1

Basic Concepts of Vibrations

1.1 Introduction

Vibration is an oscillation wherein the quantity is a parameter that defines the motion of a mechanical system. The main causes of vibration are :

1. Unbalanced forces in the machine
2. Dry friction between two mating surfaces
3. External excitation
4. Earthquakes
5. Wind-self-excited vibrations
6. Misalignment of rotating shafts
7. Loosenes in rotating machinery, loose foundations and excessive bearing clearances
8. Oil whirl in bearing.

The harmful effects of vibrations are :

1. Excessive stresses in machine parts
2. Undesirable noise, and
3. Looseness of parts and partial or complete failure of parts.

The uses of vibration phenomenon are in the following areas :

1. Musical instruments
2. Vibrating conveyors
3. Vibrating screens
4. Shakers
5. Stress relieving
6. Washing machines, and
7. Medical machinery for massaging unwanted bulges on patients.

The elimination or reduction of undesirable vibrations can be achieved by one or more of the following methods :

1. By removing the cause of vibrations
2. By Vibration isolation
3. By using shock absorbers
4. By Installing dynamic vibration absorbers.

1.2 Classification of Vibrations

The spectrum of vibrations is shown in Fig. 1.1. For the definitions of various types of vibrations, the readers may consult the Glossary of Terms given at the end of the book

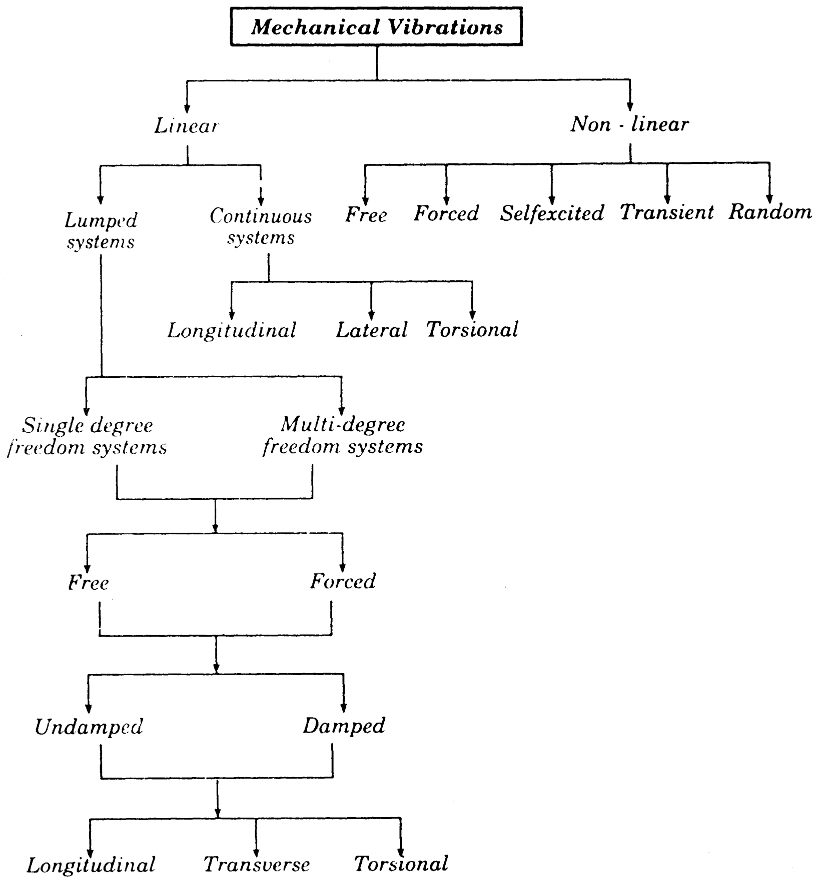

Fig. 1.1 Spectrum of mechanical vibrations.

1.3 Characteristics of Vibrations

The vibration characteristics are :

1. Displacment
2. Velocily
3. Acceleration, and
4. Phase.

1.**Displacement** - The total distance travelled by a vibrating part from one extreme position to the other extreme position is refered to as peak-to-peak displacement. It is measured in μm (10^{-6}m) or mm.

Quite often, the amplitude of displacement in simple harmonic motion is expressed by its average value, X_{avg} or the root mean square value X_{rms}.

$$X_{avg} = \frac{1}{T}\int_0^T x(t)\, dt$$

and

$$X_{rms} = \left[\frac{1}{T}\int_0^T \{x(t)\}^2\, dt\right]^{1/2}$$

where T = averege time

For simple harmonic motion, let $x(t) = X \sin \omega t$

$$\text{then } X_{avg} = \frac{1}{T}\int_0^T X \sin \omega t\, dt$$

$$= \frac{X}{T}\left|\frac{-\cos \omega t}{\omega}\right|_0^T$$

$$= \frac{X}{\omega T}[-\cos \omega T + 1]$$

$$= \frac{X}{2\pi}[-\cos 2\pi + 1] \qquad [\because \omega T = 2\pi]$$

$$= 0$$

$$X_{rms} = \left[\frac{1}{T}\int_0^T X^2 \sin^2 \omega t\, dt\right]^{1/2}$$

$$= \frac{X}{\sqrt{T}}\left[\int_0^T \frac{1}{2}(1 - \cos 2\omega t)\, dt\right]^{1/2}$$

$$= \frac{X}{\sqrt{T}}\left[\left|\frac{1}{2}\left(t - \frac{\sin 2\omega t}{2\omega}\right)\right|_0^T\right]^{1/2}$$

$$= \frac{X}{\sqrt{T}}\left[\frac{1}{2}\left(T - \frac{\sin 2\omega T}{2\omega}\right)\right]^{1/2}$$

$$= \frac{X}{\sqrt{T}}\left[\frac{1}{2}\left(T - \frac{\sin 4\pi}{2\omega}\right)\right]^{1/2}$$

$$= \frac{X}{\sqrt{T}} \times \frac{\sqrt{T}}{\sqrt{2}} = \frac{X}{\sqrt{2}}$$

2. Velocity — The speed of a vibrating mass keeps on changing continuously. The velocity is maximum at the mean position and minimum at the extrem positions. The peak value of velocity is selected for measucement.

3. Acceleration — The acceleration of a vibrating mass is maximum at the extreme positions and zero at the mean position. The acceleration is expressed in terms of acceleration due to gravity *i.e.* $g(= 9.80665 \text{ m/s}^2)$.

4. Phase — It is defined as the position of a vibrating mass with respect to a fixed point or another vibrating mass at the given instant.

The other characteristics related to vibrations are :

(*a*) Amplitude (A) — It is the maximum displacement of a vibrating mass from the mean position.

(*b*) Time period (T) — It is the time taken by a vibrating mass to complete one cycle.

(*c*) Frequency (f) — It is the number of cycles completed per second by a vibrating mass. It is measured in Hertzs (Hz).

For more difinitions, refer to Glossary of terms at the end of the book.

1.4 HARMONIC MOTION

A body is said to have simple harmonic motion when its acceleration is proportional to the displacement from the mean position and is always directed towards the mean position.

It is expressed as :

$$x(t) = A \sin \omega t$$

where $A =$ amplitude

$\omega =$ angular speed, rad/s $= \dfrac{2\pi}{T} = 2\pi f$

$T =$ time period, s

$f =$ frequency, c/s (Hz)

Velocity, $v = \dot{x}(t) = \omega A \cos \omega t$

$$= \omega A \sin\left(\omega t + \frac{\pi}{2}\right)$$

Acceleration, $a = \ddot{x}(t) = -\omega^2 A \sin\omega t = -\omega x$ ...(1.1*a*)

$= \omega^2 A \sin(\omega t + \pi)$ ...(1.1*b*)

or $a \, \alpha - x$

The simple harmonic motion of point P is shown in Fig. 1.2

Consider a point P moving along a circle with uniform angular speed ω. After time t let the point P make an angle $\theta = \omega t$ with the vertical. Drop PM perpendicular on the x- axis, as shown in Fig. 1.2 (*a*).

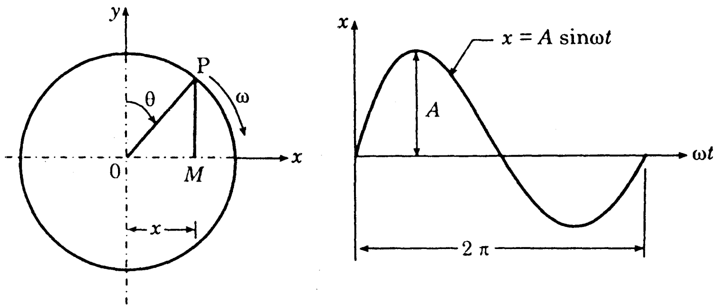

Fig. 1.2 Simple harmonic motion.

As the point P moves in a circle the motion of M is simple harmonic. The projection of OP on x-axies is,

$$OM = OP \sin\theta$$

Displacement, $x = A \sin \omega t$

where $A = OP$ = amplitude of motion.

Fig. 1.2(b) shows a sine curve depicting the variation of displacement x from mean equilibrium position plotted against θ as abscissa. The motion repeats itself after 2π radians, called the time period, T.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

where f = frequency of harmonic motion in cycles per second or Hertz (Hz).

1.4.1 Addition of Two Harmonic Motions

Let $x_1 = A_1 \sin \omega t$

$$x_2 = A_2 \sin(\omega t + \alpha)$$

The resultant motion is,

$$x = x_1 + x_2$$

$$= A_1 \sin \omega t + A_2 \sin(\omega t + \alpha)$$

$$= A_1 \sin \omega t + A_2 \sin \omega t \cos \alpha + A_2 \cos \omega t \sin \alpha$$

$$= (A_1 + A_2 \cos\alpha) \sin \omega t + A_2 \sin \alpha \cos \omega t$$

$$= A \sin(\omega t + \phi) \qquad \text{...(1.2)}$$

Resultant amplitude, $A = \left[\left(A_1 + A_2 \cos\alpha\right)^2 + \left(A_2 \sin\alpha\right)^2\right]^{0.5}$

$$= \left[A_1^2 + A_2^2 + 2 A_1 A_2 \cos\alpha\right]^{0.5} \qquad \text{...(1.3a)}$$

Phase difference, $\phi = \tan^{-1}\left[\frac{A_2 \sin\alpha}{A_1 + A_2 \cos\alpha}\right]$ ...(1.3b)

The two harmonic motions can be added graphically, as shown in Fig. 1.3.

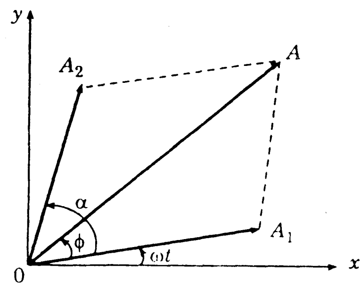

Fig. 1.3 Graphical addition of two SHM.

1.4.2 Beats

Let $x_1 = A_1 \sin \omega_1 t$

$x_2 = A_2 \sin \omega_2 t$

and $\Delta\omega = \omega_2 - \omega_1$

The resultant motion is,

$$x = x_1 + x_2$$

$$= A_1 \sin \omega_1 t + A_2 \sin (\omega_1 + \Delta\omega) t$$

$$= (A_1 + A_2 \cos \Delta\omega t) \sin \omega_1 t + A_2 \sin \Delta\omega t \cos \omega_1 t$$

$$= A \sin (\omega_1 t + \phi)$$

Resultant amplitude, $A = \left[A_1^2 + A_2^2 + 2 A_1 A_2 \cos \Delta \omega t\right]^{0.5}$ ...(1.4*a*)

Phase difference, $\phi = \tan^{-1}\left[\dfrac{A_2 \sin \Delta \omega t}{A_1 + A_2 \cos \Delta \omega t}\right]$ ...(1.4*b*)

Maximum amplitude, $A_{max} = A_1 + A_2$ ...(1.5*a*)

Minimum amplitude, $A_{min} = A_1 - A_2$ ...(1.5*b*)

Frequency of beats $= \Delta\omega$ ...(1.6)

Time period of beats $= \dfrac{2\pi}{\Delta\omega}$ ...(1.7)

The beat phenomenon is shown in Fig. 1.4.

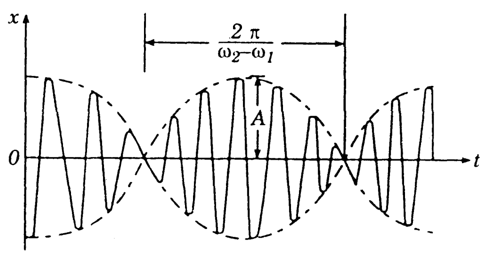

Fig. 1.4 Beat phenomenon.

Beats are formed when two simple harmonic motions differing by a small amount of frequency super impose each other.

Example 1.1 *A harmonic motion has an amplitude of 2 mm and the time period of 0.15s. Determine the maximum velocity and acceleration.*

Solution.

Let $x = A \sin \omega t$

Here $A = 2 \text{ mm}, \omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.15} = 41.889 \text{ rad/s}$

Maximum velocity, $v_{max} = \frac{dx}{dt} = A\,\omega = 2 \times 41.889 = 83.778$ mm/s

Maximum acceleration,

$$a_{max} = \frac{d^2 x}{dt^2} = A\omega^2 = 2 \times (41.889)^2 = 3509.2 \text{ mm/s}^2$$

Example 1.2 *A harmonic motion has a frequency of 10 Hz and its maximum velocity is 4.57 m/s. Determine its amplitude, period and maximum acceleration.*

Solution.

Frequency, $\omega = 2\pi f = 2\pi \times 10 = 20\,\pi$ rad/s

Maximum velocity, $v_{max} = A\omega$

or Amplitude, $A = \frac{4.57}{20\pi} = 0.07273$ m

Time period, $T = \frac{1}{f} = \frac{1}{10} = 0.1$ s

Maximum acceleration,

$$a_{max} = A\omega^2 = 0.07273 \times (20\pi)^2 = 287.12 \text{ m/s}^2$$

Example 1.3 *A harmonic displacement is $x(t) = 10 \sin (30\,t - \pi/3)$ mm, where t is in seconds, and the phase angle in radians. Find the (a) frequency and the period of motion, (b) maximum displacement, velocity, and acceleration, and (c) displacement, velocity and acceleration at t = 1.2 s.*

Solution.

Here $A = 10$ mm, $\omega = 30$ rad/s, and $\phi = \frac{\pi}{3}$

(*a*) Frequency, $f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{30}{2\pi} = 4.775$ Hz

Time period, $T = \frac{1}{f} = 0.21$ s

(*b*) Maximum displacement, $A = 10$ mm
Maximum velocity $= A\,\omega = 10 \times 30 = 300$ mm/s

Maximum acceleration $= A\omega^2 = 10 \times 900 = 9000$ mm/s^2

(*c*) At $t = 1.2$ s

$$x = 10 \sin\left(30 \times 1.2 - \frac{\pi}{3}\right) = -3.85 \text{ mm}$$

Velocity, $$v = 10 \times 30 \cos\left(30 \times 1.2 - \frac{\pi}{3}\right) = -276.84 \text{ mm/s}$$

Acceleration, $$a = -10 \times 900 \sin\left(30 \times 1.2 - \frac{\pi}{3}\right) = 3465.62 \text{ mm/s}^2$$

Example 1.4 *Find the algebraic sum of the harmonic motions :*

$$x_1 = 2 \sin\left(\omega t + \frac{\pi}{3}\right) \text{ and } x_2 = 3 \sin\left(\omega t + \frac{2\pi}{3}\right).$$

Solution.

$$x = x_1 + x_2$$

$$= 2 \sin\left(\omega t + \frac{\pi}{3}\right) + 3 \sin\left(\omega t + \frac{2\pi}{3}\right)$$

$$= 2\left[\sin \omega t \cos \frac{\pi}{3} + \cos \omega t \sin \frac{\pi}{3}\right]$$

$$+ 3\left[\sin \omega t \cos \frac{2\pi}{3} + \cos \omega t \sin \frac{2\pi}{3}\right]$$

$$= -0.5 \sin \omega t + 4.33 \cos \omega t$$

$$= A \sin (\omega t + \phi) = (A \cos \phi) \sin \omega t + (A \sin \phi) \cos \omega t$$

where

$$A = \left[(-0.5)^2 + (4.33)^2\right]^{0.5} = 4.359$$

$$\phi = \tan^{-1}\left(\frac{4.33}{-0.5}\right) = \tan^{-1}(-8.66) = -83.4°$$

Example 1.5 *Split up the harmonic motion : $x = 10 \sin\left(\omega t + \frac{\pi}{6}\right)$ into two harmonic motions one having a phase angle of zero and the other of 45°.*

Solution. Let the two harmonic motions be

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin\left(\omega t + \frac{\pi}{4}\right)$$

Then

$$A = \left[A_1^2 + A_2^2 + 2 A_1 A_2 \cos \frac{\pi}{4}\right]^{0.5}$$

$$100 = A_1^2 + A_2^2 + \sqrt{2} A_1 A_2 \qquad ...(1)$$

Also

$$\tan \frac{\pi}{6} = \frac{A_2 \sin \frac{\pi}{4}}{A_1 + A_2 \cos \frac{\pi}{4}}$$

$$\frac{1}{\sqrt{3}} = \frac{A_2}{\sqrt{2} A_1 + A_2}$$

$$\frac{A_1}{A_2} = 0.51764 \qquad ...(2)$$

Solving (1) and (2), we get

$$A_1 = 3.66 \text{ and } A_2 = 7.07$$

Hence $\quad x_1 = 3.66 \sin \omega t$ and $x_2 = 7.07 \sin\left(\omega t + \frac{\pi}{4}\right)$

Example 1.6 *A particle describes simultaneously two motions : $x_1 = 3 \sin 40t$ cm and $x_2 = 4 \sin 41t$ cm. What is the maximum and minimum amplitude of combined motion and the beat frequency ?*

Solution.

Here $\omega_1 = 40$ rad/s and $\omega_2 = 41$ rad/s, $A_1 = 3$ cm and $A_2 = 4$ cm.

Maximum amplitude $= A_1 + A_2 = 3 + 4 = 7$ cm

Minimum amplitude $= A_2 - A_1 = 4 - 3 = 1$ cm

Now $\Delta\omega = \omega_2 - \omega_1 = 41 - 40 = 1$ rad/s

Beat frequency $= \dfrac{\Delta\omega}{2\pi} = \dfrac{1}{2\pi}$ Hz

1.5 Work done by Harmonic Force

Let the hamonic force, $F = F_0 \sin \omega t$ act on a body, and make it vibrate with motion, $x = x_0 \sin(\omega t - \phi)$. Work done by the force F to displace the body through a small distance dx is :

$$dW = F \,.\, dx$$
$$= F \,.\, dx/dt \,.\, dt$$

Work done per cycle, $W = \int_0^T F \,.\, dx/dt \,.\, dt$

$$= \int_0^{2\pi/\omega} (F_0 \sin \omega t)\, [\omega \cos(\omega t - \phi)] \,.\, dt$$

$$= F_{0_3}^3\, \omega x_0 \int_0^{2\pi/\omega} \sin \omega t \cos(\omega t - \phi) \,.\, dt$$

$$= F_0^2\, \omega x_0 \int_0^{2\pi/\omega} \sin \omega t\, [\cos \omega t \cos \phi + \sin \omega t \sin \phi] \,.\, dt$$

$$= F_0\, \omega x_0 \left[\cos \phi \int_0^{2\pi/\omega} \sin \omega t \cos \omega t \,.\, dt + \sin \phi \int_0^{2\pi/\omega} \sin^2 \omega t \,.\, dt\right]$$

$$= F_0\, \omega x_0 \left[\cos \phi \int_0^{2\pi/\omega} 0.5 \sin 2\omega t \,.\, dt + \sin \phi \int_0^{2\pi/\omega} 0.5\,(1 - \cos 2\,\omega t) \,.\, dt\right]$$

$$= F_0\, \omega x_0\, [0.25 \cos \phi \;|\, -\cos 2\,\omega t \,|^{2\pi/\omega} + 0.5 \sin \phi \,|\, t - 0.5 \sin 2\omega t \,|^{2\pi/\omega}$$

$$= F_0 \omega x_0\, [\cos \phi\,(-1 + 1) + 0.5 \sin \phi\, (2\pi/\omega - 0 + 0]$$

$$= \pi x_0 F_0 \sin \phi \qquad \text{...(1.8)}$$

$W_{min} = 0$ when $\phi = 0^{\circ}$ and $W_{max} = \pi x_0 F_0$ when $\phi = 90^{\circ}$

Example 1.7 *A force $F_0 \sin \omega t$ acts on a displacement $x_0 \sin (\omega t - \pi/3)$. Taking $F_o = 50$ N, $x_0 = 25$ mm, and $\omega = 15\,\pi$ rad/s, determine the work done during (a) the first second, and (b) the first 1/20 seconds.*

Solution.

$$\text{Work done} = \int_0^{\tau} F \,.\, dx/dt \,.\, dt$$

$$= F_0\, x_0\, \omega \int_0^{\tau} \sin \omega t \cos\left(\omega t - \frac{\pi}{3}\right) . dt$$

$$= F_0\, x_0\, \omega \int_0^{\tau} 0.5\left[\sin\left(2\omega t - \frac{\pi}{2}\right) + \sin\frac{\pi}{3}\right] . dt$$

$$= 0.5\, F_0\, x_0\, \omega \left| -\frac{1}{2\omega}\cos\left(2\omega t - \frac{\pi}{3}\right) + \frac{\sqrt{3}t}{2}\right|_0^{\tau}$$

$$= 0.5\, F_0 x_0\, \omega \left[-\frac{1}{2\omega}\cos\left(2\omega\tau - \frac{\pi}{3}\right) + \frac{1}{4\omega} + \frac{\sqrt{3}\tau}{2}\right]$$

(*a*) $\tau = 1s$

Work done

$$= 0.5 \times 50 \times 25 \times 10^{-3} \times 15\,\pi \left[-\frac{1}{30\pi}\cos\left(30\pi - \frac{\pi}{3}\right) + \frac{1}{60\pi} + \frac{\sqrt{3}}{2}\right]$$

$$= 25.5 \text{ Nm}$$

(*b*) $\tau = 1/20$s

$$\text{Work done} = 29.4524\left[-\frac{1}{30\,\pi}\cos\left(1.5\,\pi - \frac{\pi}{3}\right) + \frac{1}{60\pi} + \frac{\sqrt{3}}{40}\right]$$

$$= 1.161 \text{ N.m}$$

Example 1.8 *The motion of a particle is described as : $x = 4\sin\left(\omega t + \frac{\pi}{6}\right)$. If the motion has two components, one of which $x_1 = 2\sin(\omega t - \frac{\pi}{3})$, determine the other harmonic component.*

Solution.

$$x_2 = x - x_1$$

$$= 4\sin\left(\omega t + \frac{\pi}{6}\right) - 2\sin\left(\omega t - \frac{\pi}{3}\right)$$

$$= 4\left[\sin \omega t \cos\frac{\pi}{6} + \cos \omega t \sin\frac{\pi}{6}\right]$$

$$- 2\left[\sin \omega t \cos\frac{\pi}{3} - \cos \omega t . \sin\frac{\pi}{3}\right]$$

$$= \sin \omega t\left[4 \times \frac{\sqrt{3}}{2} - 2 \times \frac{1}{2}\right] + \cos \omega t\left[4 \times \frac{1}{2} + 2 \times \frac{\sqrt{3}}{2}\right]$$

$$= (2\sqrt{3} - 1) \sin \omega t + (2 + \sqrt{3}) \cos \omega t$$

$$= 2.4641 \sin \omega t + 3.732 \cos \omega t$$

$$= A \sin (\omega t + \phi) = A \cos \phi . \sin \omega t + A \sin \phi . \cos \omega t$$

where $$A = [(2.4641)^2 + (3.732)^2]^{1/2} = 4.472$$

$$\phi = \tan^{-1}\left(\frac{3.732}{2.4641}\right) = \tan^{-1} 1.5145$$

$$= 56.564° \quad \text{or} \quad 0.3142\,\pi \text{ rad}$$

$$x_2 = 4.472 \sin(\omega t + 0.3142\,\pi)$$

Exaple 1.9 *The motion of a particle vibrating in a plane has two perpendicular harmonic components :* $x_1 = 2 \sin\left(\omega t + \frac{\pi}{6}\right)$ *and* $x_2 = 3 \sin \omega t$. *Determine the motion of the particle graphically.*

Solution.

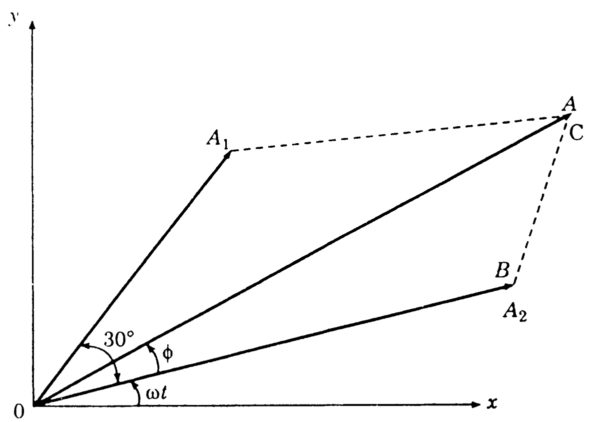

Fig. 1.5

Graphical method :

1. Draw $OA = A_1 = 2$ units on a scale of 1 cm = 0.5 units at an angle of 30° with OB drawn at an arbitrary angle ωt (Fig. 1.5).

2. Cut off $OB = A_2 = 3$units.

3. Draw AC parallel to OB and BC parallel to OA to meet at C.

4. Then $OC = A = 9.7$ cm = 4.85 units.

5. $\angle BOC = \phi = 12°$

Analytical method :

$$x = x_1 + x_2$$

$$= 2 \sin\left(\omega t + \frac{\pi}{6}\right) + 3 \sin \omega t$$

$$= 2\left[\sin \omega t \cos \frac{\pi}{6} + \cos \omega t. \sin \frac{\pi}{6}\right] + 3\sin$$

$$= \sin \omega t\left[2 \times \frac{\sqrt{3}}{2} + 3\right] + \cos \omega t\left[2 \times \frac{1}{2}\right]$$

$$= 4.732 \sin \omega t + \cos \omega t$$

$$= A \sin (\omega t + \phi)$$

$$A = \sqrt{(4.732)^2 + 1} = 4.83$$

$$\phi = \tan^{-1}\left[\frac{1}{4.732}\right] \equiv 12°$$

Example 1.10 *A particle is subjected to two harmonic motions as given below :*

$$x_1 = 15\sin\left(\omega t + \frac{\pi}{6}\right) \text{ and } x_2 = 8\cos\left(\omega t + \frac{\pi}{3}\right)$$

What extra harmonic motion should be given to the body to bring it to static equilibrium ?

Solution.

Resultant motion, $x = x_1 + x_2$

$$= 15 \sin\left(\omega t + \frac{\pi}{6}\right) + 8 \cos\left(\omega t + \frac{\pi}{3}\right)$$

$$= 15\left[\sin \omega t \cos \frac{\pi}{6} + \cos \omega t \sin \frac{\pi}{6}\right]$$

$$+ 8\left[\cos \omega t \cos \frac{\pi}{3} - \sin \omega t \sin \frac{\pi}{3}\right]$$

$$= \sin \omega t\left[15 \cos \frac{\pi}{6} - 8 \sin \frac{\pi}{3}\right]$$

$$+ \cos \omega t\left[15 \sin \frac{\pi}{6} + 8 \cos \frac{\pi}{3}\right]$$

$$= \left(15 \times \frac{\sqrt{3}}{2} - 8 \times \frac{\sqrt{3}}{2}\right) \sin \omega t + \left(15 \times \frac{1}{2} + 8 \times \frac{1}{2}\right) \cos \omega t$$

$$= 6.0262 \sin \omega t + 11.5 \cos \omega t$$

$$= A \sin(\omega t + \phi)$$

$$= (A \cos \phi) \sin \omega t + (A \sin \phi) \cos \omega t$$

$$A = [(6.062)^2 + (11.5)^2]^{1/2} = 13$$

$$\phi = \tan^{-1}\left(\frac{11.5}{6.062}\right) = \tan^{-1} 1.897 = 62.2°$$

For the body to be in static equilibrium, extra harmonic motion to be geven to body is :

$$13 \sin [\omega t + (180° + 62.2°)]$$

or $$13 \sin (\omega t + 242.2°)$$

Example 1.11 *Split up the harmonic motion :* $x = 8 \cos \left(\omega t + \frac{\pi}{4} \right)$ *into two harmonic motions, one of which has a phase angle of zero and the other a phase angle of 60°.*

Solution.

Let $$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin \left(\omega t + \frac{\pi}{3} \right)$$

$$x = x_1 + x_2$$

$$= A_1 \sin \omega t + A_2 \sin \left(\omega t + \frac{\pi}{3} \right)$$

$$= A_1 \sin \omega t + A_2 \left[\sin \omega t \cos \frac{\pi}{3} + \cos \omega t \sin \frac{\pi}{3} \right]$$

$$= \left(A_1 + A_2 \cos \frac{\pi}{3} \right) \sin \omega t + A_2 \sin \frac{\pi}{3} \cos \omega t$$

$$= (A_1 + 0.5 A_2) \sin \omega t + \frac{\sqrt{3}}{2} A_2 \cos \omega t$$

$$= A \sin (\omega t + \phi) = 8 \cos \left(\omega t + \frac{\pi}{4} \right)$$

$$= 8 \sin \left[90° + \left(\omega t + \frac{\pi}{4} \right) \right]$$

$$= 8 \sin (\omega t + 135°)$$

$$= 8 [\sin \omega t \cos 135° + \cos \omega t \sin 135°]$$

$$= 8 \left[-\frac{1}{\sqrt{2}} \sin \omega t + \frac{1}{\sqrt{2}} \cos \omega t \right]$$

$$= -5.657 \sin \omega t + 5.657 \cos \omega t$$

$$\therefore \quad A_1 + 0.5 A_2 = -5.657$$

$$\frac{\sqrt{3}}{2} A_2 = 5.657$$

$$A_2 = 6.532$$

$$A_1 = -5.657 - 0.5 \times 6.532 = 8.923$$

$$\therefore \quad x_1 = 8.923 \sin \omega t$$

$$x_2 = 6.532 \sin\left(\omega t + \frac{\pi}{3}\right)$$

If $$x_1 = A_1 \cos \omega t$$

$$x_2 = A_2 \cos\left(\omega t + \frac{\pi}{3}\right)$$

Then $$x = x_1 + x_2$$

$$= A_1 \cos \omega t + A_2\left[\cos \omega t \cos \frac{\pi}{3} - \sin \omega t \sin \frac{\pi}{3}\right]$$

$$= (A_1 + 0.5A_2) \cos \omega t - \frac{\sqrt{3}}{2} A_2 \sin \omega t$$

$$= 8 \cos\left(\omega t + \frac{\pi}{4}\right)$$

$$= 8\left[\cos \omega t \cos \frac{\pi}{4} - \sin \omega t \sin \frac{\pi}{4}\right]$$

$$= \frac{8}{\sqrt{2}} \cos \omega t - \frac{8}{\sqrt{2}} \sin \omega t$$

$$\therefore \quad A_1 + 0.5\,A_2 = \frac{8}{\sqrt{2}} = 5.657$$

$$\frac{\sqrt{3}}{2} A_2 = \frac{8}{\sqrt{2}}$$

$$A_2 = 6.532$$

$$A_1 = 5.657 - 0.5 \times 6.532 = 2.391$$

$$\therefore \quad x_1 = 2.391 \cos \omega t$$

and $$x_2 = 6.532 \cos\left(\omega t + \frac{\pi}{3}\right)$$

Example 1.12 *Find the algebraic sum of the harmonic motion* x_1 and x_2, *given below :*

$$x_1 = 2 \sin\left(\omega t + \frac{\pi}{3}\right)$$

$$x_2 = 3 \sin\left(\omega t + \frac{2\pi}{3}\right)$$

Solution. $$x = x_1 + x_2$$

$$= 2 \sin\left(\omega t + \frac{\pi}{3}\right) + 3 \sin\left(\omega t + \frac{2\pi}{3}\right)$$

$$= 2\left[\sin \omega t \cos \frac{\pi}{3} + \cos \omega t \sin \frac{\pi}{3}\right]$$

$$+ 3\left[\sin \omega t \cos \frac{2\pi}{3} + \cos \omega t \sin \frac{2\pi}{3}\right]$$

$$= \left(2 \cos \frac{\pi}{3} + 3 \cos \frac{2\pi}{3}\right) \sin \omega t$$

$$+ \left(2 \sin \frac{\pi}{2} + 3 \sin \frac{2\pi}{2}\right) \times \cos \omega t$$

$$= \left(2 \times \frac{1}{2} - 3 \times \frac{1}{2}\right) \sin \omega t + \left(2 \times \frac{\sqrt{3}}{2} + 3 \times \frac{\sqrt{3}}{2}\right) \cos \omega t$$

$$= -0.5 \sin \omega t + 4.33 \cos \omega t$$

$$= A \sin (\omega t + \phi)$$

$$= A \sin \omega t \cos \phi + A \cos \omega t \sin \phi$$

$$= (A \cos \phi) \sin \omega t + (A \sin \phi) \cos \omega t$$

$$\therefore \quad A \cos \phi = -0.5$$

$$A \sin \phi = 4.33$$

$$A = [(4.33)^2 + (-0.5)^2]^{1/2} = 4.359$$

$$\phi = \tan^{-1}\left(\frac{4.33}{-0.5}\right) = \tan^{-1}(-8.66) = -83.4°$$

$$\therefore \quad x = 4.359 \sin (\omega t - 83.4°)$$

Example 1.13 *Add the following harmonic motion analytically and check the solution graphically :*

$$x_1 = 4 \cos (\omega t + 10°)$$

$$x_2 = 6 \cos (\omega t + 60°)$$

Solution. Analytical method

$$x = x_1 + x_2$$

$$= 4 \cos (\omega t + 10°) + 6 \sin(\omega t + 60°)$$

$$= 4 [\cos \omega t \cos 10° - \sin \omega t \sin 10°]$$

$$+ 6 [\sin \omega t \cos 60° + \cos \omega t \sin 60°]$$

$$= (6 \cos 60° - 4 \sin 10°) \sin \omega t$$

$$+ (6 \sin 60° + 4 \cos 10°) \cos \omega t$$

$$= \left(6 \times \frac{1}{2} - 4 \times 0.17365\right) \sin \omega t$$

$$+ \left(6 \times \frac{\sqrt{3}}{2} + 4 \times 0.98480\right) \cos \omega t$$

$$= 2.3054 \sin \omega t + 9.1353 \cos \omega t$$

$$= A \sin(\omega t + \phi)$$

$$= (A \cos \phi) \sin \omega t + A \sin \phi) \cos \omega t$$

$$\therefore \quad A \sin\phi = 9.1353$$

$$A \cos \phi = 2.3054$$

$$A = [\,(9.1353)^2 + (2.3054)^2\,]^{1/2} = 9.42$$

$$\phi = \tan^{-1}\left(\frac{9.1353}{2.3054}\right)$$

$$= \tan^{-1} 3.9625 = 75.836°$$

$$\therefore \qquad x = 9.42 \sin(\omega t + 75.836°)$$

Graphical method :

$$x_1 = 4 \cos(\omega t + 10°)$$

$$= 4 \sin [\,90° + (\omega t + 10°)\,]$$

$$= 4 \sin(\omega t + 100°)$$

$$x_2 = 6 \sin(\omega t + 60°)$$

Refer to Fig. 1.6.

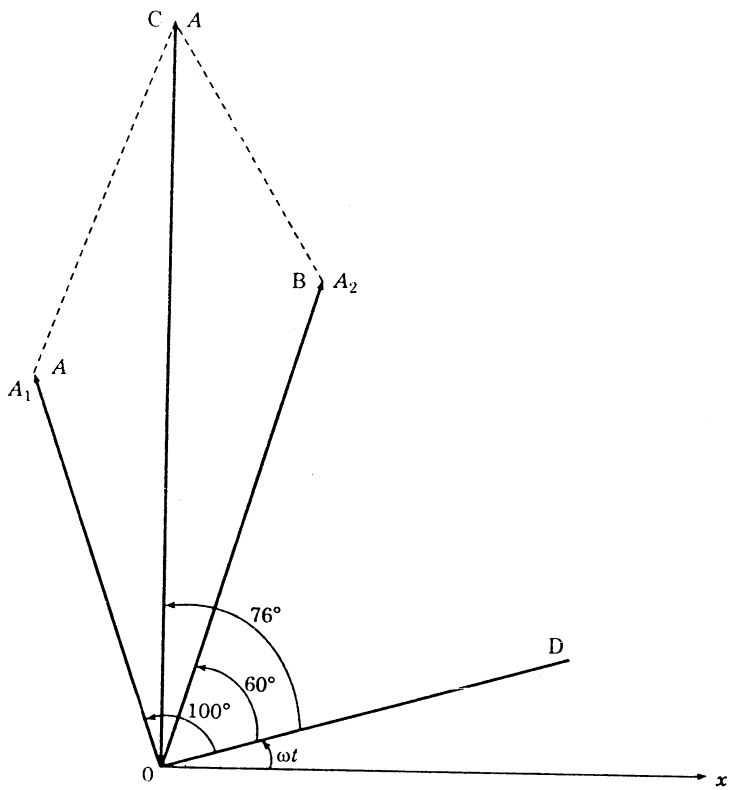

Fig. 1.6

Let $\angle DOx = \omega t$

1. Draw $\angle AOD = 100°$ and $OA = A_1 = 4$ units.
2. Draw $\angle BOD = 60°$ and $OB = A_2 = 6$ units.
3. Complete the parallelogram $OACB$.
4. $OC = A = 9.4$ units and $\angle COD = 76°$

Example 1.14 *Split the harmonic motion* $x = 10 \sin\left(\omega t + \frac{\pi}{6}\right)$ *into two harmonic motions one having a phase angle of zero and other of 45°.*

Solution.

Let $x_1 = A_1 \sin \omega t$

and $x_2 = A_2 \sin (\omega t + 45°)$

Then

$$\begin{aligned} x &= x_1 + x_2 \\ &= A_1 \sin \omega t + A_2 \sin (\omega t + 45°) \\ &= A_1 \sin \omega t + A_2 [\sin \omega t \cos 45° + \cos \omega t \sin 45°] \\ &= (A_1 + A_2 \cos 45°) \sin \omega t + A_2 \sin 45° \cos \omega t \\ &= \left(A_1 + \frac{A_2}{\sqrt{2}}\right) \sin \omega t + \frac{A_2}{\sqrt{2}} \cos \omega t \\ &= 10 \sin\left(\omega t + \frac{\pi}{6}\right) \\ &= 10\left[\sin \omega t . \cos \frac{\pi}{6} + \cos \omega t \sin \frac{\pi}{6}\right] \\ &= 10\left[\sin \omega t \times \frac{\sqrt{3}}{2} + \cos \omega t . \times \frac{1}{2}\right] \\ &= 5\sqrt{3} \sin \omega t + 5 \cos \omega t \end{aligned}$$

$\therefore$ $$A_1 + \frac{A_2}{\sqrt{2}} = 5\sqrt{3}$$

$$\frac{A_2}{\sqrt{2}} = 5$$

$\therefore$ $$A_2 = 5\sqrt{2}\ 7.07$$

$$A_1 = 5\sqrt{3} - 5 = 3.66$$

$$x_1 = 7.07 \sin \omega t$$

$$x_2 = 3.66 \sin (\omega t + 45°)$$

Example 1.15 *A harmonic motion* $x = 5 \sin (3t + \phi)$ *is to be split into two components such that one leads it by 30° and the other lags it by 90°. Find the components.*

Solution.

Let $x_1 = A_1 \sin(3t + \phi - 90°)$

and $x_2 = A_2 \sin(3t + \phi + 30°)$

$$x = x_1 + x_2$$

$$= A_1 \sin[3t + (\phi - 90°)] + A_2 \sin[3t + (\phi + 30°)]$$

$$= A_1[\sin 3t \cos(\phi - 90°) + \cos 3t \sin(\phi - 90°)]$$

$$+ A_2[\sin 3t \cos(\phi + 30°) + \cos 3t \sin(+30°)]$$

$$= A_1\Big[\sin 3t\,\{\cos\phi \cos 90° + \sin\phi \sin 90°\}$$

$$+ \cos 3t\,\{\sin\phi\cos 90° - \cos\phi\sin 90°\}\Big]$$

$$+ A_2\Big[\sin 3t\,\{\cos\phi\cos 30° - \sin\phi\sin 30°\}$$

$$+ \cos 3t\,\{\sin\phi\cos 30° + \cos\phi\sin 30°\}\Big]$$

$$= A_1[\sin 3t\,\{0 + \sin\phi\} + \cos 3t\,\{0 - \cos\phi\}]$$

$$+ A_2\left[\sin 3t\left\{\frac{\sqrt{3}}{2}\cos\phi - \frac{1}{2}\sin\phi\right\}\right.$$

$$\left. + \cos 3t\left\{\frac{\sqrt{3}}{2}\sin\phi + \frac{1}{2}\cos\phi\right\}\right]$$

$$= \sin 3t\left[A_1\{\sin\phi\} + A_2\left\{\frac{\sqrt{3}}{2}\cos\phi - \frac{1}{2}\sin\phi\right\}\right]$$

$$+ \cos 3t\left[-A_1\cos\phi + A_2\left\{\frac{\sqrt{3}}{2}\sin\phi + \frac{1}{2}\cos\phi\right\}\right]$$

$$= 5\sin(3t + \phi)$$

$$= 5[\sin 3t\cos\phi + 5\cos 3t.\sin\phi]$$

$$= 5\cos\phi \times \sin 3t + 5\sin\phi.\cos 3t$$

$\therefore$ $$A_1\sin\phi + \frac{A_2}{2}(\sqrt{3}\cos\phi - \sin\phi) = 5\cos\phi$$

and $$-A_1\cos\phi + \frac{A_2}{2}(\sqrt{3}\sin\phi + \cos\phi) = 5\sin\phi$$

Squaring and adding, we have

$$A_1^2 + \frac{A_2^2}{4}(3\cos^2\phi + \sin^2\phi - 2\sqrt{3}\sin\phi\cos\phi + 3\sin\phi$$

$$+ \cos^2\phi + 2\sqrt{3}\sin\phi\cos\phi) + A_1A_2(\sqrt{3}\sin\phi\cos\phi$$

$$- \sin^2\phi - \sqrt{3}\sin\phi\cos\phi - \cos^2\phi) = 25$$

$$A_1^2 + \frac{A_2^2}{4}(4) + A_1A_2(-1) = 25$$

Also $A_1 \sin\phi \cos\phi + \frac{A_2}{2}(\sqrt{3}\cos^2\phi - \sin\phi\cos\phi) = 5\cos^2\phi$

$$-A_1 \sin\phi\cos\phi + \frac{A_2}{2}(\sqrt{3}\sin^2\phi + \sin\phi\cos\phi) = 5\sin^2\phi$$

Adding, we have

$$\frac{A_2}{2}[\sqrt{3}] = 5$$

$$A_2 = \frac{5 \times 2}{\sqrt{3}} = \frac{10}{\sqrt{3}} = 5.773$$

$$A_1^2 + \frac{100}{3} - \frac{10}{\sqrt{3}} A_1 = 25$$

$$A_1^2 - 5.773A + 8.33 = 0$$

$$A_1 = \frac{5.773 + \sqrt{(5.773)^2 - 4 \times 8.33}}{2} = 2.93$$

$$\therefore \quad x_1 = 2.93 \sin(3t + \phi - 90°)$$

$$x_2 = 5.773 \sin(3t + \phi + 30°)$$

Example 1.16 *A harmonic force F_6 sin ωt acts on a displacement $x_0 \sin\left(\omega t - \frac{\pi}{4}\right)$. If $F_0 = 100$ N, $x_0 = 0.025$ m and $\omega = 3\pi$ rad/s, calculate the work done during (a) the first cycle, (b) the first second, and (c) the first quarter second.*

Solution .

The work done is given by,

$$W = \int_0^{\tau} F \cdot \left(\frac{dx}{dt}\right) \cdot dt$$

where $x = x_0 \sin\left(\omega t - \frac{\pi}{4}\right)$ and $F = F_0 \sin \omega t$

Now $$\frac{dx}{dt} = x_0\,\omega \cos\left(\omega t - \frac{\pi}{4}\right)$$

$$\therefore \quad W = F_0\, x_0\, \omega \int_0^{\tau} \sin\omega t \times \cos\left(\omega t - \frac{\pi}{3}\right) . dt$$

$$= \frac{1}{2} F_0\, x_0\, \omega \int_0^{\tau} \left[\sin\left(2\omega t - \frac{\pi}{4}\right) + \sin\frac{\pi}{4}\right] . dt$$

$$= \frac{1}{2} F_0\, x_0\, \omega \left| -\frac{1}{2\omega} \cos\left(2\omega t - \frac{\pi}{3}\right) + t \sin\frac{\pi}{4} \right|_0^{\tau}$$

(*a*) $\omega = 3\pi$, $\tau = \frac{2\pi}{3} s = \frac{2\pi}{3\pi} = \frac{2}{3} s$

$$W = \frac{1}{2} \times 100 \times 0.025 \times 3\pi \left[-\frac{1}{6\pi} \left\{ \cos\left(6\pi \times \frac{2}{3} - \frac{\pi}{4}\right) - \cos\frac{\pi}{4} \right\} + \frac{2}{3} \sin\frac{\pi}{4} \right]$$

$$= 11.781 \left[-\frac{1}{6\pi} \left(\cos\frac{15\pi}{4} - \cos\frac{\pi}{4} \right) + \frac{2}{3} \sin\frac{\pi}{4} \right]$$

$$= 11.781 [0 + 0.4714]$$

$$= 5.554 \text{ N.m}$$

(*b*) After 1 second,

$$W = \frac{1}{2} \times 100 \times 0.025 \times 3\pi \left[-\frac{1}{6\pi} \left\{ \cos\left(6\pi \times 1 - \frac{\pi}{4}\right) - \cos\frac{\pi}{4} \right\} + 1 \sin\frac{\pi}{4} \right]$$

$$= 11.781 \left[-\frac{1}{6\pi} \left\{ \cos\frac{23\pi}{4} - \cos\frac{\pi}{4} \right\} + \sin\frac{\pi}{4} \right]$$

$$= 11.781 [0 + 0.707]$$

$$= 8.329 \text{ N.m}$$

(*c*) For the first quarter second : $\tau = 0$ to $\frac{1}{4}$ *s*.

$$W = 11.781 \left[-\frac{1}{6\pi} \left\{ \cos\left(6\pi \times \frac{1}{4} - \frac{\pi}{4}\right) - \cos\frac{\pi}{4} \right\} + \frac{1}{4} \sin\frac{\pi}{4} \right]$$

$$= 11.781 \left[-\frac{1}{6\pi} (-0.707 - 0.707) + \frac{1}{4} \times 0.707 \right]$$

$$= 11.781 [0.075 + 0.17675]$$

$$= 2.966 \text{ N.m}$$

1.6 Fourier Series Analysis

Periodic function - If at equal intervals of abscissa x, the value of each ordinate $f(x)$ repeats itself, *i.e.* $f(x) = f(x + \alpha)$, for all x then $y = f(x)$ is called a periodic function having period α . For example : $\sin x$, $\cos x$ are periodic functions having a period 2π.

Fourier Series - Any periodic function $f(t)$ in the interval t to $t + \frac{2\pi}{\omega}$ can be represented by Fourier series as follows :

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\,\omega t + b_n \sin n\,\omega t) \qquad ...(1.9)$$

where $\omega = \frac{2\pi}{T}$ is the fundamental frequency.

$$a_o = \frac{\omega}{\pi} \int_t^{t + \frac{2\pi}{\omega}} f(t) \,.\, dt \qquad ...(1.10)$$

$$a_n = \frac{\omega}{\pi} \int_{t}^{t+\frac{2\pi}{\omega}} f(t) \cos(n\,\omega t)\, dt \qquad \text{...(1.11)}$$

$$b_N = \frac{\omega}{\pi} \int_{t}^{t+\frac{2\pi}{\omega}} f(t) \sin(n\omega t)\, dt \qquad \text{...(1.12)}$$

The harmonic of frequency ω is called the fundamental or first harmonic of $f(t)$, and the harmonic of frequency $n\omega$ is called the nth harmomic.

Some useful results for harmonic analysis

$$\int_{t}^{t+2\pi/\omega} \sin(m\omega t) \sin(n\omega t)\,.\,dt = \begin{cases} 0, \text{ for } m \neq n \\ \pi/\omega, \text{ for } m = n \end{cases}$$

$$\int_{t}^{t+2\pi/\omega} \cos(m\omega t) \cos(n\omega t)\,.\,dt = \begin{cases} 0, \text{ for } m \neq n \\ \pi/\omega, \text{ for } m = n \end{cases}$$

$$\int_{t}^{t+2\pi/\omega} \cos(m\omega t) \sin(n\omega t)\,.\,dt = 0, \text{ for all } m, n$$

$$\int_{t}^{t+2\pi/\omega} \cos(n\omega t) = 0, \text{ for all } n$$

$$\int_{t}^{t+\frac{2\pi}{\omega}} \sin(n\omega t)\,.\,dt = 0, \text{ for all } n$$

where m and n are non-zero integers.

Even and Odd Functions

A function $f(t)$ is said to be even if $f(-t) = f(t)$. For example : $\cos t$, $\sec t$, t^2 are all even functions.

A function is said to be odd if $f(-t) = -f(t)$.

For example : $\sin t$, $\tan t$, t^3 are all odd functions.

For an even function,

$$\int_{-c}^{c} f(t)\, dt = 2\int_{0}^{c} f(t)\, dt$$

and for an odd function

$$\int_{-c}^{c} f(t)\, dt = 0$$

Example 1.17 *Represent the motion shown in Fig. 1.7 by Fourier series.*

Solution.

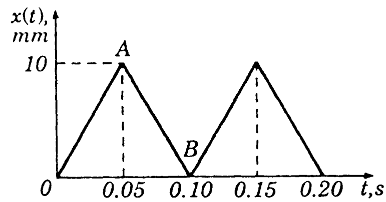

Fig. 1.7

Slope of the line $OA = \frac{10}{0.05} = 200$

Slope of the line BA

$$= -\frac{10}{0.05} = -200$$

Hence, the equations of the line segments OA and AB are :

$$x = 200t \quad \text{for} \quad 0 \le t \le 0.05 \ s$$

$$= -200t + 20 \quad \text{for} \quad 0.05 \le t \le 0.10 \ s$$

Here $T = 0.1$ s so that $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.1} = 20\,\pi$ rad/s

Hence $\qquad \frac{\omega}{2\pi} = 10$

The Fourier series is :

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

where $$a_0 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \, . \, dt$$

$$= 20 \left[\int_0^{0.05} 200\, t \, . \, dt + \int_{0.05}^{0.10} (20 - 200t) \, . \, dt \right]$$

$$= 20 \left[\left| 100\, t^2 \right|_0^{0.05} + \left| 20t - 100t^2 \right|_{0.05}^{0.10} \right] = 10$$

$$a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \cos(n\omega t) \, . \, dt$$

$$= 20 \left[\int_0^{0.05} 200t \cos(20\,\pi\, nt)\, dt + \int_{0.05}^{0.10} (20 - 200t) \times \cos(20\,\pi\, nt)\, dt \right]$$

$$= \begin{cases} -\dfrac{40}{\pi^2 n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

$$b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \sin(n\omega t)\, dt$$

$$= 20 \left[\int_0^{0.05} 200\, t \sin(20\,\pi\, nt)\,.\, dt + \int_{0.05}^{0.10} (20 - 200t) \sin(20\,\pi\, nt)\,.\, dt \right] = 0$$

$$x(t) = \frac{10}{2} - \frac{40}{\pi^2} \left[\sum_{n=1,3,5,....}^{\infty} \frac{\cos(20\,\pi\, nt)}{n^2} \right]$$

$$= 5 - \frac{40}{\pi^2} \left[\cos(20\,\pi t) + \frac{\cos(60\,\pi\, t)}{3^2} + \frac{\cos(100\,\pi t)}{5^2} + \ldots \right]$$

Example 1.18 *Represent the motion shown in Fig. 1.8 by Fourier series.*

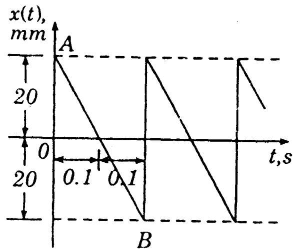

Fig. 1.8

Solution.

The equation of line AB is $x(t) = 200t + 20$ mm for $0 \le t \le 0.2$

Time period, $T = 0.2\ s$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\,\pi \text{ rad/s}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

Now $\frac{2\pi}{\omega} = 0.2$

$$a_0 = \frac{\omega}{\pi}\,.\int_0^{0.2} x(t)\,.\, dt$$

$$= 10 \int_0^{0.2} (-200t + 20)\,.\, dt$$

$$= 10 \left| -100t^2 + 20t \right|_0^{0.2}$$

$$= 10\,(-4+4) = 0$$

$$a_n = \frac{\omega}{\pi} \cdot \int_0^{0.2} x\,(t) \cos\,(n\omega t)\,.\,dt$$

$$= 10 \int_0^{0.2} (-200t + 20) \cos\,(n\omega t)\,.\,dt = 0$$

$$b_n = \frac{\omega}{\pi} \cdot \int_0^{0.2} x\,(t) \sin\,(n\omega t)\,.\,dt$$

$$= 10 \int_0^{0.2} (-200t + 20) \sin\,(n\omega t)\,.\,dt$$

$$= \frac{40}{\pi n}$$

$$x\,(t) = \frac{40}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\sin\,(10\,\pi n t)}{n}$$

$$= \frac{40}{\pi}\left[\frac{\sin(10\,\pi t)}{1} + \frac{\sin\,(20\,\pi t)}{2} + \ldots\right]$$

Example 1.19 *The periodic motion of a follower operated by a cam is saw-tooth in nature as shown in Fig. 1.9. Represent the motion by a harmonic series, if the cam rotates uniformly at 60 rpm and total lift of follower is 30 mm.*

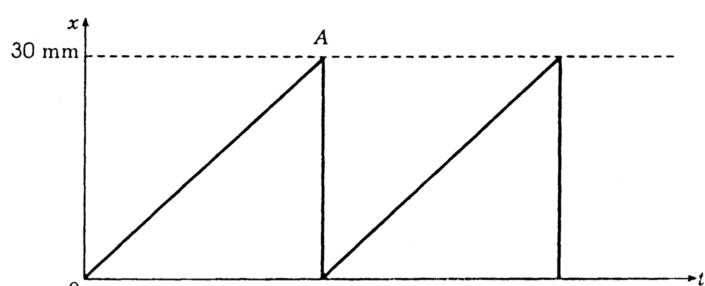

Fig. 1.9

Solution.

The equation of motion is :

$$x = \left(\frac{30}{T}\right) t \quad \text{for } 0 < t < T$$

Excitation frequency, $\omega = \dfrac{2\,\pi N}{60} = \dfrac{2\pi \times 60}{60} = 2\,\pi$ rad/s

The Fourier series is :

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

$$T = \frac{2\pi}{\omega} = 1 \, s$$

$$a_0 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t)\, dt$$

$$= 2 \int_0^{2\pi/\omega} \left(\frac{30}{T}\right) t \, dt$$

$$= \frac{60}{T} \int_0^T t\, dt = \frac{60}{T} \left| \frac{t^2}{2} \right|_0^T = 30\, T = 30 \times 1 = 30$$

$$a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \cos(n\omega t)\, dt$$

$$= 2 \int_0^T \left(\frac{30t}{T}\right) \cos(n\omega t)\, dt$$

$$= \frac{60}{T} \int_0^T t \cos(n\,\omega t)\, dt$$

$$= \frac{60}{T} \left[\left| t \times \frac{\sin(n\,\omega t)}{n\,\omega} \right|_0^T - \int_0^T 1 \times \frac{\sin(n\,\omega t)}{n\,\omega}\, dt \right]$$

$$= \frac{60}{T} \left[T\, \frac{\sin(n\,\omega T)}{n\,\omega} - \left| -\frac{\cos(n\,\omega t)}{(n\,\omega)^2} \right|_0^T \right]$$

$$= 60 \left[\frac{\cos(n\,\omega T)}{(n\,\omega)^2} - \frac{1}{(n\,\omega)^2} + \frac{\sin(n\,\omega T)}{n\,\omega} \right]$$

$$= 60 \left[\frac{\cos(2\,\pi n)}{(2\,\pi n)^2} - \frac{1}{(2\,\pi\, n)^2} + \frac{\sin(2\,\pi\, n)}{2\,\pi n} \right]$$

$$= 60 \left[\frac{1}{(2\,\pi n)^2} - \frac{1}{(2\,\pi n)^2} + 0 \right] = 0$$

$$b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \sin(n\omega t)\, dt$$

$$= 2\int_0^T \left(\frac{30t}{T}\right) \sin(n\omega t)\, dt$$

$$= \frac{60}{T}\left[\left| t \times \left\{\frac{\cos(n\,\omega t)}{n\,\omega}\right\}\right|_0^T - \int_0^T 1 \times \left\{\frac{\cos(n\,\omega t)}{n\,\omega}\, dt\right\}\right]$$

$$= \frac{60}{T}\left[-T\,\frac{\cos(n\,\omega T)}{n\,\omega} + \left|\frac{\sin(n\,\omega t)}{(n\,\omega)^2}\right|_0^T\right]$$

$$= 60\left[\frac{-\cos(2\,\pi\,n)}{(2\,\pi n)} + \frac{\sin(2\,\pi\,n)}{(2\,\pi\,n)^2}\right]$$

$$= -\frac{30}{n\,\pi}$$

$$x(t) = \frac{30}{2} + 0 + \sum_{n=1}^{\infty}\left(-\frac{30}{n\,\pi}\right)\sin(2\,\pi n t)$$

$$= 15 - \sum_{n=1}^{\infty}\left(\frac{30}{n\,\pi}\right)\sin(2\,\pi n t)$$

Example 1.20 *For a periodic "square top" wave shown in Fig. 1.10,*

$f(t) = F_0 = constant$ for *period* $0 < \omega t < \pi$

$= -F_0 = constant$ for *period* $\pi < \omega t < 2\pi$

Repersent this as a superposition of component harmonic motion.

Solution.

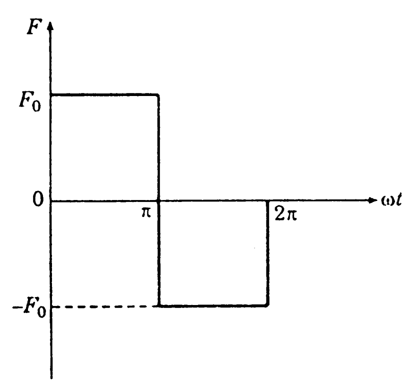

Fig. 1.10

The Foruier series is : $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\,\omega t) + b_n \sin(n\,\omega t)]$

Here $$a_0 = \frac{\omega}{\pi}\left[\int_0^{\pi/\omega} F_0\, dt + \int_{\pi/\omega}^{2\pi/\omega} (-F_0)\, dt\right] = \frac{\omega F_0}{\pi}\left[\ |t|_0^{\pi/\omega} - |t|_{\pi/\omega}^{2\pi/\omega}\ \right]$$

$$= \frac{\omega F_0}{\pi}\left[\frac{\pi}{\omega} - \left(\frac{2\pi}{\omega} - \frac{\pi}{\omega}\right)\right] = 0$$

$$a_n = \frac{\omega F_0}{\pi}\left[\int_0^{\pi/\omega} \cos(n\,\omega t)\, dt - \int_{\pi/\omega}^{2\pi/\omega} \cos(n\,\omega t)\, dt\right]$$

$$= \frac{\omega F_0}{\pi}\left[\ \left|\frac{\sin(n\,\omega t)}{n\,\omega}\right|_0^{\pi/\omega} - \left|\frac{\sin(n\,\omega t)}{n\,\omega}\right|_{\pi/\omega}^{2\pi/\omega}\ \right]$$

$$= \frac{\omega F_0}{\pi}\left[\frac{\sin(\pi n)}{n\,\omega} - \frac{\sin(2\pi n)}{n\,\omega} + \frac{\sin(\pi n)}{n\,\omega}\right]$$

$$= \frac{\omega F_0}{\pi}\left[\frac{2\sin(\pi n)}{n\,\omega} - \frac{\sin(2\pi n)}{n\,\omega}\right] = 0$$

for all even and odd values of n.

$$b_n = \frac{\omega F_0}{\pi}\left[\int_0^{\pi/\omega} \sin(n\,\omega t)\, dt - \int_{\pi/\omega}^{2\pi/\omega} \sin(n\,\omega t)\, dt\right]$$

$$= \frac{\omega F_0}{\pi}\left[\ \left|\frac{-\cos(n\,\omega t)}{n\,\omega}\right|_0^{\pi/\omega} + \left|\frac{\cos(n\,\omega t)}{n\,\omega}\right|_{\pi/\omega}^{2\pi/\omega}\ \right]$$

$$= \frac{\omega F_0}{\pi}\left[-\frac{\cos(n\,\pi)}{n\,\omega} + \frac{1}{n\,\omega} + \frac{\cos(2\,n\,\pi)}{n\,\omega} - \frac{\cos(n\,\pi)}{n\,\omega}\right]$$

$$= \frac{F_0}{\pi n}[-\cos(n\,\pi) + 1 + \cos(2\,n\,\pi) - \cos(n\,\pi)]$$

$$= \frac{F_0}{\pi n}[\cos(2\,n\,\pi) + 1 - 2\cos(n\,\pi)]$$

For even values of n, $\cos(2\,n\,\pi) = \cos(n\,\pi) = 1$

$\therefore \qquad b_n = 0$

For odd values of n, $\cos(n\,\pi) = -1$, $\cos(2\,n\,\pi) = 1$

$\therefore \qquad b_n = \frac{4F_0}{\pi n}$

$\therefore \qquad f(t) = \frac{4F_0}{\pi}\left[\frac{\sin\omega t}{1} + \frac{1}{3}\sin 3\,\omega t + \frac{1}{5}\sin 5\,\omega t + \ldots\ldots\ldots\right]$

Example 1.21 *Find the Fourier series for the triangulars wave shown in Fig. 1.11.*

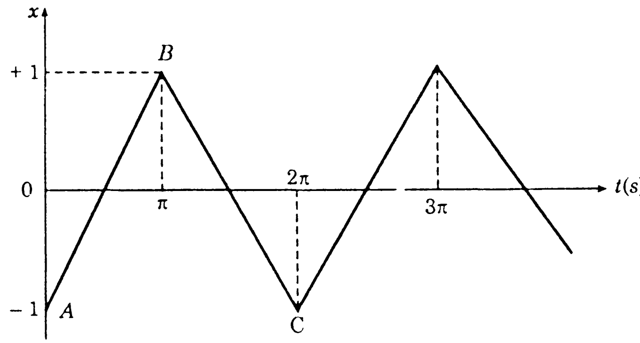

Fig. 1.11

Solution.

$$\omega = \frac{2\pi}{2\pi} = 1 \text{ rad/}s$$

Equation for line AB :

$$x = \frac{2}{\pi} t - 1 \quad \text{for } 0 \le t \le \pi$$

Line BC : $$x = -\frac{2}{\pi} t + 3 \quad \text{for } \pi \le t \le \pi$$

The Fourier series is :

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\,\omega t) + b_n \sin(n\,\omega t)]$$

$$a_0 = \frac{\omega}{\pi}\left[\int_0^{\pi}\left(\frac{2t}{\pi} - 1\right) dt + \int_{\pi}^{2\pi}\left(-\frac{2t}{\pi} + 3\right) dt\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{t^2}{\pi} - t\right|_0^{\pi} + \left|-\frac{t^2}{\pi} + 3\,t\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}[(\pi - \pi) + (-4\pi + 6\pi + \pi - 3\pi)] = 0$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi}\left(\frac{2t}{\pi} - 1\right)\cos(n\,t)\,dt + \int_{\pi}^{2\pi}\left(-\frac{2t}{\pi} + 3\right)\cos(n\,t)\,dt\right]$$

Now $$\int_0^{\pi} \cos(n\,t)\,dt = \left|\frac{\sin(n\,t)}{n}\right|_0^{\pi} = 0$$

$$\int_{\pi}^{2\pi} \cos(n\,t)\,dt = \left|\frac{\sin(n\,t)}{n}\right|_{\pi}^{2\pi} = 0$$

Also $\int_{0}^{\pi} t\cos(n\,t)\,dt = \left|t\;\frac{\sin(n\,t)}{n} - \int \sin(n\frac{t)}{n}\,dt\right|_{0}^{\pi}$

$$= \left|t\;\frac{\sin(n\,t)}{n} + \frac{\cos(n\,t)}{n^2}\right|_{0}^{\pi}$$

$$= \frac{\pi\sin(n\,\pi)}{n} + \frac{\cos(n\,\pi)}{n^2} - 0 - \frac{1}{n^2}$$

$= 0$ for n even

$= -\frac{2}{n^2}$ for n odd

Similarly $\int_{\pi}^{2\pi} t\cos(n\,t)\,dt = \frac{1}{n^2}\,[1 - \cos(n\,\pi)]$

$= 0$ for n even

$= \frac{2}{n^2}$ for n odd

$\therefore$ $a_n = -\frac{8}{\pi^2\,n^2}$ for n odd only

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi}\left(\frac{2t}{\pi} - 1\right)\sin(n\,t)\,dt + \int_{\pi}^{2\pi}\left(-\frac{2t}{\pi} + 3\right)\sin(n\,t)\,dt\right]$$

Now $\int_0^{\pi}\sin(n\,t)\,dt = \left|\frac{-\cos(n\,t)}{n}\right|_0^{\pi} = -\frac{1}{n}(-1-1) = \frac{2}{n}$ for n odd

$= 0$ for n even

$\int_{\pi}^{2\pi}\sin(n\,t)\,dt = \left|\frac{-\cos(n\,t)}{n}\right|_{\pi}^{2\pi} = -\frac{2}{n}$ for n odd

$= 0$ for n even

$$\int_0^{\pi} t\sin(n\,t)\,dt = \left|-t\,\frac{\cos(n\,t)}{n} + \int \frac{\cos(n\,t)}{n}\,dt\right|_0^{\pi}$$

$$= \left|-t\,\frac{\cos(n\,t)}{n} + \frac{\sin(n\,t)}{n^2}\right|_0^{\pi}$$

$= \frac{\pi}{n}$ for n odd

$$= -\frac{\pi}{n} \quad \text{for } n \text{ even}$$

$$\int_{\pi}^{2\pi} t \sin(n\,t)\,dt = \left| -\frac{t}{n}\cos(n\,t) + \frac{\sin(n\,t)}{n^2} \right|_{\pi}^{2\pi}$$

$$= -\frac{3\pi}{n} \quad \text{for } n \text{ odd}$$

$$= -\frac{\pi}{n} \quad \text{for } n \text{ even}$$

$$b_n = 0 \quad \text{for } n \text{ odd}$$

$$= 0 \quad \text{for } n \text{ even}$$

$$\therefore \quad f(t) = \sum_{n=1}^{\infty} -\frac{8}{\pi^2}\,\frac{\cos(n\,t)}{n^2} \quad \text{for } n \text{ odd only}$$

$$\therefore \quad f(t) = -\frac{8}{\pi^2}\left[\cos t + \frac{1}{3^2}\cos(3\,t) + \frac{1}{5^2}\cos(5t) + \ldots\right]$$

Example 1.22 *The 'square top' wave is shown in Fig. 1.12 having the values*

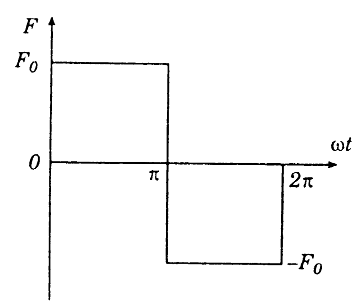

Fig. 1.12

$$f(t) = F_0 \quad \text{for} \quad 0 < \omega t < \pi$$

$$= -F_0 \quad \text{for} \quad \pi < \omega t < 2\pi$$

Represent this as a superposition of component harmonic motions.

Solution.

The Foruier series is :

$$f(t) = \frac{a_0}{2} + \sum_{i=1}^{n} [a_i \cos(i\,\omega t) + b_i \sin(i\,\omega t)]$$

$$a_0 = \frac{\omega}{\pi}\left[\int_0^{\pi/\omega} F_0\,dt + \int_{\pi/\omega}^{2\pi/\omega} (-F_0)\,dt\right]$$

$$= \frac{\omega F_0}{\pi}\left[\,|t|_0^{\pi/\omega} - |t|_{\pi/\omega}^{2\pi/\omega}\,\right]$$

$$= \frac{\omega F_0}{\pi}\left[\frac{\pi}{\omega} - \frac{\pi}{\omega}\right] = 0$$

$$a_i = \frac{\omega F_0}{\pi}\int_0^{\pi/\omega} \cos(i\,\omega t)\,dt - \frac{\omega F_0}{\pi}\int_{\pi/\omega}^{2\pi/\omega} \cos(i\,\omega t)\,dt$$

$$= \frac{\omega F_0}{i\,\omega\,\pi}\left[\,|\sin(i\,\omega t)|_0^{\pi/\omega} - |\sin(i\,\omega t)|_{\pi/\omega}^{2\pi/\omega}\,\right]$$

$$= \frac{F_0}{i\,\pi}[\,\sin(i\,\pi) - \sin(2\,i\,\pi) + \sin(i\,\pi)\,]$$

$$= \frac{F_0}{i\,\pi}[\,2\sin(i\,\pi) - \sin(2\,i\,\pi)\,] = 0$$

For all values of i.

$$b_i = \frac{\omega F_0}{\pi}\int_0^{\pi/\omega} \sin(i\,\omega t)\,dt - \frac{\omega F_0}{\pi}\int_{\pi/\omega}^{2\pi/\omega} \sin(i\,\omega t)\,dt$$

$$= -\frac{F_0}{i\,\pi}[\cos(i\,\pi) - \cos(0) - \cos(2\,i\,\pi) + \cos(i\,\pi)]$$

$$= -\frac{F_0}{i\,\pi}[2\cos(i\,\pi) - \cos(2\,i\,\pi) - 1]$$

$$b_i = 0 \quad \text{for } n \text{ even}$$

$$= \frac{4F_0}{i\,\pi} \quad \text{for } n \text{ odd}$$

$$f(t) = \frac{4F_0}{i\,\pi}\sum_{i=1,3,5,\ldots}^{n} \sin(i\,\omega t)$$

1.7 Solution Methods

Vibration problems can be solved by the following methods :

1. Newton's second law of motion
2. D'Alembert's principle
3. Energy method
4. Generalised coordinates
5. Numerical methods
6. Analogous methods.

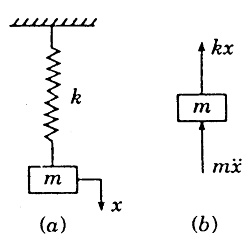

Fig. 1.13 Spring-mass system.

1.7.1 Newton's Second Law of Motion

For solving vibration problems, according to Newton's second law of motion. Inertia (or disturbing) force = $\sum$ Restoring forces

or $$m\ddot{x} = \sum F_r \qquad ...(1.13)$$

For the spring-mass system shown in Fig. 1.13, we have

$$m\ddot{x} = -kx$$

or $$m\ddot{x} + kx = 0 \qquad ...(1.14)$$

Eq. (1.14) is the equation of motion.

Natural frequency, $\omega_n = \sqrt{\dfrac{k}{m}}$ rad/s

1.7.2 D'Alembert's Principle

The D'Alembert's principle is a method to convert a dynamic system into an equivalent static system by adding the inertia force, taken in the reverse direction, to the restoring force. For the system shown in Fig. 1.13, this principle gives :

Reversed inertia force + $\sum$ Restoring force = 0

or $$m\ddot{x} + kx = 0 \qquad ...(1.15)$$

1.7.3 Energy Method

The energy method makes use of the principle of conservation of energy. A vibrating system without damping has partly kinetic energy (T) and partly potential energy (U). The total energy being constant, its rate of change with respect to time is zero.

Now $$T + U = \text{const.} \qquad ...(1.16)$$

$$\frac{d}{dt}(T + U) = 0 \qquad ...(1.17)$$

For the system shown in Fig. 1.12, we have

$$1/2 \,.\, m\dot{x}^2 + 1/2 \,.\, kx^2 = \text{const.}$$

Differentiating, we get

$$(m\ddot{x} + kx)\, x = 0$$

or $$m\ddot{x} + kx = 0$$

Thus, we find that the same equation of motion can be obtained by using the various methods. Other solution methods shall be discussed in subsequent chapters.

Review Questions

1. What are vibrations ? List their main causes.
2. What are the harmful effects of vibrations ?
3. How undesirable vibrations can be eliminated ?
4. How vibrations can be classified ?

5. Differentiate between (*a*) free and forced vibrations and (*b*) damped and undamped vibrations.
6. What are the characteristies of vibrations ?
7. Define simple harmonic motion .
8. What are beats ?
9. What is a periodc function ?
10. Write expression for the Fourier series :
11. What are the solution methods for vibration problems ?
12. State the D'Alembert's principle.

EXERCISES

1.1 Split up the harmonic motion : $x = 10 \sin\left(\omega t + \frac{\pi}{6}\right)$ into two harmonic motions one having a phase angle of zero and the other of 46°.

[**Ans.** 3.66 sin ωt, 7.197 sin (ωt + 46°)]

1.2 A harmonic displacement is given by :

$$x\ (t) = 6 \sin\left(20\,t + \frac{\pi}{3}\right) \text{ mm}$$

Where t i s in seconds and phase angle in radians. Calculate (*a*) the frequency and period of motion, (*b*) maximum velocity, and (*c*) maximum acceleration.

[**Ans.** 20 rad/s, 120 mm/s, 2400 mm/s^2]

1.3 Determine the resultant amplitude of the addition of two harmonic motions :

$$x_1 = 3 \cos (2t + 1) \text{ and } x^2 = 6 \cos (2t + 1.5)$$

[**Ans.** 10.61]

1.4 A force, $F_0 \sin \omega t$ acts on a displacement, $x_0 \sin\left(\omega t - \frac{\pi}{6}\right)$, where $F_0 = 25N$, $x_0 = 0.05m$ and $\omega = 20\,\pi$ *rad/s*.

Calculate the work done during (*a*) the first second , and

(*b*) the first $\frac{1}{40}$ *th* second.

[**Ans.**19.635 N.m, 1.032 N.m]

1.5 Subtract $x_2 = 3.6 \cos \omega t$ from $x_1 = 7.4 \sin (\omega t + 45°)$ and find the resultant amplitude and phase angle.

[**Ans.** 8.2, 59.4°]

1.6 Find the period of the functions :

(*a*) $x = 3 \sin 3t + 5 \sin 4t$

(*b*) $x = 7 \cos^2 3t$

1.7 Determine the Fourier series for the triangular wave shown in Fig. 1.14.

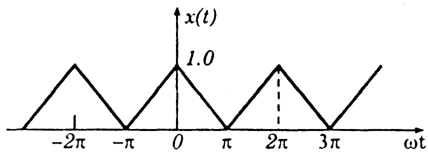

Fig. 1.14

$$\left[\textbf{Ans. } x(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos \omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\,\omega t + ...\right)\right]$$

1.8 Determine the Fourier series of the saw tooth curve shown in Fig. 1.15.

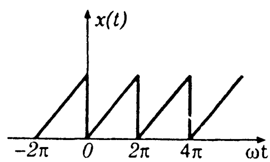

Fig. 1.15

1.9 Determine the Fourier series for the rectangular wave shown in Fig. 1.16.

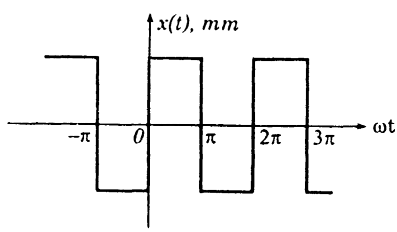

Fig. 1.16

$$\left[\textbf{Ans. } x(t) = \frac{4}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\,\omega t + ...\right)\right]$$

1.10 Represent the periodic motions shown in Figs. 1.17 by Fourier series

1.11 Represent the periodic motion shown in Fig. 1.18 by Fourier Series.

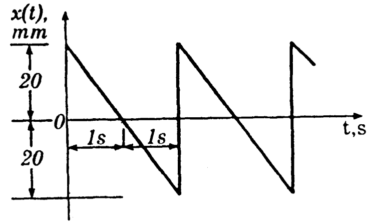

Fig. 1.17

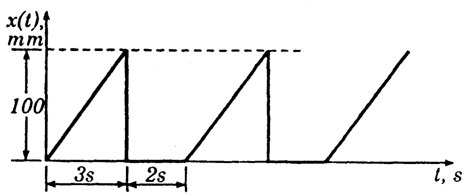

Fig. 1.18

$$\left[\textbf{Ans. } x(t) = 30 + \sum_{n=1}^{\infty} \left[\frac{100}{n\pi} \sin 1.2\pi n + \frac{88.3}{\pi^2 n^2} (\cos 1.2\pi n - 1) \right] \cos 4\pi n t \right]$$

1.12 Represent the periodic motion shown in Fig. 1.19 by Fourier Series.

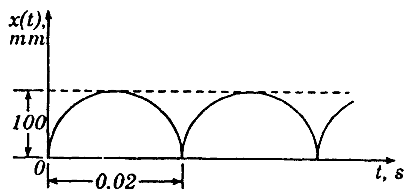

Fig. 1.19

$$\left[\textbf{Ans. } x(t) = \frac{20}{\pi} - \frac{40}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{4n^2 - 1} \right) \cos 100n\pi t \right.$$

$$\left. + \sum_{n=1}^{\infty} \left[-\frac{100}{n\pi} \cos 1.2\pi n + \frac{83.3}{\pi^2 n^2} \sin 12\pi n \right] \sin 4\pi nt \right]$$