Chapter

1 Matrices and Its Applications

1.1 CONCEPTS OF MATRICES

Now a days, the matrices is one of the most important and powerful tool in mathematics which has wide application in many areas like, economics, engineering statistics etc, to present data and set up problem in the form of reactangular array.

Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Here x, y and z are unknown and their coefficient are all numbers. Arranging the coefficients in the order in which they occur in the equations and enclosing them in square brackets, we obtain a rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This rectangular array is an example of a matrix. The horizontal lines (→) are called **rows or row vectors,** and the vertical lines (↓) are called **columns or column vectors** of the matrix. In this array there are 3 rows and 3 columns. Therefore it is a matrix of type 3 × 3. The numbers a_{11}, a_{12}......a_{32}, a_{33} etc, constituting the matrix are called its elements.

Sometimes, we also use the brackets ( ) or the double bars | | in place of the square brackets [ ] to denote matrices.

Thus, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} 2+3i & 5 \\ -3 & 1-3i \end{pmatrix}$,

$C = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ are all matrices of type 2 × 2.

1.2 DEFINITION OF MATRIX

A matrix is a rectangular arrangement of a set of mn elements (real or complex) in m rows and n column.

An m × n matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\text{.............}a_{1n} \\ a_{21} & a_{22} & a_{23}\text{...........}a_{2n} \\ \text{........} & \text{...........} & \text{..................} \\ a_{m1} & a_{m2} & a_{m3}\text{.............}a_{mn} \end{bmatrix} = [a_{ij}]$$

where i = 1, 2, 3 .....m; j = 1, 2, 3,....n. Matrices as a whole are denoted by capital letters while corresponding small letters with suffices represent its elements. The elements a_{11}, $a_{12,}a_{13}$.....a_{mn} may be real or complex. Here we have a double suffix notation in which the first suffix indicates the row and the second suffix indicates column in which the element is located. Hence a_{ij} is an element of i^{th} row and j^{th} column.

1.3 TYPES OF MATRICES

1.3.1 Row Matrix

If a matrix has only one row and any number of columns, it is called a row matrix. For example, let

A = [1 2 4 9] is a row matrix of order 1 × 4.

1.3.2 Column Matrix

If a matrix has only one column and any number of rows, it is called a column matrix. For example, let

$B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}$ is a column matrix of order 4 × 1.

1.3.3 Null Matrix or Zero Matrix

A matrix whose elements are all zero is called the null matrix or zero matrix. It is usually denoted by O. For example,

$$O_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } O_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are zero matrix of the types 3 × 4 and 3 × 3 restpectively.

Operating, $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & -6 \end{bmatrix}$$

Operating, $R_1 \to R_1 + 3R_2$; $R_3 \to R_3 + 2R_2$.

$$\sim \begin{bmatrix} -1 & 0 & -8 & 11 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here third order minors are zero while the second order minor

$$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \neq 0$$

So the rank of the matrix is 2.

Example 1.13. Find the rank of the following matrix by E-tranformations

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}.$$

Solution.

Here $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ is a 3×3 matrix

$\therefore \quad \rho(A) \leq 3$

Operating $R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Operating, $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here third order minor vanishes while the second order minor

$$\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -2 - 6 = -8 \neq 0.$$

So the rank of the matrix is 2.

Example 1.14. Find the rank of the matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Solution.

Here $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ is a 4×4 matrix

$\therefore \quad \rho(A) \leq 4$

Operating $R_4 \to R_4 - (R_1 + R_2 + R_3)$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence only fourth order minor is zero, but the third order minor

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & -3 \end{vmatrix} = 1(-4)\ (-3) = 12 \neq 0$$

So the rank of the matrix is 3.

Example 1.15. Reduce each of the matrices to normal form and hence, find their ranks.

(i) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Solution.

(i) Let $\quad A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

Operating $C_4 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 17 & 0 \end{bmatrix}$$

Operating $R_3 \to \frac{1}{16} R_3;\ R_4 \to \frac{1}{17} R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Operating $R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{bmatrix},$$

which is the required normal form. Hence Rank of A is 3.

1.12 LINEAR TRANSFORMATIONS

Let P(x, y) be a point in two dimensions with OX and OY axes. Keeping the origin O as fixed and rotate these axes and angle θ such that the new axes become OX' and OY' and the point P(x, y) shifted as P′(x', y′) respectively. Then

OA = x and AP = y

OD = x' and DP = y′

Now,

OA = OB – AB

∴ In Δ OBD, $\frac{OB}{OD} = \cos\theta$

Hence $x = x'\cos\theta - y'\sin\theta$ ...(i)

$$\frac{OB}{x'} = \cos\theta$$

∴ OB = $x'\cos\theta$

Similarly, AP = AC + PC or AP = DB + PC

In Δ DCP

$$\frac{CD}{DP} = \sin\theta$$

$$\frac{CD}{y'} = \frac{AB}{y'} = \sin\theta$$

AB = $y'\sin\theta$

In Δ DBO and In Δ PCD

$$\frac{OB}{OD} = \sin\theta \quad \frac{PC}{DP} = \cos\theta$$

$$\frac{DB}{x'} = \sin\theta \quad \frac{PC}{y'} = \cos\theta$$

DB = $x'\sin\theta$ PC = $y'\cos\theta$

Hence $y = x'\sin\theta + y'\cos\theta$ ...(ii)

From equation (i) and (ii), we get

$$Y = \begin{bmatrix} x \\ y \end{bmatrix},\ A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},\ X = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Hence a linear relation of x', y′ in terms of x, y is given by the equation (i) and (ii) and represented by

Y = AX

or, if $a_1 = \cos x$, $b_1 = -\sin x$

$a_2 = \sin x$, $b_2 = \cos x$, then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

Similarly, the relation of the form

$$x = a_1x' + b_1y' + c_1z'$$
$$y = a_2x' + b_2y' + c_2z'$$
$$z = a_3x' + b_3y' + c_3z'$$

In matrix notation form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix},$$

i.e., $$Y = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & a_3 & a_3 \end{bmatrix}\ X = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

or Y = AX, a linear transformation from (x, y, z) → (x', y′, z′) in three dimensions.

In general, the relation

Y = AX

where $$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix},\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$ define a linear transformation which carries any vector X into another vector Y over the matrix A which is called the linear operator of the transformation.

If the transformation matrix A is non-singular, i.e., if $|A| \neq 0$, Then the linear transformation is called non-singular or regular.

If the transformation matrix A is singular, i.e, if $|A| = 0$, then the linear Transformation is also called singular.

Note :

1. If A is non-singular, A^{-1} exists and for a non-singular transformation y = AX, we can write the inverse transformation, i.e., $X = A^{-1}y$.
2. If Y = AX represent a transformation from $(x_1, x_2 .... xn)$ to $(y_1, y_2 ....yn)$and the another transformation from $(y_1, y_2,...y_n)$ to $(z_1, z_2,...z_n)$ is given by z = by, then the transformation from $(x_1, x_2, x_3...xn)$ to $(z_1, z_2, z_3...z_n)$ is given by

$$Z = BY = B\,(AX)$$

or $$Z = (BA)X.$$

1.13 ORTHOGONAL TRANSFORMATIONS

A linear transformation Y = AX, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},\ A = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to be orthogonal if it transforms $y_1^2 + y_2^2 + y_3^2 + ...+y_n^2$ into $x_1^2+ x_2^2 + ...x_n^2$.

Here, the matrix A of this transformation is called an orthogonal matrix.

Now, we know that

$$X'\,X = [x_1, x_2, x_3....x_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + x_3^2+....+x_n^2$$

$$\text{Similarly, } Y'Y = [y_1, y_2, y_3....y_n] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$$= y_1^2 + y_2^2 + y_3^2+....+y_n^2$$

If Y = AX is an orthogonal transformation, then

$X'X = Y'Y = x_1^2 + x_2^2 + x_3^2+....x_n^2 = y_1^2 + y_2^2 + y_3^2+.... y_n^2 = (AX)'\,(AX)$ [$Y = AX \Rightarrow Y' = (AX)'$]

$= (X'A')\,AX$ [$(AB)' = B'A'$]

$= X'(A'A)\,X$ [$A^{|}A = I$, or $A^{-1}\,A = I$].

$= X^{|}X$

which holds only when

$A'A = I$ or when $A'\,A = A^{-1}A$

or when $A' = A^{-1}$

Hence a square matrix A is said to be orthogonal if $A'A = AA^{|} = I$.

Theorem.1 : If A is an orthogonal matrix, prove that

$|A| = \pm\,1$.

Proof : Since A is an orthogonal matrix

$\therefore\quad A'A = I$

or $|A'A| = |I|$

or $|A'|\,|A| = 1 \qquad |A'| = |A|$

or $|A|\,|A| = 1 \qquad |A| = \pm\,1$.

$|A|^2 = 1$

Theorem. 2 : Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

Proof : Since A is orthogonal, then

$AA' = I$

or $(AA')^{-1} = I^{-1}$

$\Rightarrow\ (A')^{-1}\,A^{-1} = I$ [$(AB)^{-1} = B^{-1}\,A^{-1}$]

$\Rightarrow\ (A^{-1})'\,A^{-1} = I$ [$(A'))^{-1} = (A^{-1})'$]

$\Rightarrow$ Product of A^{-1} and its transpose $(A^{-1})'$ is equal to 1

$\Rightarrow A^{-1}$ is also orthogonal.

Again, A is orgthogonal, then

$A\,A' = I$

or $(AA')' = I^{|}$

$\Rightarrow (A')'\,A' = I$, which implies that the product of A' and its transpose $(A')'$ is equal to I.

Hence A' is also orthogonal.

MULTIPLE CHOICE QUESTIONS

SET -I

1. What is the value of k, if $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ satisfies the matrix equation $A^2 - kA + 2I = 0$.
   (a) 0 (b) 1
   (c) 2 (d) 3

2. Under which one of the following condition does the system of equations $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 1 & 2 & a-4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ a \end{bmatrix}$ have a unique solution?
   (a) For all $a \in R$ (b) $a = 8$
   (c) For all $a \in Z$ (d) $a \neq 8$

3. What is the determinant of the following matrix?
$$\begin{vmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 1 & 1/2 & 0 & \dots & \dots & 0 \\ 1 & 1/2 & 1/3 & \dots & \dots & 0 \\ 1 & 1/2 & 1/3 & 1/4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1/2 & 1/3 & 1/4 & \dots & 1/n \end{vmatrix}$$
   (a) $n^2 + n + 1$ (b) $\frac{1}{n^2} + \frac{1}{n} + 1$
   (c) $\frac{1}{n!}$ (d) $\frac{n(n+1)}{2}$

4. If the system of equations $x - 2y - 3z = 1$, $(P + 2) = 3$, $(2P + 1)y + z = 2$ is inconsistent, then what is the value of P?
   (a) –2 (b) –1/2
   (c) 0 (d) 2

5. If x, y, z are in AP with common difference d and the rank of the matrix $\begin{bmatrix} 4 & 5 & x \\ 5 & 6 & y \\ 6 & k & z \end{bmatrix}$ is 2, then the value of d and k are
   (a) $d = x/2$; k is an arbitrary number
   (b) d an arbitrary number; $k = 7$
   (c) $d = k$; $k = 5$
   (d) $d = x/2$; $k = 6$

6. If C is a non-singular matrix and $B = C \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} C^{-1}$, then
   (a) $B^2 = 1$ (b) $B^2 = 0$
   (c) $B^3 = 1$ (d) $B^3 = 0$

7. If $X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the rank of X^TX, where X^T denotes the transpose of X, is
   (a) 0 (b) 2
   (c) 3 (d) 4

8. Consider the following matrix $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$. If the eigen values of A are 4 and 8, then
   (a) $x = 4, y = 0$ (b) $x = 5, y = 8$
   (c) $x = -3, y = 9$ (d) $x = -4, y = 10$

9. The eigen values of the matrix $\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ are
   (a) $3, 3 + 5i, 6 - i$ (b) $-6 + 5i, 3 + i, 3 - i$
   (c) $3 + i, 3 - i, 5 + i$ (d) $3, -1 + 3i, -1 - 3i$

10. The characteristic equations of a (3 × 3) matrix A is defined as $a(\lambda) = |\lambda| - Al = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$. If l denotes identity matrix, then the inverse of matrix A will be
   (a) $A^2 + A + 2l$ (b) $A^2 + A + l$
   (c) $-(A^2 + A + l)$ (d) $-(A^2 + A + 2l)$

11. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$, then calculate A^9.
   (a) $511A + 510I$ (b) $309A + 104I$
   (c) $154A + 155I$ (d) exp $(9A)$

12. The characteristic polynomial of 3 × 3 matrix A is $|\lambda l - A| = \lambda^3 + 3\lambda^2 + 4\lambda - 3$

   Let x = trace (A) and $y = |A|$, the determinant of A. Then,
   (a) $\frac{x}{y} = \frac{3}{4}$ (b) $\frac{x}{y} = \frac{4}{3}$
   (c) $x = y = -3$ (d) $x = 3$ and $y = -3$

13. The characteristic vector of the matrix $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ corresponding to characteristic root 1 is

(*a*) (–1, 1) (*b*) (3, 1)

(*c*) (–3, 1) (*d*) (1, –3)

14. A system of linear equations $x + 2y - z = 11$, $3x + y - 2z = 10$, $x - 3y = 5$ has

(*a*) no solution

(*b*) exactly one solution

(*c*) exactly 3 solutions

(*d*) infinitely many solution

15. The minimal polynomial of the 3 × 3 real matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$ is

(*a*) $(x - a)(x - b)$ (*b*) $(x - a)^2 (x - b)$

(*c*) $(x - a)^2 (x - A)^2$ (*d*) $(x - a)(x - b)$

16. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then the eigen values of A are

(*a*) 2, 1 and 0 (*b*) 2, $(1 + i)$ and $(1 - i)$

(*c*) 2, –1 and –1 (*d*) 1, –1 and 0

17. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be such that A has real eigen values, then

(*a*) $\theta = n\pi$ for some integer n

(*b*) $\theta = 2n\pi + \frac{\pi}{2}$ for some integer n

(*c*) there is no restriction on θ

(*d*) $\theta = 2n\pi + \frac{\pi}{4}$ for some integer n

18. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij} = 3$, $\forall\, i$ and j. Then the nullity of A is

(*a*) $n - 1$ (*b*) $n - 3$

(*c*) n (*d*) 0

19. The system of simultaneous linear equations has

$x + y + z = 0$

$x - y - z = 0$ has

(*a*) no solution in $\mathbb{R}^3$

(*b*) a unique solution in $\mathbb{R}^3$

(*c*) infinitely many solution in $\mathbb{R}^3$

(*d*) more than 2 but finitely many solutions in $\mathbb{R}^3$

20. Minimal polynomial $m(x)$ of $A_{n\times n}$ each of whose element is 1, is

(*a*) $x - 1$ (*b*) $x^2 + x$

(*c*) $x^2 + nx$ (*d*) $x^2 - nx$

21. The minimum polynomial $m(\lambda)$ of M

$$= \begin{bmatrix} 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

(*a*) $\lambda^2(\lambda - 2)^2(\lambda - 5)$ (*b*) $\lambda(\lambda - 2)^2(\lambda - 5)$

(*c*) $\lambda(1 - 2)(\lambda - 5)$ (*d*) $\lambda^2(\lambda - 2)(\lambda - 5)$

22. The eigen values of a 3 × 3 real matrix A are 1, 2 and – 3. Then,

(*a*) $A^{-1} = -\frac{1}{6}A^2$ (*b*) $A^{-1} = \frac{1}{6}(7I - A^2)$

(*c*) $A^{-1} = -\frac{1}{6}(7I - A^2)$ (*d*) $A^{-1} = \frac{1}{6}(-7I - A^2)$

23. Then, the minimal polynomial of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is

(*a*) $(x - 1)^2(x - 2)$ (*b*) $(x - 1)(x - 2)^2$

(*c*) $(x - 1)(x - 2)$ (*d*) $(x - 1)^2(x - 2)^2$

24. Let A be 3 × 3 matrix whose characteristic roots are 3, 2, –1.

If $B = A^2 - A$, then $|B|$ is

(*a*) 24 (*b*) –2

(*c*) 12 (*d*) –12

25. Let $M_1 = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$M_2 = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix}$

$M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Then, the minimal polynomial of the matrix $\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}$ is

(*a*) $x^2(x-3)^3(x+3)^2$ (*b*) $x^2(x^2-9)$
(*c*) $x^2(x^2-9)^2$ (*d*) $x(x-3)^4(x+3)^2$

26. Let A be the matrix of quadratic form $(x_1 - x_2 + 2x_3)^2$. Then, trace of A is

(*a*) 2 (*b*) 4
(*c*) 6 (*d*) 0

27. The eigen values of a 3 × 3 real matrix P are 1, –2, 3. Then,

(*a*) $P^{-1} = \frac{1}{6}(5I + 2P - P^2)$

(*b*) $P^{-1} = \frac{1}{6}(5I - 2P + P^2)$

(*c*) $P^{-1} = \frac{1}{6}(5I - 2P - P^2)$

(*d*) $P^{-1} = \frac{1}{6}(5I + 2P + P^2)$

28. The system of equations

$x + y + z = 0$
$3x + 6y + z = 0$
$\alpha x + 2y + z = 0$

has infinitely many solutions, then α is equal to

(*a*) 7 (*b*) $\frac{7}{5}$
(*c*) $\frac{5}{7}$ (*d*) 4

29. Consider the system of linear equation

$x + y + z = 3,\ x - y - z = 4,\ x - 5y + kz = 6$

Then, the value of k for which this system has an infinite number of solution is

(*a*) $k = -5$ (*b*) $k = 0$
(*c*) $k =$ (*d*) $k = 3$

30. Real matrices [A] 3 × 1, [B] 3 × 3, [C] 3 × 5, [D] 5 × 3, [E] 5 × 5 and [F] 5 × 1 are given. Matrices [B] and [E] are symmetric. Following statements are made with respect to these matrices:

(*i*) Matrix product $[F]^T [C]^T [B][C][F]$ is a scalar.
(*ii*) Matrix product $[D]^T [F] [D]$ is always symmetric. With reference to above statements, which of the following applies?

(*a*) Statement (*i*) is true but (*ii*) is false.
(*b*) Statement (*i*) is false but (*ii*) is true,
(*c*) Both the statements are true.
(*d*) Both the statements are false.

31. For any two matrices A and B, which is true?

(*a*) $AB = 0 \Rightarrow A = 0$ or $B = 0$
(*b*) $(A + B)^2 = A^2 + 2AB + B^2$
(*c*) $(AB)' = B'A'$
(*d*) $|AB| \neq |A||B|$

32. Matrix $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ is

(*a*) idempotent (*b*) nilpotent
(*c*) involutory (*d*) periodic

33. Matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ is

(*a*) unitary (*b*) orthogonal
(*c*) nilpotent (*d*) None of these

34. Match List I with List II and select the correct answer using the codes given below in the lists.

List I

A. Singular matrix
B. Non-singular matrix
C. Real symmetric
D. Orthogonal matrix

List II

1. Determinant is not defined
2. Determinant is always one
3. Determinant is zero
4. Eigen values are always real
5. Eigen values are not defined.

Codes

	A	B	C	D
(*a*)	3	1	4	2
(*b*)	2	3	4	1
(*c*)	3	2	5	4
(*d*)	3	4	2	1

35. Given an orthogonal matrix

$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, then $(AA')^{-1}$ is

(a) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(b) $\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$

36. For a matrix $[M] = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix $[M]' = [M]^{-1}$. The value of x is

(a) $-\frac{4}{5}$ (b) $-\frac{3}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$.

37. The diagonal elements of Skew-Hermitian matrix are

(a) pure real numbers or zero

(b) pure imaginary or zero

(c) complex numbers

(d) None of the above

38. The matrix $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is a

(a) Hermitain matrix

(b) skew-Hermitian matrix

(c) symmetric matrix

(d) skew-symmetric matrix

39. The matrix is a $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is a

(a) Hermitian matrix

(b) skew-Hermitian

(c) skew-symmetric

(d) symmetric

40. If A is a non-zero column vector $(n \times 1)$, then the rank of matrix AA' is

(a) zero (b) 1

(c) –1 (d) n

41. *If* P and Q are non-singular matrices, then for matrix M, which of the following is correct?

(a) Rank $(PMQ) >$ Rank M

(b) Rank $(PMQ) =$ Rank M

(c) Rank $(PMQ) <$ Rank M

(d) Rank $(PMQ) =$ Rank M + Rank (PQ)

42. Rank of singular matrix of order 4 can be at most

(a) 1 (b) 2

(c) 3 (d) 4

43. If the rank of an $n \times n$ matrix A is $(n - 1)$, then the system of equations $Ax = b$ has

(a) $(n - 1)$ parameter family of solutions

(b) one parameter family of solutions

(c) no solution

(d) a unique solution

44. Let M be a $m \times n$ $(m < n)$ matrix with rank m, then

(a) for every b in R^m, $Mx = b$ has unique solution

(b) for every b in R^m, $Mx = b$ has a solution but it is not unique

(c) there exists $b \in R^m$ for which $Mx = b$ has no solution

(d) None of the above

45. Let $M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, then the rank of M is equal to

(a) 3 (b) 4

(c) 2 (d) 1

46. Let A be a matrix of order $m \times n$ and R is non-singular matrix of order n, then

(a) rank $(RA) \neq$ rank (A)

(b) rank $(RA) \geq$ rank (A)

(c) rank $(RA) \leq$ rank (A)

(d) rank $(RA) =$ rank (A)

47. Let A be a square matrix of order n, then nullity of A is

(a) n – rank A (4) rank $A - n$

(c) n + rank A (d) None of these

48. A is a 3×4 real matrix and $AX = b$ is an inconsistent system of equations. The highest possible; rank of A is

(a) 1 (b) 2

(c) 3 (d) 4

49. The system of linear equations

$4x + 2y = 7,\ 2x + y = 6$ has

(*a*) a unique solution

(*b*) no solution

(*c*) an infinite number of solutions

(*d*) exactly two distinct solutions

50. If $x + 2y - 2u = 0,\ 2x - y - u = 0,\ x + 2z - u = 0,\ 4x - y + 3z - u = 0$ is a system of equations, then it is

(*a*) consistent with trivial solution

(*b*) consistent without trivial solution

(*c*) inconsistent with trivial solution

(*d*) inconsistent without trivial solution

51. If $3x + 2y + z = 0,\ x + 4y + z = 0,\ 2x + y + 4z = 0$ be a system of equations, then

(*a*) it is inconsistent

(*b*) it has only trivial solution

(*c*) it can be reduced to a single equation and so a solution does not exist

(*d*) the determinant of the matrix of coefficient is zero

52. $x + 2y + 2z = 1,\ 2x + y + z = -2,\ 3x + 2y + 2z = 3,\ y + z = 0$ is a system of equations, then

(*a*) no solution exists

(*b*) a unique solution exists

(*c*) an infinite number of solutions exist

(*d*) None of the above

53. How many of the following matrices have an eigen value 1?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(*a*) One (*c*) Two

(*c*) Three (*d*) Four

54. The correct set of eigen values of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ is

(*a*) –1, 2, 3 (*b*) ,1 3, –1

(*c*) –1, 2, +2 (*d*) 1, 2, 2

55. The eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are

(*a*) 8, 3, –7 (*b*) 3, 0, 15

(*c*) 3, 0, –7 (*d*) 5, 7, 8

56. If $A = \begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix}$ has the eigen values 3 and 9, then the eigen values of A^3 are

(*a*) 3,80 (*b*) 9,27

(*c*) 27,729 (*d*) 9,81

57. One of the eigen values of matrix $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ is 5, then the corresponding eigen vector is

(*a*) [1 0 2]′ (*b*) [3 0 0]′

(*c*) [4 3 2]′ (*d*) [3 –1 2]′

58. The model matrix P of $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is

(*a*) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ -5 & -3 & 2 \end{bmatrix}$ (*b*) $\begin{bmatrix} 4 & -8 & 5 \\ -8 & 7 & 0 \\ 5 & 0 & -6 \end{bmatrix}$

(*c*) $\begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 7 & 6 & -5 \\ 6 & 4 & -3 \\ -5 & -3 & 1 \end{bmatrix}$

59. The number of different $n \times n$ symmetric matrices with each element being either 0 or 1, is

(Note: Power (2, x) is same as 2^x)

(*a*) Power (2, n) (*b*) Power (2, n^2)

(*c*) Power $\left(2, \dfrac{n^2 + n}{2}\right)$ (*d*) Power $\left(2, \dfrac{n^2 - n}{2}\right)$

60. Consider the following statements:

S_1 : The sum of two singular $n \times n$ matrices may be non-singular.

S_2 : The sum of two $n \times n$ non-singular matrices may be singular.

Which of the following statements is true?

(*a*) S_1 and S_2 both are true

(*b*) S_1 is true and S_2 is false

(*c*) S_1 is false, S_2 is true

(*d*) S_1 and S_2 both are false

61. Let $Ax = b$ be a system of linear equations where A is an $m \times n$ matrix and 6 is an $m \times 1$ column vector and X is an $n \times 1$ column vector of unknown. Which of the following is false?

(*a*) The system has a solution if and only if, both A and the augmented matrix $[Ab]$ have the same rank.
(*b*) If $m < n$ and b is zero vector, then the system has infinitely many solutions.
(*c*) If $m = n$ and b is non-zero vector, them the system has a unique solution
(*d*) The system will have only a trivial solution when $m = n$, b is the zero vector and rank $(A) = n$

62. If A and B are two square matrices of same order
(*a*) $|AB| = |BA|$ (*b*) $|AB| \neq |B||A|$
(*c*) $|AB| \neq |BA|$ (*d*) None of these

63. The eigen values of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are 5 and –1. Then, the eigen values of $-2A + 3l$(l is a 2 × 2 identity matrix) are
(*a*) –7 and 5 (*b*) 7 and –5
(*c*) $-\frac{7}{5}$ and $\frac{1}{5}$ (*d*) $\frac{1}{7}$ and $-\frac{1}{2}$

64. The rank of the following $(n + 1) \times (n + 1)$ matrix where a is real number $\begin{bmatrix} 1 & a & a^2 & \ldots & a^n \\ 1 & a & a^2 & \ldots & a^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a & a^2 & \ldots & a^n \end{bmatrix}$ is
(*a*) 1 (*b*) 2
(*c*) n (*d*) Depends on 'a'

65. In a compact single dimensional array representation for lower triangular matrices of size n × n, non-zero elements of each row are stored one after another starting from the first row, the index of the (i, j) the element of the lower triangular matrix in this new representation is
(*a*) $i + j$ (*b*) $i + j - 1$
(*c*) $j + \frac{i(i-1)}{2}$ (*d*) $i + \frac{j(j-1)}{2}$

66. $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$, A is orthogonal, then
(*a*) $|A| = \pm 1$ (*b*) $|A| = 0$
(*c*) $|A^T| \neq |\text{A}|$ (*d*) None of these

67. The necessary condition to diagonalise a matrix is that
(*a*) its eigen value should be distinct
(*b*) its eigen vectors should be independent
(*c*) its eigen value should be real
(*d*) the matrix is non-singular

68. Let A be an $n \times n$ real matrix such that $A^2 = I$ and y be an n-dimensional vector. Then, the linear system of equations $Ax = y$ has
(*a*) no solution
(*b*) a unique solution
(*c*) more than one but finitely many independent solutions
(*d*) infinitely many independent solutions

69. Consider the following system of linear equations:

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and third columns of the coefficient matrix are linearly dependent. For how many values of a does this system of equations have infinite many solutions?
(*a*) 0 (*b*) 1
(*c*) 2 (*d*) Infinitely many

70. A square matrix is singular whenever
(*a*) the rows are linearly independent
(*b*) the columns are linearly independent
(*c*) the rows are linearly dependent
(*d*) None of the above

71. Let A and B are two matrices, then
(*a*) rank (AB) = rank $(B^r A^T)$
(*b*) rank (AB) = rank $(A^T B^T)$
(*c*) rank $(AB) \neq$ rank $(AB)^T$
(*d*) None of the above

72. Let rank $(AB) = 5$, then
(*a*) rank $A \leq 5$ and rank $B \geq 5$
(*b*) rank $A \leq 5$ and rank $B \leq 5$
(*c*) rank $A \geq 5$ and rank $B \geq 5$
(*d*) None of the above

73. Consider the matrix $P = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. The value of e^P is

(*a*) $\begin{bmatrix} 2e^{-2} - 3e^{-1} & e^{-1} - e^{-2} \\ 2e^{-2} - 2e^{-1} & 5e^{-2} - e^{-1} \end{bmatrix}$

(*b*) $\begin{bmatrix} e^{-1} - e^{-2} & 2e^{-2} - e^{-1} \\ 2e^{-1} - 4e^{2} & 3e^{-1} + 2e^{-2} \end{bmatrix}$

(*c*) $\begin{bmatrix} 5e^{-2}-e^{-1} & 3e^{-1}-e^{-2} \\ 2e^{-2}-6e^{-1} & 4e^{-2}+e^{-1} \end{bmatrix}$

(*d*) $\begin{bmatrix} 2e^{-1}-e^{-2} & e^{-1}-e^{-2} \\ -2e^{-1}-2e^{-2} & -e^{-1}+2e^{-2} \end{bmatrix}$

74. If A is orthogonal, then
(*a*) A^T and A^{-1} are both orthogonal
(*b*) A^T is orthogonal, but A^{-1} is not
(*c*) A^{-1} is orthogonal, but A^T is not
(*d*) None of the above

75. If A is a symmetric matrix, then
(*a*) adj (A) is a non-symmetric matrix
(*b*) adj (A) is a symmetric matrix
(*c*) adj (A) does not exist
(*d*) None of the above

76. In an $M \times N$ matrix such that all non-zero entries are covered in a row and b column. Then the maximum number of non-zero entries, such that no two are on the same row or column, is
(*a*) $\le a + b$ (*b*) $\le \max(a, b)$
(*c*) $\le \min\{M - a, N - b\}$(*d*) $\le \min\{a, b\}$

77. The system of equation $kx + y + z = 1$, $x + y + z = k$ and $x + y + kz - k^3$ does not have a solution, if k is equal to
(*a*) 0 (*b*) 1
(*c*) –1 (*d*) –2

78. The system of equations

$4x_1 + x_2 - 3x_3 - x_4 = 0$

$2x_1 + 3x_2 + x_3 - 5x_4 = 0$

$x_1 - 2x_2 - 2x_3 + 3x_4 = 0$

(*a*) no solutions
(*b*) only one solution (0, 0, 0, 0)
(*c*) infinite number of solutions
(*d*) only two solution (0,0,0,0) and (3/5, 1, 4/5, 1)

79. The system of equations

$x - y + 3z = 4$

$x + z = 2$

$x + y - z = 0$

(*a*) a unique solutions
(*b*) finitely many solutions
(*c*) infinitely many solutions
(*d*) no solution

80. Consider the equation $AX = B$, where $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then
(*a*) the equation has no solution
(*b*) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a solution of the equation
(*c*) there exists a non-zero unique solution
(*d*) the equation has infinitely many solutions

81. The system of equation

$2x + y = 5$

$x - 3y = -1$

$3x + 4y = k$ *is* consistent, when k is
(*a*) 1 (*b*) 2
(*c*) 5 (*d*) 10

82. The value of a for which the system of equations

$x + y + z = 0$

$y + 2z = 0$

$ax + z = 0$ has more than one solution is
(*a*) –1 (*b*) 0
(*c*) 1/2 (*d*) 1

83. The system of the equations

$x + 2y + z = 9$

$2x + y + 3z = 7$

can be expressed as

(*a*) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(*b*) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

(*d*) None of these

84. Let A be an $n \times$ n matrix from the set of numbers and $A^3 - 3A^2 + 4A - 6I = 0$, where I is $n \times n$ unit matrix. If A^{-1} exist, then
(*a*) $A^{-1} = A - I$ (*b*) $A^{-1} = A + 6I$
(*c*) $A^{-1} = A - 6I$ (*d*) $A^{-1} = \frac{1}{6}(A^2 - 3A + 4I)$

85. If $3x + 2y + z = 0$

$x + 4y + z = 0$

$2x + y + 4z = 0$

be a system of equations, then

(*a*) It is inconsistent

(*b*) it has only the trivial solution $x = 0, y = 0, z = 0$

(*c*) It can be reduced to a single equation and so a solution does not exist

(*d*) the determinant of the matrix of coefficient is zero

86. Let $M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Then, the rank of M is equal to

(*a*) 3 (*b*) 4

(*c*) 2 (*d*) 1

87. All the eigen value of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ lie in the disc

(*a*) $|\lambda + 1| \le 1$ (*b*) $|\lambda - 1| \le 1$

(*c*) $|\lambda + 1| \le 0$ (*d*) $|\lambda - 1| \le 2$

88. The determinant of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is

(*a*) 0 (*b*) –9

(*c*) –27 (*d*) 1

89. Let ω be a complex number such that $\omega^3 = 1$ but $\omega \ne 1$.

If $A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{bmatrix}$, then which of the following statement are ture?

(*a*) A is invertible

(*b*) rank $(A) = 2$

(*c*) 0 is an eigen value *of A*

(*d*) There exists linearly independent vectors $\nu, \omega \in C^3$ such that $A\nu = A\omega = 0$

90. Let A be a 4×4 matrix with real entries such that –1, 1, 2, –2 are its eigen values. If $B = A^4 - 5A^2 + 5I$, where I denotes the 4×4 identity matrix, then which of the following statements are correct?

(*a*) det $(A + B) = 0$ (*b*) det $(S) = 1$

(*c*) trace of $A - B$ is 0 (*d*) trace of $A + B$ is 4

91. The characteristic polynomial of the 3×3 real matrix

$$A = \begin{bmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{bmatrix} \text{ is}$$

(*a*) $\lambda^3 + a\lambda^2 + b\lambda + c$

(*b*) $(\lambda - a)(\lambda - b)(\lambda - c)$

(*c*) $(\lambda - 1)(\lambda - abc)^2$

(*d*) $(\lambda - 1)^2 (\lambda - abc)$

92. If the characteristic roots $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ are λ_1 and λ_2, the characteristic roots of $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ are

(*a*) $\lambda_1 + \lambda_2, \lambda_1 - \lambda_2$ (*b*) $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$

(*c*) $2\lambda_1$ and $2\lambda_2$ (*d*) $\lambda_1 + \lambda_2$ and $|\lambda_1 - \lambda_2|$

93. If $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Which of the following is the zero matrix?

(*a*) $A^2 - A - 5I$ (*b*) $A^2 + A - 5I$

(*c*) $A^2 + A - I$ (*d*) $A^2 - 3A + 5I$

94. Let the characteristic equation of a matrix M be $\lambda^2 - \lambda - 1 = 0$, then

(*a*) M^{-1} does not exist

(*b*) M^{-1} exists but cannot be determined from the data

(*c*) $M^{-1} = M - 1$

(*d*) $M^{-1} = M + 1$

95. If M is a 7×5 matrix of rank 3 and N is a 5×7 matrix of rank 5, then rank (MN) is

(*a*) 5 (*b*) 3

(*c*) 2 (*d*) 1

96. Consider the system of linear equations

$x + y + z = 3$, $x - y - z = 4$, $x - 5y + kz = 6$

Then, the value of k for which this system has an infinite number of solutions is

(*a*) $k = -5$ (*b*) $k = 0$

(*c*) $k = 1$ (*d*) $k = 3$

97. For which value of λ will the matrix given below become singular? $\begin{bmatrix} 8 & \lambda & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$

(*a*) 4 (*b*) 6

(*c*) 8 (*d*) 12

98. Given matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is

(*a*) 4 (*b*) 3

(*c*) 2 (*d*) 1

99. Find the value of k for which the following simultaneous equations

$x + y + z = 3; x + 2y + 3z = 4; x + 4y + kz = 6$

will not have a unique solution.

(*a*) 0 (*b*) 5

(*c*) 6 (*d*) 7

100. Solution for the system defined by the set of equations $4y + 3z = 8$; $2x - z = 2$ and $3x + 2y = 5$ is

(*a*) $x = 0, y = 1; z = 4/3$

(*b*) $x = 0; y = 1/2; z = 2$

(*c*) $x = 1; y = 1/2; z = 2$

(*d*) not exist

101. Consider the system of simultaneous equations

$x + 2y + z = 6; 2x + y + 2z = 6; x + y + z = 5$

This system has

(*a*) unique solution

(*b*) infinite number of solutions

(*c*) no solution

(*d*) exactly two solutions

102. $A_{2\times2}$ matrix which satisfy $A_2 - A = 0$, then

(*a*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(*b*) There are infinite such matrix

(*c*) There is no such matrix

(*d*) must be diagonal

103. A is any $n \times n$ matrix with entries equal to 1 then

(*d*) multiplicity of 0 is $n - 1$

(*b*) multiplicity of 0 is 1

(*c*) multiplicity of 0 is n

(*d*) multiplicity of 0 is 0

104. Consider the system $x + y + z = 0$; $x - y - z = 0$, then the system of equations have

(*a*) no solution (*b*) infinite solution

(*c*) unique solution (*d*) None of the above

105. A is any matrix which satisfy $A^3 - A^2 + A - I = 0$ and $A_{3\times3}$, then A^4 is

(a) $A = 0$ (*b*) $A = I$

(*c*) There is no such matrix

(*d*) $A^3 + A^2 - A + 1$

106. The eigen values of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ are

(*a*) 1, 4 (*b*) –1, 2

(*c*) 0, 5 (*d*) Cannot be determined

107. Which one of the following is an eigen vector of the matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

(*a*) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

108. The eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is $a + b$?

(*a*) 0 (*b*) 1/2

(*c*) 1 (*d*) 2

109. An eigen vector of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

(*a*) $[-1\ 1\ 1]^T$ (*b*) $[1\ 2\ 1]^T$

(*c*) $[1\ -1\ 2]^T$ (*d*) $[2\ 1\ -1]^T$

110. How many of the following have an eigen value 1?

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

(*a*) One (*b*) two

(*c*) three (*d*) four

111. For the matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$ one of the eigen value is 3.

The other two eigen values are

(*a*) 2, –5 (*b*) 3, – 5

(*c*) 2, 5 (*d*) 3, 5

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1. (*b*)	**2.** (*d*)	**3.** (*c*)	**4.** (*a*)	**5.** (*b*)	**6.** (*d*)	**7.** (*c*)	**8.** (*d*)	**9.** (*d*)	**10.** (*d*)
11. (*a*)	**12.** (*c*)	**13.** (*c*)	**14.** (*a*)	**15.** (*a*)	**16.** (*c*)	**17.** (*a*)	**18.** (*a*)	**19.** (*c*)	**20.** (*d*)
21. (*a*)	**22.** (*b*)	**23.** (a)	**24.** (a)	**25.** (*a*)	**26.** (*c*)	**27.** (*a*)	**28.** (*b*)	**29.** (a)	**30.** (*d*)
31. (*c*)	**32.** (*a*)	**33.** (*c*)	**34.** (*a*)	**35.** (*c*)	**36.** (*a*)	**37.** (*b*)	**38.** (*a*)	**39.** (*b*)	**40.** (a)
41. (*b*)	**42.** (*c*)	**43.** (***b***)	**44.** (*b*)	**45.** (*c*)	**46.** (*d*)	**47.** (*a*)	**48.** (*b*)	**49.** (*b*)	**50.** (*a*)
51. (*b*)	**52.** (*b*)	**53.** (*a*)	**54.** (*d*)	**55.** (*b*)	**56.** (*c*)	**57.** (*b*)	**58.** (*c*)	**59.** (*d*)	**60.** (*a*)
61. (*c*)	**62.** (*c*)	**63.** (*a*)	**64.** (*a*)	**65.** (*c*)	**66.** (*a*)	**67.** (*d*)	**68.** (*b*)	**69.** (6)	**70.** (*c*)
71. (*a*)	**72.** (*b*)	**73.** (*d*)	**74.** (*c*)	**75.** (*b*)	**76.** (*a*)	**77.** (*d*)	**78.** (*c*)	**79.** (*c*)	**80.** (*c*)
81. (*d*)	**82.** (*a*)	**83.** (*c*)	**84.** (*d*)	**85.** (*b*)	**86.** (*c*)	**87.** (*d*)	**88.** (*c*)	**89.** (*b*)	**90.** (*c*)
91. (*a*)	**92.** (*b*)	**93.** (a)	**94.** (*c*)	**95.** (*b*)	**96.** (*a*)	**97.** (*a*)	**98.** (*c*)	**99.** (*d*)	**100.** (*d*)
101. (*c*)	**102.** (*d*)	**103.** (a)	**104.** (*b*)	**105.** (*d*)	**106.** (*c*)	**107.** (*a*)	**108.** (*b*)	**109.** (*b*)	**110.** (*a*)
111. (*b*)									

EXPLANATIONS: SET-I

1. We need to find the value of *k* where the matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ satisfies the equations $A^2 - kA + 2I = 0$.

But $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$

So, $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$

Putting these values in the equation

$A^2 - kA + 2I = 0$, we get

$$\begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} - k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or $\begin{bmatrix} 1-3k+2 & -2+2k \\ 4-4k & -4+2k+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

or $1 - 3k + 2 = 0 \Rightarrow k = 1$

2. We are given that the system of equations

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 1 & 2 & a-4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ a \end{bmatrix}$$

Reduce this system-of equations to echelon form, using the operations "$R_2 \to R_2 - 2R_1$" and $R_3 \to R_3 - R_1$. These operations yields:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & -3 & -6 \\ 0 & 0 & a-8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ a-6 \end{bmatrix}$$

Also we have, this system has a unique solution so $a - 8 \neq 0$ or $a \neq 8$

3. We need to find the determinant the given matrix,

$$A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 1/3 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1/2 & 1/3 & \dots & \dots & 1/n \end{vmatrix}$$

determinant of $A = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \dots \frac{1}{n}$

or $\det(A) = \frac{1}{1 \cdot 2 \cdot 3 \dots n}$

$$\det(A) = \frac{1}{n!}$$

4. We are given that the system of equations,

$x - 2y - 3z = 1$, $(p + 2)z = 3$, $(2p + 1)y + z = 2$

is inconsistent. Then we need to find the value of p. The augmented matrix of this system of equations is, $= \begin{bmatrix} 1 & -2 & -3 & \vdots & 1 \\ 0 & 2p+1 & 1 & \vdots & 2 \\ 0 & 0 & p+2 & \vdots & 3 \end{bmatrix}$. The solution of this matrix is inconsistent if

$p + 2 = 0$

or $p = -2$

5. We are given that x, y, z are in $A.P.$ with common difference d *i.e.*,

$$y - x = d \quad \text{or} \quad y = x + d$$
$$z - y = d \quad \text{or} \quad z = d + y$$

or $z = x + 2d$

and the rank of $\begin{bmatrix} 4 & 5 & x \\ 5 & 6 & y \\ 6 & k & z \end{bmatrix}$ is z.

We need to find the values of d and k. The given matrix can be written as,

$$|A| = \begin{vmatrix} 4 & 5 & x \\ 5 & 6 & x+d \\ 6 & k & x+2d \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

$$= \begin{vmatrix} 4 & 5 & x \\ 1 & 1 & d \\ 1 & k-6 & d \end{vmatrix}$$

$\Rightarrow$ $|A| = 0$

$\Rightarrow$ $|A| = (k - 7)(4d - x) = 0$

or $d = \dfrac{x}{4}, k = 7$

6. We are given that C is a non-singular matrix and $B = C\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}C^{-1}$ or B is similar to $\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}$.

The eigen value of $\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}$ are all zeroes.

Hence, $Bk = 0$

For $k = 3$

$B^3 = 0$

7. We are given that the matrix

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We need to find the values of X^TX. The transpose of X,

$$X^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now $X^TX = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank $(X^TX) = 3$

8. We are given that the matrix $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$ has the eigen values 4 and 8. We need to find the values of x and y. We have, trace A = sum of eigen values

or $2 + y = 4 + 8$

$y = 12 - 2 = 10$

and, the determinant of the matrix A

= product of eigen values.

$2y - 3x = 4 \times 8$

or $20 - 3x = 32$

or $3x = -12$

or $x = -4$

and $y = 10$

9. We need to find the eigen values of the matrix

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

The characteristic equations of the given matrix,

$$|A - \lambda I| = 0 \begin{vmatrix} -1-\lambda & 3 & 5 \\ -3 & -1-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(-1-\lambda)^2+9]=0$$
$$(3-\lambda)(\lambda^2+2\lambda+10)=0$$
$$\lambda = 3, -1+3i, -1-3i.$$

10. We are given that the characteristic equation of a 3 × 3 matrix A is

$$\lambda^3+\lambda^2+2\lambda+1=0$$

By Cayley Hamilton theorem, this characteristic equations may be written in the matrix form as,

$$A^3+A^2+2A+I=0$$

or $A^3+A^2+2A=-I$

or $A(A^2+A+2I)=-I$

or $A^{-1}=-(A^2+A+2I.)$

11. We are given the matrix $A=\begin{bmatrix}-3 & 2\\ -1 & 0\end{bmatrix}$

The characteristic equation of the given matrix is

$$|A-\lambda I|=0$$

or $\begin{vmatrix}-3-\lambda & 2\\ -1 & 0-\lambda\end{vmatrix}=0$

or $\lambda^2+3\lambda+2=0$

By Cayley Hamilton theorem, this characteristic equation can be written in the matrix form as,

$$A^2+3A+2I=0$$

or $A^2=-(3A+2I)$

or $A^4=A^2\,.\,A^2$

$$=-(3A+2I)(-3A-2I)$$
$$=9A^2+12A+4I$$
$$=9(-3A-2I)+12A+4I$$
$$=-15\text{A}-14\text{I}$$

$A^8=A^4\,.\,A^4$

$$=(-15A-14I)(-15A-14I)$$
$$=225A^2+420A+196I$$
$$=225(-3A-2I)+420A+196I$$
$$=-255A-254I$$

$A^9=A\,.\,A^8$

$$=A(-255A-254I)$$
$$=-255A^2-254A$$
$$=-255(-3A-2I)-254A$$
$$=511A+510I$$

12. We are given that the characteristic polynomial of 3 × 3 matrix A is

$$\lambda^3+3\lambda^2+4\lambda-3=0$$

and $x = \text{trace}\,(A)$

$y = \det\,(A)$

The general characteristic polynomial of 3 × 3 matrix is given by,

$$\lambda^3 - \text{trace}\,(A)\lambda^2+\lambda+|A|=0$$

Comparing this equation by the given equation, we get

trace $=-3$

or $x=-3$

and $|A|=-3$

or $y=-3$

Hence, $x=y=-3$

13. We need to find the characteristic vector of the matrix $\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$ corresponding to the characteristic root 1.

Put $\lambda=1$ in the equation $(A-\lambda I)X=0$

$$\begin{bmatrix}2-1 & 3\\ 1 & 4-1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\ 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

or $x_1+3x_2=0$

or $x_2=k$

then $x_1=-3k$

The eigen vector corresponding the eigen value $\lambda=1$ is,

$$=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-3k\\ k\end{bmatrix} \text{ or } \begin{bmatrix}-3\\ 1\end{bmatrix}$$

14. We are given that the system of equation

$$x+2y-z=11$$
$$3x+y-2z=10$$
$$x-3y=5$$

The given system of equation can be written in the augmented matrix as,

$$[A:B]=\begin{bmatrix}1 & 2 & -1 & 11\\ 3 & 1 & -2 & 10\\ 1 & -3 & 0 & 5\end{bmatrix}$$

Applying the operations "$R_2 \to R_2 - 3R_1$" and $R_3 \to R_2 - R_1$, and these operations yield

$$=\begin{bmatrix}1 & 2 & -1 & \vdots & 11\\ 0 & -5 & 1 & \vdots & -23\\ 0 & -5 & 1 & \vdots & -6\end{bmatrix}$$

and also "$R_3 \to R_3 - R_2$", which gives,

$$= \begin{bmatrix} 1 & 2 & -1 & \vdots & 11 \\ 0 & -5 & 1 & \vdots & -23 \\ 0 & 0 & 0 & \vdots & 17 \end{bmatrix}$$

Here, rank of $(A) \neq$ rank of aug (A).

Hence, the given system of equation has no solution.

15. We need to .find the minimal polynomial of the 3×3 real matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$$

First we need to find the characteristic polynomial of the given matrix, that is

$$\text{ch}_A(x) = | I - A |$$
$$= (x - a)(x - a)(x - b)$$

Further, the minimal polynomial of the given matrix

$$= \text{1.c.m of } (x - a), (x - a), (x - b)$$

or $m_A(x) = (x - a)(x - b)$

16. We are given that the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We need to find the eigen values of A. The characteristic equations of the given matrix is, $| A - \lambda I | = 0$

or $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

or $-\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda) = 0$

or $-\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda = 0$

or $\lambda^3 - 3\lambda - 2 = 0$

Since, $\lambda = 2$ satisfies this equation. Therefore, $\lambda = 2$ is one of the root of the given matrix,

$(\lambda - 2)(\lambda^2 + 2\lambda + 1) = 0$

or $\lambda = 2, (\lambda + 1)^2 = 0, \lambda = -1, -1$

17. We are given that a matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has real eigen values. The characteristic equation of the given matrix is,

$$| A - \lambda I | = 0$$
$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

or $\cos^2\theta + \lambda^2 - 2\lambda\cos\theta + \sin^2\theta = 0$

or $\lambda^2 - 2\lambda\cos\theta + 1 = 0$

or $\lambda = \cos\theta \pm i\sin\theta$

Since, the given matrix has real eigen values, then

$$\sin\theta = 0$$

or $\theta = n\pi$ for some integer n.

18. We are given that the $n \times n$ matrix $A = [a_{ij}]$ such that $a_{ij} = 3 \ \forall \ i$ and j. Then , we need to find the nullity of the given matrix,

Let the matrix,

$$A = \begin{bmatrix} 3 & 3 & \dots & 3 \\ 3 & 3 & \dots & 3 \\ \dots & \dots & \dots & 3 \\ \dots & \dots & \dots & \dots \\ 3 & 3 & \dots & 3 \end{bmatrix}_{n \times n}$$

Reduce the matrix to echelon form using the operations "$R_2 \to R_2 - R_1$" "$R_3 \to R_3 - R_1$" ... "$R_n \to R_n - R_1$". These operations yield

$$A = \begin{bmatrix} 3 & 3 & \dots & 3 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, rank $(A) = 1$. By Sylwester's law,

$$\dim(A) = \text{rank}(A) + \text{nullity}(A)$$

or $n = 1 + \text{nullity}(A)$

or $\text{nullity}(A) = n - 1$

19. We are given that the system of simultaneous linear equations,

$$x + y + z = 0$$
$$x - y - z = 0$$

The coefficient matrix is given by,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

rank of coefficient matrix = 2. Here, the rank of matrix < no. of unknowns, therefore, the system of equation has infinitely many solution in $\mathbb{R}^3$.

20. We need to find the minimal polynomial $m(x)$ of $A_{n\times n}$, each of whose element is 1.

$$\text{Let}\quad A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{bmatrix}_{n\times n}$$

$$\text{or}\quad A^2 = \begin{bmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ n & n & \dots & n \end{bmatrix}$$

or $A^2 - nA = 0$

Hence, $m_A(x) = x^2 - nx$

21. We are given that the matrix

$$M = \begin{bmatrix} \begin{pmatrix} 2 & 8 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} (2) & 4 & 0 & 0 & 0 \\ 0 & (5) & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & (5) \end{matrix} \end{bmatrix}$$

We need to find the minimal polynomial of the given matrix. The characteristic polynomial of the given matrix is given by,

$m_m(\lambda) = (\lambda - 2)^2(\lambda - 2)(\lambda - 5)(\lambda - 0)^2(\lambda - 5)$

Hence, the minimal polynomial of the given matrix M is,

$m_m(\lambda) =$ l.c.m. of $(\lambda - 2)^2$, $(\lambda - 2)$, $(\lambda - 5)$, λ^2.

and $(\lambda - 5)$

$m_m(\lambda) = (\lambda - 2)^2, (\lambda - 5)\lambda^2$

or $m_m(\lambda) = \lambda^2(\lambda - 2)^2(\lambda - 5)$

22. We are given that the eigen values of a 3 × 3 real matrix A are 1, 2 and –3 say λ_1, λ_2 and λ_3. The characteristic equation of the given matrix is,

$x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4)x$
$- |A| = 0$

or $x^3 - (1 + 2 - 3)x^2 + (2 - 6 - 3)x - (-6) = 0$

or $x^3 - 7x + 6 = 0$

By Cayley Hamilton's theorem, we can write this equation in the matrix form as,

$A^3 - 7A + 6I = 0$

or $6I = A(-A^2 + 7I)$

or $A^{-1} = \frac{1}{6}[-A^2 + 7I]$

23. We need to find the minimal polynomial of the matrix

$$A = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} (2) & 0 \\ 0 & (2) \end{matrix} \end{bmatrix}$$

The characteristic polynomial of the given matrix is,

$ch_A(x) = (x - 1)^2(x - 2)(x - 2)$

and the minimal polynomial of the given matrix

$=$ l.c.m. of $(x - 1)^2$, $(x - 2)$ and $(x - 2)$

$= (x - 1)^2(x - 2)$

24. We are given that A be a 3 × 3 matrix whose characteristic roots are 3, 2, –1 say λ_1, λ_2 and λ_3 and $B = A^2 - A$

$\lambda_1 = 3$

$\lambda_1' = (3)^2 - 3 = 6$

$\lambda_2 = 2$

$\lambda_2' = (2)^2 - 2 = 2$

$\lambda_3 = -1$

$\lambda_3' = (-1)^2 - (-1) = 2$

The determinant of B is, $|B|$ = product of its eigen values

$= \lambda_1'\lambda_2'\lambda_3'$

$= 6 \times 2 \times 2$

$= 24$

25. We are given that,

$$M_1 = \begin{bmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 3 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}$$

The characteristic polynomial of the given matrix M is,

$ch_m(x) = (x - 3)^2(x - 3)(x + 3)^2(x - 0)^2$

And the minimal polynomial of the given matrix M is

$$m_m(x) = \text{l.c.m. of } (x-3)^2, (x-3), (x+3)^2, (x+0)^2$$

or $m_m(x) = (x-3)^2(x+3)^2 x^2$

or $m_m(x) = x^2(x-3)^2(x+3)^2$

26. We are given that A be a matrix of quadratic form

$$(x_1 - x_2 + 2x_3)^2$$

or $x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 - 4x_2x_3 + 4x_3x_1$

trace (A) = sum of coefficient of

x_1^2, x_2^2 and x_3^2.

$= 1 + 1 + 4$

$= 6$

27. We are given that the eigen values of a 3 × 3 real matrix P are 1 , –2, 3 say $\lambda_1, \lambda_2, \lambda_3$. The characteristic equation of the given matrix P is,

$$x^2 - (\text{trace } P)x^2 + (\lambda_1 \cdot \lambda_2 + \lambda_2 \cdot \lambda_3 + \lambda_3 \cdot \lambda_1)x - |P| = 0$$

or $x^3 - (1-2+3)x^2 + (-2-6+3)x - (-6) = 0$

or $x^3 - 2x^2 - 5x + 6 = 0$

By Cayley Hamilton theorem, we can write this equation in the matrix form.

$$P^3 - 2P^2 - 5p + 6I = 0$$

or $6I = 5P + 2P^2 - P^3$

or $6I = P(5I + 2P - P^2)$

or $P^{-1} = \frac{1}{6}(5I + 2P - P^2)$

28. We are given that the system of equations,

$$x + y + z = 0$$
$$3x + 6y + z = 0$$
$$\alpha x + 2y + z = 0$$

has infinitely many solutions. We need to find the value of α. The given system of equation may be written in matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 6 & 1 \\ \alpha & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the system of equation to echelon form using the operations '$R_2 \to R_2 - R_1$' and $R_3 \to R_3 - R_1$., these operations yields

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 1 \\ \alpha - 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Also applying, $R_3 \to 5R_3 - R_2$ we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 0 \\ 5\alpha - 7 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since, this system of equations has infinitely many solutions. It is possible when

$$5\alpha - 7 = 0$$

or $\alpha = \frac{7}{5}$

29. We are given that the system of equations,

$$x + y + z = 3$$
$$x - z = 4$$
$$x - 5y + kz = 6$$

has infinitely many solution and we need to find the value of k. The given system of equation may be written in the matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -5 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Reduce this system of equations to echelon form using the operations "$R_2 \to R_2 - R_1$", "$R_3 \to R_3 - R_1$" these operations yield

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -6 & k-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

are also applying "$R_3 \to R_3 - 3R_2$" we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & k+5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Since, the system has infinite solution it is possible when,

$$k + 5 = 0 \quad \text{or} \quad k = -5$$

77. We are given that the system of equation

$$kx + y + z = 1$$
$$x + ky + z = k$$
$$x + y + kz = k^3$$

does not have a solution. We need to find the value of k. The given linear system of equation can be written as,

$$\begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^3 \end{bmatrix}$$

This system of equation has no. solution if rank $(A) \neq$ rank (aug A) the augmented matrix is,

$$\begin{bmatrix} k & 1 & 1 & \vdots & 1 \\ 1 & k & 1 & \vdots & k \\ 1 & 1 & k & \vdots & k^3 \end{bmatrix}$$

This augmented matrix has no. solutions if $k = -2$.

78. We are given that the system of equation

$$4x_1 + x_2 - 3x_3 - x_4 = 0$$
$$2x_1 + 3x_2 + x_3 - 5x_4 = 0$$
$$x_1 - 2x_2 - 2x_3 + 3x_4 = 0$$

It may be written as,

$$\begin{bmatrix} 4 & 1 & -3 & -1 \\ 2 & 3 & 1 & -5 \\ 1 & -2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, the rank of the given coefficient matrix ≤ 3 and number of unknowns are four.

Hence, rank < no. of unknowns this system of equations has infinitely many solutions.

79. We are given that the system of equation,

$$x - y + 3z = 4$$
$$x + z = 2$$
$$x + y - z = 0$$

It may be written in the matrix form as,

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

Here, the coefficient of matrix is,

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

and the augmented matrix is given by,

$$\begin{bmatrix} 1 & -1 & 3 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 2 \\ 1 & 1 & -1 & \vdots & 0 \end{bmatrix}$$

Reduce the system equation in echelon form using the operations "$R_2 \to R_2 - R_1$" and $R_3 \to R_3 - R_1$. These operation yield:

$$\begin{bmatrix} 1 & -1 & 3 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & -2 \\ 1 & 1 & -1 & \vdots & -4 \end{bmatrix}$$

and also, $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -1 & 3 & \vdots & 4 \\ 0 & 1 & -2 & \vdots & -2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Here, Rank of A = Rank of aug(A) < number of unknowns.

Hence, the given system of education has infinite many solutions.

80. We are given that the equation $AX = B$, were

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1 - 4 = -3 \neq 0$$

Hence, the rank of A = 2 and, the augmented matrix is given by, $\begin{bmatrix} -1 & 2 & \vdots & 3 \\ 2 & -1 & \vdots & 1 \end{bmatrix}$

The rank of A = rank of aug (A) = 2 = number of unknowns

Hence, there exist a unique solutions.

81. We are given that the system of equations,

$$2x + y = 5 \; x - 3y = -1$$

$3x + 4y = k$ is consistent.

We need to find the value of k. The given system of equation may be written as,

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{bmatrix}$$

Since, this system of equation is consistent. Therefore,

$$\begin{vmatrix} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{vmatrix} = 0$$

or $2(-3k + 4) - 1(k + 3) + 5(4 + 9) = 0$

$$-7k + 70 = 0$$

$$k = 10$$

82. We are given that the system of equation,

$$x + y + z = 0$$
$$y + 2z = 0$$
$$\alpha x + z = 0$$

has more than one solution. We need to find the value of α. The determinant of the coefficient matrix must be zero. *i.e.*,

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ \alpha & 0 & 1 \end{vmatrix} = 0$$

or $1(1 - 0) - 1(0 - 2\alpha) + 1(0 - \alpha) = 0$

or $1 + \alpha = 0$

$\alpha = -1$

83. We are given that the system of equations,

$$x + 2y + z = 9$$
$$2x + y + 3z = 7$$

This system of equation can be expressed in the form $AX = B$ *i.e.*,

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$$

which is required matrix form.

84. We are given that A be an $n \times n$ matrix from the set of real numbers and $A^3 - 3A^2 + 4A - 6I = 0$,

where, I is $n \times n$ unit matrix.

Since,

$A^3 - 3A^2 + 4A = 6I$

or $A(A^2 - 3A + 4) = 6I$

or $A^{-1}A(A^2 - 3A + 4) = A^{-1}(6I)$

or $A^{-1} = \dfrac{1}{6}(A^2 - 3A + 4I)$

85. We are given that the system of equations,

$$3x + 2y + z = 0$$
$$x + 4y + z = 0$$
$$2x + y + 4z = 0$$

It can be written in the matrix form as,

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce this system to echelon form using the operation "$R_2 \to 3R_2 - R_1$" and "$R_3 \to 3R_3 - 2R_2$". These operations yield

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 10 & 2 \\ 0 & -1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and also "$R_3 \to 10R_3 + R_3$"

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 10 & 2 \\ 0 & 0 & 102 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives, $x = 0$, $y = 0$ and $z = 0$, Hence, this system of equations has only the trivial solution.

86. We need to find the rank of the matrix,

$$M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Reduce the matrix to echelon form using the operations "$R_2 \to R_2 + R_1$", $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 + R_1$. These operations yield.

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and also applying $R_4 \to 2R_4 - R_2$, we have,

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, rank of $M = 2$.

87. We are given that the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We need to find all the eigen values of the given matrix. Since, the characteristic equations is given by,

$|A - \lambda I| = 0$

or $(-1 - \lambda)[(1 - \lambda)^2 - 4] = 0$

$\lambda + 1 = 0, (1 - \lambda)^2 = 4$

$1 - \lambda = \pm 2 \quad \Rightarrow \quad \lambda = 3, 1$

88. We need to find the determinant of the matrix,

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The determinant of the given matrix,

$$|M| = (1)\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - (2)\begin{vmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= (1)(1)\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} - (2)(2)\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}$$

$$= (-3)\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}$$

$$= (-3)(1)\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - (-3)(2)\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= (-3)(1)(1-4) + (6)(2)(1-4) = 9 - 36 = -27$$

89. We are given that the matrix

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{bmatrix}$$

where ω be a complex number such that $\omega^3 = 1$, but $\omega \neq 1$.

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{bmatrix}$$

Applying the operations $C_3 \to C_1 + C_2 + C_3$ this operation yields.

$$A = \begin{bmatrix} 1 & \omega & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1+\omega+\omega^2 \\ \omega^2 & \omega & 1+\omega+\omega^2 \end{bmatrix} = \begin{bmatrix} 1 & \omega & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{bmatrix}$$

$\therefore$ Rank of $A = 2$ $\qquad (\because \; 1 + \omega + \omega^2 = 0)$

90. We are given that, A be a 4×4 matrix with real entries such that, –1, 1, 2, –2 are its eigen values say $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $B = A^4 - 5A^2 + 5I$.

Let the eigen values of B corresponding to

$B = (-1)^4 - 5(-1)^2 + 5 = 1 = \lambda_1^1$

$\lambda_2^1 = (1)^4 - 5(1)^2 + 5 = 1$

$\lambda_3^1 = (2)^4 - 5(2)^2 + 5 = 1$

$\lambda_4^1 = (-2)^4 - 5(-2)^2 + 5 = 1$

91. We need do determine the characteristic polynomial of the 3×3 real matrix

$$A = \begin{bmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{bmatrix}$$

which is companion matrix,

where $|A| = -c$

trace $(A) = -a$

Therefore, the characteristic polynomial is given by,

$|A - \lambda I| = 0$

$-\lambda[\lambda(a + \lambda)] - c = 0$

$-\lambda^3 - a\lambda^2 - b\lambda - c = 0$

92. We are given that the characteristic roots of $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ are λ_1 and λ_2, we need to determine the characteristic roots of $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$.

Since, the matrix $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$. Therefore, if $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ has roots λ_1, and λ_2 then, the inverse of this matrix has the characteristic roots $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$.

93. We are given that the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & 1 \\ 3 & 4 \end{bmatrix}$$

Let, $A_2 - A - 5I$

$$= \begin{bmatrix} 7 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which is required zero matrix.

94. We are given that characteristic equations of a matrix M be

$$\lambda^2 - \lambda - 1 = 0$$

It can be written in the matrix form as,

$$M^2 - M - 1 = 0$$

$$M^2 - M = 1$$

$$M^{-1}(M^2 - M) = M^{-1}$$

or $M - 1 = M^{-1}$

or $M^{-1} = M - 1$

95. We are given that M is a 7×5 matrix of rank 3 and N is a 5×7 matrix of rank 5.

or $\rho(M) = 3, \rho(N) = 5$

Since,

$$\rho(MN) \geq \rho(M) + \rho(N) - \rho(M \cap N)$$

$$\rho(NM) \leq \min(\rho(M), \rho(N))$$

or $\rho(NM) \geq 3 + 5 - 5$

$$\rho(MN) \geq 3 \quad ...(i)$$

and $\rho(MN) \leq \min(3, 5)$

$$\rho(MN) \leq 3 \quad ...(ii)$$

Hence, $\rho(MN) = 3$

96. We are given that the system of linear equation

$$x + y + z = 3$$

$$x - y - z = 4$$

$$x - 5y + kz = 6$$

has an infinite number of solution. Thus we need to find the value of k. The given system of equations may be written as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -5 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Reduce this system of echelon form using the operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$. These operations yield. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -6 & k-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ and also "$R_3 \to R_3 - 3R_2$" we have,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & k+5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Also, we have this system of equation has infinitely many solution, so,

$$k + 5 = 0$$

or $k = -5$

97. We need to determine the value of λ will the matrix given below become singular,

$$A = \begin{bmatrix} 8 & \lambda & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

If the given matrix is singular

i.e., $|A| = 0 \begin{vmatrix} 8 & \lambda & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{vmatrix} = 0$

$$8(-12) - \lambda(-24) = 0$$

or $24\lambda = 96$

or $\lambda = 4$

98. We are given that the matrix

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

We need to find the rank of the matrix. Reduce this matrix to echelon form using the operations "$R_3 \to 2R_2 - 3R_1$" and "$R_3 \to 2R_3 - R_1$". These operations yield

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

and also, $R_3 \to 5R_3 + R_2$

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, Rank of $A = 2$

99. We need to find the value of k for which the following simultaneous equations

$x + y + z = 3$

$x + 2y + 3z = 4$

$x + 4y + kz = 6$

will not have a unique solution. This system of equation may be written as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Reduce this system of equation to echelon form using the operations "$R_2 \to R_2 - R_1$" and $R_3 \to R_3 - R_1$. These operations yield

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & k-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

and also, $R_3 \to R_3 - 3R_2$ gives,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & k-7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

also this system of equation has not unique solution if,

$$k - 7 = 0$$

$$k = 7$$

100. We need to determine the solution for the system of equation

$4y + 3z = 8$

$2x - z = 2$

$3x + 2y = 5$

This system of equation may be written in matrix form as,

$$\begin{bmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$$

Reduced this system of equaton of echelon form using the operations "$R_2 \Leftrightarrow R_1$". This operations yields

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 3 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 5 \end{bmatrix}$$

and also applying "$R_3 \to 2R_3 - 3R_1$" which yields

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 3 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}$$

again, "$R_3 \to R_3 - R_2$ we get

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix}$$

Here the coefficient matrix is, $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

and augmented matrix is given by,

$$\left[\begin{array}{ccc:c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

Since the rank of coefficient matrix ≠ the rank of augmented matrix.

Therefore, the solution does not exist.

101. We are given that the system of equation

$x + 2y + z = 6$

$2x + y + 2z = 6$

$x + y + z = 5.$

The system of equation can be written in augmented matrix as,

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right]$$

Applying the operations "$R_2 \to R_2 - 2R_1$" and "$R_3 \to R_3 - R_1$" these operations yields:

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 6 \\ 2 & -3 & 2 & -6 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

also applying $R_3 \to 3R_3 - R_2$, which yields

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Since the rank of coefficient matrix ≠ the rank of augmented matrix.

Therefore, the given system of equations has no solution.

102. We are given that A be any 2×2 matrix which satisfy $A^2 - A = 0$

The characteristic equation of the given matrix is,

$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0, 1$$

There exists distinct eigen values.

Hence, A must be diagonalizable.

103. We are given that A is any $n \times n$ with entries equal to 1. Suppose, if A is 2 × 2 matrix with entries equal to 1, that is,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

multiplicity of 0 is 1.

For 3 × 3 matrix with entries equal to 1,

$$i.e., \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

multiplicity of 0 is 2.

By continuing this process, for $n \times n$ matrix with entries equal to 1, that is,

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & \dots & \dots & \dots & 1 \\ 1 & \dots & \dots & \dots & 1 \\ 1 & \dots & \dots & \dots & 1 \end{bmatrix}_{n \times n} \sim \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

multiplicity of 0 is $(n - 1)$

104. We are given that the system of equation,

$$x + y + z = 0$$

$$x - y - z = 0$$

This system of equations has rank 2 and 3 unknowns, that is, rank < no. of unknown

Hence, this system of equation has infinitely many solution.

105. We are given that, A be any 3 × 3 matrix which satisfy

$$A^3 - A^2 + A - I = 0$$

and we need to find A^4.

$$A^3 - A^2 + A - I = 0$$

or $\quad A^3 = A^2 + A - I$

or $\quad A^4 = A^3 - A^2 + A$

106. We need to find the eigen values of the matrix

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

The characteristics equation of the given matrix A,

$$|A - \lambda I| = 0$$

or $\quad \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$

or $\quad (4 - \lambda)(1 - \lambda) - 4 = 0$

or $\quad \lambda^2 - 5\lambda = 0$

$\Rightarrow \quad \lambda(\lambda - 5) = 0$

$\Rightarrow \quad \lambda = 0, 5$

The eigen values of the given matrix is 0 and 5.

107. We need to find the eigen vector of the matrix

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

The characteristic equation of the given matrix is,

$$|A - \lambda I| = 0 \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 5 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$(5 - \lambda)(5 - \lambda)[(2 - \lambda)(1 - \lambda) - 3] = 0$$

$$\lambda = 5, 5, \frac{3 \pm \sqrt{3}}{2}$$

Put $\lambda = 5$ in equation $|A - \lambda I| X = 0$ that is,

$$\begin{bmatrix} 5-5 & 0 & 0 & 0 \\ 0 & 5-5 & 5 & 0 \\ 0 & 0 & 2-5 & 1 \\ 0 & 0 & 3 & 1-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or $\quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

or $\quad 5z = 0$

$$-3z + t = 0$$

$$3z - 4t = 0$$

or $\quad z = 0, t = 0$

and also let $x = k_1, y = k_2$

Hence, the eigen vector corresponding to the eigen value $\lambda = 5$ is,

$$X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$$

Let $k_1, = 1$, and $k_2 = -2$, then

$$X = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

108. We are given that the eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. We need to find the value of $a + b$. The characteristic equation of the given matrix is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

or $(1 - \lambda)(2 - \lambda) = 0$

or $\lambda = 1, 2$

Put $\lambda = 1$ in the equation $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-1 & 2 \\ 0 & 2-1 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $a = 0$

and also put $\lambda = 2$ in the equation

$[A - \lambda]X = 0$

or $\begin{bmatrix} 1-2 & 2 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or $1 + 2b = 0$

or $b = \frac{1}{2}$

Now, $a + b = 0 + \frac{1}{2}$

$$a + b = \frac{1}{2}$$

109. We need to find the eigen vector of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

The characteristic equation of the given matrix is,

$$|A - \lambda I| = 0$$

or $\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

or $(1 - \lambda)(2 - \lambda)(3 - \lambda) = 0$

$\lambda = 1, 2, 3$

Suppose, X be the eigen vector corresponding to the eigen value $\lambda = 3$.

Hence, $(A - \lambda I)X = 0 \begin{bmatrix} 1-3 & 1 & 0 \\ 0 & 2-3 & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $-2x + y = 0$

$-y + 2z = 0$

Let $x = k$

Hence, $y = 2k$ and $z = k$

Now, the eigen vector is,

$$X = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

110. We are given that the matrices

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

say A_1, A_2 and A_3

A_1 *is a* diagonal matrix which has eigen values 1 and 0

A_2 is upper triangular matrix which has eigen values 0, 0 and A_3 is a lower triangular matrix which has eigen values –1, –1.

Hence, A_3 only one matrix which has eigen value 1.

111. We are given that the matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$ has an eigen value 3. We need to find the other two eigen values;

We have, have, trace A = sum of eigen values

$2 - 1 + 0 = \lambda_1 + \lambda_2 + \lambda_3$

$1 = 3 + \lambda_2 + \lambda_3$

or $\lambda_2 + \lambda_3 = -2$

Hence, option (b) satisfies this condition.

SET -II

1. An eigen-vector of the matrix $\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$ is
   (*a*) $(1, 2)^T$ (*b*) $(5, 0)^T$
   (*c*) $(0, 2)^T$ (*d*) None

2. Let $P = \{p_{ij}\}$ be a 50×50 matrix, where $P_{ij} = \min(i, j)$; $i, j = 1, ..., 50$. Then the rank of P equals
   (*a*) 1 (*b*) 2
   (*c*) 25 (*d*) 50

3. The system of equations
   $x + 3y + 2z = k$
   $2x + y - 4z = 4$
   $5x - 14z = 10$
   (*a*) has unique solution for $k = 2$
   (*b*) has infinitley many solutions for $k = 2$
   (*c*) has no solution for $k = 2$
   (*d*) has unique solution for any $k \neq 2$

4. If A is a 3×3 non-zero matrix such that $A^2 = 0$, then the number of non-zero eigen values of A is
   (*a*) 0 (*b*) 1
   (*c*) 2 (*d*) 3

5. Let $AX = b$ be a non-homogeneous system of linear equations. The augmented matrix $[A : b]$ is given by $\begin{bmatrix} 1 & 1 & -2 & 1 & : & 1 \\ -1 & 2 & 3 & -1 & : & 0 \\ 0 & 3 & 1 & 0 & : & -1 \end{bmatrix}$. Which of the following statement is true
   (*a*) Rank of A is 3.
   (*b*) The system has on solution
   (*c*) The system has unique solution
   (*d*) The system has infinite number of solutions

6. Consider the following system of linear equations:
   $x + y + z = 3,$
   $x + az = b,$
   $y + 2z = 3$
   This system has infinite number of solutions if
   (*a*) $a = -1, b = 0$ (*b*) $a = 1, b = 2$
   (*c*) $a = 0, b = 1$ (*d*) $a = -1, b = 1$

7. Let A be a 3×3 matrix with trace $(A) = 3$ and det $(A) = 2$. If 1 is an eigen value of A, then the eigenvalues of the matrix $A^2 - 2I$ are
   (*a*) $1, 2(i - 1), -2(i + 1)$
   (*b*) $-1, 2(i - 1), 2(i + 1)$
   (*c*) $1, 2(i + 1), -2(i + 1)$
   (*d*) $-1, 2(i - 1), -2(i + 1)$

8. The least positive integer n, such that
   $\begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}^n$ is the identity matrix of order 2 is
   (*a*) 4 (*b*) 8
   (*c*) 12 (*d*) 16

9. If A and B are 3×3 real matrices such that rank $(AB) = 1$, then rank (BA) cannot be
   (*a*) 0 (*b*) 1
   (*c*) 2 (*d*) 3

10. Let A be a 3×3 matrix with eigen values 1, –1 and 3. Then
    (*a*) $A^2 + A$ is non-singular
    (*b*) $A^2 - A$ is non-singular
    (*c*) $A^2 + 3A$ *is* non-singular
    (*d*) $A^2 - 3A$ is non-singular

11. Consider the following system of equations
    $2x + 3y + 4z = 13$
    $5x + 7y + 7z = 26$
    $9x + 13y + 15z = 13\lambda$
    The value of λ, for which the system has infinitely many solutions is
    (*a*) 1 (*b*) 2
    (*c*) 3 (*d*) 4

12. Let P be a matrix of size 3×3 with eigen values 1, 2 and 3. The P is
    (*a*) neither invertible nor diagonalilzable
    (*b*) both invertible and diagonalizable
    (*c*) invertible but not idiagonalizable
    (*d*) not invertible but diagonalizable

13. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ is
    (*a*) 1 (*b*) 2
    (*c*) 3 (*d*) 4

14. If $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then P^{50} equals

(*a*) $\begin{bmatrix} 1 & 100 & 500 \\ 0 & 1 & 100 \\ 0 & 0 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 50 & 100 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix}$

(*c*) $\begin{bmatrix} 50 & 100 & 150 \\ 0 & 50 & 100 \\ 0 & 0 & 50 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & 50 & 1275 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix}$

15. If $p_{ij} = 1$ for $1 \le i, j \le m$, then the characteristic equation of the matrix $P = \{p_{ij}\}$ is

(*a*) $\lambda^m - \lambda^{m-1} + 1 = 0$ (*b*) $\lambda^m - m = 0$

(*c*) $\lambda^m - m\lambda^{m-1} = 0$ (*d*) $\lambda^m + 1 = 0$

16. The nullity of the matrix $\begin{pmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{pmatrix}$ is

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

17. Let N be a nilpotent matrix of order 4 with real entries. Then which one of the following statements is true about eigen values of N?

(*a*) All eigen values are non-zero real numbers

(*b*) All eigen vlaues are purely imaginary

(*c*) Zero is the only eigen value

(*d*) At least one eigen value is real and atleast one eigen value has non-zero imaginary part

18. Eigen vectors of the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ corresponding to the eigen values $(1+i)$ and $(1-i)$ are respectively (where $i = \sqrt{1}$)

(*a*) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} i \\ 1 \end{pmatrix}$ (*b*) $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} i \\ -1 \end{pmatrix}$

(*c*) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ (*d*) $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} i \\ 1 \end{pmatrix}$

19. Consider the non-homogeneous system $Ax = b$ where A is a square matrix of order n. If the matrix A and the augmented matrix $[A/b]$ have the same rank $\hat{r}$ where $r < n$, then the system has

(*a*) no solution

(*b*) a unique solution

(*c*) exactly two solutions

(*d*) infinite number of solutions

20. Two eigen values of the matrix $\begin{bmatrix} 1 & 0 & 0 & -\frac{a}{2} \\ 0 & 1 & 0 & -\frac{a}{2} \\ 0 & 0 & 1 & -\frac{a}{2} \\ 0 & 0 & 0 & a \end{bmatrix}$ are

(*a*) 1 and a (*b*) 1 and $-a$

(*c*) $\frac{1}{2}$ and $-\frac{a}{2}$ (*d*) $\frac{1}{2}$ and $\frac{a}{2}$

21. Let P and Q be two $n \times n$ non zero matrices such that $P + Q = 0$. Which one of the following statements is *NEVER* true?

(*a*) P is nonsingular (*b*) $P = Q^T$

(*c*) $P = Q^{-1}$ (*d*) Rank $(P) \ne$ Rank (Q)

22. Consider the system of equations $Px = 0$, where $P =$

$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-2 & -k+20 \\ 0 & k-8 & -5k \end{bmatrix}$. The value of k for which the system will have a non trivial solution are

(*a*) 2 and –2 (*b*) 2 and –1

(*c*) –1 and –2 (*d*) 1 and –1

23. Let $P = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}$, where ω is a complex cube root of unity. Then P^{24} is

(*a*) P^2 (*b*) P

(*c*) Identity matrix (*d*) 0

24. If $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = k$, then $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$ is

(*a*) $4k + 3$ (*b*) $4k - 3$

(*c*) $2k + 1$ (*d*) k

25. For which value of μ the following system of equations is inconsistent?

$3x + 2y + z = 10$

$2x + 3y + 2z = 10$

$x + 2y + \mu z = 10$

(*a*) 0.98 (*b*) 1.4

(*c*) 1.6 (*d*) 1.8

26. Let u be a unit column vector and $A = I - 2uu^T$. Then A^T is

(*a*) $I - 2uu^T$ (*b*) $I + 2uu^T$

(*c*) $2uu^T$ (*d*) $4uu^T$

27. The system of linear equations

$$9x + 3y + z = 0$$
$$3x + z - 6w = 0$$
$$x + y + z + w = 0$$
$$-6y + z + 9w = 0 \text{ has}$$

(*a*) no solution

(*b*) infinite number of solutions

(*c*) only one solution

(*d*) more than one but finite number of solutions

28. The multiplicative inverse of the matrix

$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is given by

(*a*) $A + 4I$ (*b*) $A - 4I$

(*c*) $\frac{1}{5}(A + 4I)$ (*d*) $\frac{1}{5}(A - 4I)$

29. Let A and B be any arbitrary square matrices of order 3. Then AB and BA have

(*a*) the same eigen values and the same eigen vectors.

(*b*) the same eigen values but may have different eigen vectors.

(*c*) different eigen values but the same eigen vectors.

(*d*) different eigen values and different eigen vectors.

30. The values of a and b for which the following system of linear equations

$$ax + y + 3z = \text{a}$$
$$2x + \text{b}y - z = 3$$
$$5x + 7y + z = 7$$

has an infinite number of solutions, are

(*a*) $a = 1, b = 1$ (*b*) $0 = 1, 6 = 3$

(*c*) $a = 2, b = 3$ (*d*) $a = 2, b = 1$

31. Let $P = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$. Then

(*a*) P has two linearly independent eigen vectors

(*b*) P has an eigen vector

(*c*) P is nonsingular

(*d*) There exists a nonsingular matrix S such that $S^{-1}PS$ is a diagonal matrix

32. Let P be an $n \times n$ idempotent matrix, that is, $P^2 = P$. Which of the following is FALSE?

(*a*) P^T is idempotent

(*b*) The possible eigenvalues of P are 0 or 1

(*c*) The nondiagonal entries of P can be zero

(*d*) There are infinite number of $n \times n$ non singular matrices that are idempotent

33. Let $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The eigen vectors corresponding to the eigen values i and $-i$ are respectively

(*a*) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$

(*b*) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

(*c*) $\begin{pmatrix} -1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

(*d*) $\begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$

34. Let P be a 3×3 martix such that for same a, the linear system $Px = b$ has infinite number of solutions. Which one of the following is TRUE?

(*a*) The linear system $Px = b$ has infinite number of solutions for all b

(*b*) Rank $(P) = 3$

(*c*) Rank $(P) \neq 1$

(*d*) Rank $(P) \leq 2$

35. The inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ is

(*a*) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$

(*b*) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 00 & 0 & 0 & 1 \end{bmatrix}$

(*d*) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$

36. The number of values of λ for which the system of equations

$\lambda x + (\lambda + 3)y = 10z$

$(\lambda - 1)x + (\lambda - 2)y = 5z$

$2x + (\lambda + 4)y = lz$

has infinitely many solutions, is

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) infinite

37. Let $M = \begin{pmatrix} 1 & 1+i & 2-i \\ 1-i & 2 & 3+i \\ 2+i & 3-i & 3 \end{pmatrix}$.

If $B = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$, where $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ linearly independent eigen vectors of then the main diagonal of the matrix $B^{-1}MB$ has

(*a*) exactly on real entry

(*b*) exactly two real entries

(*c*) exactly three real entries

(*d*) no real entry

38. If ω is a non-real cube root of unity, then the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1/\omega & 1/\omega^2 \\ 1 & 1/\omega^2 & 1/\omega^4 \end{pmatrix} \text{ are}$$

(*a*) $1, -1, 0$ (*b*) $\frac{1}{3}, -\frac{1}{3}, 0$

(*c*) $1, \omega^1, \omega^2$ (*d*) $3, -3, 0$

39. The system of equations $Ax = b$, where

$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has

(*a*) no solution

(*b*) unique solution

(*c*) infinitely many solutions

(*d*) more than one but finitely many solutions

40. For real symmetric matrices *A* and *B*, which of the following is true?

(*a*) *AB* is symmetric matrix

(*b*) $AB = BA$

(*c*) All eigen values of *AB* are real if $AB = BA$

(*d*) *AB* is invertible if either *a* or *b* is invertible

41. If *A* is an $n \times n$ matrix, then the system of linear equations $Ax = 0$

(*a*) is inconsistent if rank $(A) < n$

(*b*) has exactly *r* soutions if rank $(A) = r < n$

(*c*) has infinitely many solutions if rank $(A) = n$

(*d*) has infinitely many solutions if rank $(A) < n$

42. The eigen values of the matrix

$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 1 \\ 3 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are

(*a*) 3, 2, 3, 4 (*b*) 4, 5, 1, 0

(*c*) 0, 3, 4, 5 (*d*) 1, 2, 4, 5

43. Following system of linear equations

$x_1 + 4x_2 + 3x_3 = 0$

$x_1 + 3x_2 + 4x_3 = 0$

$x_1 + 2x_2 + 5x_3 = 0$

does have

(*a*) no solution

(*b*) infinitely many solutions

(*c*) more than one but finitely many solutions

(*d*) exactly one solution

44. For the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, A^{-1} is given by

(*a*) $A^2 - 2A$ (*b*) $A^2 + 2A + 3I$

(*c*) $A^2 - 2A - I$ (*d*) $A - 3I$

45. Let A be an $n \times n$ matrix such that $x^T Ax > 0$ for every non-zero vector x in R^n. Which of the following is true?

(*a*) All eigen values of A are negative

(*b*) All eigen values of A are positive

(*c*) Exactly one eigen value of A is zero

(*d*) More than one eigen values of A are zero

46. Let $\alpha = e^{2\pi i/5}$ and the matrix

$$M = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$$

Then, the trace of the matrix $I + M + M^2$ is

(*a*) –5 (*b*) 0

(*c*) 3 (*d*) 5

47. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then A^{50} is

(*a*) $\begin{bmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 0 & 0 \\ 48 & 1 & 0 \\ 48 & 0 & 1 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & 0 & 0 \\ 24 & 1 & 0 \\ 24 & 0 & 1 \end{bmatrix}$

48. The distinct eigen values of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are

(*a*) 0 and 1 (*c*) 1 and –1

(*b*) l and 2 (*d*) 0 and 2

49. The minimal polynomial of the matrix $\begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ is

(*a*) $x(x-1)(x-6)$ (*b*) $x(x-3)$

(*c*) $(x-3)(x-6)$ (*d*) $(x(x-6)^2$

50. If the nullity of the matrix $\begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix}$ is 1, then the vlaue of k is

(*a*) –1 (*b*) 0]

(*c*) 1 (*d*) 2

51. If a 3 × 3 real skew-sysmmetric matrix has an eigen value $2i$, then one of the remaining eigen values is

(*a*) $\dfrac{1}{2i}$ (*b*) $-\dfrac{1}{2i}$

(*c*) 0 (*d*) 1

52. The minimal polynomial associated with the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is

(*a*) $x^3 - x^2 - 2x - 3$ (*b*) $x^3 - x^2 + 2x - 3$

(*c*) $x^3 - x^2 - 3x - 3$ (*d*) $x^3 - x^2 + 3x - 3$

53. Let T be the matrix (occurring in a typical transportation problem) given by $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ Then,

(*a*) Rank $T = 4$ and T is unimodular

(*b*) Rank $T = 4$ and T is not unimodular

(*c*) Rank $T = 3$ and T is unimodular

(*d*) Rank $T = 3$ and T is not unimodular

54. For the matrix $M = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$ which of the following statements are correct?

P : M is skew-Hermitian and M is Hermitian

Q : M is Hermitian and iM is skew-Hermitian

R : eigen values of iM are real

S : eigen values of iM are real

(*a*) P and R only (*b*) Q and R only

(*c*) P and S only (*a*) Q and S only

55. If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \dfrac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \dfrac{-1-i\sqrt{3}}{2} \end{bmatrix}$, then the trace of A^{102} is

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

56. Which of the following matrices is not diagonalizable?

(*a*) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (*b*) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$

(*c*) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (*d*) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

57. The number of linearly independent eigen vectors of the matrix $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ is

(*a*) 1 (*b*) 2
(*c*) 3 (*d*) 4

58. Consider the matrix $M = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$. Then,

(*a*) M has no real eigen values
(*b*) all real eigen values of M are positive
(*c*) all real eigen values of M are negative
(*d*) M has both positive and negative real eigen values

59. Let P be a $n \times n$ matrix with integral entries and $Q = P + \frac{1}{2}I$, where I denotes the $n \times n$ identity matrix, Then, Q is

(*a*) idempotent, *i.e.*, $Q^2 = Q$
(*b*) invertible
(*c*) nilpotent
(*d*) unipotent, *i.e.* $Q - l$ is nilpotent

60. Let A be a 2×2 orthogonal matrix of trace and determinant 1. Then, the angle between Au and $u (u = [1, 0]')$ is

(*a*) 15° (*b*) 30°
(*c*) 45° (*d*) 60°

61. For the matrix $\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$ the bound for the eigen the matrix values predicted by Gershgorin's theorem is

(*a*) 3 (*b*) 1
(*c*) 2 (*d*) 4

62. Let A be an $n \times n$ complex matrix whose characteristic polynomial is given by

$f(t) = t^n + c_{n-1} t^{n-1} + ... + c_1 t + c_0$. Then,

(*a*) $\det(A) = c_{n-1}$ (*b*) $\det(A) = c_0$
(*b*) $\det(A) = (-1)^n c_{n-1}$ (*d*) $\det(A) = (-1)^n c_0$

63. Let A be any $n \times n$ non-singular complex matrix and let $B = (A)^\theta A$, where $(A)^\theta$ is the conjugate transpose of A. If λ is an eigen value of B, then,

(*a*) λ is real and $\lambda < 0$ (*b*) λ is read and $\lambda \le 0$
(*c*) λ is real and $\lambda \ge 0$ (*d*) λ is real and $\lambda > 0$

ANSWERS—SET-II

1. (*d*)	**2.** (*d*)	**3.** (*b*)	**4.** (*a*)	**5.** (*d*)	**6.** (*a*)	**7.** (*d*)	**8.** (*b*)	**9.** (*d*)	**10.** (*c*)
11. (*d*)	**12.** (*b*)	**13.** (*b*)	**14.** (*d*)	**15.** (*c*)	**16.** (*b*)	**17.** (*c*)	**18.** (*d*)	**19.** (*d*)	**20.** (*a*)
21. (*d*)	**22.** (*a*)	**23.** (*c*)	**24.** (*d*)	**25.** (*b*)	**26.** (*a*)	**27.** (*c*)	**28.** (*d*)	**29.** (*a*)	**30.** (*b*)
31. (*b*)	**32.** (*d*)	**33.** (*a*)	**34.** (*d*)	**35.** (*c*)	**36.** (*b*)	**37.** (*c*)	**38.** (*d*)	**39.** (*c*)	**40.** (*c*)
41. (*d*)	**42.** (*d*)	**43.** (*b*)	**44.** (*c*)	**45.** (*b*)	**46.** (*d*)	**47.** (*c*)	**48.** (*d*)	**49.** (*d*)	**50.** (*d*)
51. (*c*)	**52.** (*a*)	**53.** (*d*)	**54.** (*b*)	**55.** (*d*)	**56.** (*d*)	**57.** (*d*)	**58.** (*d*)	**59.** (*b*)	**60.** (*d*)
61. (*a*)	**62.** (*d*)	**63.** (*d*)							

EXPLANATIONS SET -II

1. We need to find the eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$$

Since the matrix A is an upper triangular matrix. Therefore, the eigen values of A are 1, 1.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigen vector corresponding to eigen value $\lambda_i = 1$. Then $[A - 1I][x] = 0$ are equivalently.

$$\begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently $x_1 = 0$

x_2 is arbitrary.

$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 1$.

2. Let $P = \{p_{ij}\}$ be a 50×50 matrix, where $p_{ij} = \min\{i, j\}$; i, j 1, 2, ... 50. We need to find rank (P). The matrix P can be written as

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & 2 & \dots & 2 \\ 1 & 1 & 3 & 3 & 3 & \dots & 3 \\ 1 & 1 & 1 & 4 & 4 & \dots & 4 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots & 50 \end{bmatrix}$$

using $R_5 \to R_5 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, ..., $R_{50} \to R_{50} - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 2 & 2 & 2 & \dots & 2 \\ 0 & 0 & 2 & 2 & 2 & \dots & 2 \\ 0 & 0 & 0 & 3 & 3 & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 49 \end{bmatrix}$$

which is in now reduced echelon form. Therefore, the rank(P) = 50.

3. We are given that a system

$x + 3y + 2z = k$

$2x + y - 4z = 4$

$5x - 14z = 10.$

The augmented matrix is

$$[A : b] = \begin{bmatrix} 1 & 3 & 2 & \vdots & k \\ 2 & 1 & -4 & \vdots & 4 \\ 5 & 0 & -14 & \vdots & 10 \end{bmatrix}$$

Add-2 times the first row to the second row and –5 times the first row to the third row, we get

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & k \\ 0 & -5 & -8 & \vdots & 4-2k \\ 0 & -15 & -24 & \vdots & 10-5k \end{bmatrix}$$

Multiplying the second row by $\frac{1}{5}$

$$\sim \begin{bmatrix} 1 & 3 & 2 & \vdots & k \\ 0 & 1 & \frac{8}{5} & \vdots & \frac{4-2k}{5} \\ 0 & -15 & -24 & \vdots & 10-5k \end{bmatrix}$$

Hence, the system has infinitely may solution if $k = 2$.

4. We are given that a non-zero matrix A of order 3×3 such that $A^2 = 0$. We need to find the number of nonzero eigen values of A.

Since A is a nilpotent matrix of index 2. Hence, A has only zero eigen values.

5. Let $AX = b$ be a non-homogeneous system of linear equations. The augmented matrix is

$$[A : b] = \begin{bmatrix} 1 & 1 & -2 & 1 & \vdots & 1 \\ -1 & 2 & 3 & -1 & \vdots & 0 \\ 0 & 3 & 1 & 0 & \vdots & -1 \end{bmatrix}$$

Add 1 times the first row to the second row, we get

$$\sim \begin{bmatrix} 1 & 1 & -2 & 1 & \vdots & 1 \\ 0 & 3 & 1 & 0 & \vdots & 0 \\ 0 & 3 & 1 & 0 & \vdots & -1 \end{bmatrix}$$

Multiplying the second row by $\frac{1}{3}$, we get

$$\sim \begin{bmatrix} 1 & 1 & -2 & 1 & \vdots & 1 \\ 0 & 1 & \frac{1}{3} & 0 & \vdots & \frac{1}{3} \\ 0 & 3 & 1 & 0 & \vdots & -1 \end{bmatrix}$$

Add –3 times the second row to the third row, we set

$$\begin{bmatrix} 1 & 1 & -2 & 1 & \vdots & 1 \\ 0 & 1 & \frac{1}{3} & 0 & \vdots & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

which is in row reduced echelon form. Therefore, Rank $[A : b] = 2$ and Rank $A = 2$.

Hence, Rank $[A : b]$ = Rank $A = 2 < 3$. Thus system has infinite number of solutions.

6. We need to find the values of a and b such that the system of equation

$$x + y + z = 3$$

$$x + az = b$$

$$y + 2z = 3$$

has infinite number of solutions. The augmented matrix is

$$[A : b] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 0 & a & \vdots & b \\ 0 & 1 & 2 & \vdots & 3 \end{bmatrix}$$

Add–1 times the first row to the second row, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & -1 & a-1 & \vdots & b-3 \\ 0 & 1 & 2 & \vdots & 3 \end{bmatrix}$$

Using $R_2 \Leftrightarrow R_3$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 3 \\ 0 & -1 & a-1 & \vdots & b-3 \end{bmatrix}$$

Add second row to the third row, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 3 \\ 0 & 0 & a+1 & \vdots & b \end{bmatrix}$$

Which is in row reduced echelon form. Since, the system has infinite number of solution. Therefore

Rank $[A : \boldsymbol{b}]$ = Rank $A = 2$

Therefore, $a + 1 = 0$

and $6 = 0$

or $a = -1, b = 0$

7. We are given that a matrix A of order 3×3 with trace $(A) = 3$, $\det(A) = 2$ and 1 is an eigen value of A. We need to find the eigen values of the matrix $A_2 - 2I$.

Let λ_1, λ_2 be two other eigen values of matrix A. Then

trace $(A) = 3$

implies $1 + \lambda_1 + \lambda_2 = 3$

or $\lambda_1 + \lambda_2 = 2$

and det $\det(A) = 2$

implies $\lambda_1 \cdot \lambda_2 = 2$.

Now $(\lambda_1 - \lambda_2)^2 = (\lambda_1 + \lambda_2) - 4\lambda_1\lambda_2$

$= 4 - 8$

$= -4$

or equivalently

$\lambda_1 - \lambda_2 = 2i$

Solving the equations, $\lambda_1 + \lambda_2 = 2$

and $\lambda_1 - \lambda_2 = 2i$, we get

$\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$

Therefore, the eigen values of the matrix A are 1, $1 + i$, $1 - i$. Hence, the eigen values of matrix A^2 are I^2, $(1 + i)^2$, $(I - i)^2$ that is 1, $2i$, $-2i$. Therefore, the eigen values of the matrix $A^2 - 2I$ are $1 - 2$, $2i - 2i$ and $-2i - 2$ that is -1, $2i - 2$, $-2i - 2$. That is -1, $2(i - 1)$, $-2(i + 1)$.

8. We need to find the least positive integer n such that

$$A = \begin{bmatrix} \cos\dfrac{\pi}{4} & \sin\dfrac{\pi}{4} \\ -\sin\dfrac{\pi}{4} & \cos\dfrac{\pi}{4} \end{bmatrix}^n$$

is the identity matrix of order 2. Then

$$A = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Therefore, $A^2 = \dfrac{1}{2}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^8 = A^4 \cdot A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $n = 8$

9. We are given that two square matrices A and B of order 3 such that rank $(AB) = 1$. Then

$\det(AB) = 0$

or equivalently $\det(A)\det(B) = 0$

or equivalently either

$\det(A) = 0$ or $\det(B) = 0$

or Both $\det(A) = \det(B) = 0$.

Therefore, $\det(BA) = \det(B) \cdot \det(A) = 0$.

Hence, BA is singular. Thus the order BA can not be 3.

10. We are given that a matrix A of order 3×3 with eigen values 1, –1 and 3. Since 1 is the eigen value of the matrix A. Therefore,

$$|A - I| = 0$$

Hence $|A^2 - A| = |A(A - I)|$

$$= |A| \cdot |A - I| = 0$$

So, the matrix $A^2 - A$ *is* singular. Since –1 is the eigen value of the matrix A. Therefore, $|A + I| = 0$.

Now

$$|A^2 + A| = |A(A - I)|$$
$$= |A||A + I| = 0$$

So, the matrix $A^2 + A$ is singular, Since 3 is the eigen value of the matrix A. Therefore, $|A - 3I| = 0$.

Now,

$$|A_2 - 3A| = |A(A - 3)|$$
$$= |A||A - 3| = 0.$$

Hence A^2 - $3A$ is singular.

11. We need to find the value of λ for which the system of linear equations

$$2x + 3y + 4z = 13$$
$$5x + 7y + 7z = 26$$
$$9x + 13y + 1\,5z = 13\lambda$$

has infinitely many solutions. The augmented matrix is

$$[A : b] = \begin{bmatrix} 2 & 3 & 4 & \vdots & 13 \\ 5 & 7 & 7 & \vdots & 26 \\ 9 & 13 & 15 & \vdots & 13\lambda \end{bmatrix}$$

Using $R_3 \to R_3 - R_2 - 2R_1$, $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 2 & 3 & 4 & \vdots & 13 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 13\lambda - 52 \end{bmatrix}$$

Since, the system has infinitely many solutions. Therefore,

$$\text{Rank } [A : b] = \text{Rank } A = 2$$

Hence $13\lambda - 52 = 0$

or $\lambda = 4$

12. We are given that a matrix P of order 3×3 with eigen values 1, 2 and 3. Since none of the eigen value zero. Hence Pis invertible matrix and since all the eigen values of P and distinct. Therefore P is diagonalizable.

13. We need to find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

Using $R_4 \to R_4 - R_3$, $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$,

we get $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Using $R_3 \to R_3 - R_2$, $R_2 \to R_4 - R_2$,

we get $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Using $R_2 \to R_1 - R_2$, we get $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is in row reduced echelon form. Therefore, Rank of the matrix is 2.

14. We need to find P^{50}, where

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $P^2 = P \cdot P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$P^4 = P^2 \cdot P^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$\therefore \quad P^8 = P^4.P^4 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 & 36 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{16} = P^8.P^8 = \begin{bmatrix} 1 & 8 & 36 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 8 & 36 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 16 & 136 \\ 0 & 1 & 16 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad P^{32} = P^{16}.P^{16} = \begin{bmatrix} 1 & 16 & 136 \\ 0 & 1 & 16 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 16 & 136 \\ 0 & 1 & 16 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 32 & 528 \\ 0 & 1 & 32 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad P^{48} = P^{32}.P^{16} = \begin{bmatrix} 1 & 32 & 528 \\ 0 & 1 & 32 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 16 & 136 \\ 0 & 1 & 16 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 48 & 1176 \\ 0 & 1 & 48 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad P^{50}.P^2 = \begin{bmatrix} 1 & 48 & 1176 \\ 0 & 1 & 48 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 50 & 1275 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

15. We need to find the characteristic equations of the matrix $P = \{p_{ij}\}$, where

$p_{tj} = 1$ for $1 \le i, j \le m$.

Let $m = 2$. Then

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The characteristic equation of P is

$$|P - \lambda I| = 0$$

or equivalently $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

or equivalently $\lambda^2 - 2\lambda = 0$.

16. We need to find the Nullity of the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Using $R_2 \to R_2 - 2R_1$, and $R_3 \to R_3 - 7R_1$, we get

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

Using $R_2 \to R_2 - 2R_3$, we get

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

Using $R_2 \Leftrightarrow R_3$, we get

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & -21 & -14 & -29 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in row reduced echelon form.

Therefore, Rank $A = 2$.

Using Rank-Nullity theorem,

Nullity $A = 4 -$ Rank $A = 4 - 2 = 2$.

17. We need to find the eigen values of the nilpotent matrix N of the index 4 with real entries. Since N is the Nilpotent matrix of index 4. Therefore

$$N^4 = 0$$

Therefore, the eigen values λ of N are given by

$$\lambda^4 = 0$$

implies $\lambda = 0$

Thus, 0 is the only eigen value of N.

18. We need to find the eigen vector of the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ corresponding to the eigen values $(1 - i)$ and $(1 - i)$.

Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector Corresponding to the eigen value $(1 + i)$. Then

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, the corresponding system of equation is

$$-ix_1 + x_2 = 0$$

or $x_2 = ix_1$

or equivalently $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} x_1$

Thus, the eigen vector corresponding to the eigen value 1 is $\begin{bmatrix} 1 \\ i \end{bmatrix}$ or $\begin{bmatrix} i \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Let $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$ be the eigen vector corresponding to the eigen value $1 - i$. Then

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently $\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Therefore, the corresponding system of equation is

$$ix_1' + x_2' = 0$$

or equivalently

$$ix_2' = x_1'$$

or $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} x_1'$

Thus, the eigen vector corresponding to $1 - i$ is

$\begin{bmatrix} 1 \\ -i \end{bmatrix}$ or $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

Hence, option (*d*) is correct.

19. We are given that the non-homogeneous system $Ax = b$, where A is a square matrix by order n. If the matrix A and the augmented matrix $[A : b]$ have the same rank r, where $r < n$ then the system has infinite many solutions.

20. We need to find the two eigen values of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{a}{2} \\ 0 & 1 & 0 & -\frac{a}{2} \\ 0 & 0 & 1 & -\frac{a}{2} \\ 0 & 0 & 0 & a \end{bmatrix}$$ which is an upper triangular matrix.

Therefore, the eigen values are diagonal entries. Hence, 1 and a are two eigen values.

21. We are given that P and Q be two $n \times n$ non-zero matrices such that

$$P + Q = 0$$

or equivalently $P = -Q$

Thus Q is obtained by P just multiplying every row of P by -1 . But the multiplication of every row by -1 is an elementary transformations which can not change the rank of matrix.

22. We need to find the value of k for which the system of homogeneous equations $Px = 0$, where

$$P = \begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-2 & -k+20 \\ 0 & k-8 & -5k \end{bmatrix}$$

has non-trivial solution. For this det(P) = 0 or equivalently

$$(k-2)(-5k) - (k-8)(-k+2) = 0$$

or $-5k^2 + 10k + k^2 - 10k + 16 = 0$

or $-4k^2 + 16 = 0$

or $k = \pm 2$.

Thus, if $k = \pm 2$, then rank P is less than 3 and hence the system of equation $Px = 0$ has infinitely many solutions.

23. We need to find the value of P^{24} if

$$P = \begin{bmatrix} 0 & w \\ w & 0 \end{bmatrix}$$

where w is the complex cube root of unity.

Thus, $P^2 = PP = \begin{bmatrix} 0 & w \\ w & 0 \end{bmatrix} \begin{bmatrix} 0 & w \\ w & 0 \end{bmatrix}$

$$= \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix}$$

Therefore,

$$P^{24} = (P^2)^{12}$$

$$= \left(\begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix} \right)^{12}$$

$$= \begin{bmatrix} w^{24} & 0 \\ 0 & w^{24} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

24. We are given that

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Using $R_3 \to R_2 - R_3$ and $R_2 \to R_2 - R_1$, we get

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = k.$$

Therefore, $k = 0$

Now we need to find the value of determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$$

in terms of k using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_3$, we get

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{vmatrix} = 0 = k.$$

25. We need to find the value of μ for which the system of equations

$$3x + 2y + z = 10$$
$$2x + 3y + 2z = 10$$
$$x + 2y + \mu z = 10$$

is inconsistent. The augmented matrix is

$$[A : b] = \begin{bmatrix} 3 & 2 & 1 & \vdots & 10 \\ 2 & 3 & 2 & \vdots & 10 \\ 1 & 2 & \mu & \vdots & 10 \end{bmatrix}$$

Using $R_2 \to R_2 - \frac{2}{3} R_1$, $R_3 \to R_3 - \frac{1}{3} R_2$ we get

$$\sim \begin{bmatrix} 3 & 2 & 1 & \vdots & 10 \\ 0 & \frac{5}{3} & \frac{4}{3} & \vdots & \frac{10}{3} \\ 0 & \frac{4}{3} & \mu - \frac{1}{3} & \vdots & \frac{20}{3} \end{bmatrix}$$

Using $R_3 \to R_3 - \frac{4}{5} R_2$, we get

$$= \begin{bmatrix} 3 & 2 & 1 & \vdots & 10 \\ 0 & \frac{5}{3} & \frac{4}{3} & \vdots & \frac{10}{3} \\ 0 & 0 & \left(\mu - \frac{1}{3} - \frac{16}{15}\right) & \vdots & 4 \end{bmatrix}$$

The system of equations is inconsistent if

Rank $[A : b] >$ Rank A.

For thus, we have

$$\mu - \frac{1}{3} - \frac{16}{15} = 0$$

or $\mu = 1.4$

26. We need to find A^T if $A = I - 2uu^T$, where u is the unit column vector.

Since $A = I - 2uu^T$

Taking the transpose of both sides, we get

$$\begin{aligned} A^T &= (I - 2uu^T)\mathrm{T} \\ &= I^T - (2uu^T)^T \\ &= I - 2(uu^T)^T \\ &= I - 2(u^T)^T u^T \\ &= I - 2uu^T \\ &= \boldsymbol{A}. \end{aligned}$$

27. We need .to find the number of solutions of the system of homogeneous linear equations

$$9x + 3y + z = 0$$
$$3x + z - 6w = 0$$
$$x + y + z + w = 0$$
$$-6y + z + 9w = 0$$

The coefficient matrix is

$$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Using $R_1 \to R_1 - 9R_3$ and $R_2 \to R_2 - 3R_3$,

$$\sim \begin{bmatrix} 0 & -6 & -8 & -9 \\ 0 & -3 & -2 & -9 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Using $R_1 \to R_1 - 2R_2$ and $R_4 \to R_4 - 2R_2$,

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & 5 & 27 \\ 0 & 0 & -4 & 9 \end{bmatrix}$$

Using $R_4 \to R_4 - \frac{4}{5} R_3$,

$$\sim\begin{bmatrix}1 & 1 & 1 & 1\\0 & -3 & -2 & -9\\0 & 0 & 5 & 27\\0 & 0 & 0 & \frac{153}{5}\end{bmatrix}$$

which is in row reduced echelon form, Thus, Rank $A = 4$. Therefore, the system has only one solution *i.e.* trivial solution.

28. We need to find the inverse of the matrix

$$A = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$$

The adjoint of matrix A is of the form

$$adj\,(A) = \begin{bmatrix}-3 & \dots & \dots\\ \dots & -3 & \dots\\ \dots & \dots & -3\end{bmatrix}$$

and $\det(A) = 5$

Therefore,

$$A^{-1} = \frac{1}{\det(A)}\,adj\,(A)$$

$$= \frac{1}{5}\begin{bmatrix}-3 & \dots & \dots\\ \dots & -3 & \dots\\ \dots & \dots & -3\end{bmatrix}$$

$$= \frac{1}{5}[A - 4I]$$

29. Let

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

and $$B = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Then $$AB = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

and $$BA = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Thus, the eigen values of AB and BA is 1. Also the eigen vector corressponding to the eigen value i is same. Hence, option (b), (c) and (d) are incorrect. Therefore, option (a) is correct.

30. We need to find the values of a and b for which the system of equations

$$ax + y + 3z = a$$
$$2x + by - z = 3$$
$$5x + 7y + z = 7$$

has infinite number of solutions. The augmented matrix

$$[A : b] = \left[\begin{array}{ccc|c}a & 1 & 3 & a\\2 & b & -1 & 3\\5 & 7 & 1 & 7\end{array}\right]$$

Using $R_1 \to R_1 - \frac{a}{5}R_3$ and $R_2 \to R_2 - \frac{2}{5}R_3$

$$\sim\left[\begin{array}{ccc|c}0 & 41 - \frac{7a}{5} & 3 - \frac{a}{5} & \frac{-2a}{5}\\0 & b - \frac{14}{5} & \frac{-7}{5} & \frac{1}{5}\\5 & 7 & 1 & 7\end{array}\right]$$

Using $R_1 \to R_1 - \frac{\left(1 - \frac{7a}{5}\right)}{\left(b - \frac{14}{5}\right)}R_2$, and $R_1 \Leftrightarrow R_2$

$$\sim\left[\begin{array}{ccc|c}5 & 7 & 1 & 7\\0 & b - \frac{14}{5} & -\frac{7}{5} & \frac{1}{5}\\0 & 0 & \left(3 - \frac{a}{5}\right) + \frac{7}{5}\frac{\left(1 - \frac{7a}{5}\right)}{\left(b - \frac{14}{5}\right)} & -\frac{2a}{5} - \frac{1}{5}\frac{\left(1 - \frac{7a}{5}\right)}{\left(b - \frac{14}{5}\right)}\end{array}\right]$$

Thus, the system has infinite number of solutions if

Rank $[A : b]$ = Rank $A = 2$.

For this, we have

$$\left(3 - \frac{a}{5}\right) + \frac{7}{5}\frac{\left(1 - \frac{7a}{5}\right)}{\left(b - \frac{14}{5}\right)} = 0$$

and $$-\frac{2a}{5} + \frac{1}{5}\frac{\left(1 - \frac{7a}{5}\right)}{\left(b - \frac{14}{5}\right)} = 0$$

Solving for a and b, we get $a = 1$, $b = 3$

31. We are given that the matrix $P = \begin{bmatrix}1 & i\\i & -1\end{bmatrix}$

Now $$|P| = \begin{vmatrix}1 & i\\i & -1\end{vmatrix} = -1 + 1 = 0$$

Hence, P is singular. The characteristic equations of P is

$$|P - \lambda I| = 0$$

or equivalently $\begin{vmatrix} 1-\lambda & i \\ i & -1-\lambda \end{vmatrix} = 0$

Expanding the determinant, we get

$$-(1^2 - \lambda^2) - i^2 = 0$$

$$(1 - \lambda^2) = 1$$

$$\lambda^2 = 0$$

$$\lambda = 0, 0.$$

Hence, algebraic multiplicity of the eigen value $\lambda = 0$ is 2. Now let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = 0$. Then

$$\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Add-i times the first row to the second row, we get

$$\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, the corresponding system of equation is

$$x_1 + ix_2 = 0$$

or $x_1 = -ix_2$

or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Therefore $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is the only eigen vector corresponding to the eigen value $\lambda = 0$. is 1. Hence, P is not diagonalizable.

32. Let P be an $n \times n$ idempotent matrix such that P is non-singular.

Then

$$P^2 = P$$

and P^{-1} exist.

Therefore, $P^{-1}P^2 = P^{-1}P$

or equivalently $P = I$

Thus, non-singular idempotent matrix is the identity matrix. Since, there may be infinite number of $n \times n$ non-singular matrices.

33. We need to find the eigen vector corresponding to the eigen values i and $-i$ of the matrix

$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = i$. Then

$$\begin{bmatrix} 0-i & 1 \\ -1 & 0-i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Add i times the first row to the second row, we get

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, the corresponding system of equations is

$$-ix_1 + x_2 = 0$$

or $ix_1 = x_2$

If we now assign the free variable x_1 arbitrary r, Then the above equation can be written parametrically as

$$x_1 = r, x_2 = ir$$

or equivalently $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = r\begin{bmatrix} 1 \\ i \end{bmatrix}$

If we take $r = 1$, then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

let $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = -i$, then

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Add i times the first row to the second row, we set

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, the corresponding system of equation is

$$ix_1' + x_2' = 0$$

or equivalently $x_2' = ix_1'$

If we assign the free variable x_1 arbitrary values s, then the above equation can be written parametrically as

$$x_1' = s, x_2' = -is$$

or equivalently $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} s \\ -is \end{bmatrix} = s\begin{bmatrix} 1 \\ -i \end{bmatrix}$

If we take $s = -1$, then $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 \\ +i \end{bmatrix}$.

Thus, the eigen vector corresponding to the eigen values $\lambda = i, -i$ are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} -1 \\ i \end{bmatrix}$ respectively.

34. Let P be a 3 × 3 matrix such that for some b the linear system $Px = b$ has infinite number of solutions then Rank $P \le 2$.

35. We are given that the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we need to find A^{-1}

Now we want to reduce A to the identity matrix by row operations and then apply the same sequence of operations to I to produce A^{-1}. Here are the computations.

$$\left[\begin{array}{ccccc:ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Add –1 times the first row to the second, third, fourth fifth row, we get

$$\left[\begin{array}{ccccc:ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

36. We need to find the number of value of λ for which the system of equations

$$\lambda x + (\lambda + 3)y = 10z$$
$$(\lambda - 1)x + (\lambda - 2)y = 5z$$
$$2x + (\lambda + 4)y = \lambda z$$

has infinitely many solutions. The coefficient matrix of the system of equations is

$$A = \begin{bmatrix} \lambda & \lambda + 3 & -10 \\ \lambda - 1 & \lambda - 2 & -5 \\ 2 & \lambda + 4 & -\lambda \end{bmatrix}$$

Using $R_2 \to R_2 - R_1$, we get

$$\sim \begin{bmatrix} \lambda & \lambda + 3 & -10 \\ -1 & -5 & 5 \\ 2 & \lambda + 4 & -\lambda \end{bmatrix}$$

Again using $R_3 \to R_3 - R_2$ and $R_1 \to R_1 + \lambda R_2$, we get

$$\begin{bmatrix} 0 & -4\lambda + 3 & 5\lambda - 10 \\ -1 & -5 & 5 \\ 0 & \lambda - 6 & -\lambda + 10 \end{bmatrix}$$

Therefore, the determinant of A is

$$|A| = (-4\lambda + 3)(-\lambda + 10) - (\lambda - 6)(5\lambda - 10)$$
$$= -\lambda^2 - 3\lambda - 10$$

The system of equations ahs infinitely many solutions if

$$|A| = 0$$

or equivalent $\quad y - \lambda^2 - 3\lambda - 10 = 0$

or equivalently $\quad \lambda^2 + 3\lambda + 10 = 0$

which is a quadratic equation hence λ has two values.

37. Let

$$M = \begin{bmatrix} 1 & 1 + i & 2 - i \\ 1 - i & 2 & 3 + i \\ 2 + i & 3 - i & 3 \end{bmatrix}$$

if $\quad B = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$, where $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

are linearly independent eigen vectors of M. We need to find the main diagonal of the matrix $B^{-1}MB$. Since $(j, i)^{th}$ entry of M is the complex conjugate of $(j, i)^{th}$ entry of M. Therefore, M is a Hermitian matrix and hence the eigen values of M are real. Now $B^{-1}MB$ is the diagonal form of M. Thus the diagonal entries of $B^{-1}MB$ are eigen values of M and hence diagonal entries of $B^{-1}NB$ are real.

39. We are given that a system of equation $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

we need to find the number of solutions of system $Ax = b$. The augmented matrix is

$$[A : b] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 3 & 2 & 1 & \vdots & 2 \\ 4 & 3 & 2 & \vdots & 3 \end{bmatrix}$$

Add-3 times the first row to the second row and –4 times the first row to the fourth row, we get

Multiplying the second row by –1, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -2 & \vdots & -1 \\ 0 & -1 & -2 & \vdots & -1 \end{bmatrix}$$

Multiplying the second row by –1, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & -1 & -2 & \vdots & -1 \end{bmatrix}$$

Add second row to the third row, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

which k in row reduced echelon form.

Therefore,

$$\text{Rank } [A: b] = \text{Rank } A = 2 < 3.$$

So, the system $Ax = b$ has infinitely infinitely many solutions.

40. We are given that the matrices A and B are real symmetric matrices. Therefore

$$A' = A \text{ and } B' = B.$$

Now we are given that $AB = BA$.

Therefore,

$$(AB)' = (BA)' = A'B' = AB$$

Thus, AB is a real symmetric matrix and hence all eigen value of AB are real.

41. We know that the system of a linear equations in n variables has infinitely many solution if the rank of the coefficient matrix is less than n.

42. We are given that the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 1 \\ 3 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Now } |A| = 2\begin{vmatrix} 4 & 0 & 1 \\ 1 & 5 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 40$$
$$\neq 0.$$

Hence 0 can not be eigeri value of matrix A because the $|A| \neq 0$. Hence, option (*b*),(*c*) are incorrect.

Since

$$|A - 1I| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 \\ 3 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Therefore, 1 is an eigen value of A. Hence, option (*d*) is correct.

43. We are given that a system of linear homogeneous equations

The coefficient matrix of the system of linear equations is

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

$$\text{Now, } \quad |A| = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix} = 0$$

Hence, rank $A < 3$. So the system of linear equations have infinitely many solutions.

44. We are given that the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

We need to find the matrix A^{-1} in terms of matrix A. The characteristic equations of matrix A is

$$|A - \lambda I| = 0$$

or equivalently

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

By expanding the determinant, we get

$-\lambda[(1-\lambda)^2 - 1] + [2 - 1 + \lambda] = 0$

or equivalently $\quad -\lambda^3 + 2\lambda^2 + \lambda + 1 = 0$

or equivalently $\quad \lambda^3 - 2\lambda^2 - \lambda - 1 = 0$

But we know that according to the Caley Hamiltan theorem the characteristic equation of matrix A is satisfied by matrix A. Therefore, we have

$$A^3 - 2A^2 - A - I = 0$$

or equivalently $\quad I = A^3 - 2A^2 - A$

Multiplying both sides by A^{-1}, we get

$$A^{-1} \cdot I = A^{-1}A^3 - 2A^{-1}A^2 - A^{-1}A$$

or equivalently $A^{-1} = A^2 - 2A - I$

45. Let A be an $n \times n$ matrix such that $x^I Ax > 0$ for every non-zero vector x in R^n, then A is positive definite matrix and hence, all the eigen values of matrix A are positive.

46. Let $\alpha = e^{\frac{2\pi i}{5}}$ and the matrix

$$M = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$$

We need to find trace of the matrix $I + M + M^2$.

Now $\quad \alpha = e^{\frac{2\pi i}{5}}$

$$= (-1)^{\frac{2}{5}} = (1)^{\frac{1}{5}}$$

which implies $\alpha^5 = 1$

Thus α is the fifth root of unity.

Therefore, $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

and $\quad 1 \cdot \alpha \cdot \alpha^2 \cdot \alpha^3 \cdot \alpha^4 = 1$

Now $\quad M = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$

Trace $(M) = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$

Using $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$, we get

Trace $(M) = 0$

Next,

$$M^2 = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$$

$\Rightarrow \quad M^2 =$

$$\begin{bmatrix} 1 & \alpha + \alpha^2 & \alpha^2 + \alpha^3 + \alpha^4 & \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 & \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 \\ 0 & \alpha^2 & \alpha^3 + \alpha^4 & \alpha^4 + \alpha^5 + \alpha^6 & \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 \\ 0 & 0 & \alpha^4 & \alpha^5 + \alpha^6 & \alpha^6 + \alpha^7 + \alpha^8 \\ 0 & 0 & 0 & \alpha^6 & \alpha^7 + \alpha^8 \\ 0 & 0 & 0 & 0 & \alpha^8 \end{bmatrix}$$

Therefore,

Trace $(M^2) = 1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8$

$$= \frac{1(1-(\alpha^2)^5)}{1-a^2} = \frac{1 - e^{4\pi i}}{1 - e^{\frac{4\pi i}{5}}}$$

Using $\alpha = e^{\frac{2\pi i}{5}}$,

we get Trace $(M^2) = \dfrac{1-1}{1 - e^{\frac{4\pi i}{5}}} = 0$

Also trace $(I)_{5\times5} = 5$

Therefore, trace $(I + M + M^2)$ = trace (I) + trace (M) + trace (M^2)

$$= 5 + 0 + 0$$

$$= 5$$

47. We are given that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

We need to find A^{50}.

Now $\quad A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^4 = A^2 \cdot A^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A^8 = A^4 \cdot A^4$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix},$$

$$A^{16} = A^8 \cdot A^8$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix},$$

$$A^{32} = A^{16} \cdot A^{16}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 16 & 1 & 0 \\ 16 & 0 & 1 \end{bmatrix}$$

$$A^{48} = A^{32} \cdot A^{16}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 16 & 1 & 0 \\ 16 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 16 & 1 & 0 \\ 16 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 24 & 1 & 0 \\ 24 & 0 & 1 \end{bmatrix}$$

Therefore, $A^{50} = A^{48} \cdot A^2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 24 & 1 & 0 \\ 24 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 24 & 1 & 0 \\ 24 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

48. We need to find the distinct eigen values of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

which implies $\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

Expanding the determinant, we get

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 0 & -\lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & 1-\lambda \\ 0 & 0 \end{vmatrix} = 0$$

or equivalently $\lambda^2(2 - \lambda) = 0$

which implies $\lambda = 0, 2$ are distinct eigen values.

49. We need to find the minimal polynomial of the matrix

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

which implies $\begin{vmatrix} 3-\lambda & 3 & 0 \\ 3 & 3-\lambda & 0 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 0$

or equivalently $\lambda(\lambda - 6)^2 = 0$

Therefore, the minimal-polynomial of A is either $\lambda(\lambda - 6)$ or $\lambda(\lambda - 6)^2$

But $A(A - 6I) = 0$

Hence, the minimal polynomial is

$$x(x - 6)^2 = 0$$

50. We are given that the nullity of the matrix

$$\begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

is 1. We need to find the value of k. Since, the nullity of A is 1. Therefore, using Ran K-Nullity Theorem, we have

Rank (A) + Nullity (A) = 3 which implies

Rank (A) = 3 – 1 = 2

Therefore, Det (A) = 0

which implies $\begin{vmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = 0$

which implies $k = -1$.

51. We are given that a 3 × 3 real skew symmetric matrix has an eigen value $2i$. We need to find the remaining eigen values of A. Since every Skew-symmetric matrix has eigen values either complex conjugate or zero. Therefore, the eigen values of 3 × 3 real Skew symmetric matrix are $2i$, $-2i$ and 0.

52. We need to find the minimal polynomial of the matrix

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Comparing the matrix A with companion matrix

$$\begin{bmatrix} 0 & 0 & -a \\ 1 & 0 & -b \\ 0 & 1 & -c \end{bmatrix}$$

We get $a = -3, b = -2, c = -1$

Therefore, the characteristic equation is

$$\lambda^3 - \lambda^2 - 2\lambda - 3 = 0$$

Which is same as minimal polynomial of the matrix because its eigen values are distinct.

53. We are given that T is a matrix (occurring in a typical transportation Problem) given by

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in row reduced echelon form. Therefore.

Rank $(T) = 3$

and hence $\det(T) = 0$

Thus, T is not unimodular because $\det(T) \neq \pm 1$.

54. We are given that a matrix

$$M = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$$

Therefore, $\overline{M} = \begin{bmatrix} 2 & 3-2i & -4 \\ 3+2i & 5 & -6i \\ -4 & -6i & 3 \end{bmatrix}$

Thus $M^\theta = (\overline{M})^t$

$$= \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$$

Hence, M is a Hermitian matrix.

Now, $iM = \begin{bmatrix} 2i & -2+3i & -4i \\ 2+3i & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$

Therefore,

$$(\overline{iM}) = \begin{bmatrix} -2i & -2-3i & 4i \\ 2-3i & -5i & -6 \\ 4i & 6 & -3i \end{bmatrix}$$

Hence, $(iM)^\theta = (\overline{iM})^t$

$$= \begin{bmatrix} -2i & 2+3i & 4i \\ -2-3i & -5i & 6 \\ 4i & -6 & -3i \end{bmatrix} = \begin{bmatrix} 2i & -2-3i & -4i \\ 2+3i & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

$= -(iM)$

Hence iM is Skew Hermitian matrix. Hence statement (Q) is correct.

Now, we know that a complex Hermitian matrix has real eigen values. Hence, eigen values of M are real, therefore statement (R) is correct.

55. We are given that a matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$$

We need to find the trace of A102.

Since, the matrix A is a lower triangular matrix. Therefore, the rigen value of matrix A are 1, $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ that is 1, ω, ω^2, where $\omega = \frac{-1+i\sqrt{3}}{2}$ and $w^2 = \frac{-1-i\sqrt{3}}{2}$.

Therefore, trace

$$(A^{102}) = 1^{102} + \omega^{102} + (\omega^{2)102}$$

$$= 1 + (\omega^3)^{34} + (\omega^3)^{68}$$

$$= 1 + 1 + 1$$

$$= 3.$$

56. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Then eigen values of matrix A are $\lambda = 1, 1$. Therefore, the algebraic multiplicity of A is 2.

Now, $[A - 1\lambda] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Therefore, the Geometric multiplication of A is 1.

Thus, the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

57. We need to find the number of linearly independent eigen vectors of the matrix.

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Now, $$A = \left[\begin{array}{cc|cc} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Therefore, the characteristic equation of A is

$$(3 - \lambda)(4 - \lambda)(\lambda^2 - 3\lambda - 2) = 0.$$

Thus, all eigen values of matrix A are real and distinct. So, there exist unique eigen vector corresponding to every eigen values because the eigen vectors corresponding to distinct eigen values are linearly independent.

58. We are given that a matrix $M = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

Since, M is Hermitian matrix, so its eigen values are real and trace $(M) = 0$. Therefore, some eigen values are positive and some are negative, because trace (M) = sum of eigen values

59. Let P be a $n \times n$ matrix with integral entries and $Q = P + \frac{1}{2}I$, where I denote the $n \times n$ identity matrix.

Let,

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $\det(Q) = ad + \frac{1}{4} + (a + d)\frac{1}{2} - bc$

$$= ad - bc + \frac{(a + d)}{2} + \frac{1}{4}$$

$$\neq 0.$$

Therefore, Q is invertible. Hence option (b) is correct.

60. Let A be a 2×2 orthogonal matrix of trace 1 and determinant 1. We need to find the angle between Au and u (where $u = [1, 0]^t$)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

which implies $Au = \begin{bmatrix} a \\ c \end{bmatrix}$

Let v be the angle between Au and u. which implies

$$\cos\theta = a \cdot 1 + c \cdot 0 = a$$

Now $ad - bc = 1$

$a + d = 1$

Now as $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the orthogonal matrix. Then its rows are orthogonal vectors.

$$a^2 + b^2 = 1 = c^2 + a^2$$

$$ac + bd = 0$$

which implies $ac = -bd$

or equivalently $b = -\frac{ac}{d}$

or equivalently $a^2 + \frac{a^2c^2}{d^2} = 1$

or equivalently $a^2 + \frac{a^2}{d^2} - a^2 = 1$

or equivalently $a^2 = a^2$

Therefore, $a^2 = (1 - a)^2$

which implies $2a - 1 = 0$

or equivalently $a = \frac{1}{2}$

Thus $\cos\theta = \frac{1}{2}$

which implies $\theta = 60°$.

61. For the matrix $M = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$

We need to find the bounds for the eigen values predicted by Gershgorin's theorem.

Now, for columns $|\lambda| \leq \max\{1, 3, -1, 3\}$

and For rows $|\lambda| \leq \max\{1, 3, -1, 3\}$

Therefore, $|\lambda| \leq 3$.

62. Let A be an $n \times n$ complex matrix whose characteristic polynomial is

$$f(t) = t^n + c_{n-1}t^{n-1} + \ldots + c_1 t + c_0$$

$$= t^n - \text{trace}(A)t^{n-1} + \ldots + (-1)^n |A|$$

Therefore,

$$\det(A) = (-1)^n c_0$$

63. Let A be $n \times n$ non-singular complex matrix and let $B = \overline{A}^t A$ where $(\overline{A})^t$ is the conjugate transpose of A, and λ is an eigen value of B.

If λ is eigen value of B, then λ is an eigen value of A.

Thus $\lambda = \overline{\lambda}_1 \cdot \lambda_1 = |\lambda_1|^2$

which implies $\lambda > 0$ and $\lambda \neq 0$

Since A is singular matrix. So B is also singular matrix. Now as B is Hermitian matrix, therefore, it has always real eigen values.