CHAPTER 1

Introduction

This chapter deals with basic ideas about the open-loop and closed-loop control systems. The differential equations describe the dynamic operation of control systems. The Laplace transform transforms the differential equation into an algebraic equation, the solution is obtained in the transform domain. The time domain solution is determined by taking the inverse Laplace transform.

CONTENTS

CONTROL SYSTEM

A control system is a combination of elements arranged in a planned manner wherein each element causes an effect to produce a desired output. This cause and effect relationship is governed by a mathematical relation.

If the aforesaid mathematical relation is linear the control system is termed as linear control system. For a linear system the cause (independent variable or input) and the effect (dependent variable or output) are proportionally related and principle of superposition is applicable throughout the operating range of a system.

In a control system the cause acts through a control process which in turn results into an effect.

There may be variety of systems based on the principle mentioned above but all the systems have many features in common and as such common approach for the study and analysis of control systems is possible.

Control systems are used in many applications for example, systems for the control of position, velocity, acceleration, temperature, pressure, voltage and current etc.

1.1 AN EXAMPLE OF CONTROL ACTION

Control of a room temperature is achieved by switching ON and switching OFF of a power supply to a heating appliance. Thus power supply to an appliance is switched ON, when the room temperature is felt low and switched OFF, when the desired temperature is reached.

The above system can be modified, if the duration of application of power is predetermined to achieve the room temperature within desired limits.

However, a further refinement can be made by measuring the difference between the actual room temperature and the desired room temperature and this difference being the error is used to control the element which in turn controls the output, *i.e.* room temperature.

The above description indicates that in the former case the output (room temperature) has no control on the input and the control action is purely based on a sort of predetermined calibration only, where as in the latter case the control action is affected by a feedback received from the output to the input.

1.2 OPEN-LOOP CONTROL SYSTEM

Having explained the concept of control action, a control system can be described by a block diagram as shown in Fig. 1.2.1.

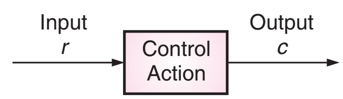

Fig. 1.2.1. Open-loop control system.

The input r controls the output c through a control action process. In the block diagram shown in Fig. 1.2.1, it is observed that the output has no effect on the control action. Such a system is termed as open-loop control system.

In an open-loop control system the output is neither measured nor fedback for comparison with the input. Faithfulness of an open-loop control system depends on the accuracy of input calibration.

1.3 CLOSED-LOOP CONTROL SYSTEM

In a closed-loop control system the output has an effect on control action through a feedback as shown in Fig. 1.3.1 and hence closed-loop control systems are also termed as feedback control systems. The control action is actuated by an error signal e which is the difference between the input signal r and the output signal c. This process of comparison between the output and input maintains the output at a desired level through control action process.

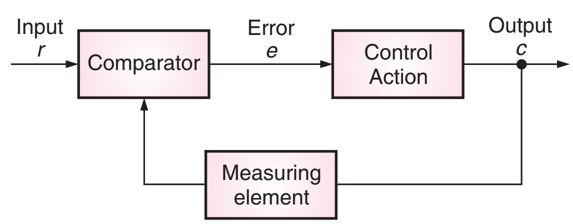

Fig. 1.3.1. Closed-loop control system.

The control systems without involving human intervention for normal operation are called automatic control systems.

A closed-loop (feedback) control system using a power amplifying device prior to controller and the output of such a system being mechanical *i.e.* position, velocity, acceleration is called servomechanism.

Comparison of Open-Loop and Closed-Loop Control System

Open-Loop	*Closed-Loop*
1. The accuracy of an open-loop system depends on the calibration of the input. Any departure from pre-determined calibration affects the output.	1. As the error between the reference input and the output is continuously measured through feedback, the closed-loop system works more accurately.
2. The open-loop system is simple to construct and cheap.	2. The closed-loop system is complicated to construct and costly.
3. The open-loop systems are generally stable.	3. The closed-loop systems can become unstable under certain conditions.
4. The operation of open-loop system is affected due to the presence of non-linearities in its elements.	4. In terms of the performance the closed-loop system adjusts to the effects of non-linearities present in its elements.

1.4 USE OF LAPLACE TRANSFORMATION IN CONTROL SYSTEMS

The control action for a dynamic control system whether electrical, mechanical, thermal, hydraulic etc. can be represented by a differential equation and the output response of such a dynamic system to a specified input can be obtained by solving the said differential equation. The system differential equation is derived according to physical laws governing a system in question.

In order to facilitate the solution of a differential equation describing a control system, the equation is transformed into an algebraic form. The differential equation wherein time being the independent variable is transformed into a corresponding algebraic equation by using Laplace transformation technique and the differential equation thus transformed is known as the equation in frequency domain. Hence, Laplace transform technique transforms a time domain differential equation into a frequency domain algebraic equation.

1.5 LAPLACE TRANSFORM

In order to transform a given function of time $f(t)$ into its corresponding Laplace transform first multiply $f(t)$ by e^{-st}, s being a complex number ($s = \sigma + j\omega$). Integrate this product w.r.t. time with limits as zero and infinity. This integration results in Laplace transform of $f(t)$, which is denoted by $F(s)$ or $\mathcal{L} f(t)$.

The mathematical expression for Laplace transform is,

$$\mathcal{L} f(t) = F(s) \qquad t \geq 0$$

or

$$F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt \qquad \text{...(1.1)}$$

The term "Laplace transform of $f(t)$" is used for the letter $\mathcal{L} f(t)$.

The time function $f(t)$ is obtained back from the Laplace transform by a process called inverse Laplace transformation and denoted as $\mathcal{L}^{-1}$ thus

$$\mathcal{L}^{-1}[\mathcal{L} f(t)] = \mathcal{L}^{-1}[F(s)] = f(t)$$

The time function $f(t)$ and its Laplace transform $F(s)$ are a transform pair.

Table 1.5 gives transform pairs of some commonly used functions and Laplace transform pairs for some functions are derived here under.

1.5.1 Derivation of Laplace transform

1. Laplace transform of e^{at}

$$\mathcal{L}\, e^{at} = \int_0^{\infty} e^{at} \cdot e^{-st}\, dt = \int_0^{\infty} e^{(a-s)t}\, dt = \frac{1}{(s-a)}$$

$\therefore$
$$\mathcal{L}\, e^{at} = \frac{1}{(s-a)} \qquad \text{...(1.2)}$$

As the inverse Laplace transform is denoted by the letter $\mathcal{L}^{-1}$ and, therefore, the inverse Laplace transform of $\frac{1}{(s-a)}$ is e^{at} and expressed as below,

$$\mathcal{L}^{-1}\left[\frac{1}{(s-a)}\right] = e^{at} \qquad \text{...(1.3)}$$

2. In the function $f(t) = e^{at}$ put $a = 0$

$\therefore$ $e^{at} = e^{0t} = 1.$ Hence, $f(t) = 1$

Therefore, using Eq. (1.2) $\mathcal{L}[1] = \frac{1}{(s-0)}$

or
$$\mathcal{L}[1] = \frac{1}{s} \qquad \text{...(1.4)}$$

and
$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 \qquad \text{...(1.5)}$$

3. In the function $f(t) = e^{at}$ put $a = j\omega$

$\therefore$ $e^{at} = e^{j\omega t}.$ Hence, $f(t) = e^{j\omega t}$

Therefore, using Eq. (1.2) $\mathcal{L}\, e^{j\omega t} = \frac{1}{(s - j\omega)}$

$\because$ $e^{j\omega t} = (\cos \omega t + j \sin \omega t)$

$\therefore$ $\mathcal{L}(\cos \omega t + j \sin \omega t) = \frac{1}{(s-j\omega)} = \frac{s + j\omega}{(s^2 + \omega^2)}$

Separating into real and imaginary parts,

$$\mathcal{L} \cos \omega t = \frac{s}{(s^2 + \omega^2)} \qquad \text{...(1.6)}$$

$$\mathcal{L} \sin \omega t = \frac{\omega}{(s^2 + \omega^2)} \qquad \text{...(1.7)}$$

and
$$\mathcal{L}^{-1}\left[\frac{s}{(s^2 + \omega^2)}\right] = \cos \omega t \qquad \text{...(1.8)}$$

$$\mathcal{L}^{-1}\left[\frac{\omega}{(s^2 + \omega^2)}\right] = \sin \omega t \qquad \text{...(1.9)}$$

4. In the function $f(t) = e^{at}$ put $a = (-\alpha + j\omega)$

$\therefore$ $e^{at} = e^{(-\alpha + j\omega)\, t}$

Hence, $f(t) = e^{(-\alpha + j\omega)\, t}$

Therefore, using Eq. (1.2)

$$\mathcal{L}\, e^{(-\alpha + j\omega)\, t} = \frac{1}{s - (-\alpha + j\omega)} = \frac{1}{(s+\alpha) - j\omega}$$

$\because$ $$e^{(-\alpha + j\omega)\, t} = e^{-\alpha t} (\cos \omega t + j \sin \omega t)$$

$\therefore$ $$\mathcal{L}\, e^{-\alpha t} (\cos \omega t + j \sin \omega t) = \frac{1}{(s+\alpha) - j\omega} = \frac{(s+\alpha) + j\omega}{(s+\alpha)^2 + \omega^2}$$

Separating into real and imaginary parts,

$$\mathcal{L} e^{-\alpha t} \cdot \cos \omega t = \frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2} \qquad ...(1.10)$$

$$\mathcal{L} e^{-\alpha t} \cdot \sin \omega t = \frac{\omega}{(s+\alpha)^2 + \omega^2} \qquad ...(1.11)$$

and $$\mathcal{L}^{-1} \left[\frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2} \right] = e^{-\alpha t} \cdot \cos \omega t \qquad ...(1.12)$$

$$\mathcal{L}^{-1} \left[\frac{\omega}{(s+\alpha)^2 + \omega^2} \right] = e^{-\alpha t} \cdot \sin \omega t \qquad ...(1.13)$$

5. In the function $f(t) = e^{at}$ put $a = 1$

$\therefore$ $e^{at} = e^{1 \cdot t} = e^t$. Hence, $f(t) = e^t$

Therefore, using Eq. (1.2) $\mathcal{L} e^t = \dfrac{1}{(s-1)}$

$\because$ $$e^t = 1 + t + \frac{t^2}{\angle 2} + \frac{t^3}{\angle 3} + \ldots$$

and $$\frac{1}{(s-1)} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + \ldots$$

Table 1.5. Table of Laplace transform pairs

S.No.	$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1	$\delta(t)$ unit impulse at $t = 0$	1
2	$u(t)$ unit step at $t = 0$	$\frac{1}{s}$
3	$u(t - T)$ unit step at $t = T$	$\frac{1}{s} e^{-sT}$
4	t	$\frac{1}{s^2}$
5	$\frac{t^2}{2}$	$\frac{1}{s^3}$
6	t^n	$\frac{\angle n}{s^{n+1}}$
7	e^{-at}	$\frac{1}{s+a}$
8	e^{at}	$\frac{1}{s-a}$

9	te^{-at}	$\frac{1}{(s+a)^2}$
10	te^{at}	$\frac{1}{(s-a)^2}$
11	$t^n e^{-at}$	$\frac{\angle n}{(s+a)^{n+1}}$
12	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
13	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
14	$e^{-\alpha t} \sin \omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
15	$e^{-\alpha t} \cos \omega t$	$\frac{(s+\alpha)}{(s+\alpha)^2+\omega^2}$
16	$\sinh \alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
17	$\cosh \alpha t$	$\frac{s}{s^2-\alpha^2}$

∴ Comparing the terms

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[t] = \frac{1}{s^2}$$

$$\mathcal{L}\left[\frac{t^2}{\angle 2}\right] = \frac{1}{s^2}$$

$$\mathcal{L}\left[\frac{t^n}{\angle n}\right] = \frac{1}{s^{n+1}} \quad \text{or} \quad \mathcal{L}[t^n] = \frac{\angle n}{s^{n+1}} \qquad \text{...(1.14)}$$

and

$$\mathcal{L}^{-1}\left[\frac{\angle n}{s^{n+1}}\right] = t^n \qquad \text{...(1.15)}$$

1.5.2 Basic Laplace Transform Theorems

Basic theorems of Laplace transform are given below :

1. Laplace transform of linear combination

$$\mathcal{L}[a\, f_1(t) + bf_2(t)] = aF_1(s) + bF_2(s) \qquad \text{...(1.16)}$$

where $f_1(t)$, $f_2(t)$ are functions of time and a, b are constants.

2. If the Laplace transform of $f(t)$ is $F(s)$, then

(i) $\mathcal{L}\,\frac{df(t)}{dt} = [s\,F(s) - f(0+)]$

(ii) $\mathcal{L}\,\frac{d^2 f(t)}{dt^2} = [s^2 F(s) - s\,f(0+) - f'(0+)]$

(iii) $\mathcal{L}\,\frac{d^3 f(t)}{dt^3} = [s^3 F(s) - s^2 f(0+) - sf'(0+) - f''(0+)]$ ...(1.17) where $f(0+), f'(0+), f''(0+)$ ... are the values of $f(t), \frac{df(t)}{dt} \frac{d^2 f(t)}{dt^2}$ ... at $t = (0+)$

3. If the Laplace transform of $f(t)$ is $F(s)$, then

$$(i)\ \mathcal{L} \int f(t) = \left[\frac{F(s)}{s} + \frac{f^{-1}(0+)}{s}\right]$$

$$(ii)\ \mathcal{L} \iint f(t) = \left[\frac{F(s)}{s^2} + \frac{f^{-1}(0+)}{s^2} + \frac{f^{-2}(0+)}{s}\right]$$

$$(iii)\ \mathcal{L} \iiint f(t) = \left[\frac{F(s)}{s^3} + \frac{f^{-1}(0+)}{s^3} + \frac{f^{-2}(0+)}{s^2} + \frac{f^{-3}(0+)}{s}\right] \quad ..(1.18)$$

where $f^{-1}(0+), f^{-2}(0+), f^{-3}(0+)$... are the values of $\int f(t), \iint f(t), \iiint f(t)$ ... at $t = (0+)$.

4. If the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L} e^{-at} f(t) = F(s+a)$$

5. If the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L} t\, f(t) = -\frac{d}{ds} F(s)$$

6. Initial value theorem

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\, \mathcal{L} f(t) \quad ...(1.19\ a)$$

or

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\, F(s) \quad ...(1.19)$$

7. Final value theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, \mathcal{L} f(t) \quad ...(1.20\ a)$$

or

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, F(s) \quad ...(1.20)$$

The final value theorem gives the final value ($t \to \infty$) of a time function using its Laplace transform and as such very useful in the analysis of control systems. However, if the denominator of $s\,F(s)$ has any root having real part as zero or positive, then the final value theorem is not valid.

1.6 SOLVED EXAMPLES

Example 1.6.1. *Find the inverse Laplace transform of the following functions :*

(*i*) $F(s) = \frac{1}{s(s+1)}$ (*ii*) $F(s) = \frac{s+6}{s(s^2+4s+3)}$

(*iii*) $F(s) = \frac{1}{s^2+4s+8}$ (*iv*) $F(s) = \frac{s+2}{s^2+4s+6}$

(*v*) $F(s) = \frac{5}{s(s^2+4s+5)}$ (*vi*) $F(s) = \frac{s^2+2s+3}{s^3+6s^2+12s+8}$.

Solution. (*i*) $F(s) = \frac{1}{s(s+1)}$

Using partial fraction expansion

$$F(s) = \frac{k_1}{s} + \frac{k_2}{s+1}$$

The coefficients can be determined as $k_1 = 1$ and $k_2 = -1$

$$\therefore \quad F(s) = \frac{1}{s} - \frac{1}{s+1}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

or

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{1}{s+1}$$

$$\therefore \quad f(t) = (1 - e^{-t}) \quad \textbf{Ans.}$$

(*ii*)

$$F(s) = \frac{s+6}{s(s^2+4s+3)}$$

The term $(s^2 + 4s + 3)$ can be factorised as $(s + 1)(s + 3)$

$$\therefore \quad F(s) = \frac{s+6}{s(s+1)(s+3)}$$

Using partial fraction expansion

$$F(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+3}$$

The coefficients can be determined as $k_1 = 2$, $k_2 = -2.5$ and $k_3 = \frac{1}{2}$.

$$\therefore \quad F(s) = 2\cdot\frac{1}{s} - 2.5\cdot\frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+3}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\left[2\cdot\frac{1}{s} - 2.5\cdot\frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+3}\right]$$

or

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} 2\cdot\frac{1}{s} - \mathcal{L}^{-1} 2.5\cdot\frac{1}{s+1} + \mathcal{L}^{-1}\frac{1}{2}\cdot\frac{1}{s+3}$$

$$\therefore \quad f(t) = (2 - 2.5\, e^{-t} + \tfrac{1}{2} e^{-3t}) \quad \textbf{Ans.}$$

(*iii*)

$$F(s) = \frac{1}{s^2+4s+8}.$$

on completing the square, the term $(s^2 + 4s + 8)$ can be expressed as $[(s + 2)^2 + (2)^2]$

$$\therefore \quad F(s) = \frac{1}{[(s+2)^2+(2)^2]}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\frac{1}{[(s+2)^2+(2)^2]}$$

or

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\frac{1}{2}\cdot\frac{2}{(s+2)^2+(2)^2}$$

$\therefore \quad f(t) = \frac{1}{2} e^{-2t} \sin(2t)$ **Ans.**

(*iv*) $$F(s) = \frac{s+2}{s^2+4s+6}$$

on completing the square the term $(s^2 + 4s + 6)$ can be expressed as $[(s+2)^2 + (\sqrt{2})^2]$

$$\therefore \quad F(s) = \frac{s+2}{(s+2)^2 + (\sqrt{2})^2}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{s+2}{(s+2)^2 + (\sqrt{2})^2}$$

$\therefore \quad f(t) = e^{-2t} \cos \sqrt{2}\, t$ **Ans.**

(*v*) $$F(s) = \frac{5}{s(s^2+4s+5)}$$

Using partial fraction expansion

$$\frac{5}{s(s^2+4s+5)} = \frac{k_1}{s} + \frac{k_2 s + k_3}{s^2+4s+5}$$

The coefficients are determined as $k_1 = 1$, $k_2 = -1$ and $k_3 = -4$

$$\therefore \quad F(s) = \frac{1}{s} - \frac{s+4}{s^2+4s+5}$$

on completing the square, the term $(s^2 + 4s + 5)$ can be expressed as $[(s + 2)^2 + (1)^2]$

$$\therefore \quad F(s) = \frac{1}{s} - \frac{s+4}{[(s+2)^2+(1)^2]}$$

$$= \frac{1}{s} - \frac{s+2}{[(s+2)^2+(1)^2]} - 2\frac{1}{[(s+2)^2+(1)^2]}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+(1)^2} - 2\frac{1}{(s+2)^2+(1)^2}\right]$$

or $$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{s+2}{[(s+2)^2+(1)^2]} - \mathcal{L}^{-1}\, 2\frac{1}{[(s+2)^2+(1)^2]}$$

$\therefore \quad f(t) = (1 - e^{-2t} \cos t - 2e^{-2t} \sin t)$ **Ans.**

(*vi*) $$F(s) = \frac{s^2+2s+3}{s^3+6s^2+12s+8}$$

The denominator $(s^2 + 6s^2 + 12s + 8)$ can be expressed as $(s + 2)^3$

$$\therefore \quad F(s) = \frac{s^2+2s+3}{(s+2)^3}$$

Using partial fraction expansion

$$F(s) = \frac{k_1}{(s+2)} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3}$$

The coefficients are determined as $k_1 = 1$, $k_2 = -2$ and $k_3 = 3$

$$\therefore \quad F(s) = \frac{1}{s+2} - \frac{2}{(s+2)^2} + \frac{3}{(s+2)^3}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\left[\frac{1}{s+2} - \frac{2}{(s+2)^2} + \frac{3}{(s+2)^3}\right]$$

or
$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\frac{1}{s+2} - \mathcal{L}^{-1}\frac{2}{(s+2)^2} + \mathcal{L}^{-1}\frac{3}{(s+2)^3}$$

$$\therefore \quad f(t) = e^{-2t} - 2t\,e^{-2t} + \tfrac{3}{2}t^2\,e^{-2t}$$

or
$$f(t) = e^{-2t}\,[1 - t(2 - \tfrac{3}{2}t)]$$ **Ans.**

Example 1.6.2. *Obtain the solution of the differential equation given below*

$$2\frac{dx}{dt} + 8x = 10; \text{ given } x(0+) = 2.$$

Solution. Taking Laplace transform on both sides the following equation is obtained :

$$\mathcal{L}\left[2\frac{dx}{dt} + 8x\right] = \mathcal{L}10 \quad \text{or} \quad \mathcal{L}\left[2\frac{dx}{dt}\right] + \mathcal{L}[8x] = \mathcal{L}10$$

$$\therefore \quad 2[s(X(s) - x(0+)] + 8[X(s)] = \frac{10}{s}$$

Substituting $x(0+) = 2$

$$2[s\,X(s) - 2] + 8[X(s)] = \frac{10}{s}$$

Simplifying,
$$X(s) = \frac{2s+5}{s(s+4)}$$

Using partial fraction expansion

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+4}$$

The coefficients can be determined as $k_1 = 1.25$ and $k_2 = 0.75$

$$\therefore \quad X(s) = \frac{1.25}{s} + \frac{0.75}{s+4}$$

Taking inverse Laplace transform on both sides

$$x(t) = (1.25 + 0.75\,e^{-4t}).$$ **Ans.**

Example 1.6.3. *Obtain the solution of the differential equation given below*

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0, \text{ given } x(0+) = 0 \text{ and } x'(0+) = 1.$$

Solution. Taking Laplace transform on both sides the following equation is obtained :

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] + \mathcal{L}\left[2\frac{dx}{dt}\right] + \mathcal{L}[2x] = 0$$

or
$$[s^2X(s) - sx(0+) - x'(0+)] + [2s\,X(s) - x(0+)] + 2\,[X(s)] = 0$$

Substituting $x(0+) = 0$ and $x'(0) = 1$

or $$[s^2X(s) - s \cdot 0 - 1] + 2[s\,X(s) - 0] + 2[X(s)] = 0$$

Simplifying, $$X(s) = \frac{1}{(s^2 + 2s + 2)}$$

On completing the square, the term $(s^2 + 2s + 2)$ can be expressed as $[(s + 1)^2 + (1)^2]$

$$\therefore \quad X(s) = \frac{1}{[(s+1)^2 + (1)^2]}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1}X(s) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2 + (1)^2}\right]$$

$$\therefore \quad x(t) = e^{-t} \sin t$$ **Ans.**

Example 1.6.4. *Find the Laplace transform of the differential equation given below and hence evaluate the time solution of the same given that $y(0+) = 0$ and $y'(0+) = 6$.*

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 12\,e^t.$$

Solution. Taking Laplace transform on both sides, the following equation is obtained :

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + \mathcal{L}\left[5\frac{dy}{dt}\right] + \mathcal{L}[6y] = \mathcal{L}[12e^t]$$

$$\therefore \quad [s^2Y(s) - sy(0+) - y'(0+)] + 5[sY(s) - y(0+)] + 6[y(s)] = \frac{12}{(s-1)}$$

Substituting $y(0+) = 0$ and $y'(0+) = 6$

$$[s^2Y(s) - s \cdot 0 - 6] + 5[sY(s) - 0] + 6[Y(s)] = \frac{12}{(s-1)}$$

or $$[s^2Y(s) - 6] + 5[sY(s)] + 6[Y(s)] = \frac{12}{(s-1)}$$

Simplifying, $$Y(s) = \frac{6s+6}{(s-1)(s^2+5s+6)}$$

The term $(s^2 + 5s + 6)$ can be expressed as $(s + 2) \,.\, (s + 3)$

$$\therefore \quad Y(s) = \frac{6s+6}{(s-1)(s+2)(s+3)}$$

Using partial fraction expansion

$$Y(s) = \frac{k_1}{s-1} + \frac{k_2}{s+2} + \frac{k_3}{s+3}$$

The coefficients can be determined as $k_1 = 1$, $k_2 = 2$ and $k_3 = -3$

$$\therefore \quad Y(s) = \frac{1}{s-1} + \frac{2}{s+2} - \frac{3}{s+3}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\left[\frac{1}{s-1} + \frac{2}{s+2} - \frac{3}{s+3}\right]$$

or $$\mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\frac{1}{s-1} + \mathcal{L}^{-1}\frac{2}{s+2} - \mathcal{L}^{-1}\frac{3}{s+3}$$

$\therefore$ $$y(t) = (e^{+t} + 2e^{-2t} - 3e^{-3t})$$ **Ans.**

Example 1.6.5. *The Laplace transformed equation for the charging current of a capacitor arranged in series with a resistance is given by,*

$$I(s) = \frac{Cs}{RCs + 1} \cdot E(s)$$

The circuit is connected to a supply voltage of E. If E = 100 V, R = 2 MΩ, C = 1 μF; calculate the initial value of the charging current.

Solution. Since $E = 100 \quad \therefore \quad E(s) = \frac{100}{s}$

Substituting the given values,

$$I(s) = \frac{1 \times 10^{-6} s}{[(2 \times 10^{6} \times 1 \times 10^{-6} s) + 1]} \cdot \frac{100}{s}$$

Applying the initial value theorem,

$$i(0+) = \lim_{t \to 0} i(t) = \lim_{s \to \infty} sI(s)$$

$$= \lim_{s \to \infty} s \cdot \frac{1 \times 10^{-6} s}{[(2 \times 10^{6} \times 1 \times 10^{-6} s) + 1]} \cdot \frac{100}{s}$$

$$= \lim_{s \to \infty} \frac{1}{\left[2 \times 10^{6} + \frac{1}{1 \times 10^{-6} s}\right]} \cdot 100 = \frac{100}{2 \times 10^{6}} = 50\ \mu\text{A}.$$ **Ans.**

Example 1.6.6. *A series circuit consisting of resistance R and an inductance of L is connected to a d.c. supply voltage of E. Derive an expression for the steady state value of the current flowing in the circuit.*

Solution. The differential equation relating the current $i(t)$ flowing in the circuit and the input voltage E is given by

$$E = Ri(t) + L\frac{di(t)}{dt}.$$

Taking Laplace transform of this equation yields,

$$E(s) = RI(s) + L[sI(s) - i(0+)]$$

because $$i(0+) = 0$$

$\therefore$ $$E(s) = RI(s) + LsI(s)$$

$\because$ E is constant (d.c. voltage)

$\therefore$ $$E(s) = \frac{E}{s} \qquad \therefore \quad \frac{E}{s} = RI(s) + Ls\,I(s)$$

or $$I(s) = \frac{1}{R + Ls} \cdot \frac{E}{s}$$

Applying the final value theorem,

$$i_{ss} = \lim_{t \to \infty} i(t) = \lim_{s \to 0} sI(s)$$

$$= \lim_{s \to 0} s \cdot \frac{1}{R + Ls} \cdot \frac{E}{s} = \lim_{s \to 0} \frac{E}{R + Ls} = \frac{E}{R}$$

$\therefore$ $$i_{ss} = \frac{E}{R} \quad \textbf{Ans.}$$

Example 1.6.7. *A system is represented by a relation given below :*

$$X(s) = R(s) \frac{100}{s^2 + 2s + 50}$$

if r(t) = 1.0 unit, find the value of x(t) when t → ∞.

Solution. Since $r(t) = 1$

$\therefore$ $R(s) = \frac{1}{s}$ $\therefore$ $X(s) = \frac{1}{s} \cdot \frac{100}{(s^2 + 2s + 50)}$

According to the final value theorem,

$$\underset{t \to \infty}{x(t)} = \lim_{s \to 0} sX(s)$$

$$= \lim_{s \to 0} s \frac{1}{s} \frac{100}{(s^2 + 2s + 50)} = \frac{100}{50} = 2.0 \text{ units.} \quad \textbf{Ans.}$$

Example 1.6.8. *The Laplace transform of the error signal in a control system is expressed by a relation*

$$E(s) = R(s) \frac{s}{s^2 + 6s + 25}$$

Calculate the steady state value of the error if r(t) = t.

Solution. Since $r(t) = t$

$\therefore$ $R(s) = \frac{1}{s^2}$ $\therefore$ $E(s) = \frac{1}{s^2} \cdot \frac{s}{(s^2 + 6s + 25)}$

According to final value theorem

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} s \frac{1}{s^2} \frac{s}{(s^2 + 6s + 25)}$$

$$= \lim_{s \to 0} \frac{1}{s^2 + 6s + 25} = \frac{1}{25} = 0.4 \text{ unit.} \quad \textbf{Ans.}$$

Example 1.6.9. *Find f (0 +), f′(0 +) and f″(0 +) for the function whose Laplace transform is given below :*

$$F(s) = \frac{4s + 1}{s(s^2 + 2)}.$$

Solution. (1) $f(0+)$ is determined as follows :

$$f(0+) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,F(s) \text{ (Initial value theorem)}$$

$$= \lim_{s \to \infty} s \cdot \frac{4s + 1}{s(s^2 + 2)} = \lim_{s \to \infty} \frac{4s + 1}{(s^2 + 2)} = \lim_{s \to \infty} \frac{4 + \frac{1}{s}}{s + \frac{2}{s}} = 0$$

$\therefore$ $f(0+) = 0.$ **Ans.**

(2) $f'(0+)$ is determined as follows :

$$\mathcal{L}f'(t) = s\mathcal{L}f(t) - f(0+) = s \cdot \frac{4s+1}{s(s^2+2)} - 0 = \frac{4s+1}{(s^2+2)}$$

$$f'(0+) = \lim_{t \to 0} f'(t) = \lim_{s \to \infty} s\mathcal{L}f'(t)$$

$$= \lim_{s \to \infty} s\frac{4s+1}{s^2+2} = \lim_{s \to \infty} \frac{4s+1}{\left(s+\frac{2}{s}\right)} = \lim_{s \to \infty} \frac{4+\frac{1}{s}}{1+\frac{2}{s^2}} = 4$$

$\therefore$ $f'(0+) = 4$. **Ans.**

(3) $f''(0+)$ is determined as follows :

$$\mathcal{L}f''(t) = s^2\,\mathcal{L}f(t) - s\,f(0) - f'(0+)$$

$$= s^2 \cdot \frac{4s+1}{s(s^2+2)} - s \cdot 0 - 4 = s \cdot \frac{4s+1}{(s^2+2)} - 4 = \frac{s-8}{(s^2+2)}$$

$$f''(0+) = \lim_{t \to 0} f''(t) = \lim_{s \to \infty} s\mathcal{L}f''(t)$$

$$= \lim_{s \to \infty} s \cdot \frac{(s-8)}{(s^2+2)} = \lim_{s \to \infty} \frac{(s-8)}{\left(s+\frac{2}{s}\right)} = \lim_{s \to \infty} \frac{\left(1-\frac{8}{s}\right)}{\left(1+\frac{2}{s^2}\right)} = 1$$

$f''(0+) = 1$. **Ans.**

Example 1.6.10. *The Laplace transform of f (t) is given by*

$$F(s) = \frac{4}{s(s+2)}$$

Find the final value using the final value theorem and verify the result by determining f (t) using inverse Laplace transform.

Solution. (1) The final value is the value of $f(t)$ as $t \to \infty$.

As per final value theorem,

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)$$

$$= \lim_{s \to 0} s\frac{4}{[s(s+2)]} = \lim_{s \to 0} \frac{4}{(s+2)} = 2$$

$\therefore$ $\lim_{t \to \infty} f(t) = 2$. **Ans.**

(2) The expression for $f(t)$ is determined as follows :

$$F(s) = \frac{4}{s(s+2)}$$

Using partial fraction expansion

$$F(s) = \frac{k_1}{s} + \frac{k_2}{s+2}$$

The coefficients can be determined as $k_1 = 2$ and $k_2 = -2$.

$\therefore$
$$F(s) = \frac{2}{s} - \frac{2}{s+2}.$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2}{s+2}\right]$$

or
$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2}{s+2}\right]$$

$\therefore$
$$f(t) = 2 - 2e^{-2t}$$

and
$$\underset{t \to \infty}{f(t)} = 2 - 0 = 2. \text{ Verified.}$$

PROBLEMS

1.1. Find the inverse Laplace transform of the functions given below :

(*i*) $F(s) = \dfrac{1}{(s+1)(s+4)}$ (*ii*) $F(s) = \dfrac{s}{s^2 + 6s + 13}$

(*iii*) $F(s) = \dfrac{3s}{(s^2+1)(s^2+4)}$ (*iv*) $F(s) = \dfrac{s + 0.25}{(s+0.5)^2}$

(*v*) $F(s) = \dfrac{1}{s} \cdot \dfrac{2}{(s+1)(s+2)}$.

1.2. Express the following differential equations in Laplace transform form :

(*i*) $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 8x = 0$, given $x(0+) = 0$ and $x'(0+) = 2$

(*ii*) $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 5x = 5$, given $x(0+) = 0$ and $x'(0+) = 0$

(*iii*) $\dfrac{d\theta}{dt} + 6\theta + 5\int \theta\, dt = 0$, given $\theta(0+) = 0$, $\theta'(0+) = 0$ and $\theta^{-1}(0+) = 0.1$

(*iv*) $2\dfrac{dx}{dt} + 4x = te^{-3t}$, $x(0+) = 0$.

1.3. By using Laplace transform determine the ratio $\dfrac{C(s)}{R(s)}$ for the differential equation given below

$$\frac{d^2c(t)}{dt^2} + 4\frac{dc(t)}{dt} + c(t) = r(t).$$

Assume all initial conditions as zero. If $r(t) = 1$ evaluate the time solution for $c(t)$.

1.4. Find the initial value of the functions having following Laplace transform

(*i*) $F(s) = \dfrac{(s+\alpha)}{[(s+\alpha)^2 + \omega^2]}$ (*ii*) $F(s) = \dfrac{2(2s+1)}{s(s^2 + 0.5s + 4)}$.

1.5. Find the final value of the functions having following Laplace transform :

(*i*) $F(s) = \dfrac{(s+\alpha)}{(s+\alpha)^2 + \omega^2}$ where $\alpha > 0$ (*ii*) $F(s) = \dfrac{2(s+1)}{s(s^2 + 4s + 5)}$.