1

Basic Equations of Elasticity

1.1 Introduction

Elasticity is the property of a deformable body due to which the body recovers its original shape upon the removal of the forces causing the deformation. Almost all engineering materials possess to a certain extent the property of elasticity. In the Theory of Elasticity, we assume the body to be perfectly elastic, homogeneous and isotropic. The first discussions of elastic phenomena occur in the writings of Hooke (1676) but the first real attempts to construct a theory of elasticity using the continuum approach, date from the first half of the eighteen century*. Since that time, the theory of elasticity has developed tremendously. In this chapter, we will study the basic equations of the Theory of Elasticity without giving their proof or deriving them†.

1.2 The State of Stress at a Point

There are two types of forces acting on a material body to produce stresses : Surface forces acting on the boundaries of the body, and body forces distributed throughout the volume of the body. These applied forces are resisted by a system of stresses throughout the body.

The state of stress at any given point of a continuous body is determined entirely by the components of stress in three mutually perpendicular planes which pass through the chosen point. The planes are usually taken perpendicular to the co-ordinate directions of some orthogonal co-ordinate system. The stress at a point on a plane section through the point is defined as the force per unit area transmitted through that section at the point. The stress components are a function of both the position of the point in the body

*For the early history of the theory of elasticity, the reader may refer to I. Todhunter and K. Pearson : A History of the Theory of Elasticity and of the Strength of Materials, 2 vols. Cambridge, 1893.

†For the derivation of these equations the reader may refer to : Singh, S., "Applied Stress Analysis", Khanna Publishers.

and the orientation of the plane passed through that point. Each of the above stress components, being a vector, can be resolved further into three components along three co-ordinate axes.

Let us refer the continuous body in a state of stress under the influence of some external forces to a Cartesian co-ordinate system x, y, z and at a given point let us consider three planes which are normal to the axes of the co-ordinates. The stress which acts on each area is resolved into three components along the axes. We denote the normal components (the normal stresses) by the symbol σ with an index which indicates the direction of the normal to the area. We denote the tangential components (the tangential stresses) by τ with two indices. The components of stress acting on an area normal to the x-axis are σ_x, τ_{xy}, τ_{xz} ; the components of stress acting on an area normal to the y-axis are τ_{yz}, σ_y, τ_{yz} ; the components of stress acting on an area normal to the z-axis are τ_{zx}, τ_{zy}, σ_z (Fig. 1.1). These nine components define the state of stress at a point given by the stress tensor :

$$[\tau_{ij}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

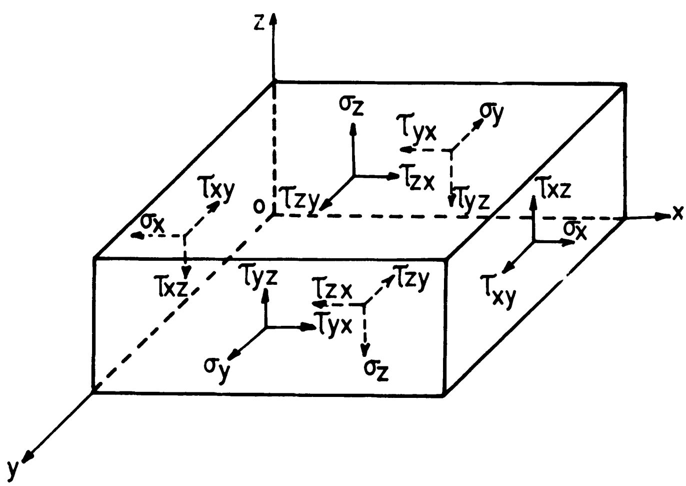

Fig. 1.1 Stress components on a rectangular parallelopiped.

The stress tensor is symmetric because $\tau_{yx} = \tau_{xy}$; $\tau_{zy} = \tau_{yz}$; $\tau_{zx} = \tau_{xz}$ (generally, $\tau_{ij} = \tau_{ji}$, $i \neq j$, where i and j denote mutually perpendicular directions).

Similarly the stress tensor in cylindrical co-ordinate system r, θ, z is given by :

$$[\tau_{ij}] = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_z \end{bmatrix}$$

where $\tau_{\theta r} = \tau_{r\theta}$; $\tau_{zr} = \tau_{rz}$; $\tau_{z\theta} = \tau_{\theta z}$, and in spherical co-ordinates r, θ, ϕ by,

$$[\tau_{ij}] = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \sigma_\phi \end{bmatrix}$$

where $\tau_{\theta r} = \tau_{r\theta}$; $\tau_{\phi r} = \tau_{r\phi}$; $\tau_{\phi\theta} = \tau_{\theta\phi}$.

If we know the stresses in three mutually perpendicular areas we can always determine the stress which acts on any area passing through the same point. We have the formulae :

$$\left.\begin{aligned} X_n &= \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) + \tau_{xz} \cos(n, z) \\ Y_n &= \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) + \tau_{yz} \cos(n, z) \\ Z_n &= \tau_{xz} \cos(n, x) + \tau_{yz}, \cos(n, y) + \sigma_z \cos(n, z) \end{aligned}\right\} \quad ...(1.1)$$

where X_n, Y_n, Z_n are components of stress which act on an area with the arbitrary normal direction $\bar{n}$.

If it is desired to express the stress components referred to co-ordinate system $0, x', y', z'$ in terms of the components of system $0, x, y, z$, then the direction cosines are determined by :

	x	y	z
x'	$a_{x'x}$	$a_{x'y}$	$a_{x'z}$
y'	$a_{y'x}$	$a_{y'y}$	$a_{y'z}$
z'	$a_{z'x}$	$a_{z'y}$	$a_{z'z}$

where $a_{x'x} = \cos(x', x)$; $a_{y'z} = \cos(y', z)$ and so on.

The stress components in the new co-ordinate system may be obtained from :

$$\tau_{ij}' = a_{il}\, a_{jm}\, \tau_{lm} \quad ...(1.2)$$

Thereby, we get

$$\begin{aligned} \sigma_x' &= \sigma_x a^2_{x'x} + \sigma_y a^2_{x'y} + \sigma_z\, a^2_{x'z} \\ &\quad + 2(\tau_{xy}\, a_{x'x} \cdot a_{x'y} + \tau_{yz}\, a_{x'y}\, a_{x'z} + \tau_{xz}\, a_{x'x}\, a_{x'z}) \quad ...(1.3) \\ \tau_{x'y'} &= \sigma_x\, a_{x'x} \cdot a_{y'x} + \sigma_y\, a_{x'y}\, a_{y'y} + \sigma_z\, a_{x'z}\, a_{y'z} \\ &\quad + \tau_{xy}(a_{x'x}\, a_{y'y} + a_{x'y}\, a_{y'x}) \\ &\quad + \tau_{yz}(a_{x'y}\, a_{y'z} + a_{x'z}\, a_{y'y}) \\ &\quad + \tau_{xz}(a_{x'x}\, a_{y'z} + a_{x'z}\, a_{y'x}) \end{aligned}$$

The rest of the stress components can be found by cyclic permutation of symbols x, y and z. The formulas for the transformation from the system x', y', z' to x, y, z, can be written as :

$$\left.\begin{aligned} \sigma_x &= \sigma_x' a^2_{x'x} + \sigma_y' a^2_{y'x} + \sigma_z' a^2_{z'x} \\ &\quad + 2\,(\tau_{x'y'} a_{x'x} a_{y'x} + \tau_{y'z'} a_{y'x} a_{z'x} \\ &\quad + \tau_{x'z'} a_{x'x} a_{x'x}) \\ \tau_{xy} &= \sigma_x' a_{x'x} a_{x'y} + \sigma_{y'} a_{y'x} a_{y'y} \\ &\quad + \sigma_{z'} a_{z'x} a_{z'y} \\ &\quad + \tau_{x'y'} (a_{x'x} a_{y'y} + a_{x'y} a_{y'x}) \\ &\quad + \tau_{y'z'} (a_{y'x} a_{z'y} + a_{y'y} a_{z'x}) \\ &\quad + \tau_{x'z'} (a_{x'x} a_{z'y} + a_{x'y} a_{z'x}) \end{aligned}\right\} \quad \ldots(1.4)$$

The expressions for the remaining stress components can be obtained by a cyclic permutation of the symbols x', y', z'.

Analogous formulas hold for other orthogonal co-ordinate systems. For example the relations between the stresses in the Cartesian and cylindrical co-ordinate systems with the same z-axis are :

		x	y	z
r	x'	$\cos\theta$	$\sin\theta$	0
θ	y'	$-\sin\theta$	$\cos\theta$	0
z	z'	0	0	1

$$\left.\begin{aligned} \sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \\ \sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta \\ \sigma_z &= \sigma_z \\ \tau_{r\theta} &= -\sigma_x \cos\theta\sin\theta + \sigma_y \sin\theta\cos\theta \\ &\qquad + \tau_{xy} (\cos^2\theta - \sin^2\theta) \\ \tau_{\theta z} &= \tau_{yz} \,.\, \cos\theta - \tau_{xz} \sin\theta \\ \tau_{rz} &= \tau_{yz} \sin\theta + \tau_{xz} \cos\theta \end{aligned}\right\} \quad \ldots(1.5)$$

and

$$\left.\begin{aligned} \sigma_x &= \sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta - 2\tau_{r\theta} \sin\theta\cos\theta \\ \sigma_y &= \sigma_r \sin^2\theta + \sigma_\theta \cos^2\theta + 2\tau_{r\theta} \sin\theta\cos\theta \\ \sigma_z &= \sigma_z \\ \tau_{xy} &= (\sigma_r - \sigma_\theta) \sin\theta\cos\theta + \tau_{r\theta} (\cos^2\theta - \sin^2\theta) \\ \tau_{yz} &= \tau_{\theta z} \cos\theta + \tau_{rz} \sin\theta \\ \tau_{xz} &= -\tau_{\theta z} \sin\theta + \tau_{rz} \cos\theta \end{aligned}\right\} \quad \ldots(1.6)$$

The importance of these stress-transformation equations is that they can be used to determine the planes on which the shear stress is zero. It can be shown that, for the most general state of stress, three such planes exist. These planes, referred to as principal planes, are mutually perpendicular. The stresses on these planes, σ_1, σ_2 and σ_3 are called principal stresses. The principal stresses are the roots of the cubic equation :

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \qquad ...(1.7)$$

where

$$\left.\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2 \\ I_3 &= \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{xz} \end{aligned}\right\} \qquad ...(1.8)$$

are three stress invariants.

Eq. (1.7) may be solved by hit and trial, Newton-Raphson method or any other numerical method and the principal stresses and principal directions may be determined.

1.3 The State of Strain at a Point

The displacements of the different points of a material body are resolved into two parts, rigid-body displacements and deformation. The first corresponds to a translation or rotation of the body as a whole, and the second corresponds to displacements of points relative to each other. It is only this later part that enters the definition of strain. In the linear theory or small deformation theory of elasticity, it is assumed that the deformations are infinitesimal and it applies to a small neighbourhood about the point in question.

The state of deformation in the neighbourhood of a given point of a continuous body is determined by six components of strain ; three components of normal strain, which we shall denote by ε, with one index, which indicates the primary direction of the segment ; three components of shear strain which we shall denote by γ with two indices. Physically, the normal strain is defined as the change in length of a small line segment divided by the length of the segment. The shearing strain is visualized as the change in angle between two initially perpendicular line segments. The components of strain determine the strain tensor,

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_x & \dfrac{\gamma_{xy}}{2} & \dfrac{\gamma_{xz}}{2} \\ \dfrac{\gamma_{xy}}{2} & \varepsilon_y & \dfrac{\gamma_{yz}}{2} \\ \dfrac{\gamma_{xz}}{2} & \dfrac{\gamma_{yz}}{2} & \varepsilon_z \end{bmatrix}$$

which, as in the case of stress tensor, is symmetric, *i.e.* $\varepsilon_{ij} = \varepsilon_{ji} : i \neq j$, and $\gamma_{ij} = 2\, \varepsilon_{ij}\, ; i \neq j$.

The components of rotation tensor are :

$$[\omega_{ij}] = \begin{bmatrix} 0 & \omega_{xy} & \omega_{xz} \\ -\omega_{xy} & 0 & \omega_{yz} \\ -\omega_{xz} & -\omega_{yz} & 0 \end{bmatrix}$$

which is skew-symmetric.

If u_x, u_y, u_z are the components of the displacement field u, then

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$$\omega_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)$$

$$\omega_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right)$$

so that the components of rigid body rotations are :

$$\left.\begin{aligned} \delta\phi_x &= \omega_{zy} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \\ \delta\phi_y &= \omega_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ \delta\phi_z &= \omega_{yx} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \end{aligned}\right\} \qquad \text{...(1.9)}$$

and the components of strain are :

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u_x}{\partial x} \\ \varepsilon_y &= \frac{\partial u_y}{\partial y} \\ \varepsilon_z &= \frac{\partial u_z}{\partial z} \\ \frac{\gamma_{xy}}{2} &= \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \frac{\gamma_{yz}}{2} &= \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{\gamma_{xz}}{2} &= \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \end{aligned}\right\} \qquad \text{...(1.10)}$$

For the transformation of strain components from one orthogonal co-ordinate system x, y, z to another orthogonal co-ordinate system x', y', z', the same laws of transformation as for the stress components hold good, *i.e.*,

$$\varepsilon'_{ij} = a_{il}\, a_{jm}\, \varepsilon_{lm} \qquad \text{...(1.11)}$$

Therefore, it is necessary to substitute ε in place of σ and $\frac{\gamma}{2}$ in place of τ in Eq. (1.3). Thus, for example

$$\left.\begin{aligned}
\varepsilon_x' &= \varepsilon_x\, a^2_{x'x} + \varepsilon_y\, a^2_{x'y} + \varepsilon_z\, a^2_{x'z} \\
&\quad + \gamma_{xy}\, a_{x'x}\, a_{x'y} + \gamma_{yz}\, a_{x'y}\, a_{x'z} \\
&\quad + \gamma_{xz}\, a_{x'x}\, a_{x'z} \\
\frac{\gamma_{x'y'}}{2} &= \varepsilon_x\, a_{x'x} \cdot a_{y'x} + \varepsilon_y \cdot a_{x'y}\, a_{y'y} + \varepsilon_z\, a_{x'z}\, a_{y'z} \\
&\quad + \frac{\gamma_{xy}}{2}\,(a_{x'x}\, a_{y'y} + a_{x'y}\, a_{y'x}) \\
&\quad + \frac{\gamma_{yz}}{2}\,(a_{x'y}\, a_{y'z} + a_{x'z}\, a_{y'y}) \\
&\quad + \frac{\gamma_{xz}}{2}\,(a_{x'x}\, a_{y'z} + a_{x'z}\, a_{y'x})
\end{aligned}\right\} \qquad \text{...(1.12)}$$

and conversely,

$$\left.\begin{aligned}
\varepsilon_x &= \varepsilon_x'\, a^2_{x'x} + \varepsilon_y'\, a^2_{y'x} + \varepsilon_z a^2_{z'x} \\
&\quad + \gamma_{x'y'} + a_{x'x}\, a_{y'x} + \gamma_{y'z'}\, a_{y'z'}\, a_{y'x}\, a_{z'y} \\
&\quad + \gamma_{x'z'}\, a_{x'x}\, a_{z'x} \\
\frac{\gamma_{xy}}{2} &= \varepsilon_x'\, a_{x'x}\, a_{x'y} + \varepsilon_y'\, a_{y'x}\, a_{y'y} \\
&\quad + \varepsilon_z'\, a_{z'x} + a_{z'y} \\
&+ \frac{\gamma_{x'y'}}{2}\,(a_{x'x}\, a_{y'y} + a_{x'y}\, a_{y'x}) \\
&\quad + \frac{\gamma_{y'z'}}{2}\,(a_{y'x}\, a_{z'y} + a_{y'y}\, a_{z'x}) \\
&\quad + \frac{\gamma_{x'z'}}{2}\,(a_{x'x}\, a_{z'y} + a_{x'y}\, a_{z'x})
\end{aligned}\right\} \qquad \text{...(1.13)}$$

other components may be written by cyclic permutation of the symbols x, y, z or x', y', z'.

As in the case of stresses, there exist three mutually perpendicular planes on which the shear strains are zero. The normal components in these directions ε_1, ε_2 and ε_3, which include the algebraically largest and smallest of all normal strains at the point, are called principal strains and the directions, principal strain directions. The principal strains are the roots of the cubic equation :

$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0 \qquad ...(1.14)$$

where

$$\left.\begin{aligned} J_1 &= \varepsilon_x + \varepsilon_y + \varepsilon_z \\ J_2 &= \varepsilon_x\,\varepsilon_y + \varepsilon_y\,\varepsilon_z + \varepsilon_x\,\varepsilon_z \\ &\quad - \frac{\gamma^2_{xy}}{4} - \frac{\gamma^2_{yz}}{4} - \frac{\gamma^2_{xz}}{4} \\ J_3 &= \varepsilon_x\varepsilon_y\varepsilon_z - \varepsilon_x\frac{\gamma^2_{xy}}{4} - \varepsilon_y\frac{\gamma^2_{xz}}{4} \\ &\quad - \varepsilon_z\frac{\gamma^2_{xy}}{4} + \frac{1}{4}\gamma_{xy}\,\gamma_{yz}\,\gamma_{xz} \end{aligned}\right\} \qquad ...(1.15)$$

are the three strain invariants.

Eq. (1.14) may be solved by hit and trial, Newton-Raphson method or any other numerical method to determine the principal strains and principal directions. The first invariant of strain J_1 is called dilatation and has the physical significance of representing the change of a unit volume of the material.

1.4 Basic Equations of Elasticity

1.4.1 Cartesian Co-ordinates

(i) Equations of equilibrium (3)

Consideration of the variation of the state of stress from point to point leads to the equilibrium equations given by

$$\left.\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + B_x &= 0 \\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + B_y &= 0 \\ \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + B_z &= 0 \end{aligned}\right\} \qquad ...(1.16)$$

where B_x, B_y and B_z are the components of the body force (in N/m^3), such as gravitational, centrifugal, or other inertia forces.

(ii) Strain-displacement relations (6)

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\partial u_x}{\partial x} \\
\varepsilon_y &= \frac{\partial u_y}{\partial y} \\
\varepsilon_z &= \frac{\partial u_z}{\partial z} \\
\gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\
\gamma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\
\gamma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}
\end{aligned}\right\} \quad \text{...(1.17)}$$

(iii) Stress-strain relations (6)

For an isotropic material, we have only two independent elastic constants λ and G, known as Lame's constants and the generalised Hooke's law gives the following stress-strain relations.

$$\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E}\,[\sigma_x - \nu(\sigma_y + \sigma_z)] \\
\varepsilon_y &= \frac{1}{E}\,[\sigma_y - \nu(\sigma_x + \sigma_z)] \\
\varepsilon_z &= \frac{1}{E}\,[\sigma_z - \nu(\sigma_x + \sigma_y)] \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{yz} &= \frac{\tau_{yz}}{G} \\
\gamma_{xz} &= \frac{\tau_{xz}}{G}
\end{aligned}\right\} \quad \text{...(1.18)}$$

where ν is Poisson's ratio and E is the modulus of elasticity

$$G = \frac{E}{2(1+\nu)} = \text{Modulus of rigidity}$$

(iv) Strain compatibility equations (6)

Given a displacement field, a unique strain field can be computed. However, given a strain field, the corresponding displacements are not necessarily unique and continuous, unless certain conditions are satisfied. These conditions, known as the compatibility equations, are expressed as,

$$\begin{aligned}
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} &= \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \\
\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} &= \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} \\
\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} &= \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} \\
2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right) \\
2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right)
\end{aligned} \qquad \text{...(1.19)}$$

(v) Stress-displacement relations (6)

The stress-strain relations using Eqs. (1.17) may be written as,

$$\begin{aligned}
\sigma_x &= 2G\frac{\partial u_x}{\partial x} + \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\
\sigma_y &= 2G\frac{\partial u_y}{\partial y} + \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\
\sigma_z &= 2G\frac{\partial u_z}{\partial z} + \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\
\tau_{xy} &= G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\
\tau_{yz} &= G\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
\tau_{zx} &= G\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)
\end{aligned} \qquad \text{...(1.20)}$$

where $\lambda = \dfrac{\nu E}{(1-\nu)(1-2\nu)}$ = Lame's constant

(vi) Equilibrium equations in terms of displacements (3)

Using the stress-displacement relations Equations (1.20), the equilibrium Equations (1.16) can be expressed in terms of displacements. These equations are known as Navier equations and are given by,

$$\nabla^2 u_x + \left(\frac{3}{1-2\nu}\right)\frac{\partial \varepsilon_m}{\partial x} + \frac{B_x}{G} = 0$$

$$\nabla^2 u_y + \left(\frac{3}{1-2\nu}\right)\frac{\partial \varepsilon_m}{\partial y} + \frac{B_y}{G} = 0 \qquad \text{...(1.21)}$$

$$\nabla^2 u_z + \left(\frac{3}{1-2\nu}\right)\frac{\partial \varepsilon_m}{\partial z} + \frac{B_z}{G} = 0$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\varepsilon_m = \frac{1}{3}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right).$$

(vii) Stress-compatibility equations (6)

Using Eqs. (1.18) and (1.16), the compatibility equations (1.19) may be expressed in terms of stress components. These equations are known as Beltrami-Michell equations and are given by :

$$\nabla^2 \sigma_x + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial x^2} = -\left(\frac{\nu}{1-\nu}\right)\nabla.\overline{B} - 2\frac{\partial B_x}{\partial x}$$

$$\nabla^2 \sigma_y + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial y^2} = -\left(\frac{\nu}{1-\nu}\right)\nabla.\overline{B} - 2\frac{\partial B_y}{\partial y}$$

$$\nabla^2 \sigma_z + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial z^2} = -\left(\frac{\nu}{1-\nu}\right)\nabla.\overline{B} - 2\frac{\partial B_z}{\partial z} \qquad \text{...(1.22)}$$

$$\nabla^2 \tau_{xy} + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial x \partial y} = -\left(\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x}\right)$$

$$\nabla^2 \tau_{yz} + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial y \partial z} = -\left(\frac{\partial B_y}{\partial z} + \frac{\partial B_z}{\partial y}\right)$$

$$\nabla^2 \tau_{zx} + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial z \partial x} = -\left(\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right)$$

where $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

$$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$$\overline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}.$$

(viii) Boundary conditions in terms of stresses (3)

When the stresses are prescribed over the entire boundary, then the boundary conditions can be expressed as :

$$
\begin{aligned}
X_n &= \sigma_x \,.\, a_{nx} + \tau_{xy} \,.\, a_{ny} + \tau_{xz}\, a_{nz} \\
Y_n &= \tau_{xy} \,.\, a_{nx} + \sigma_y \,.\, a_{ny} + \tau_{yz} \,.\, a_{nz} \\
Z_n &= \tau_{zx} \,.\, a_{nz} + \tau_{zy}\, a_{ny} + \sigma_z \,.\, a_{nz}
\end{aligned}
\qquad \text{...(1.23)}
$$

(ix) Boundary conditions in terms of displacements (3)

Using Eqs. (1.20), the Eqs. (1.23) can be expressed in terms of displacements as given below :

$$
\begin{aligned}
X_n &= 3\lambda\varepsilon_m \,.\, a_{nx} + G\left[2\frac{\partial u_x}{\partial x} a_{nx} + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) a_{ny} \right. \\
&\qquad \left. + \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) a_{nz}\right] \\
Y_n &= 3\lambda\varepsilon_m \,.\, a_{ny} + G\left[2\frac{\partial u_y}{\partial y} \,.\, a_{ny} + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) a_{nx} \right. \\
&\qquad \left. + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) a_{nz}\right] \\
Z_n &= 3\lambda\varepsilon_m \,.\, a_{nz} + G\left[2\frac{\partial u_z}{\partial z} a_{nz} + \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) a_{nx} \right. \\
&\qquad \left. + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) a_{ny}\right]
\end{aligned}
\qquad \text{...(1.24)}
$$

where $\varepsilon_m = \dfrac{1}{3}\left(\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} + \dfrac{\partial u_z}{\partial z}\right)$.

1.4.2 Cylindrical Co-ordinates

(i) Equations of equilibrium (3) (Fig. 1.2)

$$
\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0 \\
&\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + B_\theta = 0 \\
&\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + B_z = 0
\end{aligned}
\qquad \text{...(1.25)}
$$

(ii) Strain-displacement relations (6)

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

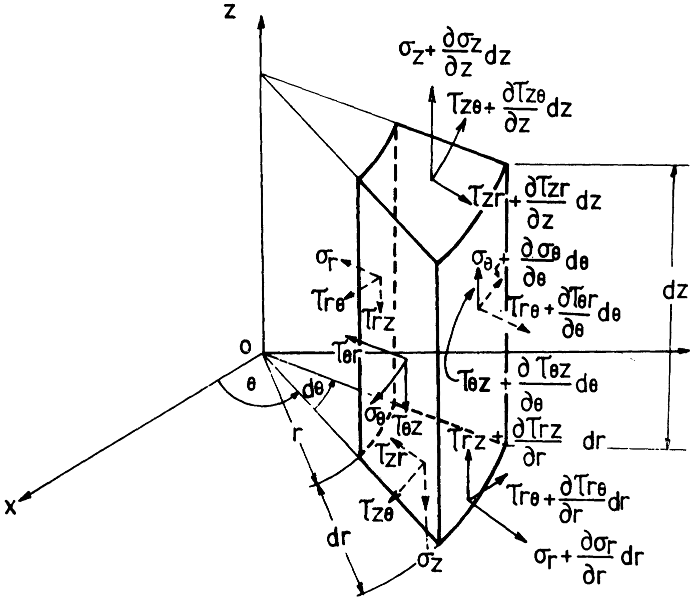

Fig. 1.2 Stresses on a cylindrical parallelopiped.

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \qquad ...(1.26)$$

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}$$

$$\gamma_{zr} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}$$

(iii) Stress-strain relations (6)

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_\theta + \sigma_r)\right]$$

$$\gamma_{r\theta} = \frac{\tau_{r\theta}}{G} \qquad ...(1.27)$$

$$\gamma_{\theta z} = \frac{\tau_{\theta z}}{G}$$

$$\gamma_{zr} = \frac{\tau_{zr}}{G}$$

(*iv*) *Strain-compatibility equations (6)*

$$\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{\theta z}}{\partial \theta} + \gamma_{zr}\right) = \frac{1}{r}\frac{\partial \varepsilon_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_z}{\partial \theta^2} + \frac{d^2 \varepsilon_\theta}{\partial z^2}$$

$$\frac{\partial^2 \gamma_{zr}}{2r\partial\theta\partial z} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \gamma_{z\theta}}{\partial z^2}\right) - \frac{\partial^2 \gamma_{r\theta}}{2\partial z^2} = \frac{\partial}{\partial r}\left(\frac{\partial \varepsilon_z}{r\partial\theta}\right)$$

$$\frac{\partial^2 \gamma_{zr}}{\partial r \partial z} = \frac{\partial^2 \varepsilon_z}{\partial r^2} + \frac{\partial^2 \varepsilon_r}{\partial z^2} \qquad \text{...(1.28)}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \gamma_{r\theta}}{\partial \theta}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_\theta}{\partial r}\right) + \left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\right)\varepsilon_r$$

$$\frac{1}{2}\cdot\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \gamma_{z\theta}}{\partial \theta}\right) + \frac{1}{r}\frac{\partial^2 \gamma_{\theta z}}{\partial\theta\partial z} - \frac{\partial^2 \gamma_{zr}}{2r^2\partial\theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varepsilon_\theta}{\partial z}\right) - \frac{1}{r}\frac{\partial \varepsilon_r}{\partial z}$$

$$-\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}r\gamma_{\theta z}\right) + \frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \gamma_{r\theta}}{\partial \theta}\right) + \frac{\partial}{\partial r}\left(\frac{1}{\partial r}\frac{\partial \gamma_{zr}}{\partial \theta}\right)$$

$$= \frac{\partial^2 \varepsilon_r}{r\partial\theta\partial z}$$

(*v*) *Stress-displacement relations (6)*

$$\sigma_r = 2G\frac{\partial u_r}{\partial r} + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right)$$

$$\sigma_\theta = 2G\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial_z}\right)$$

$$\sigma_z = 2G\frac{\partial u_z}{\partial z} + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right)$$

$$\tau_{r\theta} = G\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \qquad \text{...(1.29)}$$

$$\tau_{\theta z} = G\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

$$\tau_{zr} = G\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

(vi) Equilibrium equations in terms of displacements (3)

$$\nabla^2 u_r + \left(\frac{3}{1-2\nu}\right)\frac{\partial \varepsilon_m}{\partial r} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{B_r}{G} = 0$$

$$\nabla^2 u_\theta \left(\frac{3}{1-2\nu}\right)\frac{\partial \varepsilon_m}{r\partial \theta} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{B_\theta}{G} = 0 \quad \ldots(1.30)$$

$$\nabla^2 u_z + \left(\frac{3}{1-2\nu}\right)\frac{\partial \varepsilon_m}{\partial z} + \frac{B_z}{G} = 0$$

where $\varepsilon_m = \frac{1}{3}\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right)$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

(vii) Stress-compatibility equations (6)

In the absence of body forces, we have

$$\nabla^2 \sigma_r + \frac{2(\sigma_\theta - \sigma_r)}{r^2} - \frac{4}{r^2}\frac{\partial \tau_{r\theta}}{\partial \theta} + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial r^2} = 0$$

$$\nabla^2 \sigma_\theta - \frac{2(\sigma_\theta - \sigma_r)}{r^2} + \frac{4}{r^2}\frac{\partial \tau_{r\theta}}{\partial \theta} + \left(\frac{3}{1+\nu}\right)\left(\frac{1}{r}\frac{\partial \sigma_m}{\partial r} + \frac{\partial^2 \sigma_m}{r^2 \partial \theta^2}\right) = 0$$

$$\nabla^2 \sigma_z + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial z^2} = 0 \quad \ldots(1.31)$$

$$\nabla^2 \tau_{r\theta} - \frac{2}{r^2}\frac{\partial}{\partial \theta}(\sigma_\theta - \sigma_r) - \frac{4}{r^2}\tau_{r\theta} + \left(\frac{3}{1+\nu}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \sigma_m}{\partial \theta}\right) = 0$$

$$\nabla^2 \tau_{\theta z} - \frac{\tau_{\theta z}}{r^2} + \frac{2}{r^2}\frac{\partial \tau_{rz}}{\partial \theta} + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{r\partial \theta \partial z} = 0$$

$$\nabla^2 \tau_{zr} - \frac{\tau_{zr}}{r^2} - \frac{2}{r^2}\frac{\partial \tau_{\theta z}}{\partial \theta} + \left(\frac{3}{1+\nu}\right)\frac{\partial^2 \sigma_m}{\partial r \partial z} = 0$$

where $\sigma_m = \frac{1}{3}(\sigma_r + \sigma_\theta + \sigma_z)$

(viii) Boundary conditions in terms of stresses (3)

$$r_n = \sigma_r \,.\, a_{nr} + \tau_{r\theta}\, a_{n\theta} + \tau_{rz}\, a_{nz}$$

$$\theta_n = \tau_{r\theta} \,.\, a_{nr} + \sigma_\theta \,.\, a_{n\theta} + \tau_{\theta z}\, a_{nz} \quad \ldots(1.32)$$

$$z_n = \tau_{zr}\, a_{nr} + \tau_{\theta z}\, a_{n\theta} + \sigma_z\, a_{nz}$$

(ix) Boundary conditions in terms of displacements (3)

$$r_n = 3\lambda\varepsilon_m \,.\, a_{nr} + G\,[2\,\varepsilon_r\, a_{nr} + \gamma_{r\theta}\, a_{n\theta} + \gamma_{rz}\, a_{nz}]$$

$$\theta_n = 3\lambda\varepsilon_m \,.\, a_{n\theta} + G\,[2\varepsilon_\theta \,.\, a_{n\theta} + \gamma_{r\theta}\, a_{nr} + \gamma_{\theta z}\, a_{n\theta}] \quad \ldots(1.33)$$

$$z_n = 3\lambda\varepsilon_m \cdot a_{nz} + G\,[2\varepsilon_z \cdot a_{nz} + \gamma_{zr}\, a_{nr} + \gamma_{\theta z}\, a_{ny}]$$

For axis symmetric bodies, if z is the axis of revolution, then all stresses and strains are independent of θ.

1.5 Methods of Solution of Elasticity Problems

Given the geometry of the body, the manner in which it is loaded, *i.e.* specified boundary conditions, the distribution of body forces, and the elastic constants, it is required to determine stresses, strains and displacements as a function of location. Depending on the form of the boundary conditions, problems, can be classified in three groups :

1. *First Boundary*-Value Problem : where normal and shear stresses are prescribed on the whole boundary of the body.

2. *Second Boundary*-Value Problem : where only displacements are prescribed on the boundary.

3. *Mixed Boundary*-Value Problem : where stresses are prescribed on some parts of the boundary and displacements on other parts.

In the most general case, it is required to solve 15 equations : 3 equilibrium equations (1.16) or (1.25), 6 strain-displacement relations Eqs. (1.17) or (1.26), 6 stress-strain relations Eqs. (1.18) or (1.27)] for 15 unknowns [6 stress components, 6 strains components and 3 displacement components]. With the requirement that compatibility equations (1.19) or (1.28) and boundary conditions (1.23) or (1.32) be satisfied. The boundary conditions serve to solve for the integration constants that arise in the solution of the system of differential equations. Such a formulation of elasticity problems is known as *stress formulation.* Another approach suitable for the second boundary value problem is to solve the three equations of equilibrium in terms of displacements. Eqs. (1.21) or (1.30) for the three displacement components subject to the displacement boundary conditions, Eqs. (1.24) or (1.33). This formulation of elasticity problems is called *displacement formulation.* However, these direct methods of solving elasticity problems are too complicated to yield practical results. In order to simplify the solution methods, inverse and semi-inverse methods can be used. In the inverse method, a judicious guess of the solution is made and is checked against the basic equations and boundary conditions. If all conditions are satisfied the solution is the correct one ; otherwise, corrections are made and a revised solution is checked in the same manner. In the semi-inverse method, some of the unknowns are guessed and a simplified group of equations is solved for the remaining unknowns.

These educated guesses are based on experience, intuition, solutions of similar problems, or experiments.

1.6 Plane Stress

If a thin plate is loaded by forces applied at the boundary, parallel to the plane of the plate and distributed uniformly over the thickness (Fig. 1.3), the stress components $\sigma_z, \tau_{xz}, \tau_{yz}$ are zero on both faces of the plate, and if also assumed to be zero within the plate, then the state of stress is called plane stress.

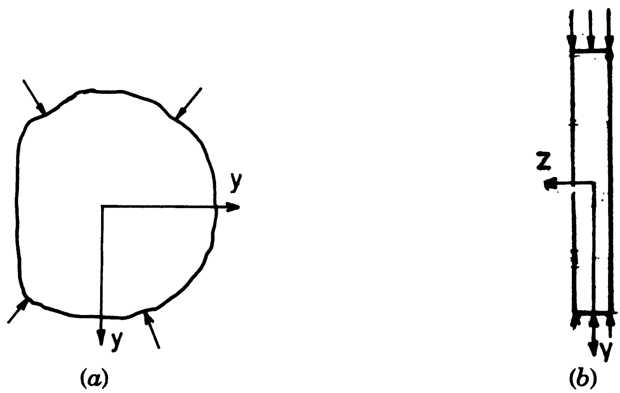

Fig. 1.3 State of plane stress.

For plane stress case, $\sigma_x, \sigma_y, \tau_{xy} = f(x, y)$ ...(1.34)

and governing equations become.

(*i*) *Equilibrium equations*

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} \frac{\partial \tau_{xy}}{\partial y} + B_x &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + B_y &= 0 \\ B_z &= 0 \end{aligned}\right\} \qquad \text{...(1.35)}$$

(*ii*) *Strain-displacement relations*

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u_x}{\partial x} \\ \varepsilon_y &= \frac{\partial u_y}{\partial y} \\ \varepsilon_z &= \frac{\partial u_z}{\partial z} \\ \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \end{aligned}\right\} \qquad \text{...(1.36)}$$

(iii) Stress-strain relations

$$\left.\begin{aligned}\varepsilon_x &= \frac{1}{E}[\sigma_x - \nu\sigma_y]\\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu\sigma_x]\\ \varepsilon_z &= -\frac{\nu}{E}[\sigma_x + \sigma_y\\ \gamma_{xy} &= \frac{\tau_{xy}}{G}\\ \gamma_{yz} &= \gamma_{zx} = 0\end{aligned}\right\} \quad \ldots(1.37)$$

(iv) Strain-compatibility equations

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} \quad \ldots(1.38)$$

(v) Stress-displacement relations

$$\left.\begin{aligned}\sigma_x &= 2G\frac{\partial u_x}{\partial x} + \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\\ \sigma_y &= 2G\frac{\partial u_y}{\partial y} + \lambda\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x}\right)\\ \tau_{xy} &= G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\end{aligned}\right\} \quad \ldots(1.39)$$

(vi) Navier equations

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u_x + \left(\frac{1}{1-2\nu}\right)\frac{\partial}{\partial x}(\varepsilon_x + \varepsilon_y) + \frac{B_x}{G} = 0\\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u_y + \left(\frac{1}{1-2\nu}\right)\frac{\partial}{\partial y}(\varepsilon_x + \varepsilon_y) + \frac{B_y}{G} = 0\end{aligned}\right\} \quad \ldots(1.40)$$

(vii) Stress-compatibility Equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

or

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) \quad \ldots(1.41)$$

(viii) Stress-boundary conditions

$$\left.\begin{aligned}X_n &= \sigma_x \,.\, a_{nx} + \tau_{xy}\, a_{ny}\\ Y_n &= \tau_{xy} \,.\, a_{nx} + \sigma_y \,.\, a_{ny}\end{aligned}\right\} \quad \ldots(1.42)$$

(ix) Boundary conditions in terms of displacements

$$\left.\begin{aligned} X_n &= \lambda\,(\varepsilon_x + \varepsilon_y)\,a_{nx} \\ &\quad + G\left[2\frac{\partial u_x}{\partial x}\,.\,a_{nx} + \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)a_{ny}\right] \\ Y_n &= \lambda\,(\varepsilon_x + \varepsilon_y)\,a_{ny} \\ &\quad + G\left[2\frac{\partial u_y}{\partial y}\,.\,a_{ny} + \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)a_{nx}\right] \end{aligned}\right\} \quad \text{...(1.43)}$$

(x) Biharmonic equation

Let there exist a body force potential V such that,

$$B_x = \frac{\partial V}{\partial x}$$

$$B_y = \frac{\partial V}{\partial y}$$

and define

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\phi}{\partial y^2} - V \\ \sigma_y &= \frac{\partial^2\phi}{\partial x^2} - V \\ \tau_{xy} &= -\frac{\partial^2\phi}{\partial x\,\partial y} \end{aligned}\right\} \quad \text{...(1.44)}$$

where ϕ = Airy's stress function.

Substituting Eq. (1.44) in Eq. (1.35), we get

$$\frac{\partial}{\partial x}\left[\frac{\partial^2\phi}{\partial y^2} - V\right] + \frac{\partial}{\partial y}\left[-\frac{\partial^2\phi}{\partial x\partial y}\right] + B_x = 0$$

$$\frac{\partial^3\phi}{\partial x\partial y^2} - \frac{\partial V}{\partial x} - \frac{\partial^3\phi}{\partial x\partial y^2} + B_x = 0$$

and

$$\frac{\partial}{\partial x}\left[-\frac{\partial^2\phi}{\partial x\partial y}\right] + \frac{\partial}{\partial y}\left[\frac{\partial^2\phi}{\partial x^2} - V\right] + B_y = 0$$

$$-\frac{\partial^3\phi}{\partial x^2\partial y} + \frac{\partial^3\phi}{\partial y\partial x^2} - \frac{\partial V}{\partial y} + B_y = 0.$$

Hence, we see that the stress components given by Eq. (1.44) satisfy the equilibrium equations.

Now substituting Eq. (1.44) in Eq. (1.41), we get

$$\nabla^2\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - 2V\right] = -(1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$\nabla^2.\nabla^2\phi - 2\nabla^2 V = -(1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$\nabla^4\phi = 2\,\nabla^2 V - (1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$= 2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) - (1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$= 2\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) - (1+\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$\nabla^4\phi = (1-\nu)\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) \qquad \text{...(1.45)}$$

Eq. (1.45) is known as the *Biharmonic equation* for plane stress problems.

where $$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

If body forces are zero, then Eq. (1.45) becomes

$$\nabla^4\phi = 0 \qquad \text{...(1.46)}$$

1.7 Plane Strain

When the length of the member in the z-direction is either very large so that no displacement is possible or the movements along the z-axis are otherwise prevented so that $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$, then the state of strain is said to be plane strain. A retaining wall with internal pressure, a culvert or tunnel and a cylindrical tube with internal pressure are some of the important problems of this kind.

The governing equations for plane strain case are :

(*i*) *Equilibrium equations*

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + B_y &= 0 \\ \frac{\partial \sigma_z}{\partial z} + B_z &= 0 \end{aligned}\right\} \qquad \text{...(1.47)}$$

(ii) Strain-displacement relations

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\partial u_x}{\partial x} \\
\varepsilon_y &= \frac{\partial u_y}{\partial y} \\
\gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\
\varepsilon_z &= \gamma_{yz} = \gamma_{zx} = 0
\end{aligned}\right\} \qquad \text{...(1.48)}$$

(iii) Stress-strain relations

$$\left.\begin{aligned}
& \varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] = 0 \\
\therefore \quad & \sigma_z = \nu(\sigma_x + \sigma_y) \\
\text{Hence} \quad & \varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\
& \varepsilon_x = \left(\frac{1+\nu}{E}\right)[\sigma_x(1-\nu) - \upsilon\sigma_y] \\
& \varepsilon_y = \left(\frac{1+\nu}{E}\right)[\sigma_y(1-\nu) - \upsilon\sigma_x] \\
& \gamma_{xy} = \frac{\tau_{xy}}{G}
\end{aligned}\right\} \qquad \text{...(1.49)}$$

(iv) Strain compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad \text{...(1.50)}$$

(v) Stress displacement relations

$$\left.\begin{aligned}
\sigma_x &= 2G\frac{\partial u_x}{\partial x} + \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) \\
\sigma_y &= 2G\frac{\partial u_y}{\partial y} + \lambda\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x}\right) \\
\sigma_z &= 2\upsilon(\lambda + G)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) \\
\tau_{xy} &= G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)
\end{aligned}\right\} \qquad \text{...(1.51)}$$

(vi) Navier equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u_x + \left(\frac{1}{1-2\upsilon}\right)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \frac{B_x}{G} = 0 \qquad \text{...(1.52 a)}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u_y+\left(\frac{1}{1-2\upsilon}\right)\frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial y}+\frac{\partial u_x}{\partial x}\right)+\frac{B_y}{G}=0 \quad \text{...(1.52 b)}$$

(*vii*) *Stress compatibility equation*

$$\nabla^2(\sigma_x+\sigma_y)=-\left(\frac{1}{1-\nu}\right)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right) \quad \text{...(1.53)}$$

(*viii*) *Biharmonic equation*

Let $\sigma_x=\dfrac{\partial^2\phi}{\partial y^2}-V,\ \ \sigma_y=\dfrac{\partial^2\phi}{\partial x^2}-V$ and $\tau_{xy}=\dfrac{-\partial^2\phi}{\partial x\partial y}$.

Substituting in Eq. (1.53), we get

$$\nabla^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-2V\right)=-\left(\frac{1}{1-\nu}\right)\left(\frac{\partial B_z}{\partial x}+\frac{\partial B_y}{\partial y}\right)$$

$$\nabla^4\phi=2\nabla^2V-\left(\frac{1}{1-\nu}\right)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)$$

$$=2\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)-\left(\frac{1}{1-\nu}\right)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)$$

$$\nabla^4\phi=\left(\frac{1-2\nu}{1-\nu}\right)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right) \quad \text{...(1.54)}$$

Eq. (1.54) is the *Biharmonic equation* for plane strain case. When body forces are zero, then

$$\nabla^4\phi=0 \quad \text{...(1.55)}$$

1.8 Spherical Coordinates

(*i*) *Equilibrium equations*

With reference to Fig. 1.4, the equilibrium equations can be written as :

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi}+\frac{1}{r}[2\sigma_r-\sigma_\theta-\sigma_\phi+\tau_{r\theta}\cot\theta]+B_r=0$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi}+\frac{1}{r}[(\sigma_\theta-\sigma_\phi)\cot\theta+3\tau_{r\theta}]+B_\theta=0$$

$$\frac{\partial\tau_{r\phi}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_\phi}{\partial_\phi}+\frac{1}{r}[3\tau_{r\phi}+2\tau_{\theta\phi}\cot\theta]+B_\phi=0 \quad \text{...(1.56)}$$

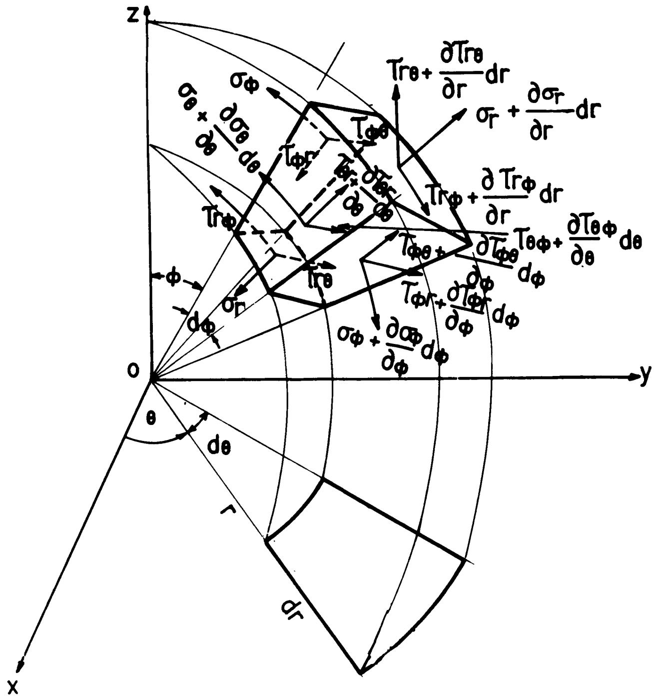

Fig. 1.4 Stress components on a spherical parallelopiped.

(ii) Strain-displacement relations

$$\left.\begin{aligned}
\varepsilon_r &= \frac{\partial u_r}{\partial}\ \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\
\varepsilon_\phi &= \frac{u_r}{r} + \frac{\partial \theta}{r}\cot\theta + \frac{1}{r\sin\theta}\cdot\frac{\partial u_\phi}{\partial \phi} \\
\gamma_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \\
\gamma_{r\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} = \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{\partial r} \\
\gamma_{\theta\phi} &= \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{\cot\theta}{r}u_\phi + \frac{1}{r\sin\theta}\cdot\frac{\partial u_\theta}{\partial \phi}
\end{aligned}\right\} \quad \text{...(1.57)}$$

(iii) Equilibrium equations in term of displacement components

$$(\lambda + 2G)\frac{1}{r}\frac{\partial e}{\partial \theta} - \frac{2G}{r\sin\theta}\left[\frac{\partial \omega_r}{\partial \phi} - \sin\theta\frac{\partial (r\omega_\phi)}{\partial r}\right] + B_\theta = 0$$

$$(\lambda + 2G)\frac{1}{r\sin\theta}\frac{\partial e}{\partial \phi} - \frac{2G}{r}\left[\frac{\partial (r\omega_\theta)}{\partial r} - \frac{\partial \omega_r}{\partial \theta}\right] + B_\phi = 0$$

$$(\lambda + 2G)\frac{\partial e}{\partial r} - \frac{2G}{r \sin\theta}\left[\frac{\partial(\omega_\phi \sin\theta)}{\partial\theta} - \frac{\partial\omega_\theta}{\partial\phi}\right] + B_r = 0 \qquad \text{...(1.58)}$$

where $$e = \frac{1}{r^2 \sin\theta}\left[\sin\theta\frac{\partial(r^2 u_r)}{\partial r} + r\frac{\partial(u_\theta \sin\theta)}{\partial\theta} + r\frac{\partial u_\phi}{\partial\phi}\right]$$

$$\omega_r = \frac{1}{2r\sin\theta}\left[\frac{\partial(u_\varphi \sin\theta)}{\partial\theta} - \frac{\partial u_\theta}{\partial\phi}\right]$$

$$\omega_\theta = \frac{1}{2r\sin\theta}\left[\frac{\partial u_r}{\partial\phi} - \sin\theta\frac{\partial(ru_\phi)}{\partial r}\right]$$

$$\omega_\phi = \frac{1}{2r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right].$$

Example 1.1 *The state of stress at a point with respect to the xyz system is*

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix} kN/m^2$$

Determine the stress tensor relative to the x'y'z' coordinate system obtained by a rotation through 30° about the z-axis.

Solution. The direction cosines are,

$$a_{x'x} = \frac{\sqrt{3}}{2},\ a_{x'y} = \frac{1}{2},\ a_{x'z} = 0$$

$$a_{y'x} = -\frac{1}{2},\ a_{y'y} = \frac{\sqrt{3}}{2},\ a_{y'z} = 0$$

$$a_{z'x} = 0,\ a_{z'y} = 0,\ a_{z'z} = 1.$$

Using Eqs. (1.2), we get

$$\sigma_{x'} = \sigma_x a^2_{x'x} + \sigma_y a^2_{x'y} + \sigma_z a^2_{x'z} + 2(\tau_{xy} a_{x'x} a_{x'y} + \tau_{yz} a_{x'y} a_{x'z} + \tau_{xz} a_{x'x} a_{x'z})$$

$$= 1000\left[3 \times \frac{3}{4} + 0 + 0 + 2\left(2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} + 0 + 0\right)\right]$$

$$= 1000\left[\frac{9}{4} + \sqrt{3}\right] = 3982 \text{ N/m}^2$$

Similarly

$$\sigma_{y'} = -982 \text{ N/m}^2,\ \sigma_{z'} = 1000 \text{ N/m}^2$$

$$\tau_{x'y'} = \sigma_x \cdot a_{x'x} a_{y'x} + \sigma_y \cdot a_{x'y} \cdot a_{y'y} + \sigma_z a_{x'z} a_{y'z} + \tau_{xy}(a_{x'x} a_{y'y} + a_{x'y} a_{y'x})$$

$$+ \tau_{yz} (a_{x'y}\, a_{y'z} + a_{x'z}\, a_{y'y})$$
$$+ \tau_{xz} (a_{x'x}\, a_{y'z} + a_{x'z}\, a_{y'x})$$

$$\tau_{x'y'} = 1000 \left[3 \times \frac{\sqrt{3}}{2} \left(-\frac{1}{2} \right) + 0 + 0 + 2 \left\{ \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} \right. \right.$$
$$\left. \left. + \frac{1}{2} \left(-\frac{1}{2} \right) + 0 + 0 \right] \right.$$
$$= 1000 \left[-\frac{3\sqrt{3}}{4} + 1 \right] = -299 \text{ N/m}^2.$$

Similarly,

$$a_{y'z'} = 134 \text{ N/m}^2 \; \tau_{x'z'} = -2232 \text{ N/m}^2.$$

Example 1.2 *The state of stress at a particular point relative to the xyz coordinate system is given by the stress matrix*

$$\begin{bmatrix} 15 & 10 & -10 \\ 10 & 10 & 0 \\ -10 & 0 & 40 \end{bmatrix} MPa$$

Determine the normal stress and the magnitude and direction of the shear stress on a surface intersecting the point and parallel to the plane given by the equation

$$2x - y + 3z = 9$$

Solution. The direction cosines of the normal to this plane are

$$a_{nx} = \frac{2}{\sqrt{14}},\; a_{ny} = \frac{-1}{\sqrt{14}},\; a_{nz} = \frac{3}{\sqrt{14}}$$

The normal stress is,

$$\sigma_n = \sigma_x . a^2_{nx} + \sigma_y a^2_{ny} + \sigma_z . a^2_{nz} + 2(\tau_{xy}\, a_{nx}\, a_{ny}$$
$$+ \tau_{yz} \,.\, a_{ny}\, a_{nz} + \tau_{xz} \,.\, a_{nx} \,.\, a_{nz})$$

$$\sigma_n = 15 \times \frac{4}{14} + 10 \times \frac{1}{14} + 40 \times \frac{9}{14}$$
$$+ 2 \left(10 \times \frac{-3}{14} + 0 - 10 \times \frac{6}{14} \right)$$
$$= \frac{1}{14} [90 + 10 + 360 - 60 - 60]$$
$$= \frac{310}{14} = 22.1428 \text{ MPa}$$

The resultant stress σ_R is

$$\sigma_R\ [(\sigma_x . a_{nx} + \tau_{xy} . a_{ny} + \tau_{xz} . a_{nz})^2 + (\tau_{xy} . a_{nx} + \sigma_y . a_{ny} + \tau_{yz} . a_{nz})^2 + (\tau_{xz} . a_{nx} + \tau_{yz} . a_{ny} + \sigma_z . a_{nz})^2]^{1/2}$$

$$= \left[\left(15 \times \frac{2}{\sqrt{14}} - \frac{10}{\sqrt{14}} - \frac{30}{\sqrt{14}} \right)^2 + \left(10 \times \frac{2}{\sqrt{14}} - 10 \times \frac{1}{\sqrt{14}} + 0 \right)^2 + \left(-10 \times \frac{2}{\sqrt{14}} + 0 + 40 \times \frac{3}{\sqrt{14}} \right)^3 \right]^{1/2}$$

$$= \left[\frac{100}{14} + \frac{100}{14} + \frac{10000}{14} \right)^{1/2} = \left(\frac{10200}{14} \right)^{1/2}$$

$$= (728.57)^{1/2} = 26.992 \text{ MPa}$$

Hence shear stress is

$$\tau = \sqrt{\sigma_R^2 - \sigma_n^2} = \sqrt{728.57 - 490.30}$$

$$= \sqrt{238.27} = 15.436 \text{ MPa} .$$

To determine the direction cosines for τ, let a_{sx}, a_{sy} and a_{sz} be the direction cosines of τ. For equilibrium of forces in the x-direction

$$\sigma_n . A . a_{nx} + \tau . A . a_{sx} = X_n . A$$

where A is the area over which these stresses act.

$$\therefore \quad a_{sx} = \frac{1}{\tau} [X_n - \sigma_n \, . \, a_{nx}]$$

$$= \frac{1}{\tau} [\sigma_x . a_{nx} + \tau_{xy} a_{ny} + \tau_{xz} . a_{nz} - \sigma_n . a_{nx}]$$

$$= \frac{1}{\tau} [(\sigma_x - \sigma_n) a_{nx} + \tau_{xy} a_{ny} + \tau_{xz} a_{nz}]$$

$$= \frac{1}{15.436} \left[(15 - 22.1428) \frac{2}{\sqrt{14}} - 10 \times \frac{1}{\sqrt{14}} - 10 \times \frac{3}{\sqrt{14}} \right]$$

$$= \frac{1}{15.436} \left[-\frac{54.2856}{\sqrt{14}} \right] = -0.9399$$

Similarly

$$a_{sy} = \frac{1}{\tau} [\tau_{xy} . a_{nx} + (\sigma_y - \sigma_n) a_{ny} + \tau_{yz} . a_{nz}]$$

$$= \frac{1}{15.436} \left[10 \times \frac{2}{\sqrt{14}} + (10 - 22.1428) \, . \left(-\frac{1}{\sqrt{14}} \right) + 0 \right]$$

$$= \frac{1}{15.436}\left[\frac{20}{\sqrt{14}} + \frac{12.1428}{\sqrt{14}}\right] = \frac{32.1428}{15.436\sqrt{14}} = 0.5565$$

$$a_{sz} = \frac{1}{15.436}\left[\tau_{xz}.a_{nx} + \tau_{yz}a_{ny} + (\sigma_z - \sigma_n)a_{nz}\right]$$

$$= \frac{1}{15.436}\left[-10 \times \frac{2}{\sqrt{14}} + 0 + (40 - 22.1428) \cdot \frac{3}{\sqrt{14}}\right]$$

$$= \frac{1}{15.436}\left[-\frac{20}{\sqrt{14}} + \frac{53.5716}{\sqrt{14}}\right]$$

$$= \frac{33.5716}{15.436\sqrt{14}} = 0.5812$$

Example 1.3 *For the stress tensor given below, determine the principal stresses and the direction cosines associated with the normals to the surface of each principal stress.*

$$\begin{bmatrix} 3000 & 1000 & 1000 \\ 1000 & 0 & 2000 \\ 1000 & 2000 & 0 \end{bmatrix} N/m^2.$$

Solution. The stress invariants are

$$I_1 = 3000$$
$$I_2 = -(1000)^2 - (2000)^2 - (1000)^2$$
$$= -[10^6 + 4 \times 10^6 + 10^6] = -6 \times 10^6$$
$$I_3 = 3000\,(-4 \times 10^6) - 1000\,(-2 \times 10^6) + 1000(2 \times 10^6)$$
$$= -12 \times 10^9 + 2 \times 10^9 + 2 \times 10^9 = -8 \times 10^9$$

The cubic equation is

$$\sigma^3 - 3000\,\sigma^2 - 6 \times 10^6\sigma + 8 \times 10^9 = 0$$

Its solution is

$$\sigma_1 = 4000 \text{ N/m}^2$$
$$\sigma_2 = 1000 \text{ N/m}^2$$
$$\sigma_3 = -2000 \text{ N/m}^2$$

To determine the direction cosines for σ_1, we have

$$[\tau_{ij} - \sigma_1 I] = \begin{bmatrix} -1000 & 1000 & 1000 \\ 1000 & -4000 & 2000 \\ 1000 & 2000 & -4000 \end{bmatrix}$$

where I is the unit matrix.

$$\therefore \quad A_1 = (16 \times 10^6 - 4 \times 10^6) = 12 \times 10^6$$
$$B_1 = -(-4 \times 10^6 - 2 \times 10^6) = 6 \times 10^6$$
$$C_1 = 2 \times 10^6 + 4 \times 10^6 = 6 \times 10^6$$

$$\therefore \qquad k_1 = \sqrt{A_1^2 + B_1^2 + C_1^2} = 10^6\sqrt{144+36+36} = 10^6\sqrt{216}$$

$$\therefore \qquad a_{nx_1} = \frac{A_1}{k_1} = \frac{12 \times 10^6}{10^6 \times \sqrt{216}} = \frac{2}{\sqrt{6}}$$

$$a_{ny_1} = \frac{6 \times 10^6}{\sqrt{216} \times 10^6} = \frac{1}{\sqrt{6}}$$

$$a_{nz_1} = \frac{6 \times 10^6}{\sqrt{216} \times 10^6} = \frac{1}{\sqrt{6}}$$

Similarly,

$$a_{nx_2} = \frac{1}{\sqrt{3}},\ a_{ny_2} = \frac{-1}{\sqrt{3}},\ a_{nz_2} = \frac{-1}{\sqrt{3}}$$

$$a_{nx_3} = 0,\ a_{ny_3} = \frac{1}{\sqrt{2}},\ a_{nz_3} = \frac{-1}{\sqrt{2}}$$

Example 1.4 *The stress components at a point are given by the following array :*

$$\begin{bmatrix} 10 & 5 & 6 \\ 5 & 8 & 10 \\ 6 & 10 & 6 \end{bmatrix} MPa$$

Calculate the principal stresses and principal planes.

Solution.

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$
$$= 10 + 8 + 6 = 24$$
$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$
$$= 10 \times 8 + 8 \times 6 + 6 \times 10 - 5^2 - 10^2 - 6^2$$
$$= 80 + 48 + 60 - 25 - 100 - 36 = 27$$
$$I_3 = \sigma_x\sigma_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2\ 2\tau_{xy}\tau_{yz}\tau_{xz}$$
$$= 10 \times 8 \times 6 - 10 \times 100 - 8 \times 36 - 6 \times 25$$
$$+ 2 \times 5 \times 10 \times 6$$
$$= 480 - 1000 - 288 - 150 + 600$$
$$= -358$$

$$\sigma^3 - 24\sigma^2 + 27\sigma + 358 = 0 \qquad \text{...(1)}$$

Now $\quad \cos 3\theta = 4\cos^3\theta - 4\cos\theta$

or $$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0 \qquad \text{...(2)}$$

Put $$\sigma = r\cos\theta + \frac{I_1}{3}$$
$$= r\cos\theta + 8$$

Eq. (1) becomes,

$$r^3\cos^3\theta + 512 + 24r^2\cos^2\theta + 192r\cos\theta - 24r^2\cos^2\theta - 1536 - 384r\cos\theta + 27r\cos\theta + 216 + 358 = 0$$

$$r^3\cos^3\theta - 165r\cos\theta - 450 = 0$$

or

$$\cos^3\theta - \frac{165}{r^2}\cos\theta - \frac{450}{r^3} = 0 \quad ...(3)$$

Comparing Eqs. (2) and (3), we have

$$\frac{165}{r^2} = \frac{3}{4}$$

or

$$r = 14.8324$$

and

$$\frac{450}{r^3} = \frac{\cos 3\theta}{4}$$

$$\cos 3\theta = \frac{450 \times 4}{(14.8324)^3} = 0.551618$$

$$3\theta = 56.5219$$

$$\theta_1 = 18.84°$$

$$\theta_2 = 138.84°, \quad \theta_3 = 258.84°$$

$$\sigma_1 = r\cos\theta_1 + 8$$
$$= 14.8324\cos 18.84° + 8$$
$$= 22.04 \text{ MPa}$$

$$\sigma_2 = r\cos\theta_2 + 8$$
$$= 14.8324\cos 138.84° + 8$$
$$= -3.17 \text{ MPa}$$

$$\sigma_3 = r\cos\theta_3 + 8$$
$$= 14.8324\cos 258.84° + 8$$
$$= 5.13 \text{ MPa.}$$

Example 1.5 *Calculate the principal stresses for the stress components given below :*

$$\begin{bmatrix} 100 & 200 & 100 \\ 200 & -200 & -300 \\ 100 & -300 & 400 \end{bmatrix} MPa.$$

Solution. $I_1 = 100 - 200 + 400 = 300$

$$I_2 = -100 \times 200 - 200 \times 400 + 100 \times 400 - 200^2 - 300^2 - 100^2$$
$$= (-2 - 8 + 4 - 4 - 9 - 1) \times 10^4$$
$$= -20 \times 10^4$$

$$I_3 = [1 \times (-8-9) - 2 \times (8+3) + 1 \times (-6+2)] \times 10^4$$
$$= [-17 - 22 - 4] \times 10^4$$
$$= -43 \times 10^4$$

$$\therefore \quad f(\sigma) = \sigma^3 - 300\,\sigma^2 - 20 \times 10^4\,\sigma + 43 \times 10^4$$

Consider the cubic equation :

$$y^3 + py^2 + qy + r = 0$$

Substitute $y = x - \frac{p}{3}$

$$\left(x - \frac{p}{3}\right)^3 + p\left(x - \frac{p}{3}\right)^2 + q\left(x - \frac{p}{3}\right) r = 0$$

$$x^3 - x^2 p + \frac{xp^2}{3} - \frac{p^3}{27} + px^2 + \frac{p^3}{9} - \frac{2}{3}xp^2 + qx - \frac{pq}{3} + r = 0$$

$$x^3 + x\left(\frac{p^2}{3} - \frac{2}{3}p^2 + q\right) - \frac{p^3}{27} + \frac{p^3}{9} - \frac{pq}{3} + r = 0$$

$$x^3 + \left(\frac{3q - p^2}{3}\right)x + \frac{2}{27}p^3 - \frac{pq}{3} + r = 0$$

$$x^3 + ax + b = 0$$

where $a = \frac{1}{3}(3q - p^2)$, $b = \frac{1}{27}(2p^3 - 9pq + 27r)$

Put $$\cos\theta = \frac{-b}{2\left(-\frac{a^3}{27}\right)^{1/2}}$$

$$g = 2\sqrt{-\frac{a}{3}}$$

then $$y_1 = g\cos\frac{\theta}{3} - \frac{p}{3}$$

$$y_2 = g\cos\left(120^\circ + \frac{\theta}{3}\right) - \frac{p}{3}$$

$$y_3 = g\cos\left(240^\circ + \frac{\theta}{3}\right) - \frac{p}{3}$$

Here $p = -300$, $q = -20 \times 10^4$, $r = 43 \times 10^4$

$$a = \frac{1}{3}[3 \times (-20 \times 10^4) - (300)^2]$$

$$= \frac{1}{3}[-60 - 9] \times 10^4 = -23 \times 10^4$$

$$b = \frac{1}{27}[2 \times (-300)^3 - 9(-300)(-20 \times 10^4) + 27 \times 43 \times 10^4]$$

$$= \frac{1}{27}[-54 - 540 + 1161] \times 10^6$$

$$= 21 \times 10^6$$

$$\cos\theta = \frac{-21 \times 10^6}{2\left(\frac{23^3}{27}\right)^{1/2} \times 10^6} = -0.49462$$

$$\theta = 119.645°$$

$$g = 2\sqrt{\frac{23 \times 10^4}{3}} = 553.775$$

$$\sigma_1 = y_1 = 553.775 \cos\left(\frac{119.645}{3}\right)° + 100$$

$$= 524.95 \text{ MPa}$$

$$\sigma_2 = y_2 = 553.775 \cos(159.8817)° + 100$$

$$= -419.986 \text{ MPa}$$

$$\sigma_3 = y_3 = 553.775 \cos(279.8817)^0 + 100$$

$$= 195.035 \text{ MPa}$$

1.9 Computer Program for Principal Stresses and Principal Planes

Nomenclature

$SX = \sigma_x$, $SY = \sigma_y$, $SZ = \sigma_z$

$SXY = \tau_{xy}$, $SYZ = \tau_{yz}$, $SXZ = \tau_{xz}$

$a_1 = 1$, $a_2 = I_2$, $a_3 = I_3$

$S(1) = \sigma_1$, $S(2) = \sigma_2$, $S(3) = \sigma_3$

$l(j) = a_{nxj}$, $m(j) = a_{nyj}$, $n(j) = a_{nzj}$

```
C     Program for principal stresses and planes
C     developed by Dr. Sadhu singh
C     Mechanical Engineering Department
C     Pantnagar University
      Dimension a(3),b(3),c(3),s(3)
      Integer x,y
      Real l(3),m(3),n(3),k(3)
      Open(unit=7,file='ps.in',status='old)
      Open(unit=8,file='ps.out',status='new')
      Write(8,5)
5     Format(1x,'Input',11,'Given Stress Components are :', //)
```

```
      Read(7,*)SX,SY,SZ,SXY,SYZ,SXZ
      Write(8,10)SX,SY,SZ
10    Format(1x,'SX=',F7.3,5X,'SY=',f7.3,5X,'SZ=',f7.3,//)
      Write(8,15)SXY,SYZ,SXZ
15    Format (1x,'SXY=',f7.3,5x,'SYZ=',F7.3,5x,'SXZ=',f7.3,//)
      a1=SX+SY+SZ
      a2 = SX*SY+XY*SZ+SX*SZ-SXY**2-SYZ**2-SXZ**2
      a3 = SX*SY*SZ+2SZY*SYZ*SXZ-SX*SYZ**2
      1-SY*SXZ**2-SZ*SXY**2
20    Write(8,20)
      Format (1x,'Ouput',//,'Principal stresses are :')
      r = (1./3.)*a1**2-a2
      t=SQRT[(1./27.)*r**3]
      q = (1./3.)*a1*a2-a3-(2./27.)*a1**3
      st=SQRT(1./3.)*r)
      alpha=acos(-q/(2.*t))
      S(1)=2.*st*(cos(alpha/3.)+(1./3.)*a1
      S(2)=2.*st*cos((alpha/3.)+2.0944)+(1./3.)*a1
      S(3)=2.*St*cos((alpha/3.)+4.1888)+(1./3.)*a1
      Do50x=1,2
      Do40y=x,3
      If(S(X).lt.S(Y))Go To 30
      Go To 40
30    temp=S(X)
      S(X) = S(Y)
      S(Y) = temp
40    Continue
50    Continue
      Write(8,60) S(1),S(2),S(3)
60    FORMAT(/,1x,'S1=',f7.3,5x,'S2=',f7.3,5x,'S3=',f7.3,/)
      Write(8,65)
65    FORMAT (/, 'Direction cosines are :', //)
      Do 80j=1,3
      a(j)=(SY-S(j))*(SZ-S(j))-SYZ**2
      b(j)= -(SXY*(SZ-S(j))-SXZ*SYZ)
      C(j)=SXY*SYZ-SXZ*(SY-S(j))
      k(j)=1./(SQRT(a(j)**2+b(j)**2+c(j)**2)
      l(j)=a(j)*k(j)
      m(j)=b(j)*k(j)
      n(j)=c(j)*k(j)
      Write(8,70)j,l(j),m(j),n(j)
70    Format(1x,'l(',i1,')=',f7.4,10x,'m(',i1,')=',
      1f7.4,10x,'n(',i1,')=',f7.4)
80    Continue
      Close(7)
      Close(8)
      Stop
```

```
END
Input
Given stress components are :
SX=21.450        SY=5.740         SZ= -6.850
SXY= -5.720      SYZ=11.750       SXZ=5.500
Output
Principal stresses are :
S1=23.428        S2=12.391        S3= -15.479
Direction Cosines are :
l(1)=0.9621      m(1)= -0.2628    n(1)=0.0728
l(2)= -0.1810    m(2)= -0.8155    n(2)= -0.5497
l(3)=0.2038      m(3)=0.5157      n(3)= -0.8321
```

EXERCISES

1.1 The state of stress at a point relative to an xyz coordinate system is

$$\begin{bmatrix} 400 & 100 & -100 \\ 100 & 0 & 200 \\ -100 & 200 & 0 \end{bmatrix} \text{N/mm}^2$$

Determine the complete state of stress relative to an $x'\ y'\ z'$ co-ordinate system if

	x	y	z
x'	30°	60°	90°
y'	120°	30°	90°
z'	90°	90°	90°

1.2 At point Q in a body the state of stress relative to a xyz co-ordinate system is

$$\begin{bmatrix} 500 & 200 & -200 \\ 200 & 0 & 400 \\ -200 & 400 & 300 \end{bmatrix} \text{MPa}$$

Using the cube shown in Fig. 1.5, determine the normal and shear stress at point Q for surface parallel to the following planes.

(*i*) *BCGF*, (*ii*) *ABEF*, and (*iii*) *BGE*.

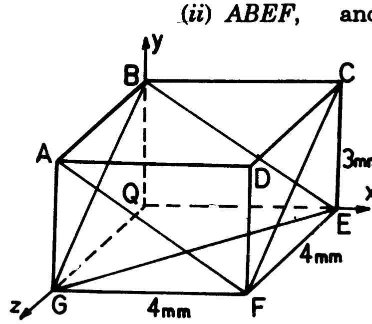

Fig. 1.5

1.3 If $\sigma_x = 4, \sigma_y = 2, \sigma_z = -2,$
$\tau_{xy} = 3, \tau_{yz} = 8, \tau_{xz} = -2$, MPa,

compute the stress vectors on planes with unit normals $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$. Compute the normal and shearing stresses on these planes.

1.4 Determine the magnitude and the direction of the principal stresses and the maximum shearing stress when,

$$\sigma_x = 1500, \quad \sigma_y = -1000,$$
$$\sigma_z = 1000, \quad \tau_{xy} = -300,$$
$$\sigma_{yz} = 0, \quad \tau_{xz} = 100 \text{ MPa}.$$

1.5 Show that

(*i*) $9\tau^2_{oct} = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$

(*ii*) $9\tau^2_{oct} = (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau^2_{xy} + \tau^2_{yz} + \tau^2_{xz})$

(*iii*) $9\tau^2_{oct} = 2I_1{}^2 - 6I_2$.

1.6 For a given displacement field

$$\bar{u} = (x^2y + 5z^2)\hat{i} + (xy^2z + y^2)\hat{j} + x^2y^2z^2\hat{k}$$

determine the strain tensor, rotation tensor and the angle of rotation at the point (2, – 1, 2).

1.7 Show that

$$\varepsilon_x = c_1(x^2 + y^2), \ \varepsilon_y = c_1(y^2 + z^2),$$
$$\gamma_{xy} = c_2xyz, \ \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0.$$

where c_1 and c_2 are constants, is not a possible state of strain.

1.8 Given the following displacement field,

$$\bar{u} = (xyz\,\hat{i} + 3x^2yz\hat{j} + 4yz\hat{k}) \times 10^{-2}$$

and a very small segment Δs having the following direction cosines before deformation :

$$a_{sx} = 0.200,$$
$$a_{sy} = 0.800,$$
$$a_{sz} = 0.555.$$

This segment is directed away from the point (2, – 1, 3). What is the new vector $\Delta s'$ after the displacement field $\bar{u}$ has been imposed ?

1.9 The state of strain at a point is given by

$$\varepsilon_x = 0.001, \ \varepsilon_y = -0.003, \ \varepsilon_z = 0.002,$$
$$\gamma_{xy} = 0.001, \ \gamma_{yz} = 0.0005, \ \gamma_{xz} = -0.002.$$

Determine the strain invariants and the principal strains.

1.10 The state of stress at a point is given by

$$\sigma_x = 200, \quad \sigma_y = -100, \quad \sigma_z = 50,$$
$$\tau_{xy} = 40, \quad \tau_{yz} = 50, \quad \tau_{xz} = 60 \text{ MPa}.$$

If $E = 2.05 \times 10^5$ N/mm^2
$G = 0.8 \times 10^5$ N/mm^2
determine the strain components.

1.11 The state of strain at a point is defined by the given strain tensor below. For a fibre with direction $(0, -1/\sqrt{5}, -2/\sqrt{5})$, calculate (*a*) the normal strain for the fibre (*b*) the magnitude of the strain vector.

$$[\varepsilon_{ij}] = \begin{bmatrix} 200 & 183 & -25 \\ 183 & 100 & -125 \\ -25 & -125 & 150 \end{bmatrix} \times 10^{-6}.$$

1.12 At a point, the stress components are :

$$\sigma_x = 600, \quad \sigma_y = 300, \quad \sigma_z = 900$$
$$\tau_{xy} = 500, \quad \tau_{yz} = 400, \quad \tau_{zx} = -200 \text{ kPa}.$$

Show that the principal directions of stress and deviative stress coincide.

1.13 The state of stress at a point is given by

$$\sigma_x = 100, \sigma_y = 200, \sigma_z = -100, \tau_{xy} = -200, \tau_{yz} = 100$$

and $\tau_{xz} = -300$ kPa.

Determine :

(*a*) the stress invariants,

(*b*) the principal stresses, and

(*c*) the direction cosines of the principal planes.

1.14 The displacement field in a body is specified as :

$$u_x = (x^2 + 3) \times 10^{-3}$$
$$u_y = 3y^2 z \times 10^{-3}$$
$$u_z = (x + 3z) \times 10^{-3}$$

Determine the strain components at a point whose co-ordinates are (1, 2, 3).

1.15 The state of stress at a point is given by

$$\begin{bmatrix} 20 & -6 & 10 \\ -6 & 10 & 8 \\ 10 & 8 & 7 \end{bmatrix} \text{MPa}.$$

Determine the principal stresses and principal directions.

1.16 The stress field in a continuous body is given by :

$$[\tau_{ij}] = \begin{bmatrix} 1 & 0 & 2y \\ 0 & 1 & 4x \\ 2y & 4x & 1 \end{bmatrix} \text{kPa}.$$

Find the stress vector at point $P(1, 2, 3)$, acting on a plane

$$x + y + z = 6.$$

1.17 The given displacement components are : $u_x = cx(y + z)^2$, $u_y = cy(x + z)^2$, $u_z = cz(x + y)^2$, where c is a constant. Find

(*a*) the components of linear strain.

(*b*) the components of rotation.

(*c*) the principal strains at a point whose co-ordinates are (1, 1, 1).

1.18 A plate whose thickness is 3 mm is stretched as shown in Fig. 1.6. Find the principal strains and the maximum shearing strain in the plate.

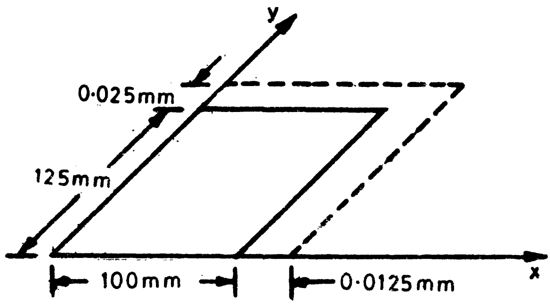

Fig. 1.6

1.19 The stress field on a body is given by

$$\sigma_x = 20x^2 + y^2 \qquad \tau_{xy} = zx$$
$$\sigma_y = 30x^3 + 200 \qquad \tau_{xz} = y^2 z$$
$$\sigma_z = 30(y^2 + z^2) \qquad \tau_{yz} = x^3 y.$$

What are the components of the body force required to ensure equilibrium ?

1.20 The state of stress at a point is given by

$$[\tau_{ij}] = 10^2 \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix} \text{MPa.}$$

Find the magnitude and direction of the stress vector acting on a plane whose normal has direction cosines $(1/2, 1/2, 1/\sqrt{2})$. What are the normal and tangential stresses acting on this plane ?

1.21 At a point P of a continuous body, the components of the stress tensor are :

$$[\tau_{ij}] = 10^3 \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 4 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix} \text{MPa.}$$

(*a*) Find the principal stresses and the principal directions.

(*b*) Find the octahedral normal and shearing stresses.

(*c*) What is the equation of the stress quadric ?

1.22 Obtain the principal stresses and the related direction cosines for the following state of stress :

$$\begin{bmatrix} 3 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 1 \end{bmatrix} \text{MPa.}$$

[**Ans.** 12.049, – 1.521, – 4.528 MPa, 0.618, 0.533, 0.577]

1.23 The state of strain at a point within a material is given by :

$$\begin{bmatrix} 200 & 100 & 0 \\ 100 & 300 & 400 \\ 0 & 400 & 0 \end{bmatrix} \times 10^{-6}$$

For $E = 200$ GPa and G = 80 GPa, ascertain the components of stress tensor. **[Ans.** 72, 99, 40, 16, 64, 0 **MPa]**

1.24 For the following state of strain, determine the principal strains.

$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & -1 & 5 \\ 2 & 5 & -4 \end{bmatrix} \times 10^{-6}.$$

[Ans. 6.01, – 1.26, – 7.72 **microns]**

1.25 The state of stress at a point is given by ;

$\sigma_x = -120$, $\sigma_y = 140$, $\sigma_z = 66$, $\tau_{xy} = 45$, $\tau_{yz} = -65$ and $\tau_{xz} = 25$ MPa.

Determine the three principal stresses and the directions associated with the three principal stresses.

[Ans. 180.2, 40.1, – 143.3 MPa,
$l_1 = 0.913, m_1 = 0.8740, n_1 = -0.4773$
$l_2 = 0.2584, m_2 = 0.4422, n_2 = 0.8589$
$l_3 = 0.9598, m_3 = -0.2062, n_3 = -0.1904$**]**

1.26 The state of stress at a joint is given by

$\sigma_x = 120$, $\sigma_y = -55$, $\sigma_z = -85$, $\tau_{xy} = -55$, $\tau_{yz} = 33$, and $\tau_{xz} = -75$ MPa.

Determine the three principal stresses and the maximum shearing stress. **[Ans.** 162.5, – 114.1, – 68.4, 138.3 **MPa]**