2Hxy=1$

we get $A=10, B=10, H=6$

Since $H^2-AB=36-100=-64<0$,

so (3) represents an ellipse with major axis along $y=-x$

and minor axis along $y=x$

Now on $y=x$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (for x=1)

152. (d)

$$\text{Let } M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then $|M-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\lambda^2=0$

$\Rightarrow \lambda=0,0,1$

$\therefore \lambda=1$ is an eigen value of M.

153. 3

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Then $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1-\lambda & 1-\lambda & 1 \\ 1-\lambda & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 3-\lambda & 1-\lambda & 1 \\ 3-\lambda & 1 & 1-\lambda \end{vmatrix} = 0$$

[by $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow (3-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3-\lambda = 0$$

$\Rightarrow \lambda = 3$ is a non-zero eigen value of A.

154. (c) $M^4 = I \Rightarrow M^{-1} M^4 = M^{-1} I \Rightarrow M^{-1} = M^3$

For $k = 1$, $M^{-1} \neq M^{4\times1+1}$, $M^{-1} \neq M^{4\times1+2}$, $M^{-1} \neq M^{4\times1}$ but $M^{-1} = M^{4\times1+3} = M^4 M^3 = M^3$ (Since $M^4 = I$)

Hence (c) is correct only.

155. (a)

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & k-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(k-\lambda) - 1 = 0$$

$$\Rightarrow \lambda^2 - (2+k)\lambda + (-1+2k) = 0$$

$$\Rightarrow \lambda = \frac{(2+k) \pm \sqrt{(2+k)^2 - 4(-1+2k)}}{2}$$

$$= \frac{(2+k) \pm \sqrt{k^2 - 4k + 8}}{2}$$

Now λ *is* $+ve \Rightarrow \lambda > 0$

$$\Rightarrow (2+k) \pm \sqrt{k^2 - 4k + 8} > 0$$

$$\Rightarrow (2+k) > \pm\sqrt{k^2 - 4k + 8}$$

$$\Rightarrow (2+k)^2 > k^2 - 4k + 8$$

$$\Rightarrow k > 1/2$$

156. 15. Since the complex eigen values occurs in conjugate pairs, so $2 + \sqrt{-1}$ *i.e.*, $2 + i$ is an eigen value implies $2 - i$ will be another eigen value. Then the eigen values are $2 + i$, $2 - i$ and 3.

Then $|P|$

= product of the eigen values

= $(2 + i)(2 - i)3 = 15$.

157. (b)

$$\Delta = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 3 \\ 5 & 9 & -6 \end{vmatrix}$$

$$= (-18 - 27) - 2(-12 - 15) - 3(18 - 15) = 0.$$

Therefore the system cannot have a unique solution. Since the system is consistent, so it must has infinite number of solutions.

Now the system has infinite number of solutions

$\Delta_3 = 0$ (by Cramer's rule)

$$\Rightarrow \begin{vmatrix} 1 & 2 & a \\ 2 & 3 & b \\ 5 & 9 & c \end{vmatrix} = 0$$

$$\Rightarrow (3c - 9b) - 2(2c - 5b) + a(18 - 15) = 0$$

$$\Rightarrow 3a + b - c = 0.$$

158. (d) The given system of equations can be re-written as: $2x + 5y = 2$ and $-4x + 3y = -30$.

Solving these two equations we get $x = 6$ and $y = -2$.

159. (a) $y = 3x + 3 = 3x + 5 \Rightarrow 3 = 5$, which is absurd. Hence the system has no solution.

160. (c)

(i) Consider the system of equations: $x + y + z = 1$, $2x + 2y + 2z = 3$ (here $m = 2 < 3 = n$).

Clearly this system doesn't have any solution. So statement I is not correct.`

(ii) Consider the system of equations: $x + y = 1$, $2x - y = 2$, $x + 2y = 1$ (here $m = 3 > 2 = n$).

Clearly this system has a solution $x = 1$ and $y = 0$. So statement II is not correct.

(iii) Consider the system of equations: $x + y = 1$, $2x - y = 2$ (here $m = 2 = n$).

Clearly this system has a solution $x = 1$ and $y = 0$. So statement III is correct.

161. (d)

$|P| =$ product of the eigen values

$$\Rightarrow \sigma^2 - \omega x = (\sigma + j\omega)(\sigma - j\omega)$$

$$= \sigma^2 - j^2\omega^2$$

$$\Rightarrow -\omega x = -j^2\omega^2$$

$$\Rightarrow x = -\omega \quad [\because j = \sqrt{-1}]$$

162. (a) Determinant of the matrix = 0 + 1 = 1 = product of the eigen value = $i \times (-i)$ [since $i^2 = -1$].

163. –6.

A has eigen value λ

$\Rightarrow A^3 - 3A^2$ has eigen value $\lambda^3 - 3\lambda^2$

$$\therefore \lambda = 1 \Rightarrow \lambda^3 - 3\lambda^2 = 1 - 3 = -2,$$

$$\lambda = -1 \Rightarrow \lambda^3 - 3\lambda^2 = -1 - 3 = -4,$$

$$\lambda = 3 \Rightarrow \lambda^3 - 3\lambda^2 = 27 - 27 = 0.$$

Therefore the eigen value of $A^3 - 3A^2$ are – 2, – 4, 0.

Hence

$tr\left(A^3 - 3A^2\right) =$ sum of the eigen values

$$= (-2) + (-4) + 0 = -6$$

164. 1.

$\lambda = 0 \Rightarrow A$ is singular

$$\Rightarrow |A| = 0$$

$$\Rightarrow \begin{vmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ 0 & 0 & x-1 \end{vmatrix} = 0 \ [by\ R_3 \to R_3 + 2R_1]$$

$\Rightarrow (x-1)(21-18) = 0$

$\Rightarrow x = 1.$

165. 1/8.

$|A|$ = product of the eigen values $=1\times 2\times 4 = 8$.

Then $|A^{-1}| = \frac{1}{|A|} = \frac{1}{8}$

$\left(\therefore AA^{-1} = I \Rightarrow |AA^{-1}| = |I| = 1 \Rightarrow |A^{-1}||A| = 1\right)$

$\therefore |(A^{-1})^T| = |A^{-1}| = \frac{1}{8}.$

166. (c)

167. (d).

$P^3 = P$

$\Rightarrow \lambda^3 = \lambda$

(by Cayley Hamilton Theorem)

$\Rightarrow \lambda^3 - \lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 1) = 0$

$\Rightarrow \lambda = 0, 1, -1$

168. 233

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - \lambda - 1 = 0.$

By Cayley Hamilton theorem, we can write

$A^2 - A - I = 0$ *i.e*; $A^2 = A + I$

$$\begin{aligned}\therefore A^4 &= A^2 \times A^2 \\ &= (A+I)\times(A+I) \\ &= A^2 + 2AI + I^2 \\ &= (A+I) + 2A + I \\ &= 3A + 2I,\end{aligned}$$

$$\begin{aligned}A^6 &= A^4 \times A^2 \\ &= (3A+2I)\times(A+I) \\ &= 3A^2 + 5AI + 2I^2 \\ &= 3(A+I) + 5A + 2I \\ &= 8A + 5I,\end{aligned}$$

$$\begin{aligned}A^{12} &= A^6 \times A^6 \\ &= (8A+5I)\times(8A+5I) \\ &= 64A^2 + 80AI + 25I^2 \\ &= 64(A+I) + 80A + 25I \\ &= 144A + 89I \\ &= 144\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + 89\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 144+89 & 144+0 \\ 144+0 & 0+89 \end{pmatrix} \\ &= \begin{pmatrix} 233 & 144 \\ 144 & 89 \end{pmatrix}\end{aligned}$$

Now $\begin{bmatrix} x[12] \\ x[11] \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{12} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{aligned}&= A^{12}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{pmatrix} 233 & 144 \\ 144 & 89 \end{pmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 233 \\ 144 \end{bmatrix}\end{aligned}$$

Comparing both sides we get, $x[12] = 233$.

169. (b) For any matrix A, rank(A^TA) = rank(AA^T) = rank(A).

170. (d)

$u = ae_1 + be_2 + ce_3$

$\Rightarrow (4,3,-3) = a(1,0,2) + b(0,1,0) + c(-2,0,1)$

$= (a,0,2a) + (0,b,0) + (-2c,0,c)$

$\Rightarrow (4,3,-3) = (a-2c, b, 2a+c)$

$\Rightarrow a - 2c = 4,\ b = 3,\ 2a + c = -3.$

Solving we get,

$a = -\frac{2}{5}, b = 3, c = -\frac{11}{5}.$

171. (c)

$M = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$ [by $R_1 \to \frac{1}{5}R_1$]

$\sim \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ [by $R_3 \to R_3 - 3R_1$]

which has two non-zero rows.

Therefore rank(M) = 2.

172. 5. Let λ be the other eigen value.

Then determinant of the matrix = product of the eigen values $\Rightarrow 50 = 10\lambda$.

So $\lambda = 5$.

173. (b) Product of the eigen values = determinant of P = 2(3-6) + 0 + 1(8 – 0) = 2.

174. (*b*).

Let $A = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$.

Then $\text{rank}(A) = 2$

($\because A$ has two non-zero rows)

$$\text{Also} \sum_{i=1}^{2}\sum_{j=1}^{2} A_{ij}^2 = A_{11}^2 + A_{12}^2 + A_{21}^2 + A_{22}^2$$

$$= 5^2 + 0^2 + 0^2 + (-5)^2$$

$$= 50.$$

Thus the matrix A satisfies the given condition.

Since A is upper triangular, so it's eigen values are the diagonal elements *i.e*; 5 and –5. So statement–I is correct. But since $|\pm 5| = 5$, which is not greater than 5, so statement–II is not correct.

175. [0] Sum of eigen values = trace(A)

$\Rightarrow \lambda_1 + \lambda_2 = 50 + 80 = 130$

$X_1^T X_2$

$$= \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}^T \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$$

$$= [70 \ \ \lambda_1 - 50] \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$$

$= [70(\lambda_2 - 80) + 70(\lambda_1 - 50)]$

$= [70(\lambda_1 + \lambda_2) - 9100]$

$= [70 \times 130 - 9100]$

$= [0]$

176. (*c*).

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 5 \\ 0 & 5-\lambda & 6 \\ 0 & -6 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} 5-\lambda & 6 \\ -6 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow \ (1-\lambda)(\lambda^2 - 10\lambda + 61) = 0$

$$\Rightarrow \ \lambda = 1, \frac{10 \pm \sqrt{10^2 - 4 \times 61}}{2}$$

$$\Rightarrow \lambda = 1, \frac{10 \pm 12j}{2} \ \ (\text{where } j = \sqrt{-1})$$

$\Rightarrow \ \lambda = 1.5 \pm 6j$

177. (*a*) Let $A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$.

Then $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5-\lambda)(1-\lambda) + 4 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$

$\Rightarrow \lambda = 3, 3$

Eigen value 3 has a multiplicity of 2.

Then $AX = \lambda X$

$$\Rightarrow \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5x - y \\ 4x + y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$$

$\Rightarrow 5x - y = 3x, \ 4x + y = 3y$

$\Rightarrow 2x - y = 0, 4x = 2y$

$\Rightarrow y = 2x$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence A has only one linearly independent eigen vector, which is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

178. (*c*)

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 15-\lambda & 2 & 3 & 4 & 5 \\ 15-\lambda & 1-\lambda & 2 & 3 & 4 \\ 15-\lambda & 5 & 1-\lambda & 2 & 3 \\ 15-\lambda & 4 & 5 & 1-\lambda & 2 \\ 15-\lambda & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

[by $C_1 \to C_1 + C_2 + C_3 + C_4 + C_5$]

$$\Rightarrow (15-\lambda)\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1-\lambda & 2 & 3 & 4 \\ 1 & 5 & 1-\lambda & 2 & 3 \\ 1 & 4 & 5 & 1-\lambda & 2 \\ 1 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow 15 - \lambda = 0$

$\Rightarrow \lambda = 15$ is a real eigen value.

179. (*d*)

$$|P| = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{2} \times 2 = 1.$$

So statement (a) is correct.

PP^T

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & 0 & -\frac{1}{2}+\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2}+\frac{1}{2} & 0 & \frac{1}{2}+\frac{1}{2} \end{bmatrix} = I$$

Hence P is orthogonal and so statement (b) is also correct.

P is orthogonal $\Rightarrow P^T = P^{-1}$. So option (c) is also correct.

Alternative method:

$$|P - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1-\lambda & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left\{\left(\frac{1}{\sqrt{2}} - \lambda\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right\} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - \sqrt{2}\lambda + 1) = 0$$

$$\Rightarrow \lambda = 1, \frac{\sqrt{2} \pm \sqrt{2}i}{2}$$

$$\Rightarrow \lambda = 1, \frac{1 \pm i}{\sqrt{2}}$$

Hence all eigen values are not real. Thus option (d) is in correct.

180. (a) Here the co-efficient matrix is, $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$. Then the characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\text{or, } \begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$\text{or, } \lambda^2 - 4\lambda - 5 = 0.$$

181. (c) If X and Y are any two eigen vectors of a symmetric matrix, then we must have $X^TY = O$.

Let $X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (given) and $Y = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

$$\text{Then } X^TY = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= [1 + 0 + (-1)] = [0]_{1\times 1}$$

$$= O \text{ (null matrix)}.$$

182. 5. We know that for a 3×3 matrix M, the characteristic polynomial is given by

$\lambda^3 -$ trace$(M) \times \lambda^2 + a\lambda - \det(M)$ ...(1)

Comparing (1) with $\lambda^3 - 4\lambda^2 + a\lambda + 30$, we get,

trace$(M) = 4$ ...(2)

and $\det(M) = -30$ ...(3)

Given that '2' is an eigen value of M. Let λ_1 and λ_2 be the other two eigen values of M.

Then equation (2)

$\Rightarrow 2 + \lambda_1 + \lambda_2 = 4$

$\Rightarrow \lambda_2 = 2 - \lambda_1$

Now equation (3)

$\Rightarrow 2\lambda_1 \lambda_2 = -30$

$\Rightarrow \lambda_1(2 - \lambda_1) = -15$

$\Rightarrow \lambda_1^2 - 2\lambda_1 - 15 = 0$

$\Rightarrow \lambda_1 = 5, -3.$

Hence the eigen values of M are 2, –3 and 5.

183. 4

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$[by\ R_1 \to R_1 + R_2 + R_3 + R_4 + R_5]$

which has four non-zero rows.

Therefore rank = 4.

184. 2.

$$P + Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & 0 \\ 8 & 8 & 8 \end{bmatrix}$$

$[by\ R_2 \to R_2 - (R_3 - R_1)]$

which has two non-zero rows.

Therefore rank$(P + Q) = 2$.

185. (c) P is the inverse of $Q \Rightarrow PQ = PQ = I$.

186. 1

Let $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then

$$A = VV^T$$

$$= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$= \begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2^2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

$$[by\, R_1 \to \frac{1}{v_1}R_1,\ R_2 \to \frac{1}{v_2}R_2,\ R_3 \to \frac{1}{v_3}R_3]$$

$$\sim \begin{bmatrix} v_1 & v_2 & v_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[by\, R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

which has one non-zero row.
Therefore rank = 1.

187. (*a*)

$$AB^T = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 3+35 & 8+20 \\ 18+14 & 48+8 \end{bmatrix} = \begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$$

188. (*c*)

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 12 - 12 = 0.$$

Therefore the given matrix is singular.

189. (*c*) We know that if $A_{n\times n}$ is an orthogonal matrix then $A^{-1} = A^T$. Here given that $Q = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ \frac{-6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & \frac{-3}{7} \end{pmatrix}$ is an orthogonal matrix.

Therefore $Q^{-1} = \begin{pmatrix} \frac{3}{7} & \frac{-6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & \frac{-3}{7} \end{pmatrix}$.

190. (*b*)

$$A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & -3 & 3 & 9 \end{pmatrix}$$

$(by\, R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - 4R_1)$

$$\sim \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ (by } R_3 \to 2R_3 - 3R_2)$$

which has two-non zero rows.
Hence rank of the given matrix = 2.

191. (*b*)

$$A = \begin{pmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

(by R3 $\to R_3 + 2R_1 - R_2$)
which has two non zero rows. So rank of the given matrix = 2.

192. (*d*) $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -4 \\ 4 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (2-2)\lambda + 12 = 0 \Rightarrow \lambda^2 + 12 = 0$$

$$\therefore \lambda = \pm i\sqrt{12} \text{ are eigen values of } A.$$

Case (*i*) : $\lambda = i\sqrt{12}$

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 2 & -4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = i\sqrt{12}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 - 4x_2 \\ 4x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} i\sqrt{12}x_1 \\ i\sqrt{12}x_2 \end{bmatrix}$$

$$\Rightarrow 2x_1 - 4x_2 = i\sqrt{12}x_1, 4x_1 - 2x_2 = i\sqrt{12}x_2$$

$$\Rightarrow (2 - i\sqrt{12})x_1 - 4x_2 = 0, 4x_1 - (2 + i\sqrt{12})x_2 = 0$$

Let $x_1 = K$. Then $x_2 = \frac{(2 - i\sqrt{12})}{4}k$

Therefore $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ \frac{(2-i\sqrt{12})}{4}k \end{bmatrix} = k\begin{bmatrix} 1 \\ \frac{(2-i\sqrt{12})}{4} \end{bmatrix}$ is an eigen vector corresponding to the eigen value $\lambda = i\sqrt{12}$ for $k \neq 0$.

Case (*ii*) : $\lambda = -i\sqrt{12}$

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 2 & -4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -i\sqrt{12}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 - 4x_2 \\ 4x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} -i\sqrt{12}x_1 \\ i\sqrt{12}x_2 \end{bmatrix}$$

$$\Rightarrow 2x_1 - 4x_2 = -i\sqrt{12}x_1, 4x_1 - 2x_2 = -i\sqrt{12}x_2$$

$$\Rightarrow (2 + i\sqrt{12})x_1 - 4x_2 = 0, 4x_1 - (2 - i\sqrt{12})x_2 = 0$$

Let $x_1 = k$. Then $x_2 = \dfrac{(2 + i\sqrt{12})}{4}k.$

Therefore $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ \dfrac{(2+i\sqrt{12})}{4}k \end{bmatrix} = k\begin{bmatrix} 1 \\ \dfrac{(2+i\sqrt{12})}{4} \end{bmatrix}$

is an eigen vector corresponding to the eigen value $\lambda = -i\sqrt{12}$ for $k \neq 0$.

Hence $\begin{bmatrix} 1 \\ \dfrac{(2-i\sqrt{12})}{4} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \dfrac{(2+i\sqrt{12})}{4} \end{bmatrix}$ are eigen vectors corresponding to the eigen vectors $i\sqrt{12}$ and $-i\sqrt{12}$ respectively.

Hence the given matrix has complex eigen values and eigen vectors.

193. (*c*) We know that for a square matrix of order '*n*', the following results hold:

(*i*) If a matrix $M_{n\times n}$ has n different eigen values, then the matrix $M_{n\times n}$ will have n linearly independent eigen vectors.

(*ii*) If a matrix $M_{n\times n}$ has some repeated eigen values then the eigen vectors may or may not be linearly independent.

(*iii*) If a matrix $M_{n\times n}$ is non-singular then the eigen values may or may not be repeated.

Thus it is clear that the statement "*S*2 implies *S*1" is correct.

194. 1. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(-2-\lambda) - 0 = 0$$

$$\Rightarrow \lambda = 1, -2, 2.$$

$\therefore$ Eigen values of A are 1, 2, – 2

If λ is an eigen value of A then the eigen value of $B = A^3 - A^2 - 4A + 5I$ is $\lambda^3 - \lambda^2 - 4\lambda + 5$.

Thus eigen values of B are $1^3 - 1^2 - 4 \times 1 + 5$ 1 , $2^3 - 2^2 - 4 \times 2 + 5$ and $(-2)^3 - (-2)^2 - 4 \times (-2) + 5$ *i.e.*, 1.

$\therefore$ $|B|$ = Product of eigen values of $B = 1 \times 1 \times 1 = 1$.

195. (*c*) The given system of equations can be represented as $AX = B$, where $A = \begin{bmatrix} 3 & 2k \\ k & 6 \end{bmatrix}$.

Then system has infinite number of solutions

$$\Rightarrow |A| = 0 \Rightarrow \begin{vmatrix} 3 & 2k \\ k & 6 \end{vmatrix} = 0 \Rightarrow 18 - 2k^2 = 0$$

$$\Rightarrow k^2 = 9 \Rightarrow k = \pm 3.$$

Case-I: $k = 3$,

$$[A : B] = \begin{bmatrix} 3 & 6 & -2 \\ 3 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

(by $R_2 \to R_2 - R_1$)

Thus rank ([A : B]) = 2. But det(A) = $\begin{vmatrix} 3 & 6 \\ 3 & 6 \end{vmatrix} = 0$. So rank($A$) = 1.

Hence rank ([A:B]) $\neq$ rank(A). Consequently the system has no solution in this case.

Case-II: $k = -3$,

$$[A : B] = \begin{bmatrix} 3 & -6 & -2 \\ -3 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 6 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

(*by* $R_2 \to R_2 + R_1$)

Thus rank ([A:B]) = 1. But det(A) = $\begin{vmatrix} 3 & 6 \\ 3 & 6 \end{vmatrix} = 0$ and so rank(A) = 1.

Hence rank ([A:B]) = rank(A) < number of variables. Consequently the system has an infinite number of solutions in this case.

196. 2. The system has infinitely many solutions

$\Rightarrow |A| = 0$.

$$\Rightarrow \begin{vmatrix} k & 2k \\ k^2 - k & k^2 \end{vmatrix} = 0 \Rightarrow k^3 - 2k^3 + 2k^2 = 0$$

$$\Rightarrow 2k^2 - k^3 = 0 \Rightarrow k^2(2-k) = 0 \Rightarrow 0, 2$$

$\therefore$ The system $AX = 0$ will have infinitely many solutions for two distinct (*i.e.*, different) real values namely, $k = 0$ and $k = 2$.

197. 25. Let A be a 3 × 3 matrix with diagonal elements –10, 5 and 0. Given that two eigen values of A are –15 and –15.

Let λ be the 3rd eigen value of A.

Then sum of the eigen values = trace of the matrix

$\Rightarrow$ $-15 - 15 + \lambda = -10 + 5 + 0 \Rightarrow \lambda = 25$.

198. 3.

$$A = UV^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \Rightarrow A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 2 = 0$

$\Rightarrow \lambda^2 - 3\lambda = 0 \Rightarrow \lambda(\lambda - 3) = 0 \Rightarrow \lambda = 0, 3.$

Therefore the largest eigen value of A is 3.

199. $\frac{11}{2}$. Given trace $(A) = 4$ and trace $(A^2) = 5$

Let λ_1, λ_2 be two eigen values of A. Then λ_1^2, λ_2^2 will be two eigen values of A^2

$\therefore$ trace $(A) = 4$ and trace $(A^2) = 5$

$\Rightarrow \lambda_1 + \lambda_2 = 4$ and $\lambda_1^2 + \lambda_2^2 = 5$

(since trace = sum of eigen values)

Now $(\lambda_1 + \lambda_2)^2 = \lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2$

$\Rightarrow 16 = 5 + 2\lambda_1\lambda_2 \Rightarrow \lambda_1\lambda_2 = \frac{11}{2}.$

$\therefore |A| = \text{product of eigen values} = \lambda_1\lambda_2 = \frac{11}{2}.$

200. (*a*) $N^2 = 0 \Rightarrow N_{3\times3}$ is a Nilpotent matrix.

We know that the eigen values of a non-zero nilpotent matrix are always zero.

Therefore the eigen values of a given matrix N are 0, 0, 0.

Questions for Practice

1. The matrix $A = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}$ is
   (*a*) Idempotent (*b*) orthogonal
   (*c*) skew symmetric (*d*) none of these

2. The matrix $A = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$ is
   (*a*) Idempotent (*b*) orthogonal
   (*c*) skew symmetric (*d*) none of these

3. If $\begin{pmatrix} 5 & x+2 \\ x+1 & -2 \end{pmatrix} = \begin{pmatrix} x+3 & 4 \\ 3 & -4 \end{pmatrix}$, then $x = ?$
   (*a*) 0 (*b*) 2
   (*c*) –2 (*d*) 1

4. If $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 10 \end{pmatrix}$, then
   (*a*) $AB = O, BA = O$ (*b*) $AB = O, BA \neq O$
   (*c*) $AB \neq O, BA = O$ (*d*) $AB \neq O, BA \neq O$

5. If $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then
   (*a*) $A^2 = B^2 = I$ (*b*) $A^2 = B^2 = -I$
   (*c*) $A^2 = I, B^2 = -I$ (*d*) $A^2 = -I, B^2 = I$

6. If A and B are skew-symmetric matrices of same order n, then $A + B$ is
   (*a*) skew-symmetric (*b*) null matrix
   (*c*) identity matrix (*d*) symmetric

7. A necessary and sufficient condition for a square matrix A to possess inverse is that
   (*a*) $A \neq O$ (*b*) adj $(A) \neq O$
   (*c*) $\det(A) \neq 0$ (*d*) $\det(A) = 0$

8. If A = $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$, then $A^{-1} = ?$
   (*a*) $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ (*b*) $\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$
   (*c*) $\begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix}$ (*d*) $\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$

9. If I_3 be the identity matrix of order 3, then $(2I_3)^{-1} = ?$
   (*a*) $\frac{1}{4}I_3$ (*b*) $2I_3$
   (*c*) I_3 (*d*) none of these

10. Non zero matrices A and B are called divisors of zero if
    (*a*) $AB \neq O$ (*b*) adj $(A) \neq O$
    (*c*) $\det(A) \neq O$ (*d*) $AB = O$

11. If A is hermitian, then iA is
    (*a*) symmetric (*b*) unitary
    (*c*) skew-hermitian (*d*) hermitian

12. If 1, ω, ω^2 are cube roots of unity, then the inverse of which of the following matrices exist?
    (*a*) $\begin{pmatrix} 1 & w \\ w & w^2 \end{pmatrix}$ (*b*) $\begin{pmatrix} \omega & \omega^2 \\ \omega^2 & 1 \end{pmatrix}$
    (*c*) $\begin{pmatrix} \omega^2 & 1 \\ 1 & \omega \end{pmatrix}$ (*d*) none of these

13. If A is an orthogonal matrix, then $A^t = ?$
    (*a*) A (*b*) A^2
    (*c*) A^{-1} (*d*) Identity matrix

14. If $A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$,, then $(A^{-1})^3 = ?$
    (*a*) $\frac{1}{27}\begin{pmatrix} 1 & -26 \\ 0 & 27 \end{pmatrix}$ (*b*) $\frac{1}{27}\begin{pmatrix} -1 & 26 \\ 0 & 27 \end{pmatrix}$
    (*c*) $\frac{1}{27}\begin{pmatrix} 1 & -26 \\ 0 & -27 \end{pmatrix}$ (*d*) $\frac{1}{27}\begin{pmatrix} -1 & -26 \\ 0 & -27 \end{pmatrix}$

15. The rank of the following $(n + 1) \times (n + 1)$ matrix is

$$\begin{pmatrix} 1 & a & a^2 & .. & .. & .. & a^n \\ 1 & a & a^2 & .. & .. & .. & a^n \\ .. & .. & .. & .. & .. & .. & .. \\ 1 & a & a^2 & .. & .. & .. & a^n \end{pmatrix}$$

(*a*) 1 (*b*) 2
(*c*) *n* (*d*) depends on *n*

16. If *A* is a square matrix of order *n*, then adj (adj *A*) = ?
(*a*) $|A|^{n-2} A$ (*b*) $|A|^{n-2}$
(*c*) $|A|^{n-1} A$ (*d*) Identity matrix

17. If $A = \begin{pmatrix} 1+i & 3-5i \\ 2i & 5 \end{pmatrix}$, then $A^{\theta} = ?$
(*a*) $\begin{pmatrix} 1-i & -2i \\ 3+5i & 5 \end{pmatrix}$ (*b*) $\begin{pmatrix} 1+i & -2i \\ 3-5i & 5 \end{pmatrix}$
(*c*) $\begin{pmatrix} 1-i & 2i \\ 3+5i & 5 \end{pmatrix}$ (*d*) $\begin{pmatrix} 1+i & -2i \\ 3+5i & -5 \end{pmatrix}$

18. If inverse of a matrix *A* exist, then *A* is
(*a*) singular (*b*) non-singular
(*c*) det(*A*) = 0 (*d*) none of these

19. If $A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 5 \\ 2 & -5 & 0 \end{pmatrix}$ then $A^{-1} = ?$
(*a*) *A* (*b*) A^2
(*c*) –*A* (*d*) doesn't exist

20. If $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ then $A^{-1} = ?$
(*a*) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (*b*) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(*c*) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (*d*) none of these

21. If $A = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}$, then *A* is
(*a*) hermitian (*b*) skew hermitian
(*c*) symmetric (*d*) unitary

22. If *A* and *B* be two square matrices of same order, then which of the followings is not true in general?
(*a*) 2*A* – 3*B* = –3*B* + 2*A* (*b*) *AB* = *BA*
(*c*) 4(*A* + *B*) = 4*A* + 4*B* (*d*) *AIB* = *AB*

23. If $A = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$ and $A + A^T = I$, then *x* = ?
(*a*) $\frac{\pi}{3}$ (*b*) $\frac{2\pi}{3}$
(*c*) $\frac{5\pi}{3}$ (*d*) 0

24. The system of equations: *x* + *y* + *z* = 2, 2*x* + *y* – *z* = 3, 3*x* + 2*y* + *kz* = 4 has a unique solution for
(*a*) *k* = 0 (*b*) $k \neq 0$
(*c*) *k* = 4 (*d*) *k* = 2

25. The value of the determinant $\begin{vmatrix} \frac{1}{a} & a & bc \\ \frac{1}{b} & 1 & ca \\ \frac{1}{c} & 1 & cb \end{vmatrix}$ is equals to:
(*a*) 0 (*b*) *abc*
(*c*) *a* + *b* + *c* (*d*) 1

26. If *A* + *B* + *C* = π, then the value of the determinant $\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$ is:
(*a*) 0 (*b*) 1
(*c*) 2 sin *B* tan *A* cos *C*
(*d*) none of these

27. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then
(*a*) $\Delta_1 = 3(\Delta_2)^2$ (*b*) $\frac{d}{dx}\Delta_1 = 3\Delta_2$
(*c*) $\frac{d}{dx}\Delta_2 = 3\Delta_1$ (*d*) $\Delta_1 = 3\Delta_2$

28. If $f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$ then $\int_0^{\frac{\pi}{2}} f(x)dx = ?$ = ?
(*a*) $\frac{1}{3}$ (*b*) $\frac{1}{2}$
(*c*) $\frac{1}{4}$ (*d*) $\frac{1}{5}$

29. For what value of *x*, $\begin{vmatrix} x+\omega^2 & \omega & 1 \\ \omega & 1+x & \omega^2 \\ 1 & \omega^2 & x+\omega \end{vmatrix} = 0$
(*a*) *x* = 0 (*b*) *x* = 1
(*c*) *x* = 2 (*d*) *x* = – 1

30. If *a* + *b* + *c* = 0, then one root of $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ is is
(*a*) *x* = 0 (*b*) *x* = 1
(*c*) *x* = 2 (7) $x = a^2 + b^2 + c^2$

31. The value of $\begin{vmatrix} 100 & 101 & 102 \\ 105 & 106 & 107 \\ 110 & 111 & 112 \end{vmatrix} = ?$
(*a*) 0 (*b*) 1
(*c*) 101 (*d*) – 1

32. If $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 1 & \alpha \\ 0 & -2 & 3 \end{pmatrix}$ and $B = A^{-1}$, then α = ?

(*a*) – 5 (*b*) 2
(*c*) – 2 (*d*) 5

33. Which of the followings is correct?

(a) Skew-symmetric matrix of even order is always singular.
(*b*) Skew-symmetric matrix of odd order is always non-singular.
(*c*) Skew-symmetric matrix of odd order is always singular.
(*d*) None of these.

34. If $A = \begin{pmatrix} 3 & 4 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$, then $(A + B)^{-1}$?

(*a*) $A^{-1} + B^{-1}$ (*b*) $A + B$
(*c*) $(A + B)^{-1}$ doesn't exist (*d*) none of these

35. Given that the rank of the matrix $A = \begin{pmatrix} -1 & 2 & 5 \\ 2 & 4 & k-4 \\ 1 & -2 & k+1 \end{pmatrix}$ is 1. Then k = ?

(*a*) 4 (*b*) 5
(*c*) –6 (*d*) 7

36. If $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, , then the rank of $X^2 + X$ is

(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 0

37. If the rank of the matrix $A = \begin{bmatrix} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \\ 1 & 1 & 1 & \lambda \end{bmatrix}$ is less than 4, then λ = ?

(*a*) 1, –3 (*b*) 1, 3
(*c*) –1, 3 (*d*) –1, –3

38. The rank of the matrix $X = \begin{pmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{pmatrix}$, is

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 4

39. The system of equations: $x + y + z = 6$, $x - y - z = -4$, $x + y - z = 0$ is/has

(*a*) consistent (*b*) inconsistent
(*c*) an unique solution (*d*) trivial solution

40. The system of equations: $x + 2y - z = 10$, $x - y - 2z = -2$, $2x + y - 3z = 8$ has

(*a*) only trivial solution
(*b*) unique non-trivial solution
(*c*) no solution
(*d*) an infinite number of solutions

41. The system of equations: $x - 5y + 3z = -1$, $2x - y - z = 5$, $5x - 7y + z = 2$ has

(*a*) no solution
(*b*) infinite number of solutions
(*c*) an unique solution
(*d*) finite number of solutions

42. The system of equations: $6x + 20y - 6z = -3$, $2x + 6y = -11$, $6y - 18z = -1$ has

(*a*) no solution
(*b*) infinite number of solutions
(*c*) an unique solution
(*d*) finite number of solutions

43. The system of equations: $x + 2y + 3z = 1$, $2x + y + 3z = 2$, $5x + 5y + 9z = 4$ has

(*a*) no solution
(*b*) infinite number of solutions
(*c*) an unique solution
(*d*) finite number of solutions

44. The system of equations:
$x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$ has

(*a*) no solution
(*b*) infinite number of solutions
(*c*) an unique solution
(*d*) only trivial solution

45. If $3x + 2y + z = 0$, $x + 4y + z = 0$, $2x + y + 4z = 0$ be the system of equations, then

(*a*) it is consistent
(*b*) it has only the trivial solution
(*c*) it can be reduced to a single equation
(*d*) the determinant of the co-efficient matrix is zero

46. The equations
$kx + y + z = 0$, $-x + ky + z = 0$, $-x - y + kz = 0$ will have a non-trivial solution if k=?

(*a*) 0 (*b*) –1
(*c*) 1 (*d*) 2

47. The system of equations
$kx + 2y - z = 1$, $(k - 1)y - 2z = 2$, $(k + 2)z = 3$ will have an unique solution if k = ?

(*a*) 0 (*b*) –1
(*c*) 1 (*d*) –2

48. The sum of eigen values of $\begin{pmatrix} 8 & -6 & 0 \\ 6 & 7 & 4 \\ -2 & 4 & 3 \end{pmatrix}$ is

(*a*) 4 (*b*) 10
(*c*) 12 (*d*) 18

49. The eigen values of A^5 if $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ are

(*a*) 1, 3, 4 (*b*) 1, 9, 16
(*c*) 1, 27, 64 (*d*) 1, 243, 1024

50. The sum and the product of the eigen values of $\begin{pmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ are respectively

(*a*) 5, 20 (*b*) 25, 30
(*c*) 5, 21 (*d*) 21, 30

51. Let A be a square matrix of order 3 with trace(A) = 9 and det(A) = 24 and if one eigen value is 2, then the other eigen values are:

(*a*) 8, 3 (*b*) 3,4
(*c*) 3, 0 (*d*) 4, 4

52. Let A be a square matrix of order n such that for some scalar λ, the matrix $A–\lambda I$ is singular. Then

(*a*) λ is a characteristic root of A
(*b*) $\lambda = 0$
(*c*) λ is not a characteristic root of A
(*d*) none of these

53. Which of the followings is true?

(*a*) a matrix may have many eigen vectors corresponding to a eigen value
(*b*) a matrix have an unique eigen vector corresponding to a eigen value
(*c*) a matrix may not have an eigen vector corresponding to a eigen value
(*d*) none of these

54. If A and B be two non-singular square matrices, then

(*a*) A and B have the same eigen values
(*b*) A^{-1} and B^{-1} have same eigen values
(*c*) $A^{-1}B$ and BA^{-1} have the same eigen values
(*d*) none of these

55. If A be a singular matrix, then which of the following can be a characteristic polynomial of A?

(*a*) $\lambda^2 + 2\lambda$ (*b*) $\lambda^3 + 2\lambda + 1$
(*c*) $\lambda + 3$ (*d*) $\lambda^2 - 4\lambda + 5$

56. If $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ is the characteristic polynomial of a square matrix A, then $A^{-1} = ?$

(*a*) $A^2 - 6A + 9I$ (*b*) $\frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}$
(*c*) $\frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}I$ (*d*) none of these

57. The eigen values of $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ are:

(*a*) 5, 7, 8 (*b*) 3, 0, –7
(*c*) 8, 3, –7 (*d*) 3, 0, 15

58. The determinant of the matrix $\begin{pmatrix} 6 & -8 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is

(*a*) 11 (*b*) – 48
(*c*) 0 (*d*) – 24

59. The minimum eigen value of the following matrix is?

$$\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

60. The rank of the matrix given below is

$$\begin{pmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 21 \end{pmatrix}$$

(*a*) 3 (*b*) 1
(*c*) 2 (*d*) 4

61. The value of the determinant $\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 \\ -1 & 4 & 7 & 4 & 0 \\ -5 & 6 & 2 & 1 & 1 \end{vmatrix}$ is

(*a*) 24 (*b*) 32
(*c*) – 112 (*d*) 0

62. Solution for the system defined by the set of equations: $4y + 3z = 8$, $2x - z = 2$, $3x + 2y - z = 5$

(*a*) $x = 0, y = 1/2, z = 2$
(*b*) $x = 0, y = 1, z = 4/3$
(*c*) $x = 1, y = 1/2, z = 2$
(*d*) non existent

63. The solution of the following set of equation $x + 2y + 3z = 20$, $7x + 3y + z = 13$, $x + 6y + 2z = 0$ is

(*a*) $x = -2, y = 2, z = 8$
(*b*) $x = 2, y = -3, z = 8$
(*c*) $x = -2, y = 3, z = -8$
(*d*) $x = 8, y = 2, z = -3$

64. The solution of the following set of equations:
$5x + 4y + 10z = 13$, $x + 3y + z = 7$, $4x - 2y + z = 0$ is

(*a*) $x = 2, y = 1, z = 1$
(*b*) $x = 1, y = 2, z = 0$
(*c*) $x = 1, y = 0, z = 2$
(*d*) $x = 0, y = 1, z = 2$

65. Consider the system of equations $-x + 2y - 3z = 2$, $x + 6y + 12z = 1$, $2x - 4y + 3kz = -4$ What is the value of k for which the system of equations has infinite number of solution?

(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 0

66. Consider $2x_1 + x_2 = 3$, $5x_1 + bx_2 = 7.5$

The system of linear combinations in two variables shown above will have infinite number of solutions, if and only if 'b' = ?

(*a*) 2.5 (*b*) 3
(*c*) 3.5 (*d*) 4

67. If $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & i & i \\ 0 & 0 & 0 & -i \end{bmatrix}$, then the matrix A^4, calculated by the use of Cayley Hamilton theorem or otherwise is

(*a*) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (*d*) none of these

68. For the matrix $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ the eigen values are

(*a*) 3, –3 (*b*) –3, –5
(*c*) 3, 5 (*d*) 5, 0

69. The number of values of 'k' for which the system of equations:

$kx + (k + 3)y = 10z$, $(k - 1)x + (k - 2)y = 5z$, $2x + (k + 4)y = kz$ has infinitely many solutions

(*a*) 2 (*b*) 3
(*c*) 5 (*d*) 0

70. If $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ and $\text{adj}(A) = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{bmatrix}$, then '$k$' = ?

(*a*) 3 (*b*) –3
(*c*) 5 (*d*) –5

71. Let N be a nilpotent matrix of order 4 with real entries. Then which one of the following statements is true about eigen values of N?

(*a*) All eigen values are non- zero real numbers
(*b*) All eigen values are purely imaginary
(*c*) Zero is the only eigen value
(*d*) At least one eigen value is real and at least one eigen value has non – zero imaginary part.

72. Let A be a 3 × 3 matrix with trace (A) = 3 and det (A) = 2. If '1' is an eigen value of A, then the eigen values of the matrix $A^2 - 2I$ are

(*a*) $1, 2(i - 1), -2(i + 1)$
(*b*) $-1, 2(i - 1), 2(i + 1)$
(*c*) $1, 2(i + 1), -2(i + 1)$
(*d*) $-1, 2(i - 1), -2(i + 1)$

73. $P = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$, and $R = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$, then which of the following statements are true?

(*a*) $PQ = PR$ (*b*) $QR = RP$
(*c*) $QP = RP$ (*d*) $PQ = QR$

74. If A be a square symmetric real valued matrix of dimension $2n$, then the eigen values of A are:

(*a*) $2n$ distinct real values
(*b*) $2n$ real values not necessarily distinct
(*c*) n distinct pairs of complex conjugate numbers
(*d*) n pairs of complex conjugate numbers not necessarily distinct

75. Which of the following is an eigen vector of the matrix $\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}$?

(*a*) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

76. The eigen values of the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ are 5 and –1. Then the eigen values of $-2A + 3I$ are

(*a*) – 7 and 5 (*b*) –7 and –5
(*c*) –7/5 and 1/5 (*d*) 1/7 and –1/2

77. Let $c_1, c_2, \ldots, c_n$ be scalars, not all zero, such that $\sum_{i=1}^{n} c_i a_i = 0$ where a_i are column vectors in R^n. Consider the set of linear equations $Ax = b$ where $A = [a_1, a_2, \ldots, a_n]$ and $b = \sum_{i=1}^{n} a_i$. The set of equations has

(*a*) *a* unique solution at $x = J_n$ where J_n denotes a *n*-dimensional vector of all 1

(*b*) no solution

(*c*) infinitely many solutions

(*d*) finitely many solutions

Answer key

1. (*b*)	2. (*a*)	3. (*b*)	4. (*b*)	5. (*c*)
6. (*a*)	7. (*c*)	8. (*b*)	9. (*a*)	10. (*d*)
11. (*c*)	12. (*d*)	13. (*c*)	14. (*a*)	15. (*a*)
16. (*a*)	17. (*a*)	18. (*b*)	19. (*d*)	20. (*c*)
21. (*a*)	22. (*b*)	23. (*a*)	24. (*b*)	25. (*a*)
26. (*a*)	27. (*b*)	28. (*a*)	29. (*a*)	30. (*a*)
31. (*a*)	32. (*d*)	33. (*c*)	34. (*d*)	35. (*c*)
36. (*c*)	37. (*a*)	38. (*c*)	39. (*a*)	40. (*d*)
41. (*a*)	42. (*a*)	43. (*c*)	44. (*d*)	45. (*b*)
46. (*a*)	47. (*b*)	48. (*d*)	49. (*d*)	50. (*c*)
51. (*b*)	52. (*a*)	53. (*a*)	54. (*c*)	55. (*a*)
56. (*c*)	57. (*d*)	58. (*b*)	59. (*a*)	60. (*a*)
61. (*a*)	62. (*c*)	63. (*b*)	64. (*b*)	65. (*b*)
66. (*a*)	67. (*a*)	68. (*c*)	69. (*a*)	70. (*d*)
71. (*c*)	72. (*d*)	73. (*a*)	74. (*b*)	75. (*a*)
76. (*a*)	77. (*c*)			

Hints

1. (*b*) Show that $AA^T = I$
2. (*a*) Show that $A^2 = A$
3. (*b*) Comparing we get, $x + 3 = 5$ and so $x = 2$
4. (*b*) Compute AB and BA.
5. (*c*)
6. (*a*) See the properties of a skew-symmetric matrix
7. (*c*)
8. (*b*) See the properties of the inverse of a matrix.
9. (*a*)
10. (*d*)
11. (*c*) See the properties of a hermitian matrix.
12. (*d*)

$$\begin{vmatrix} 1 & \omega \\ \omega & \omega^2 \end{vmatrix} = \begin{vmatrix} \omega & \omega^2 \\ \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} \omega^2 & 1 \\ 1 & \omega \end{vmatrix} = 0$$

$\left(\sin ce\ \ 1+\omega+\omega^2 = 0,\ \omega^3 = 1,\ \omega^4 = \omega\right)$

13. (*c*)
14. (*a*)
15. (*a*)
16. (*a*)
17. (*a*)
18. (*b*)
19. (*d*)
20. (*c*)
21. (*a*)
22. (*b*)
23. (*a*)

$$A + A^T = I$$

$$\Rightarrow \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} + \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} = I$$

$$\Rightarrow \begin{pmatrix} 2\cos x & 0 \\ 0 & 2\cos x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow 2\cos x = 1$$

$$\Rightarrow \cos x = \frac{1}{2} = \cos\frac{\pi}{3}$$

24. (*b*)
25. (*a*)

$$\begin{vmatrix} \frac{1}{a} & 1 & bc \\ \frac{1}{b} & 1 & ca \\ \frac{1}{c} & 1 & ab \end{vmatrix} = \frac{1}{a}\times\frac{1}{b}\times\frac{1}{c}\begin{vmatrix} 1 & a & abc \\ 1 & b & bca \\ 1 & c & abc \end{vmatrix}$$

$$= \frac{1}{abc}\times abc\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$$= 0$$

(since C_1 and C_3 are identical)

26. (*a*)

$$\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \sin\pi & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(\pi - C) & -\tan A & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan A & 0 \end{vmatrix}$$

$$= -\sin B(0 + \cos C \times \tan A) + \cos C(\tan A \times \sin B - 0)$$

$$= 0]$$

27. (*b*)
28. (*a*)

$$f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$$

$$= \cos x\,(4\cos^2 x - 1) - 2\cos x$$

$$= \cos x\left\{2(2\cos^2 x - 1) + 1\right\} - 2\cos x$$

$$= \cos x\left\{2\cos 2x + 1\right\} - 2\cos x$$

$$= 2\cos x\cos 2x + \cos x - 2\cos x$$

$$= \cos(2x + x) + \cos(2x - x) - \cos x$$

$$= \cos 3x$$

29. (*a*) Expand the determinant and use the results $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$, $\omega^4 = \omega$

30. (*a*) Use the operation $C_1 \to C_1 + C_2 + C_3$ and then take $a + b + c - x$ common from C_1

31. (*a*) Use the operation $R_3 \to R_3 + R_1 - 2R_2$.

32. (*d*)

33. (*c*)

Let, $A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$

Then A is Skew-symmetric matrix of order 3 (odd order)

Then $|A| = \begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0.$

Therefore (*c*) is satisfied.

Let $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

Then B is Skew-symmetric matrix of order 2 (even order)

$|B| = 0 + 4 = 4 \neq 0$

Therefore B is non-singular.

So (*a*) is not correct

34. (*d*)

35. (*c*) Rank(A) = 1

$\Rightarrow$ rank (A) < 3(= order of A)

$\Rightarrow$ det(A) = 0]

36. (*c*)

37. (*a*)

38. (*c*)

39. (*a*)

40. (*d*)

41. (*a*)

42. (*a*)

43. (*c*)

44. (*d*)

45. (*b*)

46. (*a*)

47. (*b*)

48. (*d*)

49. (*d*)

A is a lower triangular matrix and so it's eigen values are diagonal elements *i.e*; 1, 3, 4.

Hence the eigen values of A^5 are 1^5, 3^5, 4^5 *i.e*; 1, 243, 1024.

50. (*c*) Sum of eigen values

= trace of the matrix and determinant of the matrix

= product of the eigenvalues

51. (*b*)

Let λ_1 and λ_2 be the other two eigen values.

Then sum of eigen values of A = trace(A)

$\Rightarrow 2 + \lambda_1 + \lambda_2 = 9$

$\Rightarrow \lambda_1 + \lambda_2 = 7$ ...(1)

Again product of the eigen values of A = det(A)

$\Rightarrow 2 \times \lambda_1 \times \lambda_2 = 24$

$\Rightarrow \lambda_1\lambda_2 = 12$ ...(2)

Then solve (1) and (2)

52. (*a*) The matrix $A - \lambda I$ is singular

$\Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow \lambda$ is a characteristic root of A

53. (*a*)

54. (*c*) A and B are non-singular implies both A^{-1} and B^{-1} exist.

Now $A^{-1}B = A^{-1}BI = A^{-1}BA^{-1}A = A^{-1}(BA^{-1})A$. Since the matrices $A^{-1}(BA^{-1})\,A$ and BA^{-1} have the same eigen values, so $A^{-1}B$ and BA^{-1} have the same eigen values.

55. (*a*) A is singular $\Rightarrow \lambda = 0$ is an eigen value of A, which can be obtained by considering the characteristic polynomial $\lambda^2 + 2\lambda$.

56. (*c*)

$\lambda^3 - 6\lambda^2 + 9\lambda - 4$ is the characteristic polynomial

$\Rightarrow\ \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$ is the characteristic equation

$\Rightarrow\ A^3 - 6A^2 + 9A - 4I = O$ (using Cayley Hamilton theorem)

$\Rightarrow\ A^{-1}(A^3 - 6A^2 + 9A - 4I) = A^{-1}O = O$

$\Rightarrow\ A^2 - 6A + 9I - 4A^{-1} = O$

$\Rightarrow\ A^{-1} = \frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}I$

57. (*d*)

Sum of eigen values = trace of the matrix

58. (*b*) The matrix corresponding to the given determinant is upper triangular. Therefore given determinant = products of the all elements lying in the principal diagonal = 6 × 2 × 4 × (–1) = – 48.

59. (*a*)

Let A be the given matrix. Then it can be shown that det(A) = 0 and so one eigen value is zero, which is minimum.

60. (a)

61. (a)

The matrix corresponding to the given determinant is lower triangular .

Therefore given determinant = products of the all elements lying in the principal diagonal

$= 1 \times 2 \times 3 \times 4 \times 1 = 24.$

62. (c)

$x = 1$, $y = 1/2$, $z = 2$ satisfy all the three equations.

63. (b)

64. (b)

65. (b)

$\Rightarrow \Delta = 0$

$$\Rightarrow \begin{vmatrix} -1 & 2 & -3 \\ 1 & 6 & 12 \\ 2 & -4 & 3k \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 2 & -3 \\ 0 & 8 & 9 \\ 0 & 0 & 3k-6 \end{vmatrix} = 0$$

(by $R_1 \to R_1 + R_2, R_3 \to R_3 + 2R_1$)

$\Rightarrow (3k-6)(-8-2) = 0$

$\Rightarrow k = 2$

66. (a) System of equations has infinite number of solution $\Rightarrow$ coefficient determinant = 0.

67. (a)

$|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 & 1 \\ 0 & -1-\lambda & 0 & -1 \\ 0 & 0 & i-\lambda & i \\ 0 & 0 & 0 & -i-\lambda \end{bmatrix} = 0$$

$\Rightarrow (1-\lambda)(-1-\lambda)(i-\lambda)(-i-\lambda) = 0$

$\Rightarrow (\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i) = 0$

$\Rightarrow (\lambda^2 - 1^2)(\lambda^2 - i^2) = 0$

$\Rightarrow (\lambda^2 - 1)(\lambda^2 + 1) = 0$

$\Rightarrow \lambda^4 - 1 = 0$

$\Rightarrow \lambda^4 = 1$

Now use Cayley Hamilton's theorem.

68. (c) Sum of the eigen values = trace of the matrix = 8]

69. (a) System has infinitely many solutions $\Rightarrow$ co-efficient determinant = 0.

70. (d) Using the matrix A, find adj(A) and then compare it with given adj(A).

71. (c)

Zero is the only eigen value of a nilpotent matrix.]

72. (d), 73. (a), 74. (b), 75. (a)

76. (a)

If λ be an eigen value of A, then the eigen value of $-2A + 3I$ has the form $-2\lambda + 3$.

77. (c)

$$\sum_{i=1}^{n} c_i a_i = 0$$

$\Rightarrow$ the columns $a_1, a_2, \ldots a_n$ are not linearly independent

$\Rightarrow$ rank(A) < n

$\Rightarrow$ the system has infinite number of solutions