1

Three-Dimensional Stresses and Strains

1·1 Introduction

The forces acting on a body can be distinguished as : internal forces and external forces. The internal forces are the reactive forces which are set up due to external forces applied to the body. The stresses and strains developed in the body are due to these internal forces. The external forces are further classified as surface forces and body forces. The surface forces are distributed over the surface or boundary of the body and act from the surrounding medium, like atmospheric pressure, hydraulic pressure and contact pressure exerted by one body on another. The surface forces are described in terms of forces per unit area and are called applied stresses. The applied stresses are expressed as :

$$S_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$

$$S_y = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$$

$$S_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}$$

The body forces act throughout the body and are defined as forces per unit volume or mass. Some examples of body forces are : gravitational force, electromagnetic force, centrifugal force and inertia force. The body forces are expressed as :

$$B_x = \lim_{\Delta V \to 0} \frac{\Delta F_x}{\Delta V} \quad \text{or} \quad \lim_{\Delta m \to 0} \frac{\Delta F_x}{\Delta m}, \text{ etc.}$$

1·2 Three-Dimensional Stress System

The state of stress at a point is described by a double subscript system. The first subscript denotes the direction of the outward drawn normal on the plane on which the stress acts, and the second subscript denotes the direction towards which the stress acts. In the x, y, z coordinate system, the stress components are denoted by τ_{ij}, where $i, j = x, y, z$.

Thus, $$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

The normal components τ_{xx}, τ_{yy}, τ_{zz} are often written as σ_{xx} or σ_x, σ_{yy} or σ_y and σ_{zz} or σ_z respectively. The shear stresses are complementary. Therefore $\tau_{ij} = \tau_{ji}$, $i \neq j$. Thus we are left with six independent stress components, as given below :

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

1·3 Direction Cosines

The direction cosine is the cosine of the angle between two lines. Consider two set of rectangular axes x_1, x_2, x_3 and x_1', x_2', x_3'. Let a_{ij} be the direction cosines. Then

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) & \cos(x_1', x_3) \\ \cos(x_2', x_1) & \cos(x_2', x_2) & \cos(x_2', x_3) \\ \cos(x_3', x_1) & \cos(x_3', x_2) & \cos(x_3', x_3) \end{bmatrix}$$

The normality conditions for direction cosines are :

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \text{ and so on.}$$

The orthogonality conditions are :

$$a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \text{ and so on.}$$

1·4 Normal and Shearing Stresses

Consider a rectangular parallelopiped as shown in the Fig. 1·1. Let σ_r be the resultant stress and σ_{rx}, σ_{ry}, σ_{rz} its components along the three rectangular axes. The direction cosines of σ_r are defined as :

$$a_{rx} = \frac{\sigma_{rx}}{\sigma_r}, \quad a_{ry} = \frac{\sigma_{ry}}{\sigma_r}, \quad a_{rz} = \frac{\sigma_{rz}}{\sigma_r}$$

Let a_{nx}, a_{ny}, a_{nz} be the direction cosines of normal stress. Then

Normal stress, $\sigma_n = \sigma_r \cdot a_{nr}$

where $$a_{nr} = a_{nx} \cdot a_{rx} + a_{ny} \cdot a_{ry} + a_{nz} \cdot a_{rz}$$

$$\therefore \quad \sigma_n = \sigma_r(a_{nx} \cdot a_{rx} + a_{ny} \cdot a_{ry} + a_{nz} \cdot a_{rz})$$

$$= \sigma_{rx} \cdot a_{nx} + \sigma_{ry} \cdot a_{ny} + \sigma_{rz} \cdot a_{nz} \qquad ...(1)$$

Also

$$\left.\begin{aligned} \sigma_{rx} &= \sigma_x \cdot a_{nx} + \tau_{xy} \cdot a_{ny} + \tau_{xz} \cdot a_{nz} \\ \sigma_{ry} &= \tau_{xy} \cdot a_{nx} + \sigma_y \cdot a_{ny} + \tau_{yz} \cdot a_{nz} \\ \sigma_{rz} &= \tau_{xz} \cdot a_{nx} + \tau_{yz} \cdot a_{ny} + \sigma_z \cdot a_{nz} \end{aligned}\right\} \qquad ...(2)$$

Substituting Eqs. (2) in (1), we get

$$\sigma_n = \sigma_x . a_{nx}^2 + \sigma_y . a_{ny}^2 + \sigma_z . a_{nz}^2 + 2\left(\tau_{xy} . a_{nx}\, a_{ny} + \tau_{yz} \cdot a_{ny}\, a_{nz} + \tau_{xz} \cdot a_{nx}\, a_{nz}\right) \qquad ...(3)$$

Now $\sigma_n^2 + \tau_{ns}^2 = \sigma_r^2$

where τ_{ns} = shear stress

$$= \sqrt{\sigma_r^2 - \sigma_n^2}$$

and $\sigma_r = \sqrt{\sigma_{rx}^2 + \sigma_{ry}^2 + \sigma_{rz}^2}$

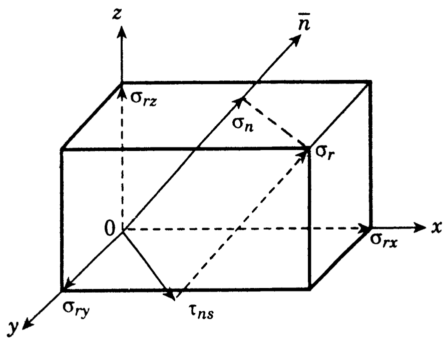


Fig. 1·1 Stresses on an oblique plane.

Let a_{sx}, a_{sy}, a_{sz} be the direction cosines of τ_{ns}. Then

$$\sigma_n \cdot a_{nx} + \tau_{ns} \cdot a_{sx} = \sigma_{rx}$$

$$a_{sx} = \frac{1}{\tau_{ns}}[\sigma_{rx} - \sigma_n \cdot a_{nx}]$$

$$= \frac{1}{\tau_{ns}}[(\sigma_x - \sigma_n)\, a_{nx} + \tau_{xy} \cdot a_{ny} + \tau_{xz} \cdot a_{nz}]$$

Similarly $a_{sy} = \dfrac{1}{\tau_{ns}}[\tau_{xy} \cdot a_{nx} + (\sigma_y - \sigma_n)\, a_{ny} + \tau_{yz} \cdot a_{nz}]$

$$a_{sz} = \frac{1}{\tau_{ns}}[\tau_{xz} \cdot a_{nx} + \tau_{yz}\, a_{ny} + (\sigma_z - \sigma_z)\, a_{nz}]$$

1·5 Equilibrium Equations

Consider an elementary rectangular parallelopiped of sides dx, dy and dz subjected to stresses as shown in Fig. 1·2. Let B_x, B_y and B_z be the body forces per unit volume. Considering the equilibrium of forces in the x-direction, we have

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) . dydz - \sigma_x . dydz + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} \cdot dy\right) . dxdz - \tau_{xy} . dxdz$$

$$+ \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} dz\right) . dxdy - \tau_{xz} . dxdy + B_x . dxdydz = 0$$

Simplifying and dividing throughout by $dxdydz$, we get

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0 \qquad ...(1)$$

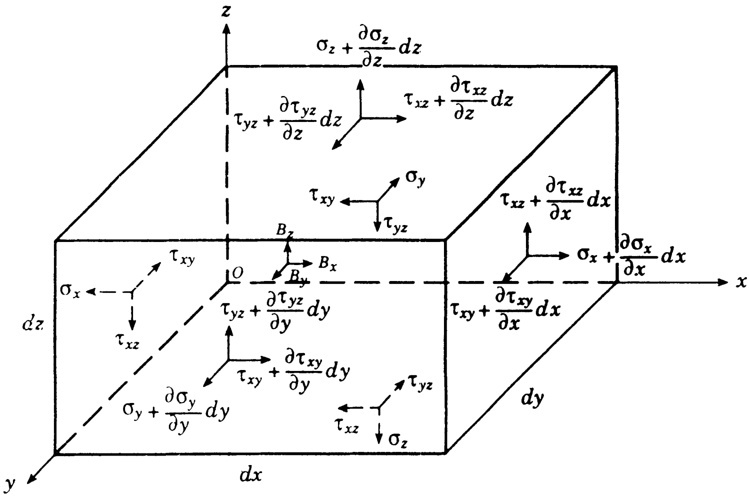


Fig. 1·2 Stresses on an elementary rectangular parallelopiped.

Similarly considering equilibrium of forces along y- and z-axes, we get

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0 \qquad \text{...(2)}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + B_z = 0 \qquad \text{...(3)}$$

The Eqs. (1), (2) and (3) are the equilibrium equations in Cartesian coordinates.

Problem 1·1 *Determine the values of A and B so that the following stress distribution represents an equilibrium state* ·

$$\sigma_x = 24x^2y, \qquad \sigma_y = Ay^3, \qquad \tau_{xy} = -Bxy^2$$

The body forces are zero.

Solution. $\frac{\partial \sigma_x}{\partial x} = 48xy$, $\frac{\partial \sigma_y}{\partial y} = 3Ay^2$, $\frac{\partial \tau_{xy}}{\partial x} = -By^2$, $\frac{\partial \tau_{xy}}{\partial y} = -2Bxy$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{gives} \quad 48xy - 2Bxy = 0$$

or $B = 24$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad \text{gives} \quad -By^2 + 3Ay^2 = 0$$

or $A = B/3 = 8$

1·6 Principal Stresses

Consider an infinitesimal tetrahedron having three faces with known stresses on the reference planes, as shown in Fig. 1·3. Assume that the inclined face ABC is a principal plane. The principal stress σ on this plane is along the normal $\bar{n}$ to this plane having direction cosines a_{nx}, a_{ny} and a_{nz}.

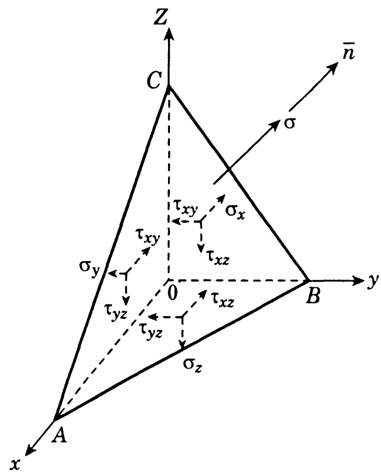


Fig. 1·3 Stress components on the faces of elementary tetrahedron.

Applying Newton's second law of motion in the z-direction and neglecting body and inertia forces, we have

$$\sigma \,.\, ABC \,.\, a_{nz} - \sigma_z \,.\, AOB - \tau_{yz} \,.\, AOC - \tau_{xz} \,.\, BOC = 0$$

Dividing throughout by area ABC, we get

$$\sigma \,.\, a_{nz} - \sigma_z \,.\, a_{nz} - \tau_{yz}\, a_{ny} - \tau_{xz} \,.\, a_{nx} = 0$$

or
$$\tau_{xz} \,.\, a_{nx} + \tau_{yz} \,.\, a_{ny} + (\sigma_z - \sigma)\, a_{nz} = 0$$

Similarly, by considering forces in the x- and y-directions, we get

$$(\sigma_x - \sigma)\, a_{nx} + \tau_{xy} \,.\, a_{ny} + \tau_{xz} \,.\, a_{nz} = 0$$

$$\tau_{xy} \,.\, a_{nx} + (\sigma_y - \sigma)\, a_{ny} + \tau_{yz} \,.\, a_{nz} = 0$$

The three equilibrium equations can be written as :

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} \begin{Bmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This is a set of three homogeneous linear equations. For a non-trivial solution, the determinant of the coefficient matrix should be equal to zero.

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

Expanding the determinant, we get

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\,\sigma^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)\,\sigma$$
$$- (\sigma_x\sigma_y\sigma_z - \sigma_x \,.\, \tau_{yz}^2 - \sigma_y \,.\, \tau_{xz}^2 - \sigma_z \,.\, \tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{xz}) = 0$$

This is a cubic equation and have three distinct roots, giving three principal stresses.

1·6·1 Stress Invariants

The combination of stresses at a point which do not vary with the orientation of axes are called stress invariants. We define :

$I_1 = \sigma_x + \sigma_y + \sigma_z$ = First invariant of stress

$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$
= Second invariant of stress

$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\,\tau_{yz}\,\tau_{xz}$
= Third invariant of stress

Thus the cubic equation can be written as :

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

This equation can be solved by Newton-Raphson method, hit-and-trial, or some numerical technique giving three principal stresses σ_1, σ_2 and σ_3.

1·6·2 Principal Planes

On the principal planes, the normal stress is either maximum or minimum. For σ_1, let

$$A_1 = \begin{vmatrix} \sigma_y - \sigma_1 & \tau_{yz} \\ \tau_{yz} & \sigma_z - \sigma_1 \end{vmatrix}, \quad B_1 = -\begin{vmatrix} \tau_{xy} & \tau_{yz} \\ \tau_{xz} & \sigma_z - \sigma_1 \end{vmatrix},$$

$$C_1 = \begin{vmatrix} \tau_{xy} & \sigma_y - \sigma_1 \\ \tau_{xz} & \tau_{yz} \end{vmatrix}$$

For non-trivial solution, let

$$\frac{a_{nx1}}{A_1} = \frac{a_{ny1}}{B_1} = \frac{a_{nz1}}{C_1} = K_1$$

where
$$K_1 = \frac{\sqrt{a_{nx1}^2 + a_{ny1}^2 + a_{nz1}^2}}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm\frac{1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$$

Thus
$$a_{nx1} = \frac{A_1}{K_1}, \quad a_{ny1} = \frac{B_1}{K_1}, \quad a_{nz1} = \frac{C_1}{K_1}$$

The principal planes for other principal stresses can be determined likewise.

1·7 Deformation of an Infinitesimal Line Element

Consider an infinitesimal line element $\underline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors and A_x, A_y, A_z are the components of $\underline{A}$ along the three coordinate axes. On infinitesimal deformation, the deformation of the line element $\underline{A}$ can be written as :

$$\begin{Bmatrix} \delta A_x \\ \delta A_y \\ \delta A_z \end{Bmatrix} = \begin{bmatrix} \partial u_x/\partial x & \partial u_x/\partial y & \partial u_x/\partial z \\ \partial u_y/\partial x & \partial u_y/\partial y & \partial u_y/\partial z \\ \partial u_z/\partial x & \partial u_z/\partial y & \partial u_z/\partial z \end{bmatrix} \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}$$

where $\left[\dfrac{\partial u_i}{\partial x_j}\right]$ = gradient of the displacement vector

Now $$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$= \varepsilon_{ij} + \omega_{ij}$$

= Strain tensor + Rotation tensor

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \dfrac{\gamma_{xy}}{2} & \dfrac{\gamma_{xz}}{2} \\ \dfrac{\gamma_{xy}}{2} & \varepsilon_y & \dfrac{\gamma_{yz}}{2} \\ \dfrac{\gamma_{xz}}{2} & \dfrac{\gamma_{yz}}{2} & \varepsilon_z \end{bmatrix}$$

$$\omega_{ij} = \begin{bmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & \omega_{yy} & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & \omega_{zz} \end{bmatrix}$$

It may be noticed that $\left.\begin{aligned} \varepsilon_{ij} &= \varepsilon_{ji} \\ \omega_{ij} &= -\omega_{ji} \\ \varepsilon_{ij} &= \gamma_{ij}/2 \end{aligned}\right\} i \neq j$

Thus strain tensor is symmetric and rotation tensor is skew-symmetric. If $\delta\phi$ is the angle of rotation, then

$$\delta\phi = \delta\phi_x\,\hat{i} + \delta\phi_y\,\hat{j} + \delta\phi_z\,\hat{k}$$

where $\delta\phi_x = \omega_{zy},\ \delta\phi_y = \omega_{xz},\ \delta\phi_z = \omega_{yx}$

1·8 Normal and Shearing Strains

The normal strain on a plane having direction cosines a_{nx}, a_{ny}, a_{nz} are :

$$\varepsilon_n = \varepsilon_x a_x^2 + \varepsilon_y a_{ny}^2 + \varepsilon_z a_{nz}^2 + \gamma_{xy}\, a_{nx}\, a_{ny} + \gamma_{yz}\, a_{ny}\, a_{nz} + \gamma_{xz}\, a_{nx}\, a_{nz}$$

Resultant strain, $\varepsilon_r = \sqrt{\varepsilon_{rx}^2 + \varepsilon_{ry}^2 + \varepsilon_{rz}^2}$

where
$$\varepsilon_{rx} = \varepsilon_x a_{nx} + \frac{\gamma_{xy}}{2} a_{ny} + \frac{\gamma_{xz}}{2} a_{nz}$$
$$\varepsilon_{ry} = \frac{\gamma_{xy}}{2} a_{nx} + \varepsilon_y \cdot a_{ny} + \frac{\gamma_{yz}}{2} \cdot a_{nz}$$
$$\varepsilon_{rz} = \frac{\gamma_{xz}}{2} a_{nx} + \frac{\gamma_{yz}}{2} a_{ny} + \varepsilon_z \cdot a_{nz}$$

Shearing strain,
$$\varepsilon_{ns} = \frac{\gamma_{ns}}{2} = \sqrt{\varepsilon_r^2 - \varepsilon_n^2}$$

1.9 Principal Strains

The principal strains are the roots of the cubic equation :
$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0$$

where
$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z = \text{First invariant of strain}$$
$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z - \frac{\gamma_{xy}^2}{4} - \frac{\gamma_{yz}^2}{4} - \frac{\gamma_{xz}^2}{4}$$
$$= \text{Second invariant of strain}$$
$$J_3 = \varepsilon_x\varepsilon_y\varepsilon_z - \frac{1}{4}(\varepsilon_x \gamma_{yz}^2 + \varepsilon_y \gamma_{xz}^2 + \varepsilon_z \gamma_{xy}^2) + \frac{1}{4} \gamma_{xy} \gamma_{yz} \gamma_{xz}.$$
$$= \text{Third invariant of strain}$$

The solution of cubic equation gives three principal strains ε_1, ε_2 and ε_3.

1·10 Principal Planes

For principal strain ε_1, let

$$A_1 = \begin{vmatrix} \varepsilon_y - \varepsilon_1 & \gamma_{yz}/2 \\ \gamma_{yz}/2 & \varepsilon_z - \varepsilon_1 \end{vmatrix}, \quad B_1 = -\begin{vmatrix} \gamma_{xy}/2 & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \varepsilon_z - \varepsilon_1 \end{vmatrix}, \quad C_1 = \begin{vmatrix} \gamma_{xy}/2 & \varepsilon_y - \varepsilon_1 \\ \gamma_{xz}/2 & \gamma_{yz}/2 \end{vmatrix}$$

Then $a_{nx1} = \dfrac{A_1}{K_1}$ where $K_1 = \dfrac{1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$ and so on.

1·11 Generalised Hooke's Law

For isotropic materials, the stress-strain relationship is :

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2G+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2G+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

where G = Shear modulus

λ = Lame's constant

$$E = \frac{G(3\lambda + 2G)}{\lambda + G} = \text{modulus of elasticity}$$

$$\nu = \frac{\lambda}{2(G+\lambda)} = \text{Poisson's ratio}$$

$$K = \lambda + \frac{2G}{3} = \text{bulk modulus}$$

1·12 Stress-Strain Relationship

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \tau_{xy}/G$$

$$\gamma_{yz} = \tau_{yz}/G$$

$$\gamma_{xz} = \gamma_{xz}/G$$

1·13 Compatibility Equations

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}, \ldots\ldots, \ldots\ldots$$

$$\frac{2\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \ldots\ldots, \ldots\ldots$$

Problem 1·2 *The Cartesian components of stress at a point are given as below :*

$\sigma_x = 15$ $\quad\sigma_y = \sigma_z = 8,$

$\tau_{xy} = 6,$ $\quad\tau_{yz} = 4,$ $\quad\tau_{xz} = 4$ *MPa.*

Determine the normal and shear stresses on a plane whose direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Solution.

Normal stress, $\sigma_n = \frac{1}{3}(15 + 8 + 8) + \frac{2}{3}(6 + 4 + 4)$

$= \mathbf{19{\cdot}67\ MPa}$

$$\sigma_{rx} = \frac{1}{\sqrt{3}}(15 + 6 + 4) = \frac{25}{\sqrt{3}} \text{ MPa}$$

$$\sigma_{ry} = \frac{1}{\sqrt{3}}(6 + 8 + 4) = \frac{18}{\sqrt{3}} \text{ MPa}$$

$$\sigma_{rz} = \frac{1}{\sqrt{3}}(4+4+8) = \frac{16}{\sqrt{3}} \text{ MPa}$$

Resultant stress, $\sigma_r = \sqrt{\frac{1}{\sqrt{3}}(625+324+256)} = 20{\cdot}04$ MPa

Shear stress, $\tau_{ns} = \sqrt{(20{\cdot}04)^2 - (19{\cdot}67)^2} = 3{\cdot}83$ MPa

Problem 1·3 *The Cartesian components of stresses at a point are given as below :*

$$\sigma_x = 7,\ \sigma_y = 6,\ \sigma_z = 5,\ \tau_{xy} = 2,\ \tau_{yz} = -2,\ \tau_{xz} = 0 \text{ MPa}$$

Determine the values of principal stresses.

Solution.

Stress invariants are :

$$I_1 = 7 + 6 + 5 = 18$$

$$I_2 = 42 + 30 + 35 - 4 - 4 - 0 = 99$$

$$I_3 = 210 - 28 - 0 - 20 + 0 = 162$$

$$\sigma^3 - 18\sigma^2 + 99\sigma - 162 = 0$$

$$\sigma = r\cos\theta + \frac{I_1}{3} = r\cos\theta + 6$$

$$r = \sqrt{\frac{4}{3}\left(\frac{324}{3} - 99\right)} = 3{\cdot}464$$

$$\cos 3\theta = \frac{4}{41{\cdot}57}\left(\frac{2}{27}\times 5832 - \frac{18\times 99}{3} + 162\right) = 0$$

$$\theta = 30°,\ 90°,\ 150°$$

$$\sigma_1 = r\cos\theta_1 + 6 = 3{\cdot}464\cos 30° + 6 = 9 \text{ MPa}$$

$$\sigma_2 = 3{\cdot}464\cos 90° + 6 = 6 \text{ MPa}$$

$$\sigma_3 = 3{\cdot}464\cos 150° + 6 = 3 \text{ MPa}$$

Problem 1·4 ***The state of stress at a point for a given reference xyz is given by the following array of terms :***

$$\begin{bmatrix} 15 & 8 & -6 \\ 8 & -12 & 5 \\ -6 & 5 & 8 \end{bmatrix} MPa$$

*Determine the **principal stresses.***

Solution.

$$I_1 = 15 - 12 + 8 = 11$$

$$I_2 = -180 - 96 + 120 - 64 - 36 - 25 = -281$$

$$I_3 = -1440 - 375 + 432 - 512 - 480 = -\mathbf{2375}$$

$$r = \sqrt{\frac{4}{3}\left(\frac{121}{3} + 281\right)} = 20{\cdot}7$$

$$\cos 3\theta = \frac{4}{8868{\cdot}3}\left(\frac{2}{27} \times 1331 + \frac{11 \times 281}{3} - 2375\right)$$

$$= -0{\cdot}562$$

$$3\theta = 124{\cdot}2°$$

$$\theta_1 = 41{\cdot}4°, \theta_2 = 71{\cdot}4°, \theta_3 = 131{\cdot}4°$$

$$\sigma_1 = 20{\cdot}7 \cos 41{\cdot}4° + \frac{11}{3} = 19{\cdot}19 \text{ MPa}$$

$$\sigma_2 = 10{\cdot}27 \text{ MPa}$$

$$\sigma_3 = -10{\cdot}02 \text{ MPa}$$

Problem 1·5 *The Cartesian components of stress at a point are given below :*

$$\sigma_x = 10, \qquad \sigma_y = 5, \qquad \sigma_z = 4,$$
$$\tau_{xy} = 2, \qquad \tau_{yz} = -4, \qquad \tau_{xz} = -6 \text{ MPa}$$

Determine the normal and shear stresses on a plane whose direction cosines are $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

Solution.

$$\sigma_n = \frac{1}{9}[10 + 20 + 16 + 2(-4 + 16 - 12)]$$

$$= 5{\cdot}1 \text{ MPa}$$

$$\sigma_{rx} = \frac{1}{3}(10 - 4 - 12) = -2$$

$$\sigma_{ry} = \frac{1}{3}(2 - 10 - 8) = -\frac{16}{3}$$

$$\sigma_{rz} = \frac{1}{3}(-6 + 8 + 8) = \frac{10}{3}$$

$$\sigma_r = \sqrt{4 + \frac{256}{9} + \frac{100}{9}} = 6{\cdot}6 \text{ MPa}$$

$$\tau_{ns} = \sqrt{(6{\cdot}6)^2 - (5{\cdot}1)^2} = 4{\cdot}17 \text{ MPa}$$

Problem 1·6 *The principal stresses at a point on a plane are*

$$\sigma_1 = 50, \qquad \sigma_2 = 40 \qquad \text{and} \qquad \sigma_3 = -20 \text{ MPa}$$

Determine the normal and shear stresses on this plane if its direction cosines are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{4}}, \frac{3}{\sqrt{14}}$.

Solution.

$$\sigma_n = \frac{1}{14}(50 + 160 - 180) = 2{\cdot}14 \text{ MPa}$$

$$\sigma_r = \sqrt{\frac{1}{14}(2500 + 6400 + 3600)} = 29{\cdot}88 \text{ MPa}$$

$$\tau_{ns} = \sqrt{(29{\cdot}88)^2 - (2{\cdot}14)^2} = 29{\cdot}80 \text{ MPa}$$

Problem 1·7 *A cube with an edge of 2·5 cm as shown in Fig. 1·4 was tested in compression. Under a load of 400 kN it failed along a plane passing through the diagonal of the top, and through the diagonals of the adjacent sides. Determine the resultant normal and shear stresses on this plane at the instant of the failure.*

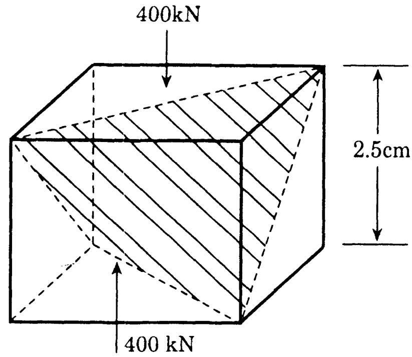


Fig. 1·4

Solution.

$$\sigma_n = -\frac{400 \times 10^3}{625} \times \frac{1}{3} = -213{\cdot}3 \text{ MPa}$$

$$\sigma_r = -\frac{400 \times 10^3}{625} \times \frac{1}{\sqrt{3}} = -369{\cdot}5 \text{ MPa}$$

$$\tau_{ns} = \sqrt{(369{\cdot}5)^2 - (213{\cdot}3)^2} = 301{\cdot}7 \text{ MPa}$$

Problem 1·8 *The strain components at a point are given by :*

$\varepsilon_x = 200$, $\varepsilon_y = 100$, $\varepsilon_z = 50$ *μ-strains, and*

$\gamma_{xy} = \gamma_{yz} = \gamma_{xz}$ *40 μ-radians.*

Calculate the normal and shearing stresses on a plane having direction cosines $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Solution.

Normal strain, $$\varepsilon_n = \frac{1}{3}(200 + 100 + 50) + \frac{1}{3}(40 \times 3) = 156{\cdot}67$$

$$\varepsilon_{rx} = \frac{1}{\sqrt{3}}(200 + 20 + 20) = \frac{240}{\sqrt{3}}$$

$$\varepsilon_{ry} = \frac{1}{\sqrt{3}}(20 + 100 + 20) = \frac{140}{\sqrt{3}}$$

$$\varepsilon_{rz} = \frac{1}{\sqrt{3}}(20 + 20 + 50) = \frac{90}{\sqrt{3}}$$

Resultant strain, $\varepsilon_r = \sqrt{\frac{1}{3}(240^2 + 140^2 + 90^2)} = \mathbf{168{\cdot}62}$

Shearing strain, $\gamma_{ns} = 2\sqrt{(168{\cdot}62)^2 - (156{\cdot}67)^2} = \mathbf{124{\cdot}72}$

Problem 1·9 *Calculate the principal strains in **Problem** 1·7.*

Solution. Strain invariants are :

$$J_1 = 200 + 100 + 50 = 350$$

$$J_2 = 200 \times 100 + 100 \times 50 + 200 \times 50 - 400 \times 3 = 33800$$

$$J_3 = 200 \times 100 \times 50 - 200 \times 400 - 100 \times 400 - 50 \times 400 + \frac{1}{4} \times 40^3$$

$$= 876000$$

$$\varepsilon^3 - 350 \times 10^{-6}\varepsilon^2 + 33800 \times 10^{-12}\varepsilon - 876000 \times 10^{-18} = 0$$

$$r = \sqrt{\frac{4}{3}\left[\frac{(350 \times 10^{-6})^2}{3} - 33800 \times 10^{-12}\right]} = \mathbf{96{\cdot}84 \times 10^{-6}}$$

$$\cos 3\theta = \frac{4}{908164 \times 10^{-18}}\left[\frac{2}{27} \times 42875000 \times 10^{-18} - \frac{350 \times 33800 \times 10^{-18}}{3} + 876000 \times 10^{-18}\right]$$

$$= 0{\cdot}47829687$$

$$\theta_1 = 20{\cdot}475°, \quad \theta_2 = 50{\cdot}415°, \theta_3 = 99{\cdot}525°$$

$$\varepsilon_1 = 96{\cdot}84 \times 10^{-6} \cos 20{\cdot}475° + \frac{350 \times 10^{-6}}{3} = \mathbf{207{\cdot}4 \times 10^{-6}}$$

$$\varepsilon_2 = -\,42 \times 10^{-6},\ \varepsilon_3 = -\,100{\cdot}6 \times 10^{-6}$$

Problem 1·10 *The stress components at a point are given by :*

$$\sigma_x = \sigma_y = \sigma_z = 50\ MPa,\ \tau_{xy} = 10,\ \tau_{yz} = 20,\ \tau_{xz} = 15\ MPa.$$

Calculate the strain components. Take E = 200 GPa and $\nu = 0{\cdot}30$.

Solution.

$$\varepsilon_x = \frac{1}{2 \times 10^5}[50 - 0{\cdot}3 \times 100] = 100 \times 10^{-6}$$

$$\varepsilon_y = \varepsilon_z = 100 \times 10^{-6}$$

$$G = \frac{E}{2(1+\nu)} = \frac{2 \times 10^5}{2 \times 1{\cdot}3} = \frac{10^5}{1{\cdot}3}\ \text{N/mm}^2$$

$$\gamma_{xy} = \frac{E}{2(1+\nu)} = \frac{2 \times 10^5}{2 \times 1{\cdot}3} = \frac{10^5}{1{\cdot}3}\ \text{N/mm}^{2\,-6}$$

$$\gamma_{yz} = \frac{20 \times 1{\cdot}3}{10^5} = 260 \times 10^{-6}$$

$$\tau_{xz} = \frac{15 \times 1{\cdot}3}{10^{-5}} = 195 \times 10^{-6}$$

Problem 1·11 *The state of stress at a point is given by : $\sigma_x = 20$, $\sigma_y = 40$, $\sigma_z = 60$, $\tau_{xy} = -20$, $\tau_{yz} = -40$ and $\tau_{xz} = 50$ MPa. Calculate the normal, shear and resultant stresses on a plane whose normal is inclined at 40° to x-axis and 54° to y-axis.*

Solution.

$$l = \cos 40° = 0{\cdot}766, \quad m = \cos 54° = 0{\cdot}588$$

$$n = \sqrt{1 - l^2 - m^2} = 0{\cdot}260$$

$$\sigma_n = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2(\tau_{xy} lm + \tau_{yz} mn + \tau_{xz} ln)$$

$$= 20 \times (0{\cdot}766)^2 + 40 \times (0{\cdot}588)^2 + 60 \times (0{\cdot}260)^2 + 2(-20 \times 0{\cdot}766 \times 0{\cdot}588 - 40 \times 0{\cdot}588 \times 0{\cdot}260 + 50 \times 0{\cdot}766 \times 0{\cdot}260)$$

$$= 19{\cdot}291 \text{ MPa}$$

$$\sigma_{xn} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$= 20 \times 0{\cdot}766 - 20 \times 0{\cdot}588 + 50 \times 0{\cdot}260 = 16{\cdot}56$$

$$\sigma_{yn} = \tau_{xy} l + \sigma_y m + \tau_{yz} n$$

$$= -20 \times 0{\cdot}766 + 40 \times 0{\cdot}588 - 40 \times 0{\cdot}260 = -2{\cdot}2$$

$$\sigma_{zn} = \tau_{xz} l + \tau_{yz} m + \sigma_z n$$

$$= 50 \times 0{\cdot}766 - 40 \times 0{\cdot}588 + 60 \times 0{\cdot}260 = 30{\cdot}38$$

$$\sigma_r = \sqrt{(16{\cdot}56)^2 + (-2{\cdot}2)^2 + (30{\cdot}38)^2} = 34{\cdot}67 \text{ MPa}$$

$$\tau_{ns} = \sqrt{\sigma_r^2 - \sigma_n^2} = \sqrt{(34{\cdot}67)^2 - (19{\cdot}291)^2} = 28{\cdot}8 \text{ MPa}$$

Problem 1·12 *The state of stress at a point is given by : $\sigma_x = 40$, $\sigma_y = 50$, $\sigma_z = -70$, $\tau_{xy} = 40$, $\tau_{yz} = 80$ and $\tau_{xz} = 30$ MPa. Determine the state of stress by rotating the axes about the x-axis through 30° in the anti-clockwise direction.*

Solution. The direction cosines after rotation are :

	x	y	z
x'	$\sqrt{3}/2$	$1/2$	0
y'	$-1/2$	$\sqrt{3}/2$	0
z'	0	0	1

Stress components after rotation are :

$$\sigma_x' = 40 \times \frac{3}{4} + 50 \times \frac{1}{4} + 0 + 2\left(40 \times \frac{\sqrt{3}}{4} + 0 + 0\right)$$

$$= 77{\cdot}14 \text{ MPa}$$

$$\sigma'_y = 40 \times \frac{1}{4} + 50 \times \frac{3}{4} + 0 + 2\left(-40 \times \frac{\sqrt{3}}{4} + 0 + 0\right)$$

$$= 12{\cdot}86 \text{ MPa}$$

$$\sigma'_z = -70 \text{ MPa}$$

$$\tau_{x'y'} = -40 \times \frac{\sqrt{3}}{4} + 50 \times \frac{\sqrt{3}}{4} + 0 + 40\left(\frac{3}{4} - \frac{1}{4}\right) + 0 + 0$$

$$= 58{\cdot}97 \text{ MPa}$$

$$\tau_{y'z'} = 80 \times \frac{\sqrt{3}}{2} - 30 \times \frac{1}{2} = 54{\cdot}28 \text{ MPa}$$

$$\tau_{x'z'} = 80 \times \frac{1}{2} + 30 \times \frac{\sqrt{3}}{2} = 65{\cdot}98 \text{ MPa}$$

Problem 1·13 *The state of stress at a point is given by : $\sigma_x = 80$, $\sigma_y = 100$, $\sigma_z = 120$, $\tau_{xy} = -60$, $\tau_{yz} = 40$ and $\tau_{xz} = 20$ MPa. Determine the principal stresses and principal planes.*

Solution.

$$I_1 = 80 + 100 + 120 = 300$$

$$I_2 = 8000 + 12000 + 9600 - 3600 - 1600 - 400 = 24000$$

$$I_3 = 960000 - 288000 - 16000 - 48000 - 96000 = 512000$$

$$\sigma^3 - 300\,\sigma^2 + 24000\sigma - 512000 = 0$$

Put $\sigma = r\cos\theta + \dfrac{I_1}{3} = r\cos\theta + 100$

$$r^3\cos^3\theta + 10^6 + 300r^2\cos^2\theta + 3\times10^4 r\cos\theta - 300(r^2\cos^2\theta + 10^4 + 200r\cos\theta) + 24000(r\cos\theta + 100) - 512000 = 0$$

$$r^3\cos^3\theta - 6000r\cos\theta - 0{\cdot}112\times10^6 = 0$$

$$\cos^3\theta - \frac{6000}{r^2}\cos\theta - \frac{0{\cdot}112\times10^6}{r^3} = 0 \qquad ...(1)$$

Now
$$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0 \qquad ...(2)$$

Comparings coefficients of Eqs. (1) and (2), we have

$$\frac{6000}{r^2} = \frac{3}{4} \qquad \text{or} \qquad r = 89{\cdot}443$$

$$\frac{0{\cdot}112\times10^6}{r^3} = \frac{1}{4}\cos 3\theta$$

or
$$\cos 3\theta = \frac{4\times0{\cdot}112\times10^6}{(89{\cdot}443)^3} = 0{\cdot}62609$$

$$3\theta_1 = 51{\cdot}2375°$$

$$\theta_1 = 17{\cdot}08°$$
$$3\theta_2 = 360° - 51{\cdot}2375 = 308{\cdot}7625°$$
$$\theta_2 = 102{\cdot}92°$$
$$3\theta_3 = 360° + 51{\cdot}2375 = 411{\cdot}2375°$$
$$\theta_3 = 137{\cdot}08°$$
$$\sigma_1 = r\cos\theta_1 + 100 = 89{\cdot}443\cos 17{\cdot}08° + 100 = 185{\cdot}5 \text{ MPa}$$
$$\sigma_2 = r\cos\theta_2 + 100 = 89{\cdot}443\cos 102{\cdot}92° + 100 = 80 \text{ MPa}$$
$$\sigma_3 = r\cos\theta_3 + 100 = 89{\cdot}443\cos 137{\cdot}08° + 100 = 34{\cdot}5 \text{ MPa}$$

Problem 1·14 *Determine whether the following strain field is compatible :*

$\varepsilon_x = 3y^2 + xy$, $\varepsilon_y = 2y + 4z + 3$, $\varepsilon_z = 3zx + 2xy$, $\gamma_{xy} = 4xy$, $\gamma_{yz} = 2xz$, *and* $\gamma_{xz} = 2xyz$.

Solution.

$$\frac{\partial^2\gamma_{xy}}{\partial x\partial y} = \frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2}$$

$$\frac{\partial\gamma_{xy}}{\partial x} = 4y, \quad \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} = 4$$

$$\frac{\partial\,\varepsilon_x}{\partial y} = \varepsilon y + x, \quad \frac{\partial^2\,\varepsilon_x}{\partial y^2} = 6$$

$$\frac{\partial\varepsilon_y}{\partial x} = 0$$

$$4 = 6 + 0$$

The strain field is not **compatible.**

1·14 Plane stress

For the plane stress case, $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ and $\sigma_x, \sigma_y, \tau_{xy} = f(x, y)$. Thus, equilibrium equations **become,**

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + B_x = 0$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + B_y = 0$$

$$B_z = 0$$

The strain-displacement **relations are** :

$$\varepsilon_x = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Stress-strains relations are :

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \gamma_{xz} = 0$$

Strain-compatibility equation is :

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Stress-compatibility equation in the absence of body forces, becomes :

$$\nabla^2 (\sigma_x + \sigma_y) = 0$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$

Airy's stress function ϕ :

Define $\sigma_x = \dfrac{\partial^2 \phi}{\partial y^2}$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

The stresses defined above in terms of Airy's stress function satisfy the equilibrium equations. When substituted in stress-compatibility equation, gives

$$\nabla^4 \phi = 0$$

where $\nabla^4 = \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$

This is called the Biharmonic equation.

1·15 Plane strain

Here $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$. The governing equations are :

Equilibrium equations :

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + B_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + B_z = 0$$

Strain-displacement relations :

$$\varepsilon_x = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Stress strain relations :

$$\varepsilon_x = \left(\frac{1+\nu}{E}\right)[(1-\nu)\,\sigma_x - \nu\sigma_y]$$

$$\varepsilon_y = \left(\frac{1+\nu}{E}\right)[(1-\nu)\,\sigma_y - \nu\sigma_x]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Strain-compatibility equation.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Stress-compatibility equation, in the absence of body forces, is

$$\nabla^2 (\sigma_x + \sigma_y) = 0$$

Biharmonic equation :

$$\nabla^4 \phi = 0$$

Problem 1·15 *Prove that the following are Airy's stress functions :*

(*a*) $\phi = Ax^2 + By^2$

(*b*) $\phi = Ax^3$

(*c*) $\phi = A(x^4 - 3x^2y^2)$

Solution.

(*a*) $$\frac{\partial \phi}{\partial x} = 2Ax, \quad \frac{\partial \phi}{\partial y} = 2By$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2A, \quad \frac{\partial^2 \phi}{\partial y^2} = 2B$$

$$\frac{\partial^4 \phi}{\partial x^4} = 0, \quad \frac{\partial^4 \phi}{\partial y^4} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2 \partial y^2} = 0$$

$$\therefore \qquad \nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Hence, it is an Airy's stress function.

(*b*) $$\frac{\partial \phi}{\partial x} = 3Ax^2, \; \frac{\partial^2 \phi}{\partial x^2} = 6Ax, \; \frac{\partial^3 \phi}{\partial x^3} = 6A, \; \frac{\partial^4 \phi}{\partial x^4} = 0$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^4 \phi}{\partial y^4} = 0$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0$$

Thus $\nabla^4 \phi = 0$ and it is an Airy's stress function.

(*c*) $$\frac{\partial \phi}{\partial x} = A(4x^3 - 6xy^2), \; \frac{\partial^2 \phi}{\partial x^2} = A(12x^2 - 6y^2)$$

$$\frac{\partial^3 \phi}{\partial x^3} = A(24x), \qquad \frac{\partial^4 \phi}{\partial x^4} = 24A$$

$$\frac{\partial \phi}{\partial y} = -\,6Ax^2y, \; \frac{\partial^2 \phi}{\partial y^2} = -\,6Ax^2$$

$$\frac{\partial^3 \phi}{\partial y^3} = 0, \qquad \frac{\partial^4 \phi}{\partial y^4} = 0$$

$$\frac{\partial^3 \phi}{\partial x^2 \partial y} = -\,12Ay, \qquad \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = -\,12A$$

$$\nabla^4 \phi = 24A - 24A + 0 = 0$$

Hence, it is an Airy's stress function.

1·16 Theories of Failure

A material is said to have failed when it is stressed beyond the elastic limit. If the material is subjected to a single type of loading, then its failure can be predicted easily. If, however the material is subjected to a complex stress system then it is not easy to predict its failure straight away. Thus the need of a theory of failure arises.

Let σ_1, σ_2, σ_3 be the three principal stresses in descending order and σ_y be the yield strength in simple tension for the material. The various theories of failure are :

(1) **Maximum Principal Stress (or Rankine's) Theory.** The material is said to have failed when the maximum principal stress σ_1 becomes equal to or more than σ_y, *i.e.*

$$\sigma_1 = \sigma_y$$

The yield strength in tension and compression is assumed to be same. It is suitable for brittle materials.

(2) **Maximum Principal strain (or St. Venant's) Theory.** The material is deemed to have failed when the maximum principal strain ε_1 becomes equal to or more than that calculated in a simple tensile test, *i.e.*

$$\frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] = \frac{\sigma_y}{E}$$

or $$\sigma_1 - \nu(\sigma_2 + \sigma_3) = \sigma_y$$

This theory over estimates the behaviour of ductile materials.

(3) **Maximum Shear Stress (or Guest-Coulomb's or Tresca's) Theory.** The failure of the material is said to have taken place when the maximum shear stress exceeds the maximum shear stress in a simple tensile test, *i.e.*

$$\frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_y}{2}$$

or $$\sigma_1 - \sigma_3 = \sigma_y$$

This theory is suitable for ductile materials. Its serious drawback is that it does not consider the intermediate principal stress.

(4) **Maximum strain Energy (or Bellrami-Haigh's) Theory.** The failure is assumed to take place when the total strain energy exceeds the strain energy determined in a simple tensile test. Thus

$$\frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1] = \frac{\sigma_y^2}{2E}$$

or $$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2$$

This theory is suitable for ductile materials.

(5) **Maximum Shear Strain Energy (or Distortion Energy) Theory.** This theory is due to Mises-Hencky. The failure is assumed to take place when the maximum shear strain energy exceeds the shear strain energy in a simple tensile test, *i.e.*

$$\frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{\sigma_y^2}{6G}$$

or $$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_4)^2 = 2\sigma_y^2$$

This theory is suitable for ductile materials.

(6) **Mohr's Theory**. According to this theory,

$$\frac{\sigma_1}{\sigma_{yt}} + \frac{\sigma_2}{\sigma_{yc}} = 1$$

where σ_{yt} and σ_{yc} are yield strengths in simple tension and compression. This theory is suitable for brittle materials.

1·16·1. Graphical Representation for Plane Stress

(1) **Maximum Principal Stress Theory.** The failure will occur when σ_1 or $\sigma_2 = \sigma_{yt}$ or σ_{yc}. These conditions are represented on $\sigma_1 - \sigma_2$ coordinates in Fig. 1·5.

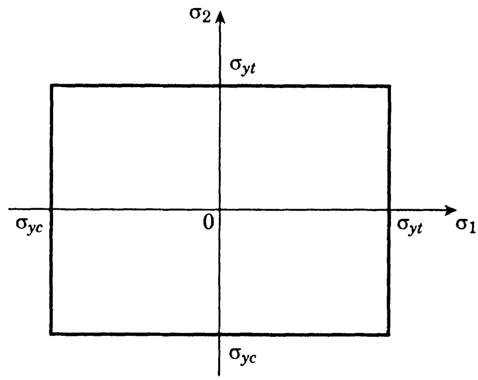


Fig. 1·5 Maximum principal stress theory.

(2) **Maximum Principal Strain Theory.** For yielding in tension,

$$\sigma_1 - \nu\sigma_2 = \sigma_{yt}$$

and for yielding in compression, with σ_2 compressive

$$\sigma_2 - \nu\sigma_1 = \sigma_{yc}$$

These equations when plotted on $\sigma_1 - \sigma_2$ coordinates produce the rhomboid failure envelope shown in Fig. 1·6.

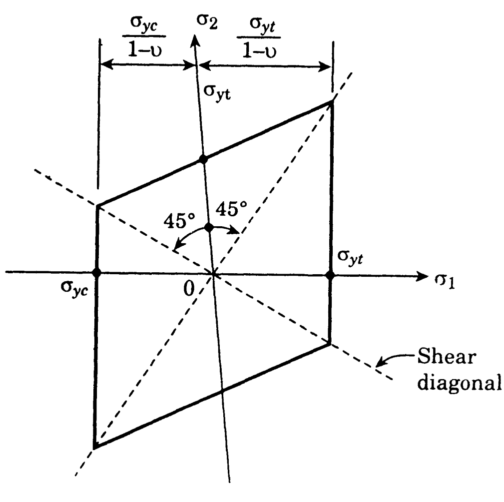


Fig. 1·6 Maximum principal strain theory.

(3) Maximum Shear Stress Theory. For yielding in tension, $\sigma_1 = \sigma_{yt}$ when $\sigma_2 = 0$ and $\sigma_2 = \sigma_{yc}$ when $\sigma_1 = 0$. When σ_1 is compressive and σ_2 is tensile in the second quadrant, then

$$\sigma_1 + \sigma_2 = \sigma_{yt}$$

and in the fourth quadrant,

$$\sigma_1 - \sigma_2 = \sigma_{yt}$$

In the third quadrant, both the stresses are negative. These equations plotted on $\sigma_1 - \sigma_2$ coordinates produce a hexagon shown in Fig. 1·7.

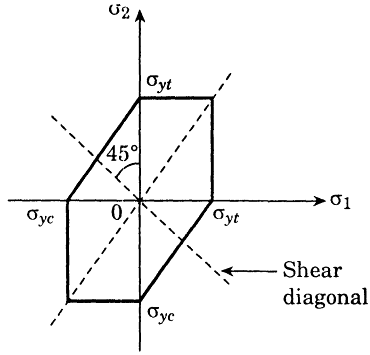


Fig. 1·7 Maximum shear stress theory.

(4) Maximum Strain Energy Theory

$$\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma_y^2$$

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - 2\nu\left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right) = 1$$

This represents an ellipse with semi-major and semi-minor axes $\frac{\sigma}{\sqrt{1-\nu}}$ and $\frac{\sigma}{\sqrt{1+\nu}}$ respectively. each at 45° to the coordinate axes, as shown in Fig. 1·8.

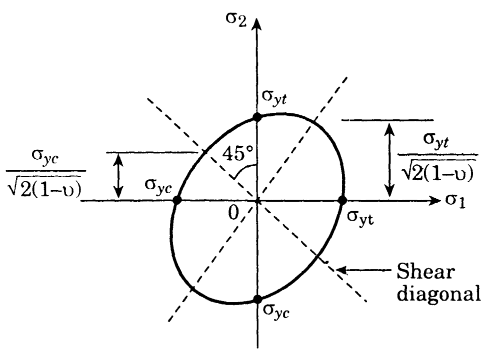


Fig. 1·8 Maximum strain energy theory.

5. Maximum Distortion Energy Theory.

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2$$

or $$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - \left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right) = 1$$

This represents an ellipse with semi-major and semi-minor axes $\sqrt{2\sigma_y}$ and $\sqrt{\frac{2}{3}}\,\sigma_y$ respectively at 45° to the coordinate axes as shown in Fig. 1·9.

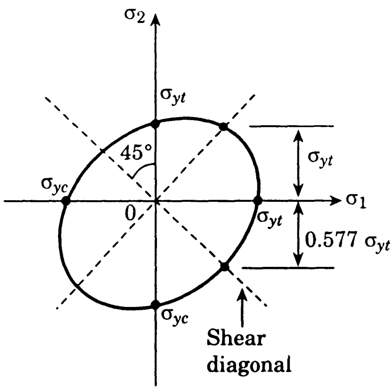


Fig. 1·9 Maximum distortion energy theory.

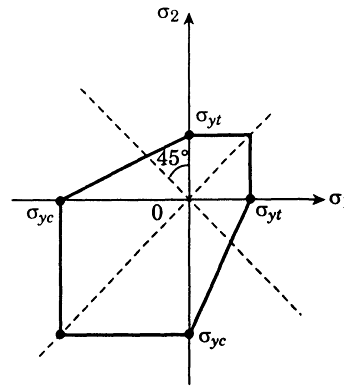


Fig. 1·10 Mohr's theory.

(6) **Mohr's Theory**

$$\frac{\sigma_1}{\sigma_{yt}} + \frac{\sigma_2}{\sigma_{yc}} = 1$$

It is shown in Fig. 1·10.

1·16·2 Derivation of Equations For Failure Theories

(*a*) Elastic strain energy due to principal stresses.

Let $\sigma_1, \sigma_2, \sigma_3$ be the principal stresses acting on a body at a point.

Work done per unit volume by $\sigma_1 = \dfrac{\sigma_1^2}{2E}$

Lateral strain produced by σ_2 and σ_3 along σ_1,

$$\varepsilon_{l1} = -\frac{\nu}{E}(\sigma_2 + \sigma_3)$$

Reduction in work done per unit volume due to lateral strain,

$$= \frac{1}{2}\sigma_1\varepsilon_{l1}$$

$$= -\frac{1}{2E}\sigma_1\nu(\sigma_2 + \sigma_3)$$

Net work done by σ_1 per unit volume,

$$w_1 = \frac{1}{2E}\left[\sigma_1^2 - \nu\sigma_1(\sigma_2 + \sigma_3)\right]$$

Similarly work done by σ_2 and σ_3 per unit volume is,

$$w_2 = \frac{1}{2E}\left[\sigma_2^2 - \nu\sigma_2(\sigma_3 + \sigma_1)\right]$$

$$w_3 = \frac{1}{2E}\left[\sigma_3^2 - \nu\sigma_3(\sigma_1 + \sigma_2)\right]$$

Total work done per unit volume by all the stresses,

$$w = w_1 + w_2 + w_3$$

$$= \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right)$$

= Total strain energy stored in the body per unit volume.

For $\sigma_3 = 0,\ u = \dfrac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 - 2\nu\,\sigma_1\sigma_2\right)$

(*b*) Shear strain energy due to principal stresses.

Shear strain energy = Total strain energy – Strain energy due to volumetric strain.

Total strain energy per unit volume,

$$u_1 = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right) \qquad ...(1)$$

Mean stress, $\sigma_m = \dfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, then

$$u_1 = \frac{1}{2E}[3\sigma^2 - 6\nu\sigma^2]$$

$$= \frac{3\sigma^2}{2E}[1 - 2\nu] = \frac{\sigma^2}{2K}$$

where K = bulk modulus

Volumetric strain energy per unit volume due to σ_m,

$$u_2 = 3\sigma_m^2\left(\frac{1-2\nu}{2E}\right)$$

$$= \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)^2\left(\frac{1-2\nu}{2E}\right)$$

$$= \left(\frac{1-2\nu}{6E}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 \qquad ...(2)$$

Shear strain energy per unit volume,

$$u_\gamma = u_1 - u_2$$

$$= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] - \left(\frac{1-2\nu}{6E}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2$$

$$= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$- \left(\frac{1-2\nu}{6E}\right)\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$= \left(\frac{1+\nu}{6E}\right)[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

If $\sigma_3 = 0$, then

$$u_\gamma = \frac{1}{6G}\left(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right)$$

Problem 1·16 *At a point in a **structural** member subjected to plane stress, the state of stress is :* $\sigma_x = 70$ ***MPa,*** $\sigma_y = -56$ *MPa,* $\tau_{xy} = -28$ *MPa. Determine which of the theories of **failure** will predict failure by yielding for this state of stress if the yield **strength** of the material in tension and compression is 250 MPa.*

Solution.

$$\sigma_{1,3} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{70-56}{2} \pm \sqrt{\left(\frac{70+56}{2}\right)^2 + (-28)^2}$$

$$= 7 \pm 68{\cdot}94 = 75{\cdot}94,\ -61{\cdot}94 \text{ MPa}$$

$$\sigma_1 = 75{\cdot}94 \text{ MPa}, \qquad \sigma_2 = 0, \qquad \sigma_3 = -61{\cdot}94 \text{ MPa}$$

(1) **Maximum normal stress theory**. For failure to occur, $\sigma_1 \geq \sigma_y$.

Here $\sigma_1 < \sigma_y$. Hence, no failure.

(2) **Maximum shear stress theory.**

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{75{\cdot}94 + 61{\cdot}94}{2} = 68{\cdot}94 \text{ MPa}$$

For failure to occur, $\tau_{max} > \dfrac{\sigma_y}{2}$

Here $\tau_{max} < \dfrac{\sigma_y}{2}$. Hence, no failure.

(3) **Maximum distortion energy theory.**

$$\left(\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2\right) < \sigma_y^2$$

$$(75{\cdot}94)^2 - (75{\cdot}94) \times (-61{\cdot}94) + (-61{\cdot}94)^2 < 250^2$$

$$14307 < 62500$$

Hence no failure.

Therefore, yielding of the structural member will not occur.

Problem 1·17 *A 200 mm diameter solid circular shaft is subjected to a torque T. The shaft is made of a material with an ultimate tensile strength of 620 MPa and an ultimate compressive strength of 820 MPa. Determine the value of maximum torque T according to the Mohr theory of failure.*

Solution.

Shear stress in shaft, $\tau = \dfrac{Tr}{J} = \dfrac{16T}{\pi d^3} = \dfrac{16T}{\pi \times (0{\cdot}2)^3} = 636{\cdot}62\ \text{TN/m}^2$

Principal stresses are :

$$\sigma_1 = 636{\cdot}62T, \qquad \sigma_2 = 0, \qquad \sigma_3 = -636{\cdot}62T$$

According to Mohr theory, we have

$$\frac{\sigma_1}{\sigma_{ut}} - \frac{\sigma_3}{\sigma_{uc}} = 1$$

$$\frac{636{\cdot}62T}{620 \times 10^6} - \frac{-636{\cdot}62T}{820 \times 10^6} = 1$$

$$1{\cdot}8032 \times 10^{-6}T = 1$$

$$T = 554{\cdot}578\ \text{kN.m}$$

Problem 1·18 *The principal stresses at a point are : 40 MPa, 30 MPa and – 20 MPa. Calculate per unit volume the (a) total strain energy, (b) volumetric strain energy, and (c) shear strain energy. E = 200 GPa, G = 80 GPa.*

Solution. Poisson's ratio, $\nu = \dfrac{E}{2G} - 1 = \dfrac{200}{2 \times 80} - 1 = 0{\cdot}25$

$$(a)\quad u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$= \frac{10^{14}}{2 \times 200 \times 10^9}[16 + 9 + 4 - 2 \times 0{\cdot}25(12 - 6 - 8)]$$

$$= \frac{30 \times 10^3}{4} = 7{\cdot}5 \text{ kN.m/m}^3$$

(*b*) $$u_v = \left(\frac{1-2\nu}{6E}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2$$

$$= \left(\frac{1 - 2 \times 0{\cdot}25}{6 \times 200 \times 10^9}\right)(40 + 30 - 20)^2 \times 10^{12}$$

$$= \frac{0{\cdot}5}{12} \times 25 \times 10^3 = 1{\cdot}04 \text{ kN.m/m}^3$$

(*c*) $$u_\gamma = u - u_v = 7{\cdot}5 - 1{\cdot}04 = 6{\cdot}46 \text{ kN.m/m}^3$$

or $$u_\gamma = \frac{1}{12\,G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{1}{12 \times 80 \times 10^9}[(40 - 30)^2 + (30 + 20)^2 + (-20 - 40)^2] \times 10^{12}$$

$$= \frac{10^2}{96}[100 + 2500 + 3600]$$

$$= \frac{620 \times 10^3}{96} = 6{\cdot}46 \text{ kN.m/m}^3$$

Problem 1·19. *A mild steel hollow shaft of* **100 mm** *external diameter and 40 mm internal diameter is subjected to* **a** *twisting moment of 6 kN.m and a bending moment of 4 kN.m. Find* **the** *direct stress, which acting alone, would produce the same (a)* **maximum** *elastic strain energy, (b) maximum elastic shear strain energy, as* **that produced** *by the principal stresses acting together. Poisson's ratio is* **0·30.**

Solution.

Torsional shear stress, $$\tau = \frac{16\,Td_0}{\pi(d_0^4 - d_i^4)}$$

$$= \frac{16 \times 6 \times 10^6 \times 100}{\pi(100^4 - 40^4)} = 31{\cdot}36 \text{ MPa}$$

Bending stress, $$\sigma_b = \frac{32\,Md_0}{\pi(d_0^4 - d_i^4)}$$

$$= \frac{32 \times 4 \times 10^6 \times 100}{\pi(100^4 - 40^4)} = 41{\cdot}81 \text{ MPa}$$

Principal stresses, $$\sigma_{1,2} = \frac{\sigma_b}{2} \pm \frac{1}{2}\sqrt{\sigma_b^2 + 4\tau^2}$$

$$= 20{\cdot}9 \pm \frac{1}{2}\sqrt{(41{\cdot}81)^2 + 4 \times (31{\cdot}36)^2}$$

$$= 20{\cdot}9 \pm 37{\cdot}69$$

$$= 58{\cdot}59 \text{ MPa}, -16{\cdot}79 \text{ MPa}$$

(a) $\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma^2$

$$(58{\cdot}59)^2 + (-16{\cdot}79)^2 - 2 \times 0{\cdot}3 \times 58{\cdot}59 \times (-16{\cdot}79) = \sigma^2$$

$$\sigma^2 = 4304{\cdot}93$$

or $$\sigma = 65{\cdot}61 \text{ MPa}$$

(b) $\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma^2$

$$(58{\cdot}59)^2 + (-16{\cdot}79)^2 - 58{\cdot}59 \times (-16{\cdot}79) = \sigma^2$$

$$\sigma^2 = 4698{\cdot}4$$

$$\sigma = 68{\cdot}54 \text{ MPa}$$

1·17 Yield Criteria

A yield criteria is a hypothesis concerning the limit of elasticity under any possible combination of stresses. The two yield criterias are : 1. Tresca's, and 2. Von-Mises-Hencky yield conditions. Both criterias neglect the effect of mean stress and require the knowledge of yield strength in simple uniaxial state of stress, assumed to be identical in tension and compression.

1·16·1 Tresca's yield Criterion

According to this criterion, yielding of a ductile metal begins when maximum shear stress reaches a definite value prescribed for a material. If $\sigma_1 > \sigma_2 > \sigma_3$, then $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$. If σ_0 is the yield strength in simple tension, then yield strength under pure shear is $\sigma_0/2$. Hence, for yielding to occur,

$$\frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_0}{2}$$

or $$\sigma_1 - \sigma_3 = \sigma_0$$

1·16·2 Von-Mises-Hencky yield criterion

This yield criterion is based upon the theory of maximum distortion energy. The yielding will occur when the distortion energy attains a critical value for a ductile material. The distortion energy under principal stresses are :

$$u = \frac{1}{12\,G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

For universal state of stress at yield,

$$u = \frac{1}{6\,G}\sigma_0^2$$

For yielding to occur,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2$$

For the plane stress case, $\sigma_3 = 0$. The Von-Mises criterion becomes,

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_0^2$$

This is known as Von-Mises ellipse, with

Major axis $= 2\sqrt{2}\,\sigma_0$

Minor axis $= \sqrt{\frac{2}{3}}\,.\,2\sigma_0$

Tresca's criterion gives,

$\sigma_1 = \sigma_0$ for σ_1 and σ_2 to be of the **same sign.**

$\sigma_1 - \sigma_2 = \sigma_0$ for σ_1 and σ_2 to be of the **opposite** sign.

This represents a hexagon. The two criterias are shown in Fig. 1·11.

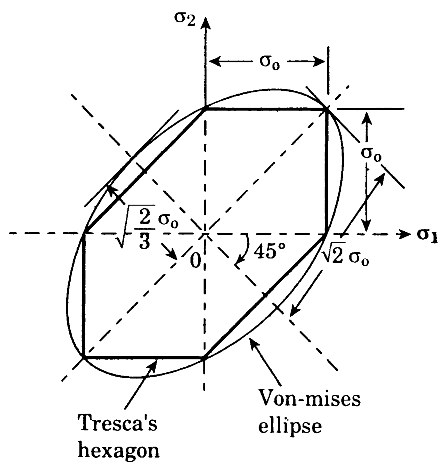


Fig. 1·11 Yield criterias for plane stress.

1·18 Ideally Plastic Solids

The materials can be classified as follows :

(1) **Rigid material**—A perfectly rigid material is one which does not deform under the action of any amount of load.

(2) **Rigid-Perfectly plastic material**—This type of material is rigid upto the yield point and then deforms plastically at constant yield stress.

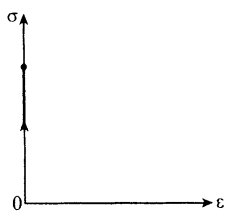
(3) **Perfectly linear elastic material**—This type of material obeys Hooke's law.

(4) **Rigid with strain hardening**—This type of material is rigid upto the yield point and then deforms plastically with strain hardening.

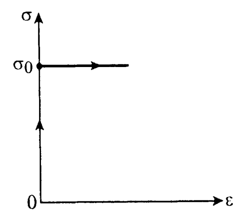
(5) **Elastic-perfectly plastic**—In such like materials, stress is proportional to strain upto the yield point and then the material deforms plastically at a constant stress.

(6) **Elasto-plastic with strain hardening**—Here the stress is proportional to strain upto the yield point and then the material deforms plastically with strain hardening.

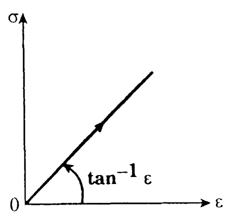
These types of materials are shown in Fig. 1.12.



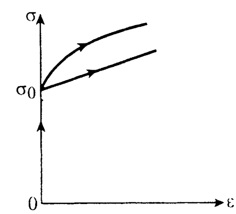
(*a*) Rigid material



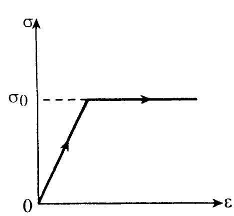
(*b*) Rigid perfectly plastic



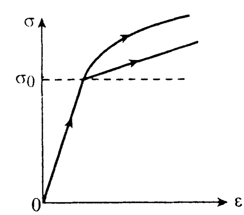
(*c*) Perfectly linear elastic



(*d*) Rigid-strain hardening



(*e*) Elasto-plastic



(*f*) Elasto-plastic with strain hardening

Fig. 1·12 Types of materials.

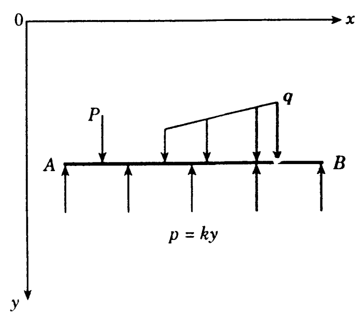
1·19 Beams on Elastic Foundation

In many practical cases the beam is supported elastically, where the elastic support is provided by a load bearing medium, referred to as the 'foundation', distributed continuously along the length of the beam. In some of these problems, the identity of the beam and the foundation can be easily established, as in the case of actual foundation structures or rail-road track. We shall study the Winkler type of foundation where the pressure in the foundation is proportional at every point to the deflection occurring at that point only.

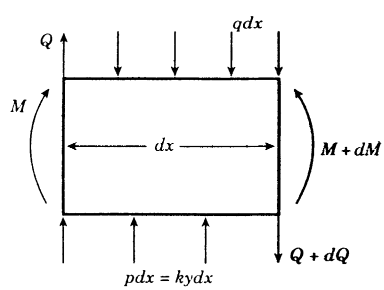
1·19·1 Differential Equation of the Elastic Line

Consider a straight beam AB supported along its entire length by an elastic medium and subjected to vertical forces as shown in Fig. 1·13 (a). Pressure p in the supporting medium is proportional to the deflection y of the beam at that point, *i.e.*, $p = ky$, where k is a constant for the supporting medium, called the ***modulus*** of the foundation. For unit width of the beam, the units of k will be N/m^2.

Let us consider an infinitely small element dx of the beam enclosed between two vertical cross-sections acted upon by a distributed load q N/m. The forces acting on this element are shown in Fig. 1·13 (b), where Q is the shear force and M the bending moment.



(a) Beam supported on entire length.



(b) Forces acting on elementary beam.

Fig. 1·13 Beam on elastic foundation.

For equilibrium of the element dx, we have

$$Q - (Q + dQ) + kydx - qdx = 0$$

or
$$\frac{dQ}{dx} = ky - q$$

Now
$$Q = \frac{dM}{dx}$$

$$\therefore \quad \frac{dQ}{dx} = \frac{d^2M}{dx^2} = ky - q$$

Also for the bending of a beam, we have

$$EI\frac{d^2y}{dx^2} = -M$$

$$\therefore \quad EI\frac{d^4y}{dx^4} = -ky + q \qquad ...(1)$$

This is the differential equation for the deflection curve of a beam supported on an elastic foundation.

Now $EI = D =$ flexural rigidity of the beam.

$$\therefore \quad \frac{d^4y}{dx^4} + \frac{k}{D}y = \frac{q}{D} \qquad ...(2)$$

The general solution of Eq. (2) can be written, as,

$$y = e^{\lambda x}(c_1 \cos \lambda x + c_2 \sin \lambda x) + e^{-\lambda x} (c_3 \cos \lambda x + c_4 \sin \lambda x) + \frac{q}{k} \qquad ...(3)$$

where
$$\lambda^4 = \frac{k}{4D}$$

and c_1 to c_4 are unknown constants.

Now

$$\left.\begin{aligned} \frac{dy}{dx} &= \tan\theta \\ EI\frac{d^2y}{dx^2} &= -M \\ EI\frac{d^3y}{dx^3} &= -Q \end{aligned}\right\} \qquad ...(4)$$

1·19·2 Infinite Beam

(a) Concentrated Load on an Infinite Beam

Consider an infinitely long beam having a single concentrated load P at the point O as shown in Fig. 1·14. Due to the symmetry of the beam we consider only right half of the beam.

As x and y tend to infinity, the deflection of the beam should be zero. Hence in Eq. (3), $c_1 = c_2 = 0$.

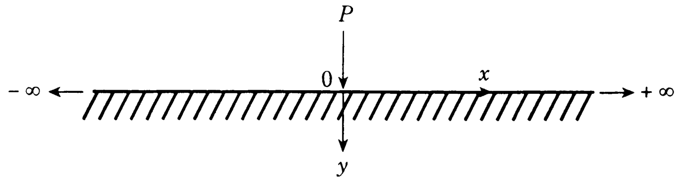


Fig. 1·14

$$\therefore \qquad y = e^{-\lambda x}(c_3 \cos \lambda x + c_4 \sin \lambda x)$$

Also $\qquad \dfrac{dy}{dx} = 0$ at $x = 0$.

Now $$\frac{dy}{dx} = e^{-\lambda x}.\lambda(-c_3 \sin \lambda x + c_4 \cos \lambda x) - \lambda e^{-\lambda x}(c_3 \cos \lambda x + c_4 \sin \lambda x)$$

$$= \lambda e^{-\lambda x}[-(c_3 + c_4)\sin \lambda x + (c_4 - c_3)\cos \lambda x]$$

or $\qquad c_4 - c_3 = 0$

or $\qquad c_3 = c_4 = c \qquad$ (say)

$$\therefore \qquad y = ce^{-\lambda x}(\cos \lambda x + \sin \lambda x)$$

Now $$P = 2\int_0^\infty ky\,dx$$

$$= 2kc\int_0^\infty e^{-\lambda x}(\cos \lambda x + \sin \lambda x)\,dx = \frac{2kc}{\lambda}$$

$$\therefore \qquad c = \frac{P\lambda}{2k}$$

$$y = \frac{P\lambda}{2k}.e^{-\lambda x}(\cos \lambda x + \sin \lambda x) \qquad ...(1)$$

At $\qquad x = 0, \quad y = \dfrac{P\lambda}{2k}$

Now $\qquad y = 0$, when $\cos \lambda x + \sin \lambda x = 0$

or $\qquad \lambda x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \ldots$, etc.

Further $$\theta = \frac{dy}{dx} = -\frac{P\lambda^2}{k}e^{-\lambda x}\sin \lambda x \qquad ...(2)$$

$$M = -EI\frac{d^2y}{dx^2} = \frac{P}{4\lambda}e^{-\lambda x}\,c\cos \lambda x - \sin \lambda x \qquad ...(3)$$

$$Q = -EI\frac{d^3y}{dx^3} = -\frac{P}{2}\,e^{-\lambda x}\cos\lambda x \qquad \ldots(4)$$

For two or more concentrated loads, the principle of superposition may be used.

(*b*) **Udl on a part of Infinite Beam.** Consider a beam of infinite length carrying Udl of intensity w over a central region BC of length l as shown in Fig. 1·15. The deflection etc. is to be calculated at point A which is at a distance 'a' and 'b' from the ends B and C respectively.

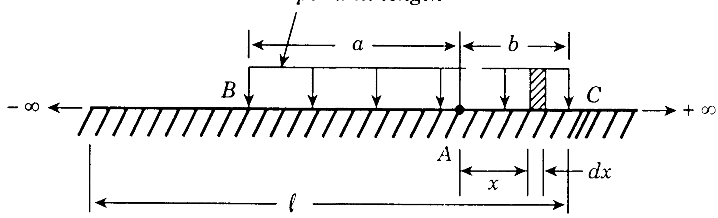


Fig. 1·15

An elementary load wdx at a distance x from A produces deflection at A,

$$dy = \frac{wdx.\lambda}{2k}.e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$$

Total deflection at point A,

$$y = \int_0^a \frac{w\lambda}{2k}.e^{-\lambda x}(\cos\lambda x + \sin\lambda x)\,dx$$

$$+ \int_0^a \frac{w\lambda}{2k}.e^{-\lambda x}(\cos\lambda x + \sin\lambda x)\,dx$$

$$= \frac{w}{2k}[2 - e^{-\lambda a}\cos\lambda a - e^{-\lambda b}\cos\lambda b]$$

If $\quad a = b = \frac{l}{2}$, then

$$y_0 = \frac{w}{k}\left[1 - e^{-\lambda l/2}.\cos\left(\frac{\lambda l}{2}\right)\right]$$

Similarly, we can get

$$M = \frac{w}{4\lambda^2}[e^{-\lambda a}\sin\lambda a + e^{-\lambda b}\sin\lambda b]$$

For $\quad a = b = l/2$, we get

$$M_0 = \frac{w}{2\lambda^2}.e^{-\lambda l/2}.\sin(\lambda l/2)$$

$$\theta = \frac{w\lambda}{2k}[e^{-\lambda a}(\cos \lambda a + \sin \lambda a) - e^{-\lambda b}(\cos \lambda b + \sin \lambda b)]$$

$$\theta_0 = 0$$

$$Q = \frac{w}{4\lambda}[e^{-\lambda a}(\cos \lambda a - \sin \lambda a) - e^{-\lambda b}(\cos \lambda b - \sin \lambda b)]$$

$$Q_0 = 0$$

(*c*) **Concentrated Moment on an Infinite Beam.** Let a concentrated moment M_0 be applied at point O on the infinitely long beam as shown in Fig. 1·16 (*a*). The concentrated moment can be regarded as a limiting case of the loading shown in Fig. 1·16 (*b*), where two concentrated loads, each equal to P, separated by a distance 'a' are applied such that as $a \to 0$, $Pa \to M_0$. This gives,

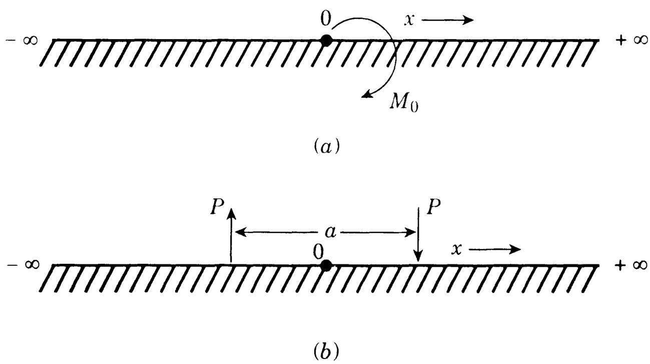


Fig. 1·16

$$y = \frac{P\lambda}{2k}[- e^{-\lambda(x+a)}\{\cos \lambda(x+a) + \sin \lambda(x+a)\} + e^{-\lambda x}(\cos \lambda x + \sin \lambda x)]$$

$$= \frac{-Pa\lambda}{2k}\left[\frac{A_{\lambda(x+a)} + A_{\lambda x}}{a}\right] \text{ for } x > 0$$

where $A_{\lambda x} = e^{-\lambda x}(\cos \lambda x + \sin \lambda x)$

Now $$\left[\frac{A_{\lambda(x+a)} + A_{\lambda x}}{a}\right]_{a \to 0} = \frac{d}{dx}(A_{\lambda x}) = -2\lambda B_{\lambda x}$$

where $B_{\lambda x} = e^{-\lambda x} \sin \lambda x$

and $(Pa)_{a \to 0} = M_0$

$\therefore$ $$y = \frac{M_0\lambda^2}{k} B_{\lambda x}$$

and $$\frac{dy}{dx} = \theta = \frac{M_0\lambda^3}{k} C_{\lambda x}$$

where $C_{\lambda x} = e^{-\lambda x} (\cos \lambda x - \sin \lambda x)$

$$-EI \frac{d^2y}{dx^2} = M = \frac{M_0}{2} D_{\lambda x}$$

where $D_{\lambda x} = e^{-\lambda x} \cos \lambda x$

$$-EI \frac{d^3y}{dx^3} = Q = -\frac{M_0 \lambda}{2} A_{\lambda x}$$

1·19·3 Semi-Infinite Beam

A semi-infinite beam may be defined as a beam which has unlimited extension of length in one direction only and having a finite end on the other side.

(*a*) **Semi-infinite Beam with Free End.**

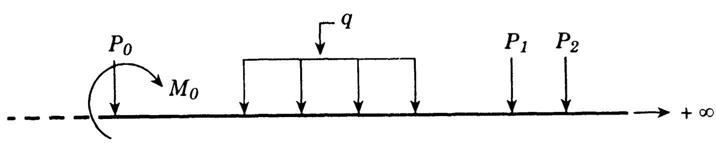


Fig. 1·17

Consider an infinitely long beam subjected to some loading due to which at point A (finite end), we have a bending moment M_A and shear force Q_A. If this moment and shear is made to vanish then the infinite beam would become a semi-infinite beam, since the end A of the semi-infinite beam has been assumed free. This can be achieved by applying at A a moment M_0 and force P_0 which will cause a bending moment $-M_A$ and a shearing force $-Q_A$ at that section as shown in Fig. 1·17.

As the end A is free, hence

$$M_A + \frac{P_0}{4\lambda} + \frac{M_0}{2} = 0$$

$$Q_A - \frac{P_0}{2} - \frac{M_0 \lambda}{2} = 0$$

$$\therefore \quad P_0 = -(\lambda M_A + Q_A)$$

$$M_0 = -\frac{2}{\lambda}(2\lambda M_A + Q_A)$$

The forces P_0 and M_0 are called the **end-conditioning forces.**

Let us consider that a force P_1 is applied at end A, as shown in Fig. 1·18, so that $M_A = 0$ and $Q_A = P_1$. Then

$$P_0 = 4Q_A = 4P_1$$

$$M_0 = -\frac{2Q_A}{\lambda} = -\frac{2P_1}{\lambda}$$

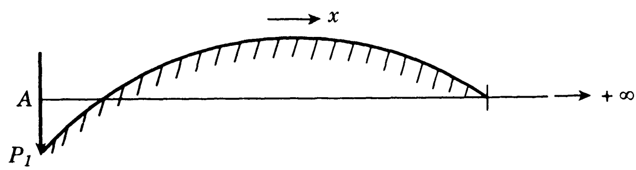


Fig. 1·18

When these are applied to the infinite beam, we get

$$y = \frac{2P_1\lambda}{k} A_{\lambda x} - \frac{2P_1\lambda}{k} B_{\lambda x}$$

$$= \frac{2P_1\lambda}{k} D_{\lambda x}$$

Similarly

$$\theta = -\frac{2P_1\lambda^2}{k} A_{\lambda x}$$

$$M = -\frac{P_1}{\lambda} B_{\lambda x}$$

$$Q = -P_1 C_{\lambda x}$$

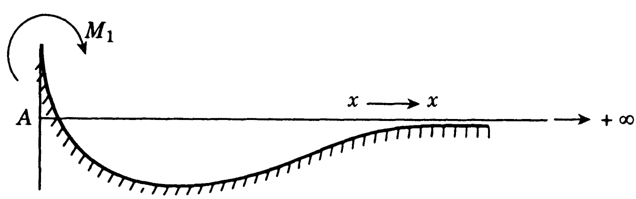


Fig. 1·19

If we apply moment M_1 at end A, as shown in Fig. 1·19, then

$$M_A = -M_1, \quad Q_A = 0$$

which gives,

$$P_0 = -4\lambda M_1$$

$$M_0 = 4M_1$$

and $y = -\dfrac{2M_1\lambda^2}{k} C_{\lambda x}$

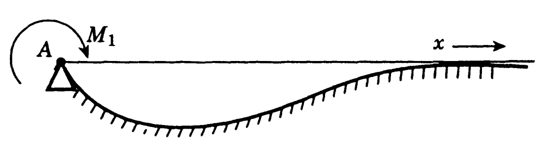


Fig. 1·20

$$\theta = \frac{4M_1\lambda^3}{k} D_{\lambda x}$$

$$M = M_1 A_{\lambda x}$$

$$Q = -2M_1\lambda B_{\lambda x}$$

(*b*) **Semi-infinite beam with hinged end**

At the hinged end A, we have

$$y = 0, M = 0$$

and $$y_A + \frac{P_0\lambda}{2k} = 0$$

$$M_A + \frac{P_0}{4\lambda} + \frac{M_0}{2} = 0$$

$\therefore$ $$P_0 = -\frac{2k}{\lambda} y_A$$

$$M_0 = \frac{k}{\lambda^2} y_A - 2M_A$$

Let us apply a moment M_1 at end A, as shown in Fig. 1·20. Then putting $y_A = 0$ and $M_A = -M_1$, we get

$$P_0 = 0, \quad M_0 = 2M_1$$

and $$y = 2M_1 \frac{\lambda^2}{k} B_{\lambda x}$$

$$\theta = 2M_1 \frac{\lambda^3}{k} C_{\lambda x}$$

$$M = M_1 D_{\lambda x}$$

$$Q = -M_1\lambda A_{\lambda x}$$

(*c*) **Semi-infinite beam with free end having udl over a finite length**

Consider a semi-infinite beam with free end carrying a *udl* as shown in Fig. 1·21. For an infinite beam,

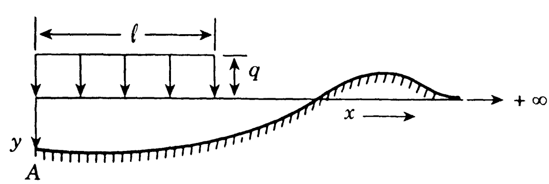


Fig. 1·21

$$M = \frac{q}{4\lambda^2} B_{\lambda l}$$

$$Q = \frac{q}{4\lambda} (1 - C_{\lambda l})$$

which gives

$$P_0 = \frac{q}{\lambda} (1 + B_{\lambda l} - C_{\lambda l})$$

$$M_0 = -\frac{q}{2\lambda^2} (1 + 2B_{\lambda l} - C_{\lambda l})$$

Applying P_0 and M_0 on the infinite beam together with q, we get

$$y = \frac{q}{2k} [(1 + B_{\lambda l} - C_{\lambda l}) A_{\lambda x} - (1 + 2B_{\lambda l} - C_{\lambda l}) B_{\lambda x} + (2 - D_{\lambda x} - D_{\lambda(l-x)})] \text{ for } 0 < x < l$$

and

$$y = \frac{q}{2k} [(1 + B_{\lambda l} - C_{\lambda l}) A_{\lambda x} - (1 + 2B_{\lambda l} - C_{\lambda l}) B_{\lambda x} + (D_{\lambda x} - D_{\lambda(x-l)})] \text{ for } x > l$$

The expressions for θ, M, Q can be obtained by differentiation.

1·19·4 Beams of Finite Length

For a beam of finite length the correct solution is the one which besides fulfilling the differential equation of the elastic line also satisfies the required conditions at both ends of the beam.

The solution is obtained by resolving the original loading into two parts a symmetrical part and an antisymmetrical part, and determining the end-conditioning forces in each of these parts. For further details the readers may refer to Theory of elasticity, 4th ed., Khanna Publishers, written by the author.

Problem 1·20 *Compute the maximum bending moment, maximum deflection and the maximum bending stress for a railroad rail subjected to a single wheel load of 100 kN. The foundation modulus* $k = 15\ MN/m^2$. *Assume that* $I = 400 \times 10^{-8}\ m^4$, $E = 200\ GN/m^2$, *the depth of the rail is 180 mm and that the distance of the centroidal axis of the cross-section of the rail from the top surface is 100 mm.*

Solution.

$$\lambda = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{\frac{15 \times 10^6}{4 \times 200 \times 10^9 \times 400 \times 10^{-8}}} = 1{\cdot}4714\ \text{m}^{-1}$$

$$M_{max} = \frac{P}{4\lambda} = \frac{100}{4 \times 1{\cdot}4714} = 16{\cdot}99\ \text{kN.m}$$

$$y_{max} = \frac{P\lambda}{2k} = \frac{100 \times 10^3 \times 1{\cdot}4714}{2 \times 15 \times 10^6} = 4{\cdot}9\ \text{mm}$$

$$\sigma_{max} = \frac{M_{max}}{z} = \frac{16{\cdot}99 \times 10^3 \times 100 \times 10^{-3}}{400 \times 10^{-8}} = 424{\cdot}75 \text{ MPa}$$

Problem 1·21 *A steel beam has a square cross-section with 50 mm sides, 20 m long and supported by an elastic Winkler foundation of modulus k = 5·6 MN/m². It is subjected to three equal loads of 20 kN each 5 m apart at the centre. Compute the deflection and the bending moment of the beam under the middle load. E = 200 GN/m².*

Solution.

$$I = \frac{5 \times 125}{12} = 52{\cdot}08 \text{ cm}^4$$

$$\lambda = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{\frac{5{\cdot}6 \times 10^4}{4 \times 200 \times 10^9 \times 52{\cdot}08 \times 10^{-8}}} = 0{\cdot}60549 \text{ m}^{-1}$$

$$x_1 = 5 \text{ m},\ x_2 = -5 \text{ m}$$

$$\lambda x_1 = 3{\cdot}02745, \quad \lambda x_2 = -3{\cdot}02745$$

$$y_0 = \frac{P\lambda}{2k}[1 + 2e^{-\lambda x}(\sin \lambda x + \cos \lambda x)]$$

$$= \frac{20 \times 10^3 \times 0{\cdot}60549}{2 \times 5{\cdot}6 \times 10^6}[1 + 2e^{-3{\cdot}02745}(\sin 3{\cdot}02745 + \cos 3{\cdot}02745)]$$

$$= 0{\cdot}989 \text{ mm}$$

$$M_0 = \frac{P}{4\lambda}[1 + 2e^{-\lambda x}(\cos \lambda x - \sin \lambda x)]$$

$$= 7{\cdot}372 \text{ kN.m}$$

Problem 1·22 *A wooden beam of cross-section 80 mm × 240 mm rests on an earth foundation. The modulus of elasticity of wood is 10 GN/m² and modulus of foundation is 5 MN/m². A uniformly distributed load of 2 kN/m acts on the middle portion of this very long beam over a span of 2 m. Compute the maximum deflection and the maximum bending stress in the beam.*

Solution.

$$I = \frac{8 \times 24^3}{12} = 9216 \times 10^{-8} \text{ m}^4$$

$$\lambda = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{\frac{5 \times 10^6}{4 \times 10^{10} \times 9216 \times 10^{-8}}} = 1{\cdot}079175 \text{ m}^{-1}$$

$$y_{max} = \frac{w}{k}[1 - e^{-\lambda l/2} \cos (\lambda l/2)]$$

$$\frac{\lambda l}{2} = \frac{1{\cdot}079175 \times 2}{2} = 1{\cdot}079175$$

$$y_{max} = \frac{2000}{5 \times 10^6}[1 - e^{-1{\cdot}079175} \cos 1{\cdot}079175] = 0{\cdot}336 \text{ mm}$$

$$\frac{\pi}{2\lambda} = \frac{\pi}{2 \times 1{\cdot}079175} = 1{\cdot}455 \text{ m}$$

$$l = 2 \text{ m}, \textit{i.e.} \quad l > \frac{\pi}{2\lambda}$$

M_{max} occurs at $\frac{\pi}{4\lambda}$ from either end.

$$\frac{\pi}{4\lambda} = 0{\cdot}7278 \text{ m}$$

$$\lambda a = 1{\cdot}079175 \times 0{\cdot}7278 = 0{\cdot}7854$$

$$\lambda b = 1{\cdot}079175 \times (2 - 0{\cdot}7278) = 1{\cdot}3729$$

$$M_{max} = \frac{w}{4\lambda^2}[e^{-\lambda a} \sin \lambda a + e^{-\lambda b} \sin \lambda b]$$

$$= \frac{2000}{4 \times (1{\cdot}079175)^2}[e^{-0{\cdot}7854} \sin 0{\cdot}7854 + e^{-1{\cdot}3729} \sin 1{\cdot}3729]$$

$$= 245{\cdot}07 \text{ N.m}$$

$$\sigma_{max} = \frac{M}{z} = \frac{245{\cdot}07 \times 12}{9216 \times 10^{-6}} = 0{\cdot}319 \text{ MPa}$$

Problem 1·23 *A long concrete footing rests on an earth foundation for which the value of the foundation modulus is 5·6 MN/m^2. The footing has a cross-section 200 mm wide and 200 mm deep. The footing supports a uniformly distributed load of 2 kN/m of length which extends over a 3 m length. Compute the value of the maximum bending moment in the footing and the maximum bending stress. E = 14 GN/m^2.*

Solution.

$$I = \frac{20^4}{12} = 13333{\cdot}3 \text{ cm}^4$$

$$\lambda = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{\frac{5{\cdot}6 \times 10^6}{4 \times 14 \times 10^9 \times 13333{\cdot}3 \times 10^{-8}}} = 0{\cdot}9306 \text{ m}^{-1}$$

$$\frac{\pi}{2\lambda} = \frac{\pi}{2 \times 0{\cdot}9306} = 1{\cdot}68793 \text{ m}$$

$$l = 3 \text{ m} > \frac{\pi}{2\lambda}$$

M_{max} occurs at

$$\frac{\pi}{4\lambda} = 0{\cdot}84396 \text{ m from either end.}$$

$$\lambda a = 0{\cdot}9306 \times 0{\cdot}84396 = 0{\cdot}7854$$

$$\lambda b = 0{\cdot}9306 \times (3 - 0{\cdot}84396) = 2{\cdot}0064$$

$$M_{max} = \frac{w}{4\lambda^2}[e^{-\lambda a} \sin \lambda a + e^{-\lambda b} \sin \lambda b]$$

$$= \frac{2000}{4 \times (0{\cdot}9306)^2}[e^{-0{\cdot}7854} \sin 0{\cdot}7854 + e^{-2{\cdot}0064} \sin 2{\cdot}0064]$$

$$= 256{\cdot}52 \text{ N.m}$$

$$\sigma_{max} = \frac{M_{max}}{z} = \frac{256{\cdot}52 \times 10}{13333{\cdot}3 \times 10^{-6}} = 0{\cdot}1924 \text{ MPa}$$