1

Fundamentals of Elasticity

1.1 Concept of Stress

When a certain system of external forces act on a body then the body offers resistance to these forces. This internal resistance offered by the body per unit area is called the stress induced in the body.

To understand the concept of stress, let us select an arbitrary system of co-ordinates, and divide the body by an arbitrary plane and reject one part of it, replacing the part by the action of the forces distributed through the cross-section as shown in Fig. 1.1. Having determined the resultant of all the forces applied to the elementary area, we define the stress as :

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

$$= \frac{dF}{dA}$$

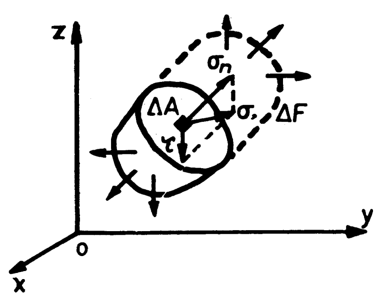

Fig. 1.1 Defining stress at a point.

The stress σ may be resolved into two components : the first perpendicular to the section under examination and the second operating in the plane of the section ; the first component will be called the normal stress σ_n, the second the shear stress τ. If a rectangular system of co-ordinates *oxyz* is selected, and the *z*-axis superimposed on the normal to the section under examination and x and y axes lie in the section planes, the shear stress can be projected onto the x and y axes, and the shear stress components τ_x and τ_y respectively, can be obtained. Therefore, at a point there is one normal stress and two shear stress components.

If an elementary parallelopiped is separated from a body as shown in Fig. 1.2, then the stresses in the body can be fully calculated from the nine stress components. The total number of independent stress components reduce to six due to the symmetry of shear stresses. The stress tensor τ_{ij} in Cartesian components then can be written as :

$$\tau_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

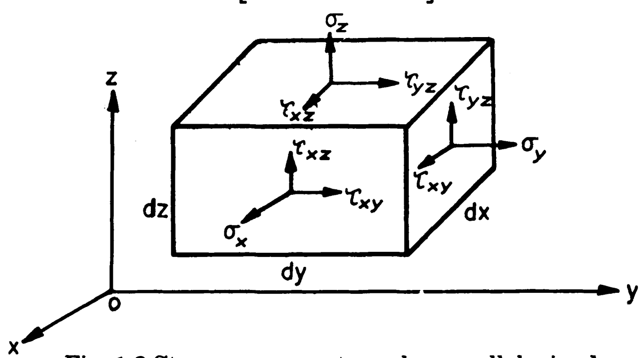

Fig. 1.2 Stresses on a rectangular parallelopiped.

In a similar way, the stress tensors in cylindrical (r, θ, z) and spherical (R, θ, ϕ) co-ordinates may be expressed as :

$$\tau_{ij} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$$

and

$$\tau_{ij} = \begin{bmatrix} \sigma_R & \tau_{R\theta} & \tau_{R\phi} \\ \tau_{R\theta} & \sigma_\theta & \tau_{\theta\phi} \\ \tau_{R\phi} & \tau_{\theta\phi} & \sigma_\phi \end{bmatrix}$$

1.2 Stress Transformation Laws

If a body under examination is broken down into elementary cubes, there will always be an infinite number of elementary tetrahedral volumes on the surface of the body. When it is of arbitrary shape ; three planes of the tetrahedron coincide with the co-ordinate axes and the fourth, coinciding with a part of the external surface of the body, will cut the co-ordinate axes at certain angles as shown in Fig. 1.3. We will express the cosines of the angles between the normal $\bar{n}$ to the surface with the co-ordinate axes by a_{nx}, a_{ny} and a_{nz} respectively. Projecting all the forces acting on the elementary tatrahedron on to the co-ordinate axes and solving the expressions thus produced in relation to projections of the resultant stress σ_R on the inclined surface, we have

$$\left.\begin{aligned}\sigma_{Rx} &= \sigma_x.a_{nx} + \tau_{xy}.a_{ny} + \tau_{xz}.a_{nz}\\ \sigma_{Ry} &= \tau_{xz}.a_{nx} + \sigma_y.a_{ny} + \tau_{yz}.a_{nz}\\ \sigma_{Rz} &= \tau_{xz}.a_{nx} + \tau_{yz}.a_{ny} + \sigma_z.a_{nz}\end{aligned}\right\} \quad ...(1.1)$$

$$\sigma_R^2 = \sigma_{Rx}^2 + \sigma_{Ry}^2 + \sigma_{Rz}^2 \quad ...(1.2)$$

The normal stress σ_n on the inclined surface *ABC* may be written in the form,

$$\begin{aligned}\sigma_n &= \sigma_{Rx}.a_{nx} + \sigma_{Ry}.a_{ny} + \sigma_{Rz}.a_{nz}\\ &= \sigma_x.a_{nx}^2 + \sigma_y.a_{ny}^2 + \sigma_z.a_{nz}^2 + 2(\tau_{xy}.a_{nx}a_{ny}\\ &\quad + \tau_{yz}.a_{ny}\,a_{nz} + \tau_{xz}.a_{xx}a_{nz})\end{aligned} \quad ...(1.3)$$

If τ_n is the shear stress on the plane *ABC* and its direction cosines are a_{sx}, a_{sy} and a_{sz} respectively, then

$$\tau_n = \sqrt{\sigma_R^2 - \sigma_n^2} \quad ...(1.4)$$

Also $\quad \sigma_{Rx} = \sigma_n.a_{nx} + \tau_n.a_{sx}$

$$\therefore \quad a_{sx} = \frac{1}{\tau_n}[\sigma_{Rx} - \sigma_n.a_{nx}]$$

$$= \frac{1}{\tau_n}[(\sigma_x - \sigma_n)a_{nx} + \tau_{xy}.a_{ny} + \tau_{xz}.a_{nz}] \quad ...(1.5)$$

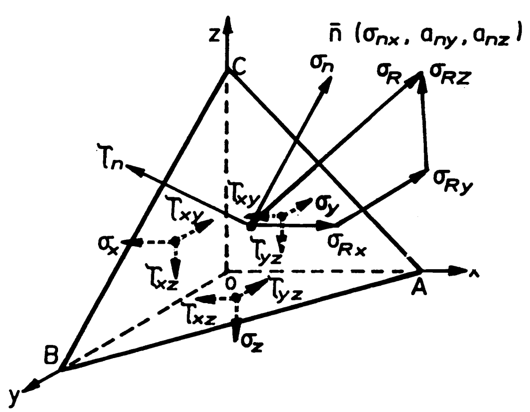

Fig. 1.3 Stresses on a tetrahedron.

Similarly, $$a_{sy} = \frac{1}{\tau_n}[\tau_{xy}.a_{nx} + (\sigma_y - \sigma_n)a_{ny} + \tau_{yz}.a_{nz}] \quad ...(1.6)$$

$$a_{sz} = \frac{1}{\tau_n}[\tau_{xy}.a_{nx} + \tau_{yz}.a_{ny} + (\sigma_z - \sigma_n)a_{nz}] \quad ...(1.7)$$

If it is desired to express the stress components with respect to a co-ordinate system obtained by rotating the original co-ordinate system xyz to the new system $x'y'z'$, the the direction cosines are determined by :

	x	y	z
x'	$a_{x'x}$	$a_{x'y}$	$a_{x'z}$
y'	$a_{y'x}$	$a_{y'y}$	$a_{y'z}$
z'	$a_{z'x}$	$a_{z'y}$	$a_{z'z}$

where $a_{x'x} = \cos(x', x)$ and so on.

The stress components in the new co-ordinate system may be obtained from,

$$\tau'_{ij} = a_{ij}\, a_{jm}\, \tau_{im} \quad ...(1.8)$$

For example,

$$\tau_{x'x'} = \sigma_x.\, a_{x'x}^{\ 2} + \sigma_y.a_{x'y}^{\ 2} + \sigma_z.a_{x'z}^{\ 2}$$
$$+ 2(\tau_{xy}.a_{x'x}.a_{x'y} + \tau_{yz}.a_{x'y}.a_{x'z} + \tau_{xz}.a_{x'x}.a_{x'z})$$
$$\tau_{x'y'} = \sigma_x a_{x'x} a_{y'x} + \sigma_y.a_{x'y}.a_{y'y} + \sigma_z.a_{x'z}.a_{y'z}$$
$$+ \tau_{xy}(a_{x'x}a_{y'y} + a_{x'y}\, a_{y'x})$$
$$+ \tau_{yz}(a_{x'y} + a_{y'z} + a_{x'z}a_{y'y})$$
$$+ \tau_{xz}(a_{x'x}\, a_{y'z} + a_{x'z}\, a_{y'x})$$

Other terms can be written in a similar fashion.

From the transformation law, we can determine the planes on which the shear stress is zero, *i.e.* the principal planes. The principal stresses are the roots of the cubic equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \quad ...(1.9)$$

where $I_1 = \sigma_x + \sigma_y + \sigma_z$

$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$

$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\,\tau_{xz}$

are the three *stress invariants*. The solution of this equation may be obtained either by hit and trial method or by using a numerical technique like the Newton-Raphson method.

1.3 Spherical and Deviator Stress Tensors

We can write that

$$\tau_{ij} = \tau'_{ij} + \tau''_{ij}$$

where $\tau'_{ij} = \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix} = \tau_{ij} - \delta_{ij}\sigma_m$

$$= \begin{bmatrix} \dfrac{2\sigma_x - \sigma_y - \sigma_z}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \dfrac{2\sigma_y - \sigma_x - \sigma_z}{3} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \dfrac{2\,\sigma_z - \sigma_x - \sigma_y}{3} \end{bmatrix} \quad ...(1.10)$$

is the *Deviator* stress tensor' and

$$\tau''_{ij} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \delta_{ij}\,\sigma_m \quad ...(1.11)$$

is the *spherical* or *hydrostatic* stress tensor

where $\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$ is the mean stress.

The deviator stress tensor consists of only pure shear stress components. It will be seen subsequently that plastic deformation can take place only, when deviatorial stresses, shear stresses, or both of a certain combined magnitude are present.

1.4 Equilibrium Equations

1.4.1 Cartesian Co-ordinates

Considering the equilibrium of an elementary rectangular parallelopiped subjected to a generalised state of stress, it can be shown that the equilibrium equations become :

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + B_z = 0 \end{aligned}\right\} \quad ...(1.22)$$

where B_x, B_y and B_z are the body forces per unit volume along the three co-ordinate axes. These equations hold good both in the elastic and plastic range.

1.4.2 Cylindrical Co-ordinates

In cylindrical co-ordinates (r, θ, z), the equilibrium equations can be obtained as :

$$\left.\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + B_\theta = 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + B_z = 0 \end{aligned}\right\} \quad \text{...(1.13)}$$

For problems having axial symmetry, $\tau_{\theta z} = \tau_{r\theta} = 0$, and we get

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + B_z = 0 \quad \text{...(1.14)}$$

1.4.3 Spherical Co-ordinates

In spherical co-ordinates (R, θ, ϕ), the equilibrium equations are :

$$\left.\begin{aligned} &\frac{\partial \sigma_R}{\partial R} + \frac{1}{R}\frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \tau_{R\phi}}{\partial \phi} \\ &\qquad + \frac{1}{R}(2\sigma_R - \sigma_\theta - \sigma_\phi + \tau_{R\theta}\cot\theta) + B_R = 0 \\ &\frac{\partial \tau_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} \\ &\qquad + \frac{1}{R}[(\sigma_\theta - \sigma_\phi)\cot\theta + 3\tau_{R\theta}] + B_\theta = 0 \\ &\frac{\partial \tau_{R\phi}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \sigma_\phi}{\partial \phi} \\ &\qquad + \frac{1}{R}(3\tau_{R\phi} + 2\tau_{\theta\phi}\cot\theta) + B_\phi = 0 \end{aligned}\right\} \quad \text{...(1.15)}$$

For spherical symmetry, when $\tau_{R\theta} = \tau_{R\phi} = \tau_{\theta\phi} = 0$ and $\sigma_\theta = \sigma_\phi$, we get

$$\left.\begin{aligned} \frac{\partial \sigma_R}{\partial R} + \frac{2}{R}(\sigma_R - \sigma_\theta) = 0 \\ \frac{\partial \sigma_\theta}{\partial \theta} = \frac{\partial \sigma_\phi}{\partial \phi} = 0 \end{aligned}\right\} \quad \text{...(1.16)}$$

1.5 Octahedral Stresses

A plane which is equally inclined to the three axes of reference, is called the octahedral plane, *i.e.* its direction cosines are $\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}} \pm \frac{1}{\sqrt{3}}$. The normal and shearing stress acting on this plane are called the octahedral stresses. Using Eq. (1.3), the normal octahedral stress in the principal co-ordinate system becomes,

$$(\sigma_n)_{oct} = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \qquad ...(1.17)$$

The resultant stress on the octahedral plane, by using Eq. (1.2) becomes,

$$(\sigma_R)_{oct} = \sqrt{\tfrac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)} \qquad ...(1.18)$$

Hence the octahedral shear stress becomes,

$$\tau_{oct} = \tfrac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \qquad ...(1.19)$$

Now $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$

$$= 2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$
$$= 2I_1^2 - 6I_2$$

Hence $\quad 9\tau_{oct}^2 = 2I_1^2 - 6I_2 \qquad ...(1.20)$

For a generalised state of stress, substituting the values of I_1 and I_2, we get

$$\tau_{oct} = \tfrac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)} \qquad ...(1.21)$$

The effective shear stress

$$\tau_{eff} = \frac{\sqrt{2}}{2}\tau_{oct} \qquad ...(1.22)$$

Example 1.1 *The stress tensor at a point is given by*

$$\tau_{ij} = \begin{bmatrix} 50 & 50 & 150 \\ 50 & 100 & 100 \\ 150 & 100 & 150 \end{bmatrix} N/mm^2$$

Calculate for the plane having direction cosines

$$a_{nx} = \frac{1}{\sqrt{6}},\ a_{ny} = \frac{1}{\sqrt{3}},\ a_{nz} = \frac{1}{\sqrt{2}}$$

(a) total stresses, (b) normal stress, and (c) shear stress and its direction.

Solution.

(*a*)

$$\sigma_{Rx} = \sigma_x a_{nx} + \tau_{xy} a_{ny} + \tau_{xz} a_{nz}$$

$$= 50 \times \frac{1}{\sqrt{6}} + 50 \times \frac{1}{\sqrt{3}} + 150 \times \frac{1}{\sqrt{2}}$$

$$= 155.346 \text{ N/mm}^2$$

$$\sigma_{Ry} = \tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{yz} a_{nz}$$

$$= 50 \times \frac{1}{\sqrt{6}} + 100 \times \frac{1}{\sqrt{3}} + 100 \times \frac{1}{\sqrt{2}}$$

$$= 148.858 \text{ N/mm}^2$$

$$\sigma_{Rz} = \tau_{xz} a_{nx} + \tau_{yz} a_{ny} + \sigma_z a_{nz}$$

$$= 150 \times \frac{1}{\sqrt{6}} + 100 \times \frac{1}{\sqrt{3}} + 150 \times \frac{1}{\sqrt{2}}$$

$$= 225.038 \text{ N/mm}^2$$

Total stress,

$$\sigma_R = \sqrt{\sigma_{Rx}^2 + \sigma_{Ry}^2 + \sigma_{Rz}^2}$$

$$= 311.34 \text{ N/mm}^2$$

(*b*) Normal stress is given by,

$$\sigma_n = \sigma_{Rx} a_{nx} + \sigma_{Ry} a_{ny} + \sigma_{Rz} a_{nz}$$

$$= 155.346 \times \frac{1}{\sqrt{6}} + 148.858 \times \frac{1}{\sqrt{3}} + 225.038 \times \frac{1}{\sqrt{2}}$$

$$= 308.488 \text{ N/mm}^2$$

(*c*) Shear stress is given by

$$\tau_n = \sqrt{\sigma_R^2 - \sigma_n^2}$$

$$= 42.05 \text{ N/mm}^2$$

Let a_{sx}, a_{sy} and a_{sz} be the direction cosines of τ_n. Then

$$a_{sx} = \frac{1}{\tau_n} [(\sigma_x - \sigma_n)a_{nx} + \tau_{xy} a_{ny} + \tau_{yz} a_{nz}]$$

$$= \frac{1}{42.05} \left[(50 - 308.488) \times \frac{1}{\sqrt{6}} + 50 \times \frac{1}{\sqrt{3}} + 150 \times \frac{1}{\sqrt{2}} \right]$$

$$= 0.6993$$

Similarly $a_{sy} = -0.6955$ and $a_{sz} = 0.0238$.

Example 1.2 *The state of stress at a point is given by the following stress tensor :*

$$\tau_{ij} = \begin{bmatrix} 50 & 50 & -40 \\ 50 & -30 & 30 \\ -40 & 30 & -100 \end{bmatrix} MPa$$

(a) Calculate the stress invariants, (b) magnitude and direction of the principal stresses, and (c) spherical and deviator stress tensors.

Solution.

(*a*)
$$I_1 = \sigma_x + \sigma_y + \sigma_z$$
$$= 50 - 30 - 100 = -80$$
$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$
$$= 50(-30) + (-30)(-100) + 50(-100) - (50)^2 - (-30)^2 - (40)^2$$
$$= -85 \times 10^2$$
$$I_3 = | \tau_{ij} |$$
$$= [50(30 - 9) - 50(-50 + 12) - 40(15 - 12)]10^2$$
$$= 283 \times 10^3$$

(*b*) Principal stresses are the roots of the equation,
$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$
$$\sigma^3 + 80\,\sigma^2 - 85 \times 10^2\,\sigma - 283 \times 10^3 = 0$$

We shall use Newton-Raphson method to determine the roots of this equation.

Let σ_i be a root of this equation, then the better approximated root can be obtained from the iteration process, given by
$$\sigma_{i+1} = \sigma_i - \frac{f(\sigma_i)}{f'(\sigma_i)}$$

Let us assume

$\sigma_I = 60$ MPa, then first iteration gives,
$$\sigma_i = 60 - \frac{-289 \times 10^3}{1.19 \times 10^3}$$
$$= 84.2 \text{ MPa}$$

Second iteration gives
$$\sigma_{II} = 84.2 - \frac{1.653 \times 10^5}{2{,}624.092}$$
$$= 77.9 \text{ MPa}$$

Third iteration gives
$$\sigma_{III} = 77.9 - \frac{0.13 \times 10^5}{2{,}216.923} = 77.314 \text{ MPa}$$

Further iterations do not improve upon this value by appreciable amount. Hence we take $\sigma_1 = 77.314$ MPa. Dividing the cubic equation by $(\sigma - 77.314)$, we get
$$\sigma^2 + 157.3\,\sigma + 36604.63 = 0$$

The roots of this quadratic equation are :

$$\sigma_2 = -28.396 \text{ MPa}$$

$$\sigma_3 = -128.90 \text{ MPa}$$

To determine the principal directions, the following procedure may be adopted :

(*i*) Determine the stress tensor $\tau_{ij} - \sigma_i I$, where I is the identity or unit matrix.

(*ii*) Determine the cofactors of elements of the first row of the new stress tensor obtained in (*i*) above. Let these cofactors be denoted by A_i, B_i and C_i.

(*iii*) The principal directions are then given by,

$$a_{nx_i} = \frac{A_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$$

$$a_{ny_i} = \frac{B_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$$

$$a_{nz_i} = \frac{C_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$$

For σ_1 to be substituted, we find that

$$\tau_{ij} - \sigma_1 I = \begin{bmatrix} -27.314 & 50 & -40 \\ 50 & -107.314 & 30 \\ -40 & 30 & -177.314 \end{bmatrix}$$

$$A_1 = 107.314 \times 177.314 - 30 \times 30 = 18{,}125.71$$

$$B_1 = -(-50 \times 177.314 + 40 \times 30) = 7{,}665.70$$

$$C_1 = 50 \times 30 - 107.314 \times 40 = -2{,}792.56$$

$$K_1 = \sqrt{A_1^2 + B_1^2 + C_1^2}$$

$$= 19.877 \times 10^3$$

Hence $$a_{nx_1} = \frac{A_1}{K_1} = 0.9119$$

$$a_{ny_1} = \frac{B_1}{K_1} = 0.3856$$

$$a_{nz_1} = \frac{C_1}{K_1} = -0.1405$$

Similarly, we find that

$$a_{nx_2} = -0.2717,\ a_{ny_2} = 0.8240,\ a_{nz_2} = 0.4970$$

$$a_{nx_3} = 0.3074,\ a_{ny_3} = -0.4151,\ a_{nz_3} = 0.8562$$

(c) $$\sigma_m = \frac{1}{2}(\sigma_x + \sigma_y + \sigma_z)$$
$$= \frac{1}{3}(50 - 10 - 100) = -26.667 \text{ MPa}$$

Deviator stress tensor

$\tau'_{ij} = \tau_{ij} - \delta_{ij}\,\sigma_m$. Hence

$$\tau'_{ij} = \begin{bmatrix} 23.337 & 50 & -40 \\ 50 & -56.667 & 30 \\ -40 & 30 & -126.667 \end{bmatrix} \text{MPa}$$

Spherical stress tensor

$$\tau''_{ij} = \begin{bmatrix} -26.667 & 0 & 0 \\ 0 & -26.667 & 0 \\ 0 & 0 & -26.667 \end{bmatrix} \text{MPa}$$

1.6 Concept of Strain

Consider the deformation of a rectangular parallelopiped. As a result of deformation the length of the edges of the parallelopiped and the angles between them change. The element itself is displaced from one position to another. It is only this later part that enters the definition of strain. In the small deformation theory, it is assumed that the deformations are infinitesimal. The state of deformation in the neighbourhood of a point of a continuous body is determined by six components of strain, three normal and three shear. Physically, the normal strain is defined as the change in length of a small line segment per unit original length of the segment. Whereas the shearing strain is considered as the change in angle between two initial perpendicular line segments.

1.6.1 Strain Components in Cartesian Co-ordinates

The strain tensor in Cartesian co-ordinates is given by

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_z \end{bmatrix}$$

where $\varepsilon_x = \dfrac{\partial u_x}{\partial x}$, $\varepsilon_y = \dfrac{\partial u_y}{\partial y}$, $\varepsilon_z = \dfrac{\partial u_z}{\partial z}$

$$\left.\begin{aligned} \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \gamma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \gamma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \end{aligned}\right\} \quad ...(1.23)$$

and u_x, u_y, u_z are the components of the displacement field.

The infinitesimal strain tensor can be expressed as :

$$d\varepsilon_{ij} = \begin{bmatrix} d\varepsilon_x & \frac{d\gamma_{xy}}{2} & \frac{d\gamma_{xz}}{2} \\ \frac{d\gamma_{xy}}{2} & d\varepsilon_y & \frac{d\gamma_{yz}}{2} \\ \frac{d\gamma_{xz}}{2} & \frac{d\gamma_{yz}}{2} & d\varepsilon_z \end{bmatrix} \quad ...(1.24)$$

Similarly the strain-rate tensor can be written as :

$$\dot{\varepsilon}_{ij} = \begin{bmatrix} \dot{\varepsilon}_x & \frac{\dot{\gamma}_{xy}}{2} & \frac{\dot{\gamma}_{xz}}{2} \\ \frac{\dot{\gamma}_{xy}}{2} & \dot{\varepsilon}_y & \frac{\dot{\gamma}_{yz}}{2} \\ \frac{\dot{\gamma}_{xz}}{2} & \frac{\dot{\gamma}_{yz}}{2} & \dot{\varepsilon}_z \end{bmatrix} \quad ...(1.25)$$

where $\cdot = \frac{d}{dt}$

1.6.2 Strain Components in Cylindrical Co-ordinates

The strain components in cylindrical co-ordinates (r, θ, z) are :

$$\varepsilon_r = \frac{\partial u_r}{\partial r}\ ;\ \varepsilon_\theta = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)\ ;\ \varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\left.\begin{aligned} \gamma_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\ \gamma_{\theta z} &= \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \\ \gamma_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \end{aligned}\right\} \quad ...(1.26)$$

For axial symmetry with $u_\theta = 0$, we get

$$\left.\begin{aligned} &\varepsilon_r = \frac{\partial u_r}{\partial r}\ ; \quad \varepsilon_\theta = \frac{u_r}{r}\ ; \quad \varepsilon_z = \frac{\partial u_z}{\partial z} \\ &\gamma_{r\theta} = \gamma_{\theta z} = 0\ ;\ \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \end{aligned}\right\} \quad ...(1.27)$$

1.6.3 Strain Components in Spherical Co-ordinates

The strain components in spherical co-ordinates (R, θ, ϕ) are :

$$\varepsilon_R = \frac{\partial u_R}{\partial R} ; \varepsilon_\theta = \frac{1}{R}\left(\frac{\partial u_\theta}{\partial \theta} + u_R\right)$$

$$\varepsilon_\phi = \frac{1}{R \sin\phi}\left(\frac{\partial u_\phi}{\partial \phi} + u_R \sin\phi + u_\theta \cos\theta\right)$$

$$\left.\begin{aligned} \gamma_{R\theta} &= \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R} + \frac{1}{R}\frac{\partial u_R}{\partial \theta} \\ \gamma_{\theta\phi} &= \frac{1}{R\sin\theta}\left(\sin\theta\frac{\partial u_\phi}{\partial \theta} - u_\phi \cos\theta + \frac{\partial u_\theta}{\partial \phi}\right) \\ \gamma_{R\phi} &= \frac{1}{R\sin\theta}\cdot\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R} \end{aligned}\right\} \quad ...(1.28)$$

For spherical symmetry with $u_\theta = u_\phi = 0$, we have

$$\left.\begin{aligned} &\varepsilon_R = \frac{\partial u_R}{\partial R} ; \varepsilon_\theta = \frac{u_R}{R} ; \varepsilon_\phi = \frac{u_R}{R} \\ &\gamma_{R\theta} = \gamma_{\theta\phi} = \gamma_{R\phi} = 0 \end{aligned}\right\} \quad ...(1.29)$$

1.7 Deviator and Spherical Strain Tensors

Now $\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon''_{ij}$

= Deviator strain tensor + Spherical strain tensor

where $\varepsilon'_{ij} = \varepsilon_{ij} - \delta_{ij}\varepsilon_m$

$$= \begin{bmatrix} \frac{2\varepsilon_x - \varepsilon_y - \varepsilon_z}{3} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \frac{2\varepsilon_y - \varepsilon_z - \varepsilon_x}{3} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \frac{2\varepsilon_z - \varepsilon_x - \varepsilon_y}{3} \end{bmatrix} \quad ...(1.30)$$

$$\varepsilon''_{ij} = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix} \quad ...(1.31)$$

and $\varepsilon_m = \frac{1}{2}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$ is the mean strain.

If the volume remains constant during the deformation, then

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = 0 \quad ...(1.32)$$

Hence $\varepsilon''_{ij} = 0$

Therefore, $\varepsilon_{ij} = \varepsilon'_{ij}$ ...(1.33)

1.8 Strain Transformation Laws

Analogous to the stress transformation laws, the strain transformation laws may be expressed as :

$$\varepsilon'_{ij} = a_{il}\, a_{jm}\, \varepsilon_{lm} \qquad ...(1.34)$$

The normal strain on a plane whose direction cosines are a_{nx}, a_{ny} and a_{nz} is given by :

$$\varepsilon_n = \varepsilon_x a_{nx}^2 + \varepsilon_y\, a_{ny}^2 + \varepsilon_z . a_{nz}^2 + \gamma_{xy} . a_{nx}\, a_{ny} + \gamma_{yz} . a_{ny}\, a_{nz} + \gamma_{xz} . a_{nx}\, a_{nz} \qquad ...(1.35)$$

The components of strain acting on the above plane along the co-ordinate axes are :

$$\varepsilon_{Rx} = \varepsilon_x a_{nx} + \frac{1}{2}(\gamma_{xy} . a_{ny} + \gamma_{xz} . a_{nz})$$

$$\varepsilon_{Ry} = \varepsilon_y . a_{ny} + \frac{1}{2}(\gamma_{xy} . a_{nx} + \gamma_{yz} . a_{nz})$$

$$\varepsilon_{Rz} = \varepsilon_z . a_{nz} + \frac{1}{2}(\gamma_{xz} . a_{nx} + \gamma_{yz} . a_{ny})$$

The resultant strain on the plane becomes,

$$\varepsilon_R = \sqrt{\varepsilon_{Rx}^2 + \varepsilon_{Ry}^2 + \varepsilon_{Rz}^2} \qquad ...(1.36)$$

Hence the shear strain on the plane becomes,

$$\frac{\gamma_n}{2} = \sqrt{\varepsilon_R^2 - \varepsilon_n^2} \qquad ...(1.37)$$

The direction cosines of the shearing strain may be obtained from,

$$a_{sx} = \frac{2}{\gamma_n}(\varepsilon_{Rx} - \varepsilon_n\, a_{nx})$$

$$a_{sy} = \frac{2}{\gamma_n}(\varepsilon_{Ry} - \varepsilon_n\, a_{ny}) \qquad ...(1.38)$$

$$a_{sz} = \frac{2}{\gamma_n}(\varepsilon_{Rz} - \varepsilon_n\, a_{nz})$$

The principal strains are the roots of the cubic equation,

$$\varepsilon^3 - J_1\, \varepsilon^2 + J_2\, \varepsilon - J_3 = 0 \qquad ...(1.39)$$

where $J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)$$

$$J_3 = \varepsilon_x\varepsilon_y\varepsilon_z - \frac{1}{4}(\varepsilon_x\gamma_{yz}^2 + \varepsilon_y\gamma_{xz}^2 + \varepsilon_z\, \gamma_{xy}^2 - \gamma_{xy}\gamma_{yz}\gamma_{xz})$$

are the three strain invariants.

The solution of Eq. (1.39) may be obtained by hit-and-trial method or any numerical technique like the Newton-Raphson method.

1.9 Octahedral Strains

The normal and shearing strains acting on a plane whose direction cosines are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$ are called octahedral strains. In the principal co-ordinate system, the normal octahedral strain becomes,

$$(\varepsilon_n)_{oct} = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \qquad \text{...(1.40)}$$

and the octahedral shearing strain is

$$\frac{\gamma_{oct}}{2} = \frac{1}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2} \qquad \text{...(1.41)}$$

or

$$\frac{9}{4}\gamma^2_{oct} = 2J_1^2 - 6J_2 \qquad \text{...(1.42)}$$

The resultant strain on the octahedral plane is

$$(\varepsilon_R)_{oct} = \sqrt{\tfrac{1}{3}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)} \qquad \text{...(1.43)}$$

In the general co-ordinate system, Eq. (1.42) can be written as :

$$\frac{9}{4}\gamma_{oct}^2 = (\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2) \qquad \text{...(1.43)}$$

and

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2}\gamma_{oct} \qquad \text{...(1.44)}$$

Example 1.3 *The strain tensor at a point is given by*

$$\varepsilon_{ij} = \begin{bmatrix} 0.001 & 0.0005 & 0.002 \\ 0.0005 & 0.002 & 0.001 \\ 0.002 & 0.001 & 0.003 \end{bmatrix}$$

Determine the magnitude and direction of the principal strains.

Solution. $J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$$= 6 \times 10^{-3}$$

$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z - \tfrac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)$$

$$= 10^{-6}(2 + 6 + 3) - \tfrac{1}{4} \times 10^{-6}(0.25 + 1 + 4)$$

$$= 5.75 \times 10^{-6}$$

$$J_3 = | \varepsilon_{ij} |$$
$$= 10^{-9}[(6-1) - 0.5(1.5-2) + 2(0.5-4)]$$
$$= -1.75 \times 10^{-9}$$

The cubic equation becomes

$$\varepsilon^3 - 6 \times 10^{-3}\varepsilon^2 + 5.75 \times 10^{-6}\varepsilon + 1.75 \times 10^{-9} = 0$$

Solving this equation by Newton-Raphson method, let

$$\varepsilon = 1.2 \times 10^{-3}$$

First iteration gives

$$\varepsilon = 1.2 \times 10^{-3} - \frac{1.738 \times 10^{-6}}{-4.33 \times 10^{-3}} = 1.6 \times 10^{-3}$$

Second iteration gives $\varepsilon = 1.545 \times 10^{-3}$

Further iterations do not improve the value appreciably. Hence, we take the root to be equal to 1.545×10^{-3}. Dividing the cubic equation by $(\varepsilon - 1.545 \times 10^{-3})$ and solving the quadratic equation, we get the other two roots to be, 4.696×10^{-3} and -0.241×10^{-3}. Hence the principal strains in descending order are :

$$\varepsilon_1 = 4.696 \times 10^{-3}, \quad \varepsilon_2 = 1.545 \times 10^{-3};$$
$$\varepsilon_3 = -0.241 \times 10^{-3}.$$

The principal strain directions are determined in the same manner as discussed in Example 1.2 for stresses. Following this procedure, we get

$$a_{nx_1} = 0.479, \quad a_{ny_1} = -0.382, \quad a_{nz_1} = 0.790$$
$$a_{nx_2} = -0.245, \quad a_{ny_2} = 0.923, \quad a_{nz_2} = 0.297$$
$$a_{nx_3} = 0.751, \quad a_{ny_3} = 0.455, \quad a_{nz_3} = 0.477.$$

Example 1.4 *The strain tensor at a point is given by*

$$\varepsilon_{ij} = \begin{bmatrix} 0.002 & -0.005 & 0.003 \\ -0.005 & 0.003 & 0.002 \\ 0.003 & 0.002 & 0.004 \end{bmatrix}$$

Determine the deviator and spherical strain tensors.

Solution. $\varepsilon_m = \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$

$$= \frac{1}{3}(0.002 + 0.003 + 0.004) = 0.003$$

Deviator strain tensor

$$\varepsilon'_{ij} = \varepsilon_{ij} - \delta_{ij}\varepsilon_m$$

$$\varepsilon'_{ij} = \begin{bmatrix} -0.001 & -0.005 & 0.003 \\ -0.005 & 0 & 0.002 \\ 0.003 & 0.002 & 0.001 \end{bmatrix}$$

Spherical strain tensor is,

$$\varepsilon''_{ij} = \delta_{ij}\,\varepsilon_m$$

$$\varepsilon''_{ij} = \begin{bmatrix} 0.003 & 0 & 0 \\ 0 & 0.003 & 0 \\ 0 & 0 & 0.003 \end{bmatrix}$$

Example 1.5 *The principal strains at a point in a body are given by :*

$$\varepsilon_1 = 0.002\,;\quad \varepsilon_2 = 0.0002\,;\quad \varepsilon_3 = -0.001$$

Determine the octahedral normal and shearing strains.

Solution. $(\varepsilon_n)_{oct} = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$

$$= 0.0004$$

$$(\varepsilon_R)_{oct} = \sqrt{\frac{1}{3}\,(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)}$$

$$= 10^{-3}\sqrt{\frac{1}{3}\,(4 + 0.04 + 1)}$$

$$= 1.296 \times 10^{-3}$$

$$\gamma_{oct} = 2\sqrt{(\varepsilon_R)_{oct}^2 - (\varepsilon_n)_{oct}^2}$$

$$= 2 \times 10^{-3}\sqrt{(1.296)^2 - (0.4)^2}$$

$$= 2.466 \times 10^{-3}.$$

1.10 Generalised Hooke's Law

According to Hooke's law, within the elastic (proportional) limits, stress is proportional to strain. In its general form, for a homogeneous, linear and isotropic material, this can be expressed as :

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2G & \lambda & 0 & 0 & 0 \\ & & \lambda + 2G & 0 & 0 & 0 \\ & \text{Symmetric} & & G & 0 & 0 \\ & & & & G & 0 \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad ...(1.45)$$

where $$\lambda = \frac{\nu E}{(1-\nu)(1-2\nu)}$$

$$G = \frac{E}{2(1+\nu)}$$

are the Lame's constant. E is the modulus of elasticity and ν is the Poisson's ratio. There are two independent elastic constants for a homogeneous, linear and isotropic material.

The deviator stress and strain tensors can be related as :

$$\tau'_{ij} = 2G\varepsilon'_{ij} \qquad ...(1.46)$$

Similarly, the spherical stress and strain tensors can be related as :

$$\tau''_{ij} = 3K\varepsilon''_{ij} \qquad ...(1.47)$$

where $$K = \frac{E}{3(1-2\nu)}$$

is the Bulk Modulus.

For volume constancy,

$$\varepsilon''_{ij} = 0.$$

Hence $$\varepsilon_{ij} = \varepsilon'_{ij}$$

Therefore, Eq. (1.46) becomes,

$$\tau'_{ij} = 2G\,\varepsilon_{ij} \qquad ...(1.48)$$

The stresses in terms of strains can be written as :

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{1}{G} & 0 & 0 \\ & \text{Symmetric} & & & \frac{1}{G} & 0 \\ & & & & & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \qquad ...(1.49)$$

1.11 Elastic Stain Energy

The elastic strain energy per unit volume is given by :

$$U = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$$

$$= \frac{1}{2}\left[\frac{\sigma_1(2\sigma_1 - \sigma_2 - \sigma_3)}{6G} + \frac{\sigma_2(2\sigma_2 - \sigma_3 - \sigma_1)}{6G} + \frac{\sigma_3(2\sigma_3 - \sigma_1 - \sigma_2)}{6G} + \frac{(\sigma_1 + \sigma_2 + \sigma_3)^2}{9K}\right]$$

$$= \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{12G} + \frac{(\sigma_1 + \sigma_2 + \sigma_3)^2}{18K} \qquad ...(1.50)$$

The strain energy of distortion U' can be obtained from deviator stress components.

$$U' = \frac{1}{2}\left[\frac{(2\sigma_1 - \sigma_2 - \sigma_3)^2}{18G} + \frac{(2\sigma_2 - \sigma_3 - \sigma_1)^2}{18G} + \frac{(2\sigma_3 - \sigma_1 - \sigma_2)^2}{18G}\right]$$

$$= \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{3}{4G}\tau_{oct} \qquad \text{...(1.51)}$$

The strain energy of dilatation U'' can be obtained from the spherical stress and strain tensors.

$$U'' = \frac{(\sigma_1 + \sigma_2 + \sigma_3)^2}{18K} \qquad \text{...(1.52)}$$

Thus $$U = U' + U'' \qquad \text{...(1.53)}$$

1.12 Compatibility Equations

We have seen that there are six strain components, which are obtained from three displacement components by taking the derivatives. If the displacement relations are such that a given system of strains gives rise to a continuous single-valued system of displacements, then certain restrictions have to be imposed on the strain components. The resulting equations are known as the compatibility equations. In Cartesian co-ordinates, these equations are given by

$$\left.\begin{aligned}
\frac{\partial^2\gamma_{xy}}{\partial x\partial y} &= \frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2}\\
\frac{\partial^2\gamma_{yz}}{\partial y\partial z} &= \frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2}\\
\frac{\partial^2\gamma_{xz}}{\partial x\partial z} &= \frac{\partial^2\varepsilon_x}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial x^2}\\
\frac{2\partial^2\varepsilon_x}{\partial y\partial z} &= \frac{\partial}{\partial x}\left[-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{xz}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right]\\
\frac{2\partial^2\varepsilon_y}{\partial x\partial z} &= \frac{\partial}{\partial y}\left[-\frac{\partial\gamma_{xz}}{\partial y} + \frac{\partial xy}{\partial z} + \frac{\partial yz}{\partial x}\right]\\
\frac{2\partial^2\varepsilon_z}{\partial x\partial y} &= \frac{\partial}{\partial z}\left[-\frac{\partial\gamma_{xy}}{\partial z} + \frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{xz}}{\partial y}\right]
\end{aligned}\right\} \qquad \text{...(1.54)}$$

1.13 Theories of Strength

1. Maximum Principal Stress Theory (Rankine's Theory). According to this theory, the maximum principal stress in the material determines failure regardless of what the other principal stresses may be, provided that the latter are absolutely smaller. This theory cannot be applied as a criterion of yielding because three equal tensile or compressive stresses as such belong to those states of stress which should cause failure according to this theory, in contradiction to the experimental experience, according to which such states cannot produce a plastic distortion in compact solid material. The surface representing this theory is a cube. The faces of the cube are six planes symmetrically spaced around the origin of the σ_1, σ_2, σ_3 system of rectangular co-ordinates and parallel to the co-ordinate planes. Parts of this surface, however have served as criteria of the cleavage fracture of materials.

2. Maximum Principal Strain Theory (St. Venant's Theory). According to this theory, the maximum positive elastic extension of the material determines failure of either kind. Now the maximum principal strain is given by,

$$\varepsilon_1 = \frac{1}{E}\,[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

and the strain under uniaxial tension corresponding to the yield point is, $\frac{\sigma_0}{E}$. Hence for the material to yield.

$$\sigma_1 - \nu(\sigma_2 + \sigma_3) = \sigma_0 \qquad \text{...(1.55)}$$

This theory assumes same yield strain in tension and compression, which is possible only if $(\sigma_0)_t = \nu\,(\sigma_0)_c$, where the subscripts t and c stand for tension and compression respectively. This is not the case in many materials.

3. Constant Elastic Energy of Deformation Theory (Beltrami Theory). Total strain energy of deformation is given by

$$U = \frac{1}{2E}\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

and the strain energy under simple tension is $U = \frac{\sigma_0^2}{2E}$

Hence for the material to yield,

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_0^2\,. \qquad \text{...(1.56)}$$

The total elastic energy stored in a material before it reaches the plastic state can have no significance as a limiting condition, since under high hydrostatic pressure, large amount of strain energy may be stored without causing either fracture or permanent deformation.

4. Distortion Energy Theory (Von-Mises-Hencky Theory). The energy of distortion is given by

$$U = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

Under simple tension, $U = \dfrac{\sigma_0^2}{6G}$.

At the plastic limit, the elastic energy of distortion reaches a constant value, which may be expressed as :

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2 \qquad \text{...(1.57)}$$

Also $\quad \tau_{oct} = \dfrac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$

Hence $\quad 9\tau_{oct}^2 = 2\sigma_0^2$

$\therefore \quad \tau_{oct} = \dfrac{\sqrt{2}}{3}\sigma_0 = \text{const.} = 0.471\,\sigma_0 \qquad \text{...(1.58)}$

For simple or pure shear, according to this theory, the principal stresses have the values,

$$\sigma_1 = -\sigma_2 = \frac{\sigma_0}{\sqrt{3}} = 0.577\,\sigma_0,\ \sigma_3 = 0 \qquad \text{...(1.59)}$$

For ductile materials, it expresses the equation of the limiting surface of yielding. This surface is a straight circular cylinder. Its axis coincides with the space diagonal in the positive quadrant of the σ_1, σ_2, σ_3 system of rectangular coordinates. The radius of the cylinder is equal to $\sqrt{\frac{2}{3}}\,\sigma_0$. For $\sigma_3 = 0$, we get

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_0^2 \qquad \text{...(1.60)}$$

which is called the "plasticity ellipse".

5. Maximum Shear Stress Theory (Tresca's Theory). Assuming $\sigma_1 > \sigma_2 > \sigma_3$, the maximum shear stress is,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

Also under simple tension,

$$\tau_{max} = \frac{\sigma_0}{2}$$

Hence condition of yielding, according to this theory becomes,

$$\sigma_1 - \sigma_3 = \sigma_0 \qquad \text{...(1.61)}$$

For simple or pure shear

$$\sigma_1 = -\sigma_3 = \frac{\sigma_0}{\sqrt{2}} \qquad \text{...(1.62)}$$

and $\quad \tau_{oct} = 0.5\,\sigma_0. \qquad \text{...(1.63)}$

In general form, $\sigma_1 - \sigma_3 = \pm\,\sigma_0$, $\sigma_2 - \sigma_1 = \pm\,\sigma_0$, $\sigma_3 - \sigma_2 = \pm\,\sigma_0$.

These represent a set of three parallel planes. Each set is perpendicular to one co-ordinate plane and parallel to the plane bisecting at an angle of 45°, the angle of the two co-ordinate axes which appear in the corresponding equation of the two parallel planes. These planes cut segments equal to σ_0 on the co-ordinate axes. Consequently, these intersect the plane expressed by

$$\sigma_1 + \sigma_2 + \sigma_3 = 0$$

and form a hexagon on it with apexes at the points $\sigma_1 = \pm\frac{2}{3}\sigma_0$, $\sigma_2 = \pm\frac{2}{3}\sigma_0$, $\sigma_3 = \pm\frac{2}{3}\sigma_0$, $\sigma_2 = \sigma_3 = \pm\frac{1}{3}\sigma_0$, $\sigma_3 = \sigma_1 = \pm\frac{1}{3}\sigma_0$, $\sigma_1 = \sigma_2 = \pm\frac{1}{3}\sigma_0$. Length of each side of hexagon = $\sqrt{\frac{2}{3}}\,\sigma_0$. The surface represented by this theory is a hexagonal prism. This theory assumes equal yield stress in tension and compression, which is not verified for brittle materials.

6. Mohr's theory. A material may fail either through plastic slip or by fracture when either the shearing stress τ in the planes of slip has increased to a certain value which in general will depend also on the normal stress σ acting across the same planes or when the largest tensile normal stress has reached a limiting value dependent on the properties of the material.

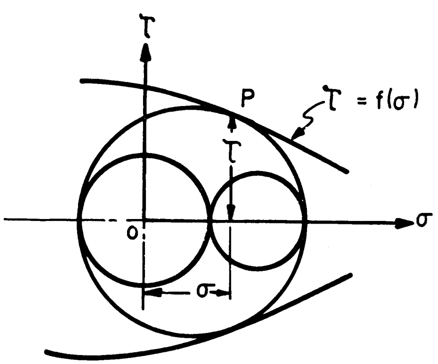

Fig. 1.4 Mohr's envelope.

Let $\tau = f(\sigma)$. The circle which is tangent to the limit curve $\tau = f(\sigma)$ at point P (Fig. 1.4) is the one representing the state of stress causing slip at the prescribed values of σ and τ. The limiting curve $\tau = f(\sigma)$ must, therefore, be the envelope of all largest principal stress circles and the points of tangency of the major stress circles along the enveloping curve $\tau = f(\sigma)$ must represent in their abscissa σ and ordinates τ, the normal and shearing stresses in the plane of slip. Now

$$\left(\sigma - \frac{\sigma_1 + \sigma_3}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2$$

Let
$$p = \frac{1}{2}(\sigma_1 + \sigma_3)$$

$$\tau_m = \frac{1}{2}(\sigma_1 - \sigma_3)$$

then
$$(\sigma - p)^2 + \tau^2 = \tau_m^2$$

This equation represents the family of major principal stress circles in parameter form. The equation of their envelope is obtained by partially differentiating with respect to p.

$$\therefore \qquad \sigma = p + \tau_m \cdot \frac{d\tau_m}{dp}$$

Hence
$$\tau = \tau_m \sqrt{1 - \left(\frac{d\tau_m}{dp}\right)^2} \qquad ...(1.64)$$

This is the equation of Mohr's envelope of the major principal stress in parameter form.

7. *Octahedral Shearing Stress Theory.* The octahedral shearing stress τ_{oct} at the limit of yielding is a function of the octahedral normal stress $(\sigma_n)_{oct}$, *i.e.*

$$\tau_{oct} = f(\sigma_n)_{oct}$$

Now
$$(\sigma_n)_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \sigma_m$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \qquad ...(1.65)$$

Therefore, this equation expresses a condition of flow in solids in which the limiting value of the shearing stress τ_{oct} in the octahedral planes depends on mean normal stress σ_m. This represents a surface of revolution in a system of rectangular co-ordinates, whose axis is equally inclined with respect to the co-ordinate axes σ_1, σ_2 and σ_3.

The theories of elastic failure for the two dimensional case have been represented graphically in Fig. 1.5.

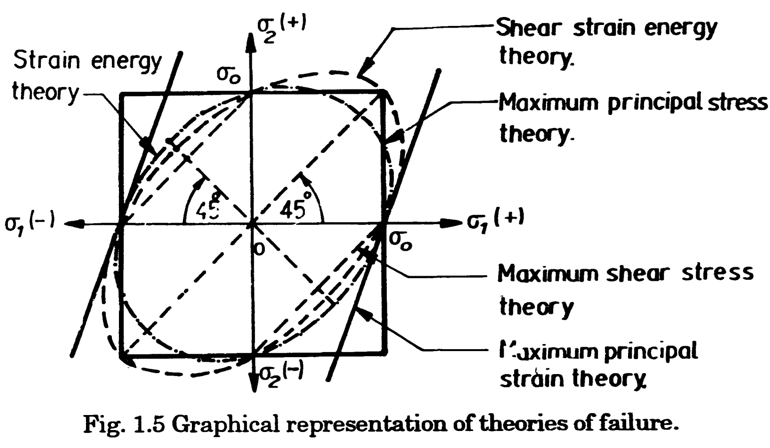

Fig. 1.5 Graphical representation of theories of failure.

EXERCISES

1.1 The stress components at a point are :

$$\sigma_x = 20,\ \sigma_y = 10,\ \sigma_z = 15,\ \tau_{xy} = \tau_{yz} = 10,$$
$$\tau_{xz} = -10 \text{ MPa}.$$

Determine the normal and shearing stresses on the octahedral plane and the direction of the shearing stress.

1.2 The stress components at a point are :

$$\sigma_x = -50,\ \sigma_y = 30,\ \sigma_z = 20,\ \tau_{xy} = -60,$$
$$\tau_{yz} = 40,\ \tau_{xz} = 50 \text{ MPa}.$$

Determine the principal stresses and the principal directions.

1.3 The state of stress at a point is given by the following array of terms in the *xyz* co-ordinate system :

$$\tau_{ij} = \begin{bmatrix} 10 & 15 & 20 \\ 15 & 25 & 15 \\ 20 & 15 & 30 \end{bmatrix} \text{MPa}$$

If this system of axes is rotated by 30° about the *z*-axis in the anti-clockwise direction, determine the new stress tensor.

1.4 The stress tensor at a point is given by :

$$\tau_{ij} = \begin{bmatrix} 60 & 40 & 50 \\ 40 & 40 & 20 \\ 50 & 20 & 80 \end{bmatrix} \text{MPa}$$

Calculate the deviatoric and spherical stress tensors.

1.5 The state of stress at a point is given by :

$$\sigma_x = x^3yz + x^2y^2,\quad \tau_{xy} = x^2yz$$
$$\sigma_y = xy^2z + yz,\quad \tau_{yz} = xy^2z$$
$$\sigma_z = x^2y^2z^2 + xz,\quad \tau_{xz} = xyz^2.$$

In the absence of the body forces, determine whether the equilibrium equations are satisfied or not at the point (3, – 4, 2).

1.6 The strain tensor at a point is given by

$$\varepsilon_{ij} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix} \times 10^{-3}$$

Calculate the deviatoric and spherical strain tensors.

1.7 The strain components at a point are given by the following array of terms :

$$\varepsilon_{ij} = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \\ 4 & -2 & 3 \end{bmatrix} \times 10^{-3}$$

Calculate the principal strains and the principal directions.

1.8 For the strain tensor given in Exercise 1.7, determine the new strain tensor by rotating the axes about the *z*-axis through 45° in the anti-clockwise direction.

1.9 The principal strains at a point are given by

$$\varepsilon_1 = 2 \times 10^{-3}, \varepsilon_2 = -3 \times 10^{-3}, \varepsilon_3 = -4 \times 10^{-3}.$$

Calculate the octahedral normal and shearing strains.

1.10 The principal stresses at a point are

$$\sigma_1 = 30, \sigma_2 = 25, \sigma_3 = 20 \text{ MPa}.$$

Calculate (*a*) strain energy, and (*b*) distortion energy. Assume $\nu = 0.3$, $E = 210$ GPa.

1.11 The strain tensor at a point in a body is given by

$$\varepsilon_x = 0.005, \varepsilon_y = 0.006, \varepsilon_z = 0.004$$

$$\gamma_{xy} = -0.002, \gamma_{yz} = 0.003, \gamma_{xz} = -0.001.$$

Calculate the stress tensor at this point assuming

$$E = 200 \text{ GPa}, G = 84 \text{ GPa}.$$

1.12 Explain the different theories of strength and give their limitations.

1.13 Derive the equilibrium equation in Cartesian coordinates.

1.14 What is the necessity of compatibility equations ? Write the compatibility equations in Cartesian coordinates.

1.15 Explain the theories of elastic failure and give their limitations.