Unit-I

Chapter 1

Analysis

Field Properties

The real number system is of all a set $\{a, b, c, ...\}$ on which the operations of addition and

multiplication are defined so that every pair of real numbers has a unique sum and product, both real numbers, with the following properties.

(i) Commutative laws: $a + b = b + a$ and $ab = ba$.

(ii) Associative laws: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

(iii) Distributive law: $a(b + c) = ab + ac$.

(iv) There are distinct real numbers 0 and 1, $a + 0 = a$ and $a1 = a$ for all a.

(v) For each a there is a real number $-a \ni a + (-a) = 0$, and if $a \neq 0$, there is a real number $1/a \ni a(1/a) = 1$.

The Order Relation

The real number system is ordered by the relation <, which has the following properties.

(i) For each pair of real numbers a and b, exactly one of the following is true: $a = b$ or $a < b$ or $b < a$.

(ii) Transitive: If $a < b$ and $b < c$, then $a < c$.

(iii) If $a < b$, then $a + c < b + c$ for any c and if $0 < c$, then $ac < bc$.

- **The Triangle Inequality:** If a and b are any two real numbers, then $|a + b| \leq |a| + |b|$.
- If a and b are any two real numbers, then $|a - b| \geq ||a| - |b||$ and $|a + b| \geq ||a| - |b||$.

Supremum of a Set

A set K of real numbers is bounded above if there is a real number $b \ni x \leq b$ whenever $x \in K$. In this case b is an upper bound of K.

If β is an upper bound of K, but no number less than β then β is a supremum of K, $\beta = \sup K$.

- **The Archimedean Property:** If a and b are positive number, then $na > b$ for some integer n.

Infimum of a Set

A set K of real numbers is bounded below if there is a real number $a \ni x \geq a$ whenever $x \in K$. In this case a is a lower bound of K.

If α is a lower bound of K, but no number greater than α, then α is an infimum of K, $\alpha = \inf K$.

- A set K is said to be bounded if it is bounded above as well as bounded below.

Open Set and Closed Set

If x_0 is a real number and $\varepsilon > 0$, then the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is an ε-neighborhood of x_0. If a set K contains an ε-neighborhood of x_0, then K is a neighborhood of x_0, and x_0 is an interior point of K.

The set of interior points of K is the interior of K, denoted by K^0. If every point of K is an interior point, then K is open.

- A set K is closed if K^c is open.

Deleted Neighborhood

A deleted neighborhood of a point x_0 is a set that contains every point of some neighborhood of x_0 except for x_0 itself.

- The union of open sets is open
- The intersection of closed sets is closed

Limit Point, Boundary Point, Isolated Point

Let K be a subset of R. Then

(i) x_0 is a limit point of K if every deleted neighborhood of x_0 contains a point of K.

(ii) x_0 is a boundary point of K if every neighborhood of x_0 contains at least one point in K and one not in K. The set of boundary points of K is the boundary of K, denoted by ∂K. The closure of K is denoted by $\overline{K}$, $\overline{K} = K \cup \partial K$

(iii) x_0 is an isolated point of K, if $x_0 \in K$ and there is a neighborhood of x_0 that contains no other point of K.

(iv) x_0 is an exterior to K, if x_0 is in the interior of K^c. The collection of such points is the exterior of K.

- *A* set is closed iff it contains all its limit points.

Dense and Perfect Sets

- A subset K of R is said to be dense in R if every point of R is a limit point of K.
- A set K is said to be dense in itself if every point of K is a limit point of K.
- A subset K of R is said to be a perfect set if it is closed and dense in itself.

Open Coverings

A collection A of open sets is an open covering of a set K if every point in K is contained in a set A belonging to A, (i.e), if $K \subset \cup \{A \mid A \in \text{A}\}$.

- **Heine–Borel Theorem:** If A is an open covering of a closed and bounded subset K of the real line, then K has an open covering $\widetilde{A}$ consisting of finitely many open sets belonging to A.
- **Bolzano–Weierstrass Theorem:** Every bounded infinite set of real numbers has at least one limit point.

Countable and Uncountable Sets

An infinite set K is said to be countably infinite or denumerable or enumerable if it is equivalent to the set N of Natural numbers. Otherwise it is uncountable.

- Countable union of countable set is countable.
- Every subset of a countable set is countable.
- The set of all rational numbers is countable.

Limit

$f(x)$ approaches the limit l as x approaches x_0 and write $\lim_{x \to x_0} f(x) = l$

- If $\lim_{x \to x_0} f(x)$ exists, then it is unique.(i.e) if $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} f(x) = l_2$, then $l_1 = l_2$.
- If $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$ then

$$\lim_{x \to x_0} (f+g)(x) = l_1 + l_2,$$

$$\lim_{x \to x_0} (f-g)(x) = l_1 - l_2$$

$$\lim_{x \to x_0} (fg)(x) = l_1 l_2$$

- and if $l_2 \neq 0$ $\lim_{x \to x_0} \left(\dfrac{f}{g}\right)(x) = \dfrac{l_1}{l_2}$.

Continuous

(i) A function g is continuous at x_0, if g is defined on an open interval (a, b) containing x_0 and
$\lim_{x \to x_0} g(x) = g(x_0)$

(ii) A function g is continuous from the left at x_0, if g is defined on an open interval (a, x_0) and
$g(x_{0^-}) = g(x_0)$

(iii) A function g is continuous from the right at x_0, if g is defined on an open interval (x_0, b) and
$g(x_{0^+}) = g(x_0)$

Piecewise Continuous

A function g is piecewise continuous on $[a, b]$ if

(i) $g(x_{0^+})$ exists for all x_0 in $[a, b)$

(ii) $g(x_{0^-})$ exists for all x_0 in $(a,b]$

(iii) $g(x_{0^+}) = g(x_{0^-}) = g(x_0)$ for all but finitely many points x_0 in (a, b).

If (iii) fails to hold at some x_0 in (a,b), g has a jump discontinuity at x_0. Also, g has a jump discontinuity at a if $g(a+) \neq g(a)$ or at b if $g(b-) \neq g(b)$

- If f and g are continuous on a set K, then so are $f+g$, $f-g$ and fg.

  f/g is continuous at each x_0 in $K \ni g(x_0) \neq 0$.

Removable Discontinuities

- Let g be defined on a deleted neighborhood of x_0 and discontinuous at x_0. A function g has a removable discontinuity at x_0 if $\lim_{x \to x_0} g(x)$ exists.

Uniform Continuity

A function g is uniformly continuous on a subset K of its domain if, $\forall\, \varepsilon > 0$, there is a $\delta > 0 \ni |g(x) - g(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$ and $x, x_0 \in K$.

- If g is continuous on a closed and bounded interval $[a, b]$, then g is uniformly continuous on $[a, b]$.
- If g is continuous on a set K, then g is uniformly continuous on any finite closed interval contained in K.
- If g is monotonic and non constant on $[a, b]$, then g is continuous on $[a,b]$ iff its range $R_g = \{g(x) | x \in [a, b]\}$ is the closed interval with endpoints $g(a)$ and $g(b)$.
- If g is continuous on a finite closed interval $[a,b]$, then g is bounded on $[a, b]$.
- **Intermediate Value Theorem:** Suppose that f is continuous on $[a, b]$, $f(\mathrm{a}) \neq f(b)$, and μ is between $f(a)$ and $f(b)$. Then $f(c) = \mu$ for some c in (a, b).

Derivative

A function f is differentiable at an interior point x_0 of its domain if the difference quotient

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, x \neq x_0 \text{ limit exists}$$

- If f is differentiable at x_0, then f is continuous at x_0.
- If f and g are differentiable at x_0, then so are $f+g$, $f-g$ and fg with

  (i) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

  (ii) $(f-g)'(x_0) = f'(x_0) - g'(x_0)$

  (iii) $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

  The quotient f/g is differentiable at x_0 if $g(x_0) \neq 0$, with

  (iv) $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{[g(x_0)]^2}$

- **The Chain Rule:** Suppose that g is differentiable at x_0 and f is differentiable at $g(x_0)$. Then the composite function $h = f \circ g$, defined by $h(x) = f(g(x))$, is differentiable at x_0, with

$$h'(x_0) = f'(g(x_0))\, g'(x_0).$$

Right- and Left-Hand Derivatives

If f is defined on $[x_0, b)$ the right-hand derivative of f at x_0 is defined to be

$$f'_+(x_0) = \lim_{x \to x_{0+}} \frac{f(x) - f(x_0)}{x - x_0} \text{ the limit exists,}$$

If f is defined on $(a, x_0]$ the left-hand derivative of f at x_0 is defined to be

$$f_-'(x_0) = \lim_{x \to x_{0-}} \frac{f(x) - f(x_0)}{x - x_0} \text{ the limit exists.}$$

Differentiable on the Closed Interval

(i) f is differentiable on the closed interval $[a,b]$ if f is differentiable on the open interval (a, b) and $f_+'(a)$ and $f_-'(b)$ both exist.

(ii) f is continuously differentiable on $[a, b]$ if f is differentiable on $[a, b]$, f' is continuous on (a, b), $f_+'(a) = f'(a+)$ and $f_-'(b) = f'(b-)$ both exist.

- **Rolle's Theorem:** Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a,b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some c in the open interval (a, b).
- **Intermediate Value Theorem for Derivatives:** Suppose that f is differentiable on $[a, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c) = \mu$ for some c in (a, b).
- **Generalized Mean Value Theorem:** If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then $[g(b) - g(a)]\, f'(c) = [f(b) - f(a)]\, g'(c)$ for some $c \in (a, b)$.
- **Mean Value Theorem:** If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a,b), then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some $c \in (a, b)$.
- If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).
- If f' exists and does not change sign on (a, b), then f is monotonic on (a, b), increasing, non decreasing, decreasing, or non increasing as $f'(x) > 0$, $f'(x) \geq 0$, $f'(x) < 0$, or $f'(x) \leq 0$, respectively, for all $x \in (a, b)$.
- If $|f'(x)| \leq M$, $a < x < b$, then $|f(x) - f(x_0)| \leq M\,|x - x_0|$, $x, x_0 \in (a, b)$.
- **L'Hospital's Rule:** Suppose that f and g are differentiable and g' has no zeros on (a, b). Let $\lim\limits_{x \to b-} f(x) = \lim\limits_{x \to b-} g(x) = 0$ or $\lim\limits_{x \to b-} f(x) = \pm\infty$ and $\lim\limits_{x \to b-} g(x) = \pm\infty$, and suppose that $\lim\limits_{x \to b-} \dfrac{f'(x)}{g'(x)} = l$ (finite or $\pm\infty$). Then $\lim\limits_{x \to b-} \dfrac{f(x)}{g(x)} = l.$

Sequence

- A function $S: N \to R$ is known as real sequence and its denoted by $\{S_n\}$.

Bounded Sequences

A sequence $\{S_n\}$ is bounded above if there is a real number $b \ni S_n \le b$ for all *n,* bounded below if there is a real number $a \ni S_n \ge a$ for all *n,* bounded if there is a real number $r \ni |S_n| \le r$ for all *n.*

Convergence of a Sequence

A sequence $\{S_n\}$ converges to a limit l if $\forall\ \varepsilon > 0$ there is an integer $m \ni |S_n - l| < \varepsilon$ if $n \ge m$.

$\{S_n\}$ is convergent if $\lim\limits_{n\to\infty} S_n = l.$

A sequence that does not converge is diverges, or is divergent.

- Every convergent sequence is bounded.
- The limit of a convergent sequence is unique.
- Every bounded sequence has a limit point.
- The set of the limit points of a bounded sequence has the greatest and the least members.
- A necessary and sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.
- A sequence $\{S_n\}$ is said to be a cauchy sequence if $\forall \varepsilon > 0 \exists\ m \in N \ni |S_{n+p} - S_n| < \varepsilon$ $\forall\ n \ge m \forall\ p \ge 1$
- **Sandwich theorem**: If $\{a_n\}$ and $\{b_n\}$ are two sequences converging to same limit l then $\{c_n\}$ is a sequence $\ni a_n \le c_n \le b_n\ \forall\ n \ge m, m \in N$, then limiting value of $c_n = l$
- **Cauchy's first theorem on limits**: If $\lim\limits_{n\to\infty} a_n = l$ then $\lim\limits_{n\to\infty}\left(\dfrac{a_1 + a_2 + \ldots + a_n}{n}\right) = l$
- If a sequence $\{a_n\}$ of positive terms converges to a positive limit l, then $\lim\limits_{n\to\infty}\left(a_1 a_2 \ldots a_n\right)^{1/n} = l$
- **Cesaro's theorem**: If the sequences $\{a_n\}$ and $\{b_n\}$ converge to finite limit a and b respectively, then $\lim\limits_{n\to\infty} \dfrac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n} = ab$
- **Cauchy's second theorem on limits:** If $\{a_n\}$ is a positive term sequence if $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists then so does $\lim\limits_{n\to\infty} (a_n)^{1/n}$, and the two limits are equal.

A sequence $\{u_n\}$ is decreasing if $u_n \ge u_{n+1}$ for all *n*, or increasing if $u_n \le u_{n+1}$ for all *n*. A monotonic sequence is a sequence that is either increasing or decreasing. If $u_n > u_{n+1}$ for all *n*, then $\{u_n\}$ is decreasing, while if $u_n < u_{n+1}$ for all *n*, $\{u_n\}$ is increasing.

- (i) If $\{u_n\}$ is increasing, then $\lim\limits_{n\to\infty} u_n = \sup\{u_n\}$.

  (ii) If $\{u_n\}$ is decreasing, then $\lim\limits_{n\to\infty} u_n = \inf\{u_n\}$.

SUBSEQUENCE OF A SEQUENCE

A sequence $\{x_k\}$ is a subsequence of a sequence $\{x_n\}$ if $x_k = x_{n_k}$, $k \geq 0$ where $\{x_{n_k}\}$ is an increasing infinite sequence of integers in the domain of $\{x_n\}$.

- If $\lim\limits_{n\to\infty} x_n = x \quad (-\infty \leq x \leq \infty)$, then $\lim\limits_{k\to\infty} x_{n_k} = x$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.
- If $\{x_n\}$ is monotonic and has a subsequence $\{x_{n_k}\} \ni \lim\limits_{k\to\infty} x_{n_k} = x \quad \{-\infty \leq x \leq \infty\}$, then $\lim\limits_{n\to\infty} x_n = x.$
- A point $\bar{x}$ is a limit point of a set S iff there is a sequence $\{x_n\}$ of points in $S \ni x_n \neq \bar{x}$ for $n \geq 1$, and $\lim\limits_{n\to\infty} x_n = \bar{x}$.
- If $\{x_n\}$ is bounded, then $\{x_n\}$ has a convergent subsequence.

SERIES

- A necessary condition for convergence of an infinite series $\sum u_n$ converges, then $\lim\limits_{n\to\infty} u_n = 0$
- **Cauchy's Convergence Criterion for Series:** A series $\sum u_n$ converges iff for every $\varepsilon > 0$ there is an integer $m \ni |u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon \ \forall n \geq m$ and $p \geq 1$.
- If a series $\sum u_n$ of positive monotonic decreasing terms converges then not only $u_n \to 0$ but also $nu_n \to 0$ as $n \to \infty$.
- The positive term geometric series $1 + r + r^2 + \ldots$ converges for $r < 1$ and diverges for $r \geq 1$.
- A positive term series $\sum \frac{1}{n^p}$ is converges iff $p > 1$
- **The Comparison Test**: If $0 \leq u_n \leq v_n \ \forall n \geq m, m \in N$, then
  (i) $\sum u_n$ is convergent if $\sum v_n$ is convergent
  (ii) $\sum v_n$ is divergent if $\sum u_n$ is divergent
- Suppose that $u_n \geq 0$ and $v_n > 0$ for $n \geq m$, and $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$ where, $0 < l < \infty$. Then $\sum u_n$ and $\sum v_n$ converge or diverge together.
- **Cauchy's Root Test** : If $u_n \geq 0$ for $n \geq m, m \in N$ such that $\lim\limits_{n\to\infty} (u_n)^{1/n} = l$ then the series
  (i) converges if $l < 1$, (ii) diverges if $l > 1$, (iii) test fails if $l = 1$
- **D'Alembert's Ratio Test:** If $u_n \geq 0$ for $n \geq m, m \in N$ such that $\lim\limits_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = l$ then the series
  (i) converges if $l < 1$, (ii) diverges if $l > 1$, (iii) test fails if $l = 1$
- **Raabe's Test:** If $u_n \geq 0$ for $n \geq m, m \in N$ such that $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ then the series
  (i) converges if $l > 1$, (ii) diverges if $l < 1$, (iii) test fails if $l = 1$

- **Logarithmic Test:** If $u_n \geq 0$ for $n \geq m, m \in N$ such that $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = l$ then the series (i) converges if $l > 1$, (ii) diverges if $l < 1$, (iii) test fails if $l = 1$
- **The Integral Test:** Let $u_n = f(n),\ n \geq a,$ where f is positive, non increasing, and locally integrable on $[a, \infty)$. Then $\sum u_n$ and $\int_a^\infty f(x)\,dx$ converges or diverge together.
- **Alternating Series Test** : The series $\sum(-1)^n u_n$ converges if $0 \leq u_{n+1} \leq u_n$ and $\lim_{n\to\infty} u_n = 0.$
- **Absolute Convergence:** A series $\sum u_n$ converges absolutely if $\sum|u_n|$ is convergent.
- **Abel's Test:** If $u_{n+1} \leq u_n$ for $n \geq N$, $\lim_{n\to\infty} u_n = 0$ and $\sum u_n$ is a convergent series, then the series $\sum u_n b_n$ is also converges.
- **Dirichlet's Test:** If b_n is a positive, monotonic decreasing function with limit zero, and if for the series $\sum u_n$, the sequence $\{S_n\}$ of partial sums is bounded then the series $\sum u_n b_n$ is convergent.

Sequence and Series of functions

Suppose that $\{f_n\}$ is a sequence of functions, defined on an interval I and the sequence of values $\{f_n(x)\}$ converges for each x in I. $\{f_n\}$ converges pointwise on I to the limit function f is defined by

$$f(x) = \lim_{n\to\infty} f_n(x), x \in S.$$

- If the series $\sum f_n$ converges for every point $x \in I$ and $f(x) = \sum_{n=0}^{\infty} f_n(x), \forall x \in [a, b]$ the function f is called the sum of the series $\sum f_n$ on $[a,b]$.

Uniform Convergence

A sequence $\{f_n\}$ is said to converge uniformly on an interval $[a,b]$ to a function f if for any $\varepsilon > 0$ and for all $x \in [a, b]$ there is an integer N (independent of x but depend on ε) such that for all $x \in [a, b]$, $|f_n(x) - f(x)| < \varepsilon,\ \forall n \geq N.$

- Uniform convergence implies pointwise convergence. Converse not true.
- **Cauchy's Uniform Convergence Criterion:** A sequence of functions $\{f_n\}$ converges uniformly on $[a,b]$ iff for each $\varepsilon > 0$ there is an integer N such that $|f_{n+p}(x) - f_n(x)| < \varepsilon$ $\forall n \geq N, p \geq 1.$
- Let $\{f_n\}$ be a sequence of functions, such that $\lim_{n\to\infty} f_n(x) = f(x), x \in [a, b]$ and let $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on $[a,b]$ iff $M_n \to 0$ as $n \to \infty$.
- **Weierstrass's Test :** The series $\sum f_n$ converges uniformly on $[a,b]$ if there exist a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a, b]$, $|f_n(x)| \leq M_n, \forall n,$
- If $\{f_n\}$ converges uniformly to f on $[a,b]$ and each f_n is continuous on $[a,b]$ then so is f. (*i.e.*,) a uniform limit of continuous functions is continuous.

- **Abel's Test:** If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of x in the interval $[a,b]$ and $b_n(x)$ is bounded for all values of n and x concerned, and if the series $\sum u_n(x)$ is uniformly convergent on $[a,b]$, then so also is the series $\sum b_n(x)u_n(x)$
- **Dirichlet's Test:** If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of x in $[a,b]$ and $b_n(x)$ tends uniformly to zero for $a \le x \le b$, and if there is a number $M>0$ independent of x and n such that for all values of x in $[a,b]$, $\left|\sum_{r=1}^{n} u_r(x)\right| \le M, \forall n$ then the series $\sum b_n(x)u_n(x)$ is uniformly convergent on $[a,b]$

Monotonic Functions

A function f is non decreasing on an interval I if

$$f(x) \le f(y) \text{ whenever } x \text{ and } y \text{ are in } I \text{ if } x < y \qquad ...(1)$$

or non-increasing on I if

$$f(x) \ge f(y) \text{ whenever } x \text{ and } y \text{ are in } I \text{ and } x < y \qquad ...(2)$$

In either case, f is monotonic on I. If $\le$ can be replaced by $<$ in (1), f is increasing on I. If $\ge$ can be replaced by $>$ in (2), f is decreasing on I. In either of these two cases, f is strictly monotonic on I.

- Suppose that g is monotonic on (a, b) and define $\alpha = \inf_{a<x<b} f(x)$ and $\beta = \sup_{a<x<b} f(x)$.

(i) If g is non decreasing, then $g(a+) = \alpha$ and $g(b-) = \beta$.

(ii) If g is non increasing, then $g(a+) = \beta$ and $g(b-) = \alpha$.

- A function $f(x)$ is said to be convex in $[x_1, x_2]$ if $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2)$, $0 \le \lambda \le 1$ in interval $[x_1, x_2]$ and $f''(x) \ge 0$.
- A function $f(x)$ is said to be concave in $[x_1, x_2]$ if $f(\lambda x_1 + (1-\lambda)x_2) \ge \lambda f(x_1) + (1-\lambda)f(x_2)$, $0 \le \lambda \le 1$ in interval $[x_1, x_2]$ and $f''(x) \le 0$.
- A point x is said to be fixed point of the curve if $f(x) = x$.

Functions of Bounded Variation

- A finite set P of points, $x_0, x_1, x_2, ..., x_n$ where $a = x_0 \le x_1 \le x_2 \le ... \le x_n = b$ is called a partition of the interval $[a, b]$.
- Corresponding to partition P, $|f(x_i) - f(x_{i-1})|$ is known as length of the i^{th} sub interval and $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ is known as the variation of the function in $[a, b]$.
- Supremum of the sum is known as variation or total variation of f on $[a, b]$. (i.e) $V(f, a, b) = \underset{P}{Sup} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.
- The function f is said to be bounded variation on $[a, b]$ iff its total variation $V(f, a, b)$ is finite.

- A bounded monotonic function is a function of bounded variation.
- The sum (difference) of two functions of bounded variation is also of bounded variation.
- The variation function of a function f of bounded variation is continuous iff f is a continuous function.

Riemann Integrable

- Corresponding to the partition P, Upper sum and Lower sum are defined as follows;
  $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$ and $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ respectively, where M and m denotes supremum and infimum of $f(x)$ in i^{th} sub interval.
- The upper and lower integral on $[a, b]$ are defined as $\overline{\int_a^b} fdx = \inf U$ and $\underline{\int_a^b} fdx = \sup L.$
- $\overline{\int_a^b} fdx = \underline{\int_a^b} fdx = \int_a^b fdx$, we say that f is Riemann integrable on $[a, b]$.
- A necessary and sufficient condition for the integrability of a bounded function f is that $\lim\{U(P, f) - L(P, f)\} = 0$ when the norm of the partition P tends to 0
- If f is bounded on $[a, b]$, the oscillation of f on $[a, b]$ is defined by
  $$W_f[a, b] = \sup_{a \le x, x' \le b} |f(x) - f(x')|$$
- A bounded function f is integrable on $[a, b]$ iff $\forall \varepsilon > 0 \, \exists$ a partition $P \ni U(P, f) - L(P, f) < \varepsilon$.
- If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.
- If f is monotonic on $[a, b]$, then f is integrable on $[a, b]$.
- A bounded function f having a finite number of points of discontinuity on $[a, b]$ is integrable on $[a, b]$.
- If f and g are integrable on $[a, b]$, then so is $f + g$
- If f is integrable on $[a, b]$ and m is a constant, then mf is integrable on $[a, b]$.
- If f and g are integrable on $[a, b]$ and $f(x) \le g(x)$ for $a \le x \le b$, then $\int_a^b f(x)dx \le \int_a^b g(x)\, dx.$
- If f is integrable on $[a, b]$, then so is $|f|$.
- If f and g are integrable on $[a, b]$, then so is the product fg.
- **Fundamental Theorem of Calculus:** If f is continuous on $[a, b]$, then f has an antiderivative on $[a, b]$. If F is any anti derivative of f on $[a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a).$

Improper Integrals

- If f is locally integrable on $[a, b)$, define $\int_a^b f(x)\, dx = \lim_{c \to b-} \int_a^c f(x)\, dx$ provided that the limit exists (finite).

- If f is locally integrable on $(a, b]$, define $\int_a^b f(x)\,dx = \lim_{c \to a+} \int_c^b f(x)\,dx$ provided that the limit exists (finite).
- **Comparison Test:** If h and g are locally integrable on $[a, b)$ and $0 \le h(x) \le g(x)$, $a \le x < b$, then

  (i) $\int_a^b h(x)dx < \infty$ if $\int_a^b g(x)\,dx < \infty$ and

  (ii) $\int_a^b g(x)\,dx = \infty$ if $\int_a^b h(x)dx = \infty$.

Absolutely Integrable

- g is absolutely integrable on $[a, b)$ if g is locally integrable on $[a, b)$ and $\int_a^b |g(x)|\,dx < \infty$ then $\int_a^b g(x)\,dx$ converges absolutely, or is absolutely convergent.
- If g is locally integrable on $[a, b)$ and $\int_a^b |g(x)|\,dx < \infty$, then $\int_a^b g(x)\,dx$ converges, (i.e) an absolutely convergent integral is convergent.
- **Dirichlet's Test:** Suppose that h is continuous and its antiderivative $F(x) = \int_a^x h(t)\,dt$ is bounded on $[a, b)$. Let g' be absolutely integrable on $[a, b)$, and suppose that $\lim_{x \to b-} g(x) = 0$. Then $\int_a^b h(x)\,g(x)\,dx$ converges.

Lebesgue Measure

- If $A \subseteq R$ then outer measure of A, $m^*(A) = \inf \sum_{i=1}^{n} l(A_i)$ where each A_i are open set and union of A_i contains set A (i.e) $A \subseteq \bigcup_{i=1}^{n} A_i$
- If $A \subseteq R$ then inner measure of A, $m_*(A) = \sup \sum_{i=1}^{n} l(A_i)$ where each A_i are closed set $A \supseteq \bigcup_{i=1}^{n} A_i$ and A_i's are non overlapping.
- A set $A \subseteq R$ is measurable if $m^*(A) = m_*(A) = m(A)$
- A singleton set is measurable with measure zero.
- Every countable set is measurable with measure zero.
- Corresponding to the partition P, Upper and Lower Lebesgue sum are defined as follows; $U(P, f) = \sum_{i=1}^{n} \sup_{A_i} f(x) m(A_i)$ and $L(P, f) = \sum_{i=1}^{n} \inf_{A_i} f(x) m(A_i)$ respectively.

- The upper and lower Lebesgue integral on $[a, b]$ are defined as $L\overline{\int_a^b} fdx = \inf U(P, f)$ and $L\underline{\int_a^b} fdx = \sup L(P, f)$.
- $L\overline{\int_a^b} fdx = L\underline{\int_a^b} fdx = L\int_a^b fdx$, we say that f is Lebesgue integrable on $[a, b]$.
- Every bounded Riemann integrable function over $[a, b]$ is Lebesgue integrable.
- A bounded function f is integrable on a finite interval $[a, b]$ iff the set A of discontinuities of f in $[a, b]$ is of Lebesgue measure zero.
- **Bounded Convergence Theorem:** Let $\{f_n\}$ be a sequence of functions measurable on a measurable subset $A \subseteq [a, b] \ni \lim_{n\to\infty} f_n(x) = f(x)$. Then if $\exists$ a constant $M \ni |f_n(x)| \le M, \forall n$ and for all x, we have $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$
- **Monotone Convergence Theorem:** Let A be any measurable subset of $[a, b]$ with finite measure. Let $\{f_n\}$ be a sequence of measurable functions such that for $x \in A$, $0 \le f_1(x) \le f_2(x) \le \ldots \le f_n(x) \le \ldots$ if $\lim_{n\to\infty} f_n(x) = f(x)$, then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.
- **Classical Lebesgue dominated Convergence Theorem:** Let $\{f_n\}$ be a sequence of measurable functions on $[a, b]$ $\ni \lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere on $[a, b]$. If $\exists$ a Lebesgue integrable function g on $[a, b]$ such that for each $n \in N$, $|f_n(x)| \le g(x)$ almost everywhere on $[a, b]$. Then f is Lebesgue integrable and $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Power Series

- An infinite series of the form $\sum_{n=0}^{\infty} a_n(x - x_0)^n$.where, x_0 and $a_0, a_1, \ldots,$ are constants, is called a power series in $x - x_0$.
- The radius of convergence of $\sum a_n(x - x_0)^n$ is given by $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ if the limit exists in the extended reals.
- If $g(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, |x - x_0| < R$, then, $a_n = \frac{g^{(n)}(x_0)}{n!}$.

- **Uniqueness of Power Series :** If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n$ for all x in some interval $(x_0 - r, x_0 + r)$, then $a_n = b_n, n \geq 0$.

FUNCTIONS OF SEVERAL VARIABLES

If $x_1, x_2, \ldots, x_n$ are n independent variables and Z is the only dependent variable then $Z = f(x_1, x_2, \ldots, x_n)$ are explicit functions of n independent variables $x_1, x_2, \ldots, x_n$

- The set of values x_1, y_1 other than a, b that satisfy the conditions $|x_1 - a| < \delta, |y_1 - b| < \delta$, were δ is an arbitrarily small positive number, is said to form a neighbourhood of the point (a, b).
- A point (ξ, η) is called a limit point of condensation of a set of points S, if for every neighbourhood of (ξ, η) contains an infinite number of points of S. The limit point itself may or may not be a point of the set.
- A function f tends to a limit l, when (x, y) tends to (a, b) if for every positive number ε, corresponds a neighbourhood N of $(a, b) \ni |f(x, y) - l| < \varepsilon$, for every point (x, y) other than (a, b) of the neighbourhood N.

  (i.e) $\lim_{(x,y)\to(a,b)} f(x, y) = l.$
- If a function f is defined in some neighbourhood of (a, b), then the limit $\lim_{y \to b} f(x, y)$, if it exists, is a function of x, say $\phi(x)$. If then the limit $\lim_{x\to a} \phi(x)$ exists and is equal to λ, we write $\lim_{x\to a} \lim_{y \to b} f(x, y) = \lambda$ and say that λ is a repeated limit of f as $y \to b, x \to a$. If we change the order of taking the limits, we get the other repeated limit $\lim_{y\to b} \lim_{x\to a} f(x, y) = \lambda'$ when first $x \to a$, and then $y \to b$. These two limits may or may not be equal.
- A function f is said to be continuous at a point (a, b) of its domain of definition, if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

- Partial derivative of $f(x, y)$ with respect to x and y are denoted by $\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$ and $\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$
- A sufficient condition that a function f be continuous at (a, b) is that one of the partial derivatives exists and is bounded in a neighbourhood of (a, b) and that the other exists at (a, b).
- The function f is said to be differentiable at (x, y) if the change δf can be expressed in the form $\delta f = A\delta x + B\delta y + \delta x \phi(\delta x, \delta y) + \delta y \psi(\delta x, \delta y)$, where A and B are constants independent of $\delta x, \delta y$ and ϕ, ψ are functions of $\delta x, \delta y$ tending to zero as $\delta x, \delta y$ tend to zero simultaneously.

- If (a, b) be a point of the domain of definition of a function f such that f_x is continuous at (a, b), f_y exists at (a, b) then f is differentiable at (a, b).

JACOBIANS

- If $u_1, u_2, \ldots, u_n$ be n differentiable functions of n variables $x_1, x_2, \ldots, x_n$ then the determinant

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian or the Functional determinant of the functions $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ and is denoted by $\dfrac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}$ or $J\left(\dfrac{u_1, u_2, \ldots, u_n}{x_1, x_2, \ldots, x_n}\right)$

METRIC SPACE

A metric space is a nonempty set X together with a real-valued function d defined on $X \times X \to R$ э if x, y and z are arbitrary members of X, then

(i) $d(x, y) \geq 0$, with equality iff $x = y$,

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

d is a metric on X.

NORMED VECTOR SPACE

A normed vector space is a vector space X together with a real-valued function N defined on X э if x and y are arbitrary vectors in X and a is a real number, then

(i) $N(x) \geq 0$ with equality iff $x = 0$

(ii) $N(ax) = |a| N(x)$

(iii) $N(x + y) \leq N(x) + N(y)$.

N is a norm on X, and (X, N) is a normed vector space.

- If (X, N) is a normed vector space, then $d(x, y) = N(x - y)$ is a metric on X.

METRIC ON R^N

If $p \geq 1$ and $X = (x_1, x_2, \ldots, x_n)$, let $\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

- The metric induced on R^n by this norm is $d_p(X, Y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$

Open and closed Spheres

- Let (X, d) be any metric spaces, and $a \in X$. Then for any $r > 0$, the set $S_r(a) = \{x \in X : d(x, a) < r\}$ is called an open sphere of radius r centered at a.
- The set $S_r(a) = \{x \in X : d(x, a) \leq r\}$ is called a closed sphere of radius r centered at a.
- Let (X, d) be any metric spaces, and $a \in X$. A subset N_a of X is called a neighbourhood of a point $a \in X$, if there exists an open sphere $S_r(a)$ centered at a and contained in N_a.(i.e) $S_r(a) \subseteq N_a$, for some $r > 0$
- A subset A of a metric space (X, d) is said to be open in X with respect to the metric d, if A is a neighbourhood of each of its points.
- Let A be any subset of a metric space (X, d). A point a of X is called an adherent pint of A, if every open sphere centered at a contains a point of A.
- An adherent point a of a subset A of X is called an isolated point if every open sphere centered at a contains no point of A other than a itself.
- An adherent point a of a subset A of X is said to be limit point of A if every open sphere centered at a contains at least one member of A other than a.
- The set of all limit pints of A is called the derived set of A and is denoted by A'.
- A subset A of a metric space (X, d) is said to be closed if A contains all its limit points.

Interior, Exterior, Frontier and Boundary Points

- Let A be any subset of a metric space (X, d). A point a in A is an interior point of A if there exists $r > 0 \ni a \in S_r(a) \subseteq A$.
- A point $x \in X$ is said to be an exterior of A, if it is an interior point of the complement of A.
- A point $x \in X$ is said to be a frontier of $A \subseteq X$ if it is neither an interior nor an exterior point of A.
- If the frontier point belongs to A then it is called a boundary pint of A.
- A subset A of a metric space (X, d) is said to be dense in X, if the closure of A is X.
- A metric space X is said to be separable if there is a countable subset of X, which is dense in X.

Convergence

A sequence $\{x_n\}$ in a metric space (X, d) converges to $x \in X$ if $\lim_{n \to \infty} d(x_n, x) = 0$.
$(i.e.)$ $\lim_{n \to \infty} x_n = x$.

- The limit of a convergent sequence is unique

- If $\lim_{n\to\infty} x_n = x$ then every subsequence of $\{x_n\}$ converges to x.

Cauchy Sequence

A sequence $\{x_n\}$ in a metric space (X, d) is a Cauchy sequence if for every $\varepsilon > 0$ there is an integer

$$N \ni d(x_n, x_m) < \varepsilon \quad \text{and} \quad m, n > N.$$

- If a sequence $\{x_n\}$ in a metric space (X, d) is convergent; then it is a Cauchy sequence.
- **Complete**: A metric space (X, d) is complete if every Cauchy sequence in X has a limit.

Bounded

The diameter of a nonempty subset S of X is $d(S) = \sup\{d(x, y) \mid x, y \in S\}$. If $d(S) < \infty$ then S is bounded.

- A set T is compact if it has the Heine-Borel property.
- An infinite subset T of X is compact iff every infinite subset of T has a limit point in T.
- A subset T of a metric X is compact iff every infinite sequence $\{t_n\}$ of members of T has a subsequence that converges to a member of T
- If T is compact then every Cauchy sequence $\{t_n\}_{n=1}^{\infty}$ in T converges to a limit in T.
- If T is compact, then T is closed and bounded.
- If T is compact, then T is totally bounded.
- If (X, d) is complete and T is closed and totally bounded, then T is compact.
- Let (X, d) and (Y, d') be any two metric spaces. A function $f: X \to Y$ is said to be homeomorphism if f is both one-one and onto, f and f^{-1} are both continuous.

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SOLVED EXAMPLES

Ex.1. Let us consider the following subsets of R^2

(i) The set of complex $Z \ni |Z| < 1$ is open and bounded.

(ii) The set of complex $Z \ni |Z| \le 1$ is closed, perfect and bounded.

(iii) A non empty finite set is closed and bounded.

(iv) The set of all integers are closed.

(v) The set consisting of the numbers $\frac{1}{n}$ $(n = 1, 2, \ldots)$. This set has a limit point and bounded.

(vi) The segment (a, b) is bounded.

Ex.2. Find the infimum and supremum of a function $f : (0,1) \to R$ defined by $f(x) = \frac{1}{x}$

Sol: Consider a function $f : (0,1) \to R \ni f(x) = \frac{1}{x}$. The range of f is $(1, \infty)$.

$\therefore$ f is not bounded above on $(0,1)$, but it is bounded below. f does not have a maximum or a minimum on $(0,1)$. However, $\inf f = 0$. If we replace $(0,1)$ by $(0,1]$ then the range of f is $[1, \infty)$. So f is bounded below and has a minimum which is also the infimum equal to 0. But f is not bounded above.

Ex.3. Consider the power series $\sum_{n=1}^{\infty} \frac{1}{2^n n} x^n$

(i) For which values of $x \in R$ does the series converge (absolutely? conditionally?)

Sol: (i) The ratio of the absolute value of successive terms is $\frac{n}{n+1} \frac{|x|}{2} \to \frac{|x|}{2}$ as $n \to \infty$.

Hence the series converges absolutely for $|x| < 2$. If $x = +2$ the series is the harmonic series, $\sum \frac{1}{n}$ which diverges by comparison or the integral test. If $x = -2$ the series converges, but only conditionally, by the Leibnitz test.

Ex.4. Determine the function $f(x) = \begin{cases} 3x - 2 & if\ x < 1 \\ x^3 & if\ x \ge 1 \end{cases}$ is differentiable at $x = 1$. Find the derivative.

Sol: $f(1) = 1^3 = 1$

The limit from the right is

$$\lim_{x \to 1^+} \frac{f(x) - (1)}{x - 1} = \lim_{x \to 1^+} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1^+} \frac{(x-1)(x^2 + x + 1)}{x - 1}$$

$$= \lim_{x \to 1^+} (x^2 + x + 1) = 3$$

The limit from the left is

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{3x-2-1}{x-1}=\lim_{x\to 1^-}3=3$$

Since, the limt from the left equals the limit from the right, the limit exists and is equal to 3. So $f'(1)=3$.

Ex.5. Find $\lim_{x\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{2n}\right\}$

Sol: Divide the interval $[1,2]$ into n equal pieces. The right hand Riemann sum for the function

$f(x)=\frac{1}{x}$ is $\frac{1}{n}\left\{\frac{n}{n+1}+\frac{n}{n+2}+....\frac{1}{2}\right\}$ and

$\therefore$ *f(x)* is continuous, hence integrable the sum converges to $\int_1^2\frac{1}{x}dx=\log 2$

Ex.6. Is the series $\sum_{n=1}^{\infty}\frac{nx}{1+n^4x^2}$ converges uniformly on $[a,\infty)$ for $a>0$

Sol:
$$\frac{nx}{1+n^4x^2}=\frac{1}{x}.\frac{1}{1/x^2+n^4}\le\frac{1}{a}.\frac{n}{n^4}=\frac{1}{a}.\frac{1}{n^3}$$

As $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges by the p - test, $\sum_{n=1}^{\infty}\frac{nx}{1+n^4x^2}$ converges by the Comparison test.

Ex.7. Is $\int_{\pi}^{\infty}\left|\frac{\sin x}{x}\right|dx$ converges

Sol: We approximate $\left|\frac{\sin x}{x}\right|$ from below by a step function ϕ. For $n>0$ an odd integer $\left|\sin(n\pi/2)\right|=1$

and
$$\left|x-n\pi/2\right|<\frac{\pi}{6}\Rightarrow\left|\sin x\right|>\frac{1}{2}\quad\Rightarrow\quad\left|\frac{\sin x}{x}\right|\ge\frac{1}{(n+1)\pi}$$

Define
$$\phi(x)=\begin{cases}\dfrac{1}{(n+1)\pi} & \text{for } n \text{ odd } \dfrac{n\pi}{2}-\dfrac{\pi}{6}\le x\le\dfrac{n\pi}{2}+\dfrac{\pi}{6}\\ 0 & \text{otherwise}\end{cases}$$

Then,
$$\int_{\pi}^{\infty}\left|\frac{\sin x}{x}\right|dx\ge\int_{\pi}^{\infty}\phi(x)dx\ \ge\sum_{k=1}^{\infty}\frac{2\pi}{6}\frac{1}{(2k+1+1)\pi}=\infty$$

$\therefore\ \int_{\pi}^{\infty}\left|\frac{\sin x}{x}\right|dx$ is diverges

Ex. 8. Let $f_n:R\to R$ be the function defined by $f_n(x)=\sum_{k=1}^{n}\frac{1}{2k^2+k\cos(kx)}$

(i) Prove that f_n converges uniformly to a continuous function $f : R \to R$
(ii) Prove that f is differentiable

Sol: (i) Note that $0 < \dfrac{1}{2k^2 + k\cos(kx)} \le \dfrac{1}{k^2}$ for all $x \in R$ and all $k \ge 1$.

$\therefore$ The series $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$ converges it follows from the Weierstrass M - test that the sequence f_n converges uniformly on R to a fuction f.

$\therefore$ Every f_n is continuous, so is f.

(ii)
$$\left.\frac{d}{dx}\right|_x \frac{1}{(2k^2 + k\cos(kx))} = \frac{\sin(kx)}{(2k + k\cos(kx))^2}$$

and that the absolute value of the term on the right hand side is bounded by $\dfrac{1}{k^2}$ for all $x \in R$ and all $k \ge 1$. It follows from the Weierstrass M- test that the sequence f_n' converges uniformly to a function $g : R \to R$. $\because$ Every f_n' is continuous, so is g. It follows from this that f is differentiable with derivative $g(x)$ at x.

Ex.9. Is the following functions Riemann integrable on [0, 1]?

$$f(x) = \begin{cases} x, & x \in Q \\ 0 & x \notin Q \end{cases}$$

Sol: No, on any interval $[a,b] \subset [0,1]$. $\inf\{f(x) : a \le x \le b\} = 0$ and $\sup\{f(x) : a \le x \le b\} = b > 0$.

$\therefore$ All lower sums $L(f, P) = 0$ and all upper sums are the same as for the integrable function $g(x) = x$, hence $\inf\{U(f,P) : P\} = \dfrac{1}{2}$. Therefore f is not integrable.

Ex.10. Define the sequence $\{a_n\}_{n=0}^{\infty}$ of numbers as follows. Let $a_0 = 2$ for $n > 0$, define a_n by

$$a_n = \frac{1}{2} + \frac{a_{n-1}}{2}$$

(i) Show that $1 < a_n < a_{n-1}$ for all n.

(ii) Show that the sequence $\{a_n\}_{n=0}^{\infty}$ is convergent and find its limit.

Sol:

(i) By induction assume $a_{n-1} > 1$. Then a_n is the sum of 1/2 and a number greater than 1/2 and thus is greater than 1. The inequality $a_n < a_{n-1}$ can be rewritten as $\dfrac{1}{2} < \dfrac{a_{n-1}}{2}$ or $a_{n-1} > 1$ which again holds by the induction hypothesis.

(ii) By completeness of the reals, a decreasing sequence of numbers bounded below has a limt L.

Taking limits in both sides of the recursion $a_n = \frac{1}{2} + \frac{a_{n-1}}{2}$ as $n \to \infty$ gives $L = \frac{1}{2} + \frac{L}{2}$ which is solved by $L = 1$.

Ex.11. Consider the series $f(x) = \sum_{k=1}^{\infty} \frac{e^{-kx}}{k}$

(i) For what values of $x \in R$ does it converge?

Sol: (i) The ratio of the absolute value of the successive terms is $\frac{k}{k+1} e^{-x}$ which tends to e^{-x} as $k \to \infty$. Hence the series converges absolutely for $x > 0$ and diverges for $x < 0$.

For $x = 0$ the series is the harmonic series which diverges.

Ex.12. Consider the sequence $f_n(x) = \sqrt{x^2 + \left(\frac{1}{n}\right)}$

(i) Prove the sequence $f_n(x)$ converges uniformly to $f(x) = |x|$ on *R*.

(ii) Prove $f_n'(x)$ converges pointwise on *R* and that for $x \neq 0$, $\lim_{n\to\infty} f_n'(x) = f'(x)$

Sol:

(i) Both f_n and f are even functions. Let $g(x) = f_n(x) - f(x)$. For $x > 0$ we have

$$g'(x) = \frac{x}{\sqrt{x^2 + \left(\frac{1}{n}\right)}} - 1 < 0$$

hence g is decreasing on $(0, \infty)$. Thus $|f_n(x) - f(x)| \leq 1/\sqrt{n}$,

So $f_n(x)$ converges uniformly to f.

(ii) For $x \neq 0$

$$\lim_{n\to\infty} \frac{x}{\sqrt{x^2 + \left(1/n\right)}} = \frac{x}{\sqrt{x^2 + \lim\left(1/n\right)}} = \frac{x}{|x|} = f'(x)$$

So $f_n(x)$ converges pointwise to $f(x)$ for $x \neq 0$.

Ex.13. Show that the function $f(x) = \frac{1}{1+x^2}$ is uniformly continuous on *R*.

Sol: Let $\varepsilon > 0$ be given. Without loss of generality, we may assume that $\varepsilon < 2$. Then for $M = \sqrt{\frac{2}{\varepsilon} - 1}$

We have $\left|\frac{1}{1+x^2}\right| \leq \frac{1}{1+M^2} < \frac{\varepsilon}{2}$ Hence for all $|x|, |y| > M$

We have $\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| \leq \left|\frac{1}{1+x^2}\right| + \left|\frac{1}{1+y^2}\right| < \frac{2}{1+M^2} < \varepsilon$.

Now , since *f* is continuous on *R*, it is continuous and thus uniformly continuous on [- *M* - 1, *M* + 1].

For above $\varepsilon > 0$, $\exists$ $\delta_1 > 0$ $\ni$ for all $x, y \in [-M-1, M+1]$, we have $|f(x) - f(y)| < \varepsilon$. Then for $\delta = \min\{1, \delta_1\} > 0$, when $x, y \in R$ $\ni$ $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

Therefore $f(x) = \dfrac{1}{1+x^2}$ is uniformly continuous on *R*.

Ex.14. Find the radius of convergence *R* of each of the power series. Discuss the convergence of the power series at the points $|x - t| = R$.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n}$

Sol: (i)

$$\lim_{n\to\infty} \left| \frac{\dfrac{x^{2(n+1)-1}}{(2(n+1)-1)!}}{\dfrac{x^{2n-1}}{(2n-1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+1)(2n)} \right| = 0$$

So, by the ratio test, the series converges absolutely for all *x*, and so $R = \infty$.

(ii)

$$\lim_{n\to\infty} \left| \frac{\dfrac{(n+1)(x-1)^{n+1}}{2^{n+1}}}{\dfrac{n(x-1)^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \, \frac{x-1}{2} \right| = \left| \frac{x-1}{2} \right|$$

So, by the ratio test, the series converges absolutely if $|x-1| < 2$ and diverges if $|x-1| > 2$.

Thus, $R = 2$. If $|x-1| = 2$ then the power series diverges.

Ex.15. Prove the following series converge conditionally

(i) $1 + \dfrac{1}{\sqrt{2}} - \dfrac{2}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \dfrac{1}{\sqrt{5}} - \dfrac{2}{\sqrt{6}} + \ldots.$ (ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^{(1-n)/n}$

Sol: (i) Let $\{a_n\}$ be the sequence $1, 1, -2, 1, 1, -2, \ldots..$

Let $b_n = \dfrac{1}{\sqrt{n}}$. Note that the sequence of partial sums of $\displaystyle\sum_{n=1}^{\infty} a_n$ is bounded and $\{b_n\}$ is a decreasing sequence with limit 0.

By Dirichlet's Test $1 + \dfrac{1}{\sqrt{2}} - \dfrac{2}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \dfrac{1}{\sqrt{5}} - \dfrac{2}{\sqrt{6}} + \ldots. = \displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges.

(ii) Let $a_n = (-1)^n n^{-1}$ and Let $b_n = n^{\frac{1}{n}}$. Note that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges by Alternating series Test and that $\{b_n\}$ is a bounded monotone sequence for *n* sufficiently large. We apply Abel's Test

$$\sum_{n=1}^{\infty}(-1)^n n^{(1-n)/n} = \sum_{n=1}^{\infty}(-1)^n n^{-1} n^{\frac{1}{n}} = \sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

Ex.16. (i) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$

(ii) Prove that $f(x) = x^3$ is not uniformly continuous on R.

Sol: (i) Given $\varepsilon > 0$ let $\delta = \varepsilon^2 . \left|\sqrt{x} - \sqrt{y}\right| \le \left|\sqrt{x} + \sqrt{y}\right|$.

Hence, if $|x - y| < \delta = \varepsilon^2$ then we have $\left|\sqrt{x} - \sqrt{y}\right|^2 \le \left|\sqrt{x} - \sqrt{y}\right|\left|\sqrt{x} + \sqrt{y}\right| = |x - y| < \varepsilon^2$

Hence, $\left|\sqrt{x} - \sqrt{y}\right| < \varepsilon$. This shows that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

(ii) Pick $\varepsilon = 1$. Given any $\delta > 0$ let $x > 0 \ni \dfrac{3\delta x^2}{2} > 1$. Then $d\left(x + \delta/2, x\right) < \delta$ but we have

$$d\left(f\left(x + \delta/2\right), f(x)\right) = \left|\left(x + \delta/2\right)^3 - x^3\right| = \left|\frac{3\delta x^2}{2} + \frac{3\delta^2 x}{2^2} + \frac{\delta^3}{2^3}\right| \ge \frac{3\delta x^2}{2} > 1$$

This shows that $f(x) = x^3$ is not uniformly continuous.

Ex.17. Is the set $Q(\sqrt{2}) = \{a + b\sqrt{2} : \ a, b \in Q\}$ countable or uncountable?

Sol: The set of rational numbers is countable. Hence for any given *a*, the set $\left\{a + b\sqrt{2} : b \in Q\right\}$ is also countable. The set $Q\left(\sqrt{2}\right)$ is the union over all rational numbers *a* of the countable set $\left\{a + b\sqrt{2} : b \in Q\right\}$. Since the countable union of countable set is countable. We conclude that $Q\left(\sqrt{2}\right)$ is countable.

Ex.18. Let $f(x) = \begin{cases} 1 - x^2 & \text{if} \quad 0 \le x < 1 \\ \dfrac{(1+x)}{2} & \text{if} \quad 1 \le x < 2 \end{cases}$. Show that f is not continuous in an interval $[0, 2]$.

Sol: Consider that $\lim_{x \to 1^-} f(x) = 0$ and $\lim_{x \to 1^+} f(x) = 3/2 \quad \therefore \lim_{x \to 1} f(x)$ does not exists. Hence f is not continuous at $x = 1$. Hence f is not continuous on $[0, 2]$.

Ex.19. Determine which of the following series converge.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$ (ii) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{[n(\log n)(\log\log n)]}$

Sol: (i) The integral $\int \frac{1}{\sqrt{x}\log x}dx$ is equalent to $\int \frac{1}{\log u}du$ and $\int \frac{e^{\omega}}{\omega}\,d\omega$. Which do not have elementary antiderivatives (i.e, the antiderivative is not expressible as an algebraic combination of polynomials, exponentials, logarithms, trig functions and inverse trig functions). So using the integral test would be problematic. So, using comparison test $\sqrt{n} > \log n$ for all $n > 0$.

Hence, $\frac{1}{\log n} > \frac{1}{\sqrt{n}}$ and so $\frac{1}{\sqrt{n}\log n} > \frac{1}{\sqrt{n}\sqrt{n}} = \frac{1}{n} > 0.$

Since, $\sum \frac{1}{n}$ diverges. $\therefore \sum \frac{1}{\sqrt{n}\log n}$ diverges.

(ii) We can integrate $\int \frac{1}{x\log x\log(\log x)}dx = \log(\log(\log x)) + c$

Notice that $\sum_{n=4}^{\infty} \frac{1}{n\log n\log(\log n)} > \int_4^{\infty} \frac{1}{x\log x\log(\log x)}dx = \lim_{b\to\infty} \log\log\log b - \log\log\log 4 = \infty$

Hence, $\sum_{n=4}^{\infty} \frac{1}{n\log n\log(\log n)}$ diverges.

Ex.20. Is the following functions uniformly continuous?

(i) $f(x) = 3x + 11$ on R (ii) $f(x) = x^2$ on $[0,3]$

Sol: (i) Given $\varepsilon > 0$, let $\delta = \varepsilon/3$. Then $|x - y| < \delta = \frac{\varepsilon}{3}$

$\Rightarrow 3|x-y| = |3x - 3y| = |3x + 11 - (3y + 11)| = |f(x) - f(y)| < \varepsilon$. So, the function $f(x) = 3x + 11$ on R is uniformly continuous.

(ii) Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{6}$ Then $|x - y| < \delta = \frac{\varepsilon}{6} \Rightarrow |x^2 - y^2| = |x-y||x+y| \le |x-y|.6 < \frac{\varepsilon}{6}(6) = \varepsilon$.

So the function is uniformly continuous on $[0,3]$.

Ex.21. Find the interior of each set

(i) $[0,3] \cup (3,5)$ (ii) $\{r \in Q:\ 0 < r < \sqrt{2}\}$ (iii) $\{r \in Q:\ r \ge \sqrt{2}\}$

Sol: (i) $[0,3] \cup (3,5)$. The interior is $(0,5)$.

(ii) $\{r \in Q:\ 0 < r < \sqrt{2}\}$. The interior is ϕ. Since the irrationals are dense in *R*.

(iii) $\{r \in Q:\ r \ge \sqrt{2}\}$. The interior is ϕ.

Ex.22. Determine which of the following subsets of R is compact.

(i) $[1,3)$ (ii) $\{x \in Q : 0 \le x \le 2\}$

Sol: (i) Let $f = \{A_n\}$, where $A_n = \left(0, 3 - \frac{1}{n}\right)$ for $n \in N$. Then f is open cover for $[1,3)$ which has no finite subcover. So $[1,3)$ is not compact.

(ii) Because the rationals are dense in R. We can construct an infinite collection of rational points that get close to $\sqrt{2}$, but never reach it. Let $x_1 = \frac{1}{2}$, $\because$ x_1 & $\sqrt{2}$ are real numbers, there is a rational number between them, call it x_2. Now, since $x_2 \in R$, there is some rational number between x_2 and $\sqrt{2}$, call it x_3. Continue to get $x_1 < x_2 < x_3 ...... < x_n < .....\sqrt{2}$.

Now Let $A_n = (1, x_n)$. Then $J = \{A_n\}$ is an open cover of our set, but it has no finit subcover.

$\{x \in Q : \ 0 \le x \le 2\}$ is not compact.

Ex.23. Determine whether each series converges conditionally, converges absolutely or diverges.

(i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log n}$ (ii) $\sum \frac{(-2)^n}{n^2}$

Sol: (i) $a_n = \frac{(-1)^n}{\log n}$ so $|a_n| = \frac{1}{\log n}$. Notice that $\log n < n$, So $\frac{1}{\log n} > \frac{1}{n}$. Thus by the comparison test, $\sum |a_n|$ diverges. On the other hand $\lim |a_n| = 0$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log n}$ converges conditionally.

(ii) $a_n = \frac{(-2)^n}{n^2}$ and $\lim |a_n| = \lim \frac{2^n}{n^2} == \lim \frac{2^n \log 2}{2n} = \lim \frac{2^n \log 2}{2} = \infty$ .Thus the series diverges.

Ex.24. Find the radius of convergence R and the interval of convergence C for each series.

(i) $\sum \frac{n^2}{2^n} x^n$

Sol: $R = \lim \frac{n^2}{2^n} \frac{2^{n+1}}{(n+1)^2} = \lim \frac{2n^2}{(n+1)^2} = 2$.

At $x = -2$ our series is $\sum \frac{n^2}{2^n} (-2)^n = \sum (-1)^n n^2$ which diverges.

At $x = 2$ our series is $\sum \frac{n^2}{2^n} (2)^n = \sum n^2$ Which diverges. Thus $C = (-2, 2)$.

OBJECTIVE TYPE QUESTIONS (PART-B)

1. $f_n = \sum_n \frac{\sin(nt)}{n^2}$

$g_n = \sum_n (-1)^n 2^{-n^2}$

(a) f_n converges absoluted but g_n is not converges

(b) g_n converges absoluted but f_n is not converges

(c) both f_n and g_n are converges absoluted.

(d) both f_n and g_n are diverges.

2. Let M be a compact subset of a complete metric space (X, d). Then

(a) $x \to dist(x, M) = \inf_{y \in M} d(x, y)$ is a continuous function on X

(b) $x \to dist(x, M) = d(x, y)$ is not a continuous function on X

(c) $x \to dist(x, M) = d(x, y)$ is a continuous function on X

(d) $x \to dist(x, M) = \inf_{y \in M} d(x, y)$ is not a continuous function on X

3. Let h and j be differentiable functions on an open interval I. Suppose that $a, b \in I$ satisfy $a < b$ and $h(a) = h(b) = 0$. Then

(a) $h'(x) + h(x) j'(x) = 0$ for some $x \in (a, b)$

(b) $j(x) + h(x) j'(x)$ for some $x \in [a, b)$

(c) $h'(x) + j'(x)$ for all $x \in [a, b]$

(d) $j'(x) + h'(x) j(x) = 0$ for some $x \in [a, b]$

4. Suppose that g is integrable on $[a, b]$ and that there exists $k > 0 \ni g(x) \geq k$ for all $x \in [a, b]$. Then

(a) $1/g$ is integrable on $[a, b]$

(b) $1/g$ is not integrable on $[a, b]$

(c) $1/g$ is not continuous on $[a, b]$

(d) $1/g$ is continuous on $[a, b]$

5. Let g be defined as

$$g(x) = \begin{cases} 2^n (x - n) + 1 & \text{if } n - 2^{-n} \leq x \leq n \\ 2^n (n - x) + 1 & \text{if } n \leq x \leq n + 2^{-n} \\ 0 & \text{otherwise} \end{cases}$$

Then

(a) $\int_1^\infty g$ converges but $\sum_1^\infty g$ diverges

(b) $\int_1^\infty g$ diverges but $\sum_1^\infty g$ converges

(c) $\int_1^\infty g$ and $\sum_1^\infty g$ both are converges

(d) $\int_1^\infty g$ and $\sum_1^\infty g$ both are diverges.

6. The closed unit ball $B = \{x \in \ell^\infty : \|x\|_\infty \leq 1\}$ is

(a) closed

(b) bounded

(c) compact

(d) closed and bounded but not compact

7. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be bounded sequence such that every $n, x_n \leq y_n \leq z_n$ and such that $\limsup z_n \leq \liminf x_n$. Then

(a) $\lim x_n = \lim y_n = \lim z_n$

(b) $\lim x_n \leq \lim y_n$

(c) $\lim x_n \leq \lim z_n$

(d) $\lim x_n < \lim y_n < \lim z_n$

8. If x_n is a decreasing and $\sum x_n$ converges then $\lim n x_n = 0$ which of the following imply true?

(a) $\sum_n \frac{1}{n^s}$ converges if $0 \le S \le 1$

(b) $\sum_n \frac{1}{n^s}$ diverges if $0 \le S \le 1$

(c) $\sum_n \frac{1}{n^s}$ converges if $S \le 1$

(d) $\sum_n \frac{1}{n^s}$ diverges if $S \ge 1$

9. $\lim_{\theta \to \pi/2} (1 - 5\cot\theta)^{\tan\theta}$

(a) 0 (b) 1

(c) e^{-5} (d) $\log 5$

10. $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \cos\left(\frac{\pi k}{2n}\right)$

(a) 2π (b) 1 (c) 0 (d) $2/\pi$

11. Let $C'(R)$ denote the set of all continuously differentiable real valued functions defined on the real line. Define

$$A = \{ f \in C'(R) \mid f(0) = 1, \quad f(1) = 1, |f'(x)| \le 1/2 \text{ for all } x \in R \}$$

where f' denotes the derivative of the function *f*. Pick out the true statement.

(a) A is an infinite set
(b) A is an empty set
(c) A is a finite and non empty set
(d) None of these

12. The function $g(x) = \sqrt{|x|}$ with domain $[-1,1]$ is

(a) uniformly continuous but does not satisfy a Lipschitz condition.
(b) uniformly continuous and satisfy a Lipschitz condition.
(c) continuous and compact
(d) not uniformly continuous

13. A function $g:[0,1] \to R$

$$g(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$$

(a) $g(x)$ is bounded variation
(b) $g(x)$ is not a bounded variation
(c) $g'(x)$ exists
(d) None of these

14. The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n+3}$

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

15. Find the sets of points of discontinuity for the function $f:[-1,1] \to R$ defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational} \end{cases}$$

(a) $[-1,1]$ (b) $(-1,1)$

(c) $[-1,1] \setminus \{0\}$ (d) ϕ

16. Let $f:[0,1] \to R$ be twice differentiable with $f''(x) > 0$ for all $x \in [0,1]$. If $f(0) > 0$ and $f(1) = 1$. Then

(a) $f(d) = d$ for some $d \in [0,1]$ iff $f'(1) < 1$

(b) $f(d) = d$ for some point $d \in [0,1]$ iff $f'(1) > 1$

(c) $f(d) \ne d$ iff $f'(1) < 1$

(d) $f'(1) < 1$ iff $f(d) > d$ for all $d \in [0,1]$

17. An absolutely continuous function is Lipschitz only if

(a) $|g|$ is bounded

(b) $|g'|$ is bounded

(c) g is continuous but not bounded

(d) None of these

18. Find the sets of points of discontinuity for the function $f:[0,\infty)\to R$ defined by

$$f(x)=\begin{cases}(x) & \text{if } [x] \text{ is even}\\ 1-(x) & \text{if } [x] \text{ is odd}\end{cases}$$

where $[x]$ is the largest integer less than or equal to x and $(x)=x-[x]$.

(a) $[0,\infty)$ (b) $[-1,1]$

(c) ϕ (d) $[-1,1]\setminus\{0\}$

19. Pick out the true statements

(a) $|\sin x-\sin y|\le|x-y|$ for all $x,y\in R$

(b) $|\sin 2x-2y|\le|x-y|$ for all $x,y\in R$

(c) $|\sin^2 x-\sin^2 y|\ge|x-y|$ for all $x,y\in R$

(d) $|\sin^3 x-\sin^3 y|\ge|x-y|$ for all $x,y\in R$

20. Let g be a continuous function from R to R.

(a) $\{x:g(x)=0\}$ is a open subset of R.

(b) $\{x:g(x)>c\}$ is a closed subset of R.

(c) $\{x:g(x)<c\}$ is a closed subset of R.

(d) $\{x:g(x)=0\}$ is a closed subset of R.

21. Let g be a function of bounded variation on $[a,b]$. Then

(a) $\int_a^b|g'|\le V_a^b g$ where $V_a^b g$ is the variation of g.

(b) $\int_a^b|g'|\ge V_a^b g$ where $V_a^b g$ is the variation of g.

(c) $\int_a^b|g'|\ne V_a^b g$

(d) $\int_a^b|g'|=0$

22. Suppose that $g:R\to R$ has the property that $|g(x)-g(y)|\le|x-y|^2$ for all $x,y\in R$

(a) g is not a constant

(b) g is not differentiable

(c) g is constant

(d) $g'(y)\ne 0$

23. Let $\{f_n\}$ be a sequence of functions defined on $[0,1]$. Determine $\lim\limits_{n\to\infty} n^2x(1-x^2)^n$

(a) 1 (b) 0

(c) ∞ (d) 2

24. Let $\{f_n\}$ be a sequence of functions defined on $[0,1]$. Evaluate $\lim\limits_{n\to\infty} x(1-x^2)^n$

(a) 0 (b) 1

(c) log 2 (d) e

25. Let g be a bounded function on $[a,b]$, so that there exists $M>0 \ni |g(x)|\le M$ for all $x\in B$. Then

(a) $U(g,P)-L(g,P)\le 2M[U(g^2,P)-L(g^2,P)]$ for all partitions P of $[a,b)$

(b) $[U(g^2,P)-L(g^2,P)]=U(g,P)-L(g,P)$ for all partitions P of $(a,b]$

(c) $U(g,P)-L(g,P)\ge M^2[U(g^2,P)-L(g^2,P)]$ for all partitions P of $[a,b)$

(d) $U(g^2,P)-L(g^2,P)\le 2M[U(g,P)-L(g,P)]$ for all partitions P of $[a,b]$

26. Find the sets of points of discontinuity for the function $f:[-1,1]\to R$ defined by

$$f(x)=\begin{cases}1 & \text{if } x \text{ is irrational}\\ 0 & \text{if } x \text{ is rational}\end{cases}$$

(a) $(-1,1)$ (b) $[-1,1]$

(c) ϕ (d) $[-1,\infty)$

27. Let $\{f_n\}$ be a sequence of functions defined on $[0,1]$ Find $\lim\limits_{n\to\infty} nx(1-x^2)^n$

(a) 1 (b) ∞

(c) 0 (d) $e^{-1/2}$

28. Let $g:(-1,1)\rightarrow R$ be a differentiable function such that there exists a limit

$$\lim_{x\to 0}\frac{g(x)}{x^2}=L\in R$$

(a) second derivative $g''(0)$ exists and equals L

(b) second derivative $g''(0)$ does not exists

(c) second derivative $g''(0)$ exists and equals to $2L$

(d) second derivative $g''(0)$ exists and equal to 0.

29. If $|h_n|\leq g\in L'(\Omega)$ then h_n are uniformly integrable on Ω.

(a) There exist a family of functions h_n that is uniformly integrable but there is no integrable function g so that $|h_n|\leq g$.

(b) There exist a family of functions h_n that is not integrable but there is a integrable function g so that $|h_n|\leq g$.

(c) Does not exist any family of function.

(d) h_n is uniformly integrable and $|h_n|\leq g$.

30. Let $f:[0,1]\rightarrow R$ be a continuous function.

Then $\lim_{n\to\infty}\frac{1}{n}\sum_{K=0}^{n} f\left(\frac{K}{n}\right)=?$

(a) 0 (b) $\int_0^1 f(t)dt$

(c) 1 (d) $\int_0^1 f^1(t)dt$

31. Let g be a twice differentiable function on the interval (a,∞)

(a) $\left(\sup_{x\in(a,\infty)}\{|g'(x)|\}\right)^2\leq 4\left(\sup_{x\in(a,\infty)}\{|g(x)|\}\right)\left(\sup_{x\in(a,\infty)}\{|g''(x)|\}\right)$

(b) $\left(\sup_{x\in(a,\infty)}\{|g(x)|\}\right)^2\leq 4\left(\sup_{x\in(a,\infty)}\{g(x)\}\right)$

(c) $\sup_{x\in(a,\infty)}|g'(x)|\leq \sup_{x\in(a,\infty)}|g''(x)|$

(d) $\left(\sup_{x\in(a,\infty)}\{|g''(x)|\}\right)^2\leq 4\left(\sup_{x\in(a,\infty)}|g(x)||g'(x)|\right)$

32. Suppose that $g:R\rightarrow R$ is a differentiable at a and that $g(a)=0$. Then which of the following imply true.

(a) If $h(x)=|g(x)|, h$ is differentiable at a then $g'(a)>0$

(b) If $h(x)=g(x),$ h is differentiable at a iff $g'(a)=0$

(c) If $h(x)=g(x),$ and $g'(a)=0$ then h is not differentiable at a

(d) if h is not differentiable at a then $g'(a)>0$ $g(x)=|h(x)|$

33. What is the radius of convergence of the following series $\sum_{n=0}^{\infty}\frac{z^n}{n!}$

(a) 1 (b) -1

(c) ∞ (d) 0

34. Let $K\in[0,\infty)$ be a real number. Define

$$f_K(t)=\begin{cases} t^K\sin {}^{1}\!/_{t} & t\neq 0\\ 0 & t=0\end{cases}$$

$A=\{K\in[0,\infty)\,|\,f_K \text{ is differentiable}\}$

then $A=?$

(a) $(0,\infty)$ (b) $(1,\infty)$

(c) $(-\infty,\infty)$ (d) $[0,\infty)$

35. Suppose that $f:[a,b]\rightarrow R$ is Riemann integrable on $[a,b]$

(a) $\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{i=1}^{n} f\left(a+i\frac{b-a}{n}\right)$

(b) $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+i(b-a)\right)$

(c) $\int_a^b f(x)dx \le \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{a}{b}+i\frac{b}{n}\right)$

(d) $\int_a^b f(x)dx \ge \lim_{n\to\infty} \frac{b-a}{n}\sum_{i=1}^{n} f(b)$

36. Let *f(x)* be continuous with

$\int_1^x f(t)dt = K + e^{(x-1)^2}$ Find *f* and the constant *K*

(a) $f(x) = 2(x-1)e^x, K = 1$

(b) $f(x) = 2(x-1)e^{(x-1)^2}, K = -1$

(c) $f(x) = 2xe^{(x-1)}, K = 0$

(d) $f(x) = 4e^{(x-1)^3}, K = 1$

37. Let a smooth function *f* have the properties $f(0)=4,\ f(1)=0,\ f(3)=6.$ Then which of the following is true?

(a) at some point $0<c<3$ one has $f''(c)>0$

(b) at some point $0<c<1$ one has $f''(c)<0$

(c) at $0<c<3$ one has $f''(c)=0$

(d) at $0<c<1$ one has $f''(c)<0$

38. Suppose that *f* is real valued, bounded on $[a,b]$ and f^3 is Riemann integrable on $[a,b]$. Then

(a) f^2 is Riemann integrable.

(b) f^2 is not Riemann integrable

(c) $\int_0^1 f(x)dx = \lim_{c\to 0}\int_c^1 f(x)dx$

(d) f^2 is discontinuous

39. What is the least value of $K>0 \ni$ $|\sin^2 x - \sin^2 y| \le K|x-y|$ for all real numbers x and y?

(a) $\frac{1}{2}$ (b) 2

(c) 1 (d) -1

40. Consider the functions $K(x) = x^2\sin\left(\frac{1}{x}\right)$ and $h(x) = x^2\sin\left(\frac{1}{x^2}\right)$

(a) $h(x)$ is of bounded variation on $[-1,1]$

(b) $K(x)$ is of bounded variation on $[-1,1]$

(c) both $h(x)$ and $K(x)$ are bounded variation on $[-1,1]$

(d) $K(x)$ is not a bounded variation on $[-1,1]$

41. Let $\{g_k\}$ be a sequence of functions of bounded variation on $[a,b]$. If $V[g_k;a,b]\le M<+\infty$ for all *k* and $g_k \to g$ pointwise on $[a,b]$.

(a) g is not a bounded variation

(b) g is of bounded variation and $V[g;a,b]\ge M$

(c) g is of bounded variation and $V[g;a,b]\le M$

(d) g is of bounded variation and $V[g;a,b]>M$

42. In the plane R^2 is provided with the Lebesgue measure, what is the measure of the set

$A=\{(x,y)\in R^2 \mid x^2+y^2=1\}$

(a) 1 (b) 0

(c) (x,y) (d) ∞

43. Let *f* be a real valued function on *R*. Consider the functions $w_j(x)=\sup\{|f(u)-f(v)|\}$:

$u,v\in\left[x-\frac{1}{j},x+\frac{1}{j}\right]$ where *j* is a positive integer and $x\in R.$ Define next $A_{j,n}=\left\{x\in R: w_j(x)<\frac{1}{n}\right\},\quad n=1,2,...$ and

$$A_n = \bigcup_{j=1}^{\infty} A_{j,n} \quad n = 1, 2, ...$$

Now let $C = \{x \in R : f$ is continuous at $x\}$

Express C interms of the set A_n

(a) $C = \bigcup_{n=1}^{\infty} A_n$ (b) $C = \bigcap_{n=1}^{\infty} A_n$

(c) $C = A_n$ (d) $C = 0$

44. Let f be a continuous real valued function on R and n a positive integer find $\frac{d}{dx}\int_0^x (2x-t)^n f(t)dt$

(a) $2n\int_0^x (2x-t)^{n-1} f(t)dt + x^n f(x)$

(b) $\int_0^x (2x-t)^n f(t)dt + f(x)$

(c) $\int_0^x (2x-t)^{n-1} f(t)dt + x^n f(x)$

(d) $\int_0^x (2x-t)^n f(t)dt$

45. Suppose that (f_n) is a sequence of decreasing functions $f_n : [1,3] \to R$ which converges pointwise to 0. Then

(a) (f_n) converges uniformly to -1

(b) (f_n) does not converges uniformly

(c) (f_n) converges uniformly to 0

(d) (f_n) converges to 1

46. Suppose that $g : (a,b) \to R$ and that g'' exists everywhere

(a) $g''(x) \geq 0$ for all $x \in (a,b)$ iff g is convex

(b) $g''(x) \leq 0$ for all $x \in (a,b)$ iff g is compact

(c) $g''(x) = 0$ for all $x \in (a,b)$ iff g is integrable

(d) $g'(x) \geq 0$ for all $x \in (a,b)$ then g is bounded

47. A real valued function on an interval $[a,b]$ is said to be a function of bounded variation if $\exists\ M > 0 \ni$ for any finite set of points $a = a_0 < a_1 < a_2 < ... < a_n = b$ we have $\sum_{i=0}^{n-1} |f(a_i) - f(a_{i+1})| < M$. Which of the following statements are necessarily true?

(a) Any monotone function on $[0,1]$ is not of bounded variation.

(b) If $f : R \to R$ is continuously differentiable, then its restriction to the interval $[-n, n]$ is bounded variation on that interval, for any positive integer n.

(c) Any continuous function on $[0,1]$ is of bounded variation

(d) $f : R \to R$ is not differentiable

48. $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = 0$ which of the following imply true?

(a) g is differentiable at $x = 0$ and $g'(0) = 0$

(b) g' is continuous at $x = 0$

(c) g is not differentiable at $x = 0$

(d) g' is continuous for all x.

49. Which of the following is true?

(a) $1 + x^2 \geq e^{x^2}$ for all $x \in R$

(b) $1 + x^2 \leq e^{x^2}$ for all $x \in R$

(c) $x^2 < e^{x^2} + 1$ for all $x \in R$

(d) $e^{x^2} \leq x + 1$ for all $x \in R$

50. Let $g : [0,1] \to R$ be a continuous function with continuous second derivative g'' and $g(0) = g'(1) = 0$.

(a) If $\int_0^1 g'(x)g''(x)dx = 0$ then $g \equiv 0$

(b) If $g \equiv 0$ then $\int_0^1 g(x)g''(x) = 0$

(c) If $\int_0^1 g(x)g''(x)dx = 0$ then $g \equiv 0$

(d) If $g \neq 0$ then $\int_0^1 g(x)g''(x) = 0$

51. If $g : R \to R$ satisfies

$d\left(g(x), g(y)\right) \leq d(x, y)^2$ for all $x, y \in R$. Then

(a) g is constant (b) g is polynomial

(c) g is nonconstant (d) None of these

52. Let f be a differentiable function of one variable and let g be the function of two variables given by $g(x, y) = f(ax + by)$ where a, b are fixed non zero numbers. Write down a partial differential equation satisfied by the function g

(a) $b\dfrac{\partial g}{\partial x} = a\dfrac{\partial g}{\partial y}$ (b) $a\dfrac{\partial g}{\partial x} = b\dfrac{\partial g}{\partial y}$

(c) $b\dfrac{\partial g}{\partial x} = -a\dfrac{\partial g}{\partial y}$ (d) $\dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial y}$

53. Let g be a continuous function on $[a, b]$ and assume that $g(a) < 0 < g(b)$. Let $W = \{x \in [a, b], g(x) < 0\}$ and let $w = \sup W$.

(a) $g(w) \neq 0$ (b) $g(w) > 0$

(c) $g(w) = 0$ (d) $g(w) < 0$

54. Let K be a fixed positive integer. Find R_K the radius of convergence of the power series

$\Sigma\left(\dfrac{n+1}{n}\right)^{n^2} z^{Kn}$

(a) 1 (b) 0

(c) $e^{1/K}$ (d) ∞

55. The series $\sum_{n=1}^{\infty}(-1)^n \dfrac{1}{2n+3}$ is

(a) convergent

(b) conditionally convergent

(c) absolutely convergent

(d) divergent

56. Let (r_n) be an enumeration of the set Q, then there exists a subsequence (r_{n_s}) such that

(a) $\lim_{s\to\infty} r_{n_s} = +\infty$ (b) $\lim_{s\to\infty} r_{n_s} = -\infty$

(c) $\lim_{s\to\infty} r_{n_s} = 0$ (d) $\lim_{s\to\infty} r_{n_s}^0 = 1$

57. Suppose that $g(x)$ is a continuous function on the interval $[a, b] \ni g(x) > 0$ for all x.

(a) $\int_b^a g^1(x)dx < 0$ (b) $\int_a^b g(x)g^1(x)dx = 0$

(c) $\int_a^b g(x)dx > 0$ (d) $\int_a^b dx = 0$

58. The series $\sum_{n=1}^{\infty}(-1)^n \dfrac{n}{n+2}$

(a) convergent

(b) conditionally convergent

(c) absolutely convergent

(d) divergent

59. The series $\sum_{n=1}^{\infty}(-1)^n \dfrac{n\log n}{e^n}$

(a) convergent

(b) conditionally convergent

(c) absolutely convergent

(d) divergent

60. Let $\{x_n\}$ be a sequence of nonnegative numbers and

(a) If $\lim x_n = 0$ then $\lim\sqrt{x_n} = 0$

(b) If $\lim x_n = 0$ then $\lim\sqrt{x_n} \neq 0$

(c) If $\lim x_n \neq 0$ then $\lim\sqrt{x_n} \neq 0$

(d) If $\lim\sqrt{x_n} = 0$ then $\lim x_n \neq 0$

61. Suppose that there exists $N_o \ni a_n \leq b_n$ for all $n > N_o$

(a) if $\lim a_n = +\infty$ then $\lim b_n = +\infty$

(b) if $\lim a_n = -\infty$ then $\lim b_n = +\infty$

(c) if $\lim a_n = +\infty$ then $\lim b_n = -\infty$

(d) if $\lim a_n \neq -\infty$ then $\lim b_n = -\infty$

62. The sequence of functions $\sin(nx), n \geq 1$ is
(a) equicontinuous in $c[0,1]$
(b) bounded in $c[0,1]$
(c) compact in $c[0,1]$
(d) not a equicontinuous in $c[0,1]$

63. Let h_n be a function $\ni$ $h_n : (0,1) \to R$. Then

(a) If $h_n \to h$ and $h_n' \to f$ then h is continuous on $(0,1)$ and $h^1 = f$

(b) If $h_n \to h$ then h is not continuous on $(0,1)$

(c) If $f_n \to f$ then h_n' does not exists

(d) If $h_n \to h$ and $h_n' \to f$ then h is not continuous on $(0,1)$ and $h' = f$

64. Suppose $G : R \to C$. There is a constant M such that $|G(x) - G(y)| \leq M|x - y|$ for all $x, y \in R$ iff
(a) G is continuous almost everywhere
(b) G is absolutely continuous and $|G'| \geq M$ almost everywhere
(c) G is absolutely continuous and $|G'| \leq M$ almost everywhere
(d) G is continuous and G' does not exist.

65. Let $x > 0$. Define $f(x) = \int_0^x \frac{\sin xy}{y} dy$ Evaluate $f'(x)$ as a function of x.

(a) $\frac{\sin x}{x}$ (b) $2\frac{\sin(x^2)}{x}$

(c) $\cos x$ (d) 1

66. Find the coefficient of x^7 in the Taylor series expansion of the function $f(x) = \sin^{-1} x$ around 0 in the interval $-1 < x < 1$

(a) $\frac{1}{15}$ (b) $\frac{5}{112}$

(c) 1 (d) 103

67. If f is an increasing real-valued function of a real variable then
(a) f has at most countable number of discontinuities.
(b) f does not have discontinuities
(c) f has at most countable number of continuities
(d) None of these

68. The minimum value of the function

$f(x, y) = x^2 + 5y^2 - 6x + 10y + 6$ is
(a) 10 (b) 6
(c) 5 (d) -8

69. The interval of convergence of the series

$(x+1) - \frac{(x+1)^2}{4} + \frac{(x+1)^3}{9} - \frac{(x+1)^4}{16} + \ldots$ is

(a) $-2 \leq x \leq 0$ (b) $-1 < x < 0$

(c) $x \leq 0$ (d) $-2 \leq x \leq 1$

70. Which of the following series is diverges

(a) $\sum \frac{1}{\sqrt{n!}}$ (b) $\sum \left(\frac{1}{2}\right)^n \left(50 + \frac{2}{n}\right)$

(c) $\sum \frac{1}{(2^n + n)}$ (d) $\sum_{n=2}^{\infty} \frac{\log n}{n}$

71. The function $g(x) = x^3$ is not uniformly continuous on

(a) $[0,1]$ (b) $(0,1)$

(c) R (d) $[0,3]$

72. A differentiable real valued function on R with bounded derivative is
(a) continuous
(b) uniformly continuous
(c) bounded
(d) not continuous

73. Evaluate $\lim_{n \to \infty} \frac{n^2}{n!}$

(a) 0 (b) 1
(c) e (d) ∞

74. Find $\lim_{n\to\infty}(1+a+...+a^n)$ for $|a|<1$

(a) 1 (b) $\frac{1}{1-a}$

(c) 0 (d) ∞

75. Determine the interval of convergence of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$

(a) $(0,1)$ (b) $(0,2]$

(c) $(-\infty,\infty)$ (d) $[1,2]$

76. What is the cardinality of the following set

$A=\{f\in C^1[0,1]: f(0)=0, f(1)=1,$

$|f^1(t)|\le 1$ for all $t\in[0,1]\}$

(a) ∞ (b) 0

(c) $-\infty$ (d) 1

77. Evaluate $\lim_{n\to\infty}\frac{1}{n}\sum_{K=1}^{\left[\frac{n}{2}\right]}\cos\left(\frac{K\pi}{n}\right)$ where $\left[\frac{n}{2}\right]$ denotes the largest integer not exceeding $n/2$.

(a) $1/\pi$ (b) $\pi/2$

(c) 2π (d) 0

78. Let $f(x)=\frac{1}{1+x^2}$ consider its Taylor expansion about a point $a\in R$, given by

$f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$. What is the radius of convergence of this series?

(a) 1 (b) $\sqrt{a^2+1}$

(c) ∞ (d) a

79. Let $f:[-1,1]\to R$ be continuous. Assume that $\int_{-1}^{1}f(t)dt=1$. Evaluate $\lim_{n\to\infty}\int_{-1}^{1}f(t)\cos^2 nt\,dt$

(a) 0 (b) $1/2$

(c) 1 (d) $\pi/2$

80. Let $f:R\to R$ be a continuously differentiable function $\ni f'(0)=0$. Define x and $y\in R$

$g(x,y)=f\left(\sqrt{x^2+y^2}\right)$.

Pick out the true statements

(a) g is differentiable only on $R^2\setminus\{(0,0)\}$

(b) g is differentiable function on R^2 iff $f(0)=0$

(c) g is differentiable function on R^2

(d) None of these

81. The sequence of functions $\left\{\frac{nx}{1+xn}\right\}$ over the interval $[0,2]$ is

(a) convergent

(b) uniformly convergent

(c) not uniformly convergent

(d) absolutely convergent

82. The series $\sum_{n=1}^{\infty}\frac{\sin x}{n^2+1}$ over R is

(a) convergent

(b) uniformly convergent

(c) absolutely convergent

(d) divergent

83. Let $\{g_n\}$ be a sequence of functions of X which converges pointwise on X. Let $G\subset X$ be finite

(a) $\{g_n\}$ converges uniformly on G

(b) $\{g_n\}$ does not converges

(c) $\{g_n\}$ converges but not uniformly converges on G

(d) None of the above

84. Pick out the sequences $\{f_n\}$ which are uniformly convergent

(a) $f_n(x) = \frac{\sin x}{\sqrt{n}}$ on R

(b) $f_n(x) = x^n$ on $[0,1]$

(c) $f_n(x) = nx\overline{e}^{nx}$ on $(0,\infty)$

(d) All the above

85. Let $\{a_n\}$ be a sequence of positive real numbers $\ni \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$ find $\lim\limits_{n\to\infty} a_n$

(a) 1 (b) 0

(c) ∞ (d) a

86. The series $\sum\limits_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+5/4}}$ is

(a) convergent (b)uniformly convergent

(c) absolutelyconvergent (d) divergent

87. *L(x)* is Lipschitz, if there is a constant $M \ni$ for any $|L(x)-L(y)| < M(x-y)|$ for any x and y. Which of the following imply true?

(a) If the function $G(x)$ is absolutely continuous on a closed interval $[a,b]$ then the function $L(G(x))$ is also absolutely continuous on $[a,b]$

(b) If the function $L(G(x))$ is absolutely continuous on $[a,b]$ then the function $G(x)$ is absolutely continuous on a closed interval $[a,b]$

(c) If the function $G(x)$ is continuous on (a,b) then the function $L(G(x))$ is absolutely continuous on $[a,b]$

(d) The function $L(G(x))$ is absolutely continuous on $[a,b]$ then the function $G(x)$ is continuous on (a,b)

88. Let $\{g_n\}$ be a sequence of real valued measurable functions on $X \ni$ for each $n \in N$

$$\mu\left(\left\{x \in X : |g_n(x) - g_{n+1}(x)| > \frac{1}{2^n}\right\}\right) < \frac{1}{2^n}$$

(a) $\{g_n\}$ is converges but not pointwise convergent

(b) $\{g_n\}$ is pointwise convergent μ almost everywhere on *X*.

(c) $\{g_n\}$ diverges

(d) $\{g_n\}$ is not a cauchy sequence

89. The space $C([a,b])$ equipped with L^1 norm $\|\bullet\|_1$ defined by $\|g\|_1 = \int_a^b |g(x)|\,dx$ is

(a) compact

(b) convergent sequence

(c) uniformly convergent sequence

(d) incomplete

90. Consider the sequence of continuous functions given by $g_n(x) = \frac{1+2\cos^2(nx)}{\sqrt{n}}$

(a) $g_n(x)$ converges to the constant 0 function on *R*

(b) $g_n(x)$ converges uniformly to the constant 0 function on *R*

(c) $g_n(x)$ converges to $[0,1]$

(d) $g_n(x)$ does not converges

91. Evaluate $\lim\limits_{n\to\infty} \sum\limits_{K=1}^{n} \frac{1}{\sqrt{4n^2 - K^2}}$

(a) π (b) $\pi/6$

(c) 2π (d) 0

92. Evaluate $\lim\limits_{n\to\infty} n\sin(2\pi e n!)$

(a) 4π (b) 8π

(c) 2π (d) ∞

93. Which of the followig functions are uniformly continuous?

(a) $f(x)=\sin^2 x,\ \ x\in R$

(b) $f(x)=1/x,\qquad x\in(0,1)$

(c) $f(x)=x^2,\qquad x\in R$

(d) None of these

94. The series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right)$ is

(a) convergent
(b) uniformly convergent
(c) absolutely convergent
(d) divergent

95. Let f and g be differentiable functions

(a) If $f'(x)=g'(x)$ for all x, then $f(x)=g(x)+k$

(b) If $f'(x)=g'(x)$ for all x, then $f(x)=k$

(c) If $f'(x)=0$ for all x, then $g(x)=x$

(d) If $g'(x)=f(x)$ for all x, then $g(x)=k+g'(x)$

96. Let $\ell\in R$ and let $\{f_n\}$ be a real sequence

(a) If $f_n\to a$ then the sequence $\{\ell.f_n\}$ converges to $\ell.a$

(b) If $f_n\to a$ then the sequence $\{\ell.f_n\}$ does not converges to $\ell.a$

(c) If f_n converges to zero then the sequence $\{f_n\}$ converges to ℓ.

(d) If the sequence $\{\ell.x_n\}$ converges to $\ell.a$ then $f_n\not\to a$

97. Which of the following sets is not a discrete subset of R

(a) $S=\left\{\frac{1}{n}\mid n\in z_+\right\}$

(b) $T=\left\{\frac{1}{n}\mid n\in z_+\right\}\cup\{0\}$

(c) Z

(d) Every finite subset of R

98. Evaluate $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right\}^{1/n}$

(a) 0 (b) 1

(c) $4/e$ (d) ∞

99. Evaluate $\lim_{n\to\infty}\left(\frac{1}{1+n^3}+\frac{4}{8+n^3}+...\frac{n^2}{n^3+n^3}\right)$

(a) 1 (b) 0

(c) $\frac{1}{3}\log 2$ (d) ∞

100. Find the points in R where the following function is differentiable

$$f(x)=\begin{cases}\tan^{-1}x & \text{if } |x|\le 1\\ \frac{\pi}{4}Sgn(x)+\frac{|x|-1}{2} & \text{if } |x|>1\end{cases}$$

where sgn *(x)* equals $+1$ if $x>0$, -1 if $x<0$ and is equal to zero if $x=0$ and $\tan^{-1}(x)$ takes its values in the range $\left(-\pi/2,\pi/2\right)$ for real numbers x.

(a) R (b) Q

(c) N (d) $R\setminus\{-1\}$

101. Let $f,f_n:[0,1]\to R$ be continuous functions. Complete the following sentence ∍ both statements (i) and (ii) below are true. Let $f_n\to f$ ....

(i) $\lim_{n\to\infty}\int_0^1 f_n(x)dx=\int_0^1 f(x)dx$

(ii) $\lim_{n\to\infty}\lim_{n\to 0}f_n(x)=\lim_{x\to 0}\lim_{n\to\infty}f_n(x)$

(a) continuous on $[0,1]$

(b) uniformly on $[0,1]$

(c) uniformly on $(0,1)$

(d) None of the above

102. Evaluate $\iint\limits_{[0,1]\times[0,1]} \max\{x,y\}\,dxdy$

(a) 1 (b) 0

(c) $2/3$ (d) log 2

103. Let $f \in C[-1,1]$ Evaluate $\lim\limits_{h\to 0}\frac{1}{h}\int_{-h}^{h} f(t)dt$

(a) 2 (b) $2f(0)$

(c) $f^1(0)$ (d) 0

104. Let $f \in C^1[-1,1]$. Evaluate $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{K=1}^{n} f^1\left(\frac{K}{3n}\right)$

(a) $3\left[f\left(1/3\right)-f(0)\right]$

(b) $3\left[f\left(1/3\right)+f(0)\right]$

(c) $f^1\left(1/3\right)+f^1(0)$

(d) 0

105. Let $f \in C[-\pi,\pi]$ Evaluate $\lim\limits_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos nt\ dt$

(a) 0 (b) -1

(c) 2π (d) ∞

106. Let $f \in C[-\pi,\pi]$ Evaluate $\lim\limits_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos^2 nt\ dt$

(a) 0 (b) $\frac{1}{2}\int_{-\pi}^{\pi} f(t)dt$

(c) $\int_{0}^{\infty} f(t)dt$ (d) $\cos^2 nt$

107. The sequence $f_n(x)=\frac{nx}{1+nx}, x\in(0,\infty)$

(a) uniformly convergent
(b) not uniformly convergent
(c) absolutely covergent
(d) None of the above

108. The series $\sum_{n=1}^{\infty}\frac{n\sin nx}{e^n}, x\in[0,\pi]$

(a) uniformly convergent
(b) not uniformly convergent
(c) absolutely covergent
(d) None of the above

109. The sequence $f_n(x)=\frac{x^n}{1+x^n}, x\in[0,2]$

(a) uniformly convergent
(b) not uniformly convergent
(c) absolutely covergent
(d) None of the above

110. Compute $F'(x)$ where $F(x)=\int_{-x}^{x}\frac{1-\bar{e}^{xy}}{y}dy, x>0$

(a) e^{x^2} (b) $2(e^{x^2}-\bar{e}^{x^2})/x$

(c) $\cos x^2$ (d) 0

111. Let $a>0$ and let $K\in N.$

Evaluate $\lim\limits_{n\to\infty} a^{-nK}\prod_{j=1}^{K}\left(a+\frac{j}{n}\right)^n$

(a) $e^{\frac{K(K+1)}{2a}}$ (b) $\log K(K+1)$

(c) 0 (d) a^n

112. $\lim\limits_{n\to\infty}\sin\left[\left(2n\pi+1/2n\pi\right)\sin\left(2n\pi+1/2n\pi\right)\right]$ is

(a) 0 (b) ∞

(c) $\sin 1$ (d) $\log(\sin n)$

113. $\lim\limits_{n\to\infty}\frac{1}{n}[(n+1)(n+2)...(n+n)]^{1/n}$ **is**

(a) 0 (b) ∞

(c) $4/e$ (d) log 4

114. Find the points where the following function is differentiable

$$f(x)=\begin{cases}\tan^{-1}x & \text{if } |x|\le 1\\ \frac{\pi}{4|x|}+\frac{|x|-1}{2} & \text{if } |x|>1\end{cases}$$

(a) $R\setminus\{-1\}$ (b) $(0,\infty)$

(c) $(-\infty,\infty)$ (d) C

115. For no distinct *p, q* is

(a) $L^p \supset L^q$ (b) $L^q = L^p$

(c) $L^p \subset L^q$ (d) $L^p = 0$

116. Let (X,d) be a metric space

(a) a subset $M \subset X$ is unbounded if the function d is bounded on $M \times M$

(b) a subset $M \subset X$ is bounded if the function d is bounded on $M \times M$

(c) a subset $M \subset X$ is bounded if the function d is unbounded on $M \times M$

(d) a subset $M \subset X$ is unbounded if the function d is unbounded on M.

117. Let $g_n : R \to R$ be defined by

$$g_n(x) = \frac{1}{n^3\left(x-\left(\frac{1}{n}\right)^2\right)+1} \text{ and let } g(x)=0$$

(a) $g_n(x) \to g(x)$ for each x then g_n converges uniformly to g.

(b) $g_n(x) \to g(x)$ for each x then g_n converges to zero

(c) $g_n(x) \to g(x)$ for each x but g_n doesn't converge uniformly to g.

(d) $g_n(x) \to 0$ for each x then g_n converge to zero.

118. Let a_n be a sequence of non negative real numbers and suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Then

(a) $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges

(b) $\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges

(c) $\sum_{n=N+1}^{\infty} a_n$ converges

(d) $\sum_{n=1}^{\infty} a_n^{\,2}$ converges

119. Suppose that $g : R \to R$ is continuous on R and that $g(r) = 0$ for every rational number r. Then

(a) $g(x) = 0$ for all $x \in R$

(b) $g(x) > 0$ for all $x \in R$

(c) $g(x) < 0$ for all $x \in R$

(d) $g(x) \neq 0$ for all $x \in R$

120. Which of the following sequences / series of functions are uniformly convergent on $[0,1]$?

(a) $f_n(x) = n^2 x(1-x^2)^n$

(b) $f_n(x) = \left(\cos(\pi n! x)\right)^{2n}$

(c) $\sum_{m=1}^{\infty} \frac{\cos\left(m^6 x\right)}{m^3}$

(d) All the above

121. Let $f \in C'[0,1]$. For a partition

$(P): 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ define

$S(P) = \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$. Compute the supremum of *S(P)* taken over all possible partitions *P*

(a) $\int_0^1 |f'(t)|\,dt$ (b) $\int_0^1 f(t)dt$

(c) 0 (d) 1

122. Let D_n be the open disc of radius n with centre at the point $(n,0) \in R^2$. Then there exist a function $f : R^2 \to R$ of the form *f(x, y)=ax+by* $\ni \bigcup_{n=1}^{\infty} D_n = \{(x,y) \mid f(x,y) > 0\}$. The value of *a* and *b* is

(a) $a=1, b=0$ (b) $a=0, b=1$

(c) $a=-1, b=0$ (d) $a=0, b=-1$

123. Find the sum of the series $\sum_{K=1}^{\infty}\frac{K^2}{K!}$

(a) 2^n (b) $2e$

(c) $(K+1)^n$ (d) K^{-n}

124. The radius of convergence of the power series

$\sum_{K=1}^{\infty}\sqrt{\log K}x^k$ is

(a) K (b) 1

(c) 0 (d) ∞

125. What are the values of $\alpha \in R$ for which the following series is convergent $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^\alpha}$

(a) $\alpha \geq 0$ (b) $0<\alpha<\infty$

(c) $\alpha \geq 1$ (d) $\alpha = 1$

126. $\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2+1}\right)$

(a) 1 (b) 0

(c) $1/2$ (d) ∞

127. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}}$ is

(a) 1 (b) 0

(c) $\sqrt{2}$ (d) $\log(1+\sqrt{2})$

128. Let $f: R \to R$ be a given function. Which of the following is uniformly continuous

(a) $f(x)=\frac{e^x}{x}$ on $(0,2)$

(b) $f(x)=\frac{1}{x^2}$ on $(0,1)$

(c) for all x and $y \in R$

$|f(x)|=|f(y)|\leq|x-y|^{1/2}$

(d) $f(x)=\sum_{n=1}^{\infty}\frac{g(x-n)}{2^n}, x\in R$

129. Let $f(x)=[x]+(x+[x])^2$ for $x \in R$, where $[x]$ denotes the largest integer not exceeding x. what is the set of all values taken by the function f?

(a) Q (b) R

(c) $R\setminus\{0\}$ (d) $R\setminus\{-1\}$

130. Let n be a positive integer. Let

$f(x)=x^{n+2}\sin\frac{1}{x}$ if $x\neq 0$ and let $f(0)=0$.

For what value of n will f be twice differentiable but with its second derivative discontinuous at $x=0$.

(a) 1 (b) 2

(c) 0 (d) 3

131. What is the coefficient of x^8 in the expansion of $x^2(\cos x^2)$ around $x=0$

(a) 1 (b) 0

(c) -1 (d) 2

132. Let $g: R\to R$ be a differentiable function $\ni |g'(x)|\leq M$ for all $x\in R$. For what values of ε will the function $f(x)=x+\varepsilon g(x)$ be necessarily one to one?

(a) $\varepsilon>\frac{1}{M}$ (b) $\varepsilon<1/M$

(c) $\varepsilon = M$ (d) $\varepsilon = 1$

133. If $f(x)=\int_x^{x^2}e^{t^2}dx, x>1.$ then $f'(x)=$

(a) $2xe^{x^4}-e^{x^2}$ (b) $e^{x^2}+e^{2x}$

(c) $2xe^{2x}$ (d) 0

134. Pick out the true statments.

(i) $f(x)\geq 0$ for $x\geq 0$ and $f(x)\leq 0$ for $x\leq 0$

(ii) f is a decreasing function on the entire real line

(a) (i) and (ii) both are true

(b) (i) is true but (ii) is not

(c) (ii) is true but (i) is not

(d) both (i) and (ii) are false

135. Let $f:[-1,2]\to R$ be given by

$f(x)=2x^3-x^4-10$. What is the value of x where f assumes its minimum value

(a) 1 (b) -1
(c) 2 (d) 0

136. $\int_0^1 \log x dx = ?$

(a) 0 (b) 1
(c) -1 (d) ∞

137. Let $\{X,d\}$ be a metric space and let $E\subset X$. For $x\in X$ define $d(x,E)=\inf_{y\in E} d(x,y)$

Pick out the true statements

(a) $|d(x,E)-d(y,E)|\le d(x,y)$ for all $x,y\in X$

(b) $d(x,E)=d(x,y_0)$ for some $y_0\in E$

(c) $|d(x,E)|=|d(x_0,y_0)|$

(d) All the above

138. Find $\lim_{n\to\infty}\int_{-1}^{1}\sin^n x dx$

(a) 0 (b) 1
(c) -1 (d) $\pi/2$

139. Find $\lim_{n\to\infty}\left(\frac{3}{2}-\frac{3}{4}+\frac{3}{8}-\frac{3}{16}+...+(-1)^n\frac{3}{2^n}\right)$

(a) 1 (b) -1
(c) 0 (d) ∞

140. Let $[x]$ denote the largest integer less than, or equal to $x\in R$. The function

$f(x)=[x^2]\sin \pi x, x>0$ is

(a) continuous at $x=\sqrt{n},\ n\in N$

(b) uniformly continuous at $x=\sqrt{n},\ n\in N$

(c) continuous everywhere

(d) discontinuous at $x=\sqrt{n},\ n\in N,\ n\neq K^2$

141. Let $[x]$ denote the largest integer less than or equal to $x\in R$, The function

$f(x)=[x]+(x-[x])^{[x]},\ x\ge \frac{1}{2}$ is

(a) continuous at $x=\frac{1}{2},\ n\in N$

(b) uniformly continuous at $x=\frac{1}{2},\ n\in N$

(c) continuous everywhere

(d) discontinuous

142. Evaluate $\sum_{K=1}^{n} Ke^{Kx},\ x\in R\setminus\{0\}$

(a) $\dfrac{ne^{(n+2)x}-(n+1)e^{(n+1)x}+e^x}{(e^x-1)^2}$

(b) $\dfrac{ne^{(n+2)x}}{(e^x-1)^2}$

(c) $\dfrac{e^x}{e^x-1}$

(d) $\dfrac{ne^{(n+2)x}+e^x}{(e^x-1)}$

143. If $\sum X_n$ converges absolutely then

(a) $\sum X_n^2$ converges

(b) $\sum X_n^2$ diverges

(c) $\sum X_n^2$ neither converges nor diverges

(d) $\sum X_n^{-2}$ converges

144. Let $\sum a_n(x-t)^n$ and $\sum b_n(x-t)^n$ have raddi of converges M_1 and M_2 respectively. If $|x-t|<M$ where $M=\min\{M_1,M_2\}$ then

(a) $\sum(a_n)(x-t)^n$ converges and is equal to $\sum a_n(x-t)^n+\sum b_n(x-t)^n$

(b) $\sum(a_n+b_n)(x-t)^n$ converges absolutely and is equal to $\sum a_n(x-t)^n+\sum b_n(x-t)^n$

(c) $\sum a_n(x-t)^n$ diverges and is equal to $\sum a_n(x-t)^n$

(d) None of the above

145. Let M be a metric space such that M is a finite set. Then
(a) Every subset of M is closed
(b) Every subset of M is open
(c) Every subset of M is both open and closed
(d) Every subset of M is not closed

146. Let k^{∞} be the set of all real sequences $\{a_n\} \ni |a_n| \le 1$ for every n. Then

(a) $\sum \frac{|a_n - b_n|}{2^n}$ converges

(b) $\sum \frac{a_n}{1+a^n}$ converges

(c) $\sum \frac{|a_n - b_n|}{2^n}$ diverges

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ diverges

147. let $\{x_n\} \in \ell^2,\ g : \ell^2 \to R$ by $g(\{y_n\}) = \sum_n x_n y_n$.
(a) g is continuous
(b) g is not continuous
(c) g is unbounded
(d) g is constant

148. Any compact metric space M is
(a) not separable
(b) complete
(c) continuous
(d) closed

149. If g is a continuous, periodic function from R to a metric spece M, then
(a) g is not continuous on R
(b) g is uniformly continuous on R
(c) g is contraction mapping from R to R
(d) M is separable

150. Let $f : R \to R$ be a function which is differentiable at $x = a$. Evaluate

$$\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a}$$

(a) $a^n f'(a) - nf(a)$

(b) $a^n f'(a) - na^{n-1} f(a)$

(c) $a^n f(a) - a^{n-1} f'(a)$

(d) $a^n f'(a)$

151. Let $f : R \to R$ be a function which is differentiable at $x = a$ Evaluate

$$\lim_{n \to \infty} n\left[\sum_{j=1}^{K} f\left(a + {j}/{n}\right) - Kf(a)\right]$$

(a) $\frac{K^2}{2} f'(a)$ (b) $(K+1) f'(a)$

(c) $\frac{K(K+1)}{2} f'(a)$ (d) $f'(a)$

152. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Pick out the cases which imply that the sequence is cauchy.

(a) $|x_n - x_{n+1}| \ge \frac{1}{2^n}$ for all n

(b) $|x_n - x_{n+1}| \le {1}/{n}$ for all n

(c) $|x_n - x_{n+1}| \le {1}/{n^2}$ for all n

(d) $|x_n - x_{n+1}| \ge {1}/{n^2}$ for all n

153. Let $h_n(x) = \frac{1}{1+n^2x^2}$ and $g_n(x) = nx(1-x)^n$ for $x \in [0,1]$.

(a) $\{h_n\}$ converges

(b) $\{g_n\}$ does not converges

(c) $\{h_n\}$ and $\{g_n\}$ converges pointwise but not uniformly on $[0,1]$

(d) $\{h_n\}$ and $\{g_n\}$ converges pointwise and also uniformly continuous on $[0,1]$

154. Suppose that $g^1(x)$ exists and is bounded. Then
(a) g is continuous
(b) g is uniformly continuous
(c) g is not continuous
(d) g is constant

155. The series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}\sin \frac{1}{n}$ is

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

156. The series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{(n+1)(n+2)}$ is

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

157. The function $f(x) = x\sin \frac{1}{\sqrt{x}}$ if $x \neq 0$ and $f(0)=0$ is

(a) continuously differentiable
(b) differentiable but not continuously differentiable
(c) not differentiable
(d) None of the above

158. The function $f(x) = |x|^{3/2}$ is

(a) continuously differentiable
(b) differentiable but not continuously differentiable
(c) not differentiable
(d) None of the above

159. Let $\{g_n\}$ be a sequence of real valued continuous functions defined on $[0,1]$ and assume that $g_n \to g$ uniformly on $[0,1]$. Then

(a) $\lim_{n\to\infty}\int_{1/n}^{1} g_n(x)dx = \int_0^1 g(x)dx$

(b) $\lim_{n\to\infty}\int_{1/n}^{1} g_n(x)dx = 0$

(c) $\lim_{n\to\infty}\int_{1/n}^{1} g_n(x)dx > \int_0^1 g(x)dx$

(d) $\int_0^1 g(x)dx = 1$

160. $\lim_{n\to\infty}\int_{-1}^{1} e^{x^2/n}\,dx$

(a) 2 (b) 1
(c) 0 (d) ∞

161. Evaluate $\lim_{n\to\infty}\int_{-1}^{3}\frac{nx^2+3}{x^3+nx}dx$

(a) 4 (b) 2
(c) 1 (d) 0

162. The function $f(x) = x\sin|x|$ is

(a) continuously differentiable
(b) differentiable but not continuously differentiable
(c) not differentiable
(d) None of the above

163. Sum of the following infinite series

$$\frac{1}{6}+\frac{5}{6.12}+\frac{5.8}{6.12.18}+\frac{5.8.11}{6.12.18.24}+\ldots$$

(a) $5^{1/2}-1$ (b) $\frac{1}{2}\left(4^{1/3}-1\right)$

(c) $\sqrt{2}-1$ (d) 0

164. $\lim_{x\to 0}(1+3x^2)^{5\cot x+\frac{\operatorname{cosec} x}{x}}$

(a) e^6 (b) $\log 6$
(c) 1 (d) 0

165. The series $\sum_{n=1}^{\infty}\left(\sqrt[3]{n^3+1}-n\right)$ is

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

166. The series $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\frac{7}{4.5.6}+\ldots$ is

(a) convergent
(b) uniformly convergent
(c) conditionally convergent
(d) divergent

167. Let g, h be continuous on a closed interval $[a,b]$ with $|h(x)|>0$ for $x\in[a,b]$ suppose

$g_n \to g$ and $h_n \to h$ as $n \to \infty$ uniformly on $[a,b]$. Then

(a) $\frac{1}{h_n}$ is defined for large n and $\frac{g_n}{h_n} \to \frac{g}{h}$ uniformly on (a,b) as $n \to \infty$

(b) $\frac{1}{h_n}$ is defined for large n and $\frac{g_n}{h_n} \to \frac{g}{h}$ uniformly on $[a,b]$ as $n \to \infty$

(c) $\frac{g_n}{h_n} \to \frac{g}{h}$ is continuous on (a,b) as $n \to \infty$

(d) $\frac{g_n}{h_n} \to \frac{g}{h}$ is unbounded on (a,b)

168. The geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

(a) converges on (a,b)
(b) converges uniformly on any closed interval $[a,b] \subset (-1,1)$
(c) diverges
(d) converges on $[a,b] \subset (-1,1)$

169. $\lim_{(x,y)\to(0,0)} \frac{\sin x \sin y}{x^2+y^2}$ is

(a) 0 (b) 1
(c) 2 (d) does not exist

170. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^2+2y^4}$ is

(a) 0 (b) 1
(c) 2 (d) limit does not exist

171. $g(x,y) = \begin{cases} \dfrac{x^2+y^2}{\sin\sqrt{x^2+y^2}}, & 0 < \|(x,y)\| < \pi \\ 0 & (x,y) = (0,0) \end{cases}$ is

(a) differentiable at $(0,0)$
(b) continuous at $(0,0)$
(c) not differentiable at $(0,0)$
(d) limit exist

172. $\lim_{x\to\infty} x\left(\log\left(1+x/2\right) - \log x/2\right)$

(a) 2 (b) 1
(c) 0 (d) ∞

173. $\int_0^{\pi/2} \log\tan\theta \, d\theta =$

(a) 1 (b) 0
(c) $\pi/2$ (d) $\pi/4$

174. The series $\sum_{n=1}^{\infty} (n^{1/n} - 1)^n$ is

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

175. The series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ is

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

176. The series $\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{n}{n+1}}$ is

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

177. If $\alpha > 1/2$ then

$$g(x,y) = \begin{cases} |xy|^{\alpha} \log(x^2+y^2) & ,(x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is

(a) differentiable at $(0,0)$
(b) not differentiable at $(0,0)$
(c) limit does not exist
(d) none of the above

178. $g(x,y)=\sqrt{|xy|}$ is

(a) differentiable at $(0,0)$

(b) not differentiable at $(0,0)$

(c) limit exist at $(0,0)$

(d) continuous at $(0,0)$

179. $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is

(a) converges (b)diverges

(c) not bounded (d) decreasing sequence

180. Let $f:A\to R$ and assume that g is bounded function on $A\subseteq R$.

(a) If $\lim_{x\to c} f(x)g(x)=0$ then $\lim_{x\to c} g(x)=0$

(b) If $\lim_{x\to c} f(x)=0$ then $\lim_{x\to c} f(x)g(x)=0$

(c) If $\lim_{x\to c} f(x)=g(x)$ then $\lim_{x\to c} g(x)=c$

(d) If $\lim_{x\to c} f(x)\neq g(x)$ then $\lim_{x\to c} g(x)=0$

181. $g(x)=1/x$ is

(a) uniformly continuous on $[0,1]$

(b) not uniformly continuous on $(0,1]$

(c) uniformly continuous on $(0,1]$

(d) continuous on $[0,1]$

182. Suppose that g is an infinitely differentiable function which satisfies $g''+g'-g=0$. If $g(0)=g(a)=0$, then

(a) $g(x)=0$ for all $x\in[0,a]$

(b) $g(x)\neq 0$ for all $x\in[0,a]$

(c) $g(x)<0$ for $x\in(0,a)$

(d) $g(x)>0$ for $x\in(0,a)$

183. If g is differentiable on an interval where $g^1(x)\neq 1$ then

(a) g has two fixed point

(b) g have at most one fixed point

(c) g has three fixed points

(d) g has infinitely many fixed points.

184. Let $\{a_1,a_2,...\}$ be an enumeration of Q. For each $n\in\mathbb{N}$, define

$$f_n(x)=\begin{cases} 1/2^n & \text{if } x>a_n \\ 0 & \text{if } x\le a_n \end{cases}$$

and $g(x)=\sum_{n=1}^{\infty} f_n(x)$. Then which of the following imply true?

(a) $g(x)$ is uniformly convergent on R

(b) $g(x)$ is not monotone in R

(c) $g(x)$ is continuous for $x\in Q$

(d) $g(x)$ is divergent.

185. Find te coefficient x^7 in the Taylor series expansion of the function $f(x)=\log\left(x+\sqrt{1+x^2}\right)$ about the origin

(a) -5/112 (b) 1/10

(c) 1 (d) 104

186. Let $f:R\to R$ be differentiable at $x=a$.

Evaluate $\lim_{n\to\infty}\left\{f\left(a+1/n^2\right)+f\left(a+2/n^2\right)+\dots +f\left(a+n/n^2\right)-nf(a)\right\}$

(a) $f(0)$ (b) $\frac{1}{2}f^1(a)$

(c) $f^1(a)$ (d) 1

187. Let $f:R\to R$ be continuously differentiable

Evaluate: $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f^1\left(k/n\right)$.

(a) $f(1)$ (b) $f(1)-f(0)$

(c) 1 (d) ∞

188. Assume that $b>0$. Let $x_1=\sqrt{b}$ and let $x_{n+1}=\sqrt{b+x_n}$ for all $n\ge 1$. Then

(a) $x_n < 1+\sqrt{b}$ and that (x_n) is increasing

(b) (x_n) is diverges

(c) limit of (x_n) is 0

(d) (x_n) is decreasing

189. Let $a, b \in R$ with $a < b.$ Let $\langle g_n \rangle$ be sequence of continuous real valued functions that converges uniformly on $[a,b].$ For every $n \in w$ let $h_n(x) = \int_a^x g_n$

(a) $\langle h_n \rangle$ converges uniformly on $[a,b]$

(b) $\langle h_n \rangle$ diverges

(c) $\langle h_n \rangle$ converges pointwise on (a,b)

(d) $\langle h_n \rangle$ converges on (a,b)

190. For every $n \in w$ and every $x \geq 0$ but

$$g_n(x) = \frac{1}{1+x^n}.$$

(a) $\langle g_n \rangle$ converges pointwise but not uniformly on $[0,\infty)$

(b) $\langle g_n \rangle$ converges uniformly but not poinwise on $(0,\infty)$

(c) $\langle g_n \rangle$ does not converges

(d) $\langle g_n \rangle$ converges on $(0,\infty)$

191. $a \in A$ is isolated iff

(a) if there is $\varepsilon > 0 \ni B(a;\varepsilon) \cap A = \{a\}$

(b) if there is $\varepsilon < 0 \ni B(a;\varepsilon) \cup A = A$

(c) if there is $\delta > 0 \ni B(a;\delta) \setminus A = \{a\}$

(d) if there is $\delta < 0 \ni B(a;\delta) = \{a\}$

192. A differentiable real valued function on R with bounded derivative is

(a) continuous

(b) discontinuous

(c) uniformly continuous

(d) jump continuous

193. $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x^2}$

(a) $\log 6$ (b) 1

(c) 0 (d) $e^{-1/6}$

194. $\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2 - k^2}$

(a) $\pi/2$ (b) $\pi/4$

(c) 1 (d) 0

195. Compute $f(x)$, $f_n(x) = \lim_{n \to \infty} n^2 x(1-x^2)^n$ where $0 \leq x \leq 1$

(a) 1 (b) 0

(c) 2 (d) ∞

196. The interval of convergence of the series $\sum_{n=1}^{\infty} \frac{\log(n+1)}{\sqrt{n+1}} (x-5)^n$ is

(a) $4 \leq x < 6$ (b) $2 \leq x \leq 6$

(c) $5 \leq x \leq 6$ (d) $4 \leq x \leq 6$

197. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x dx}{\sin x + \cos x}$

(a) 1 (b) $\frac{1}{\sqrt{2}} \log\left(\sqrt{2}+1\right)$

(c) 0 (d) 2π

198. If g and h are measurable functions then

(a) the set $\{x : g(x) \neq h(x)\}$ is measurable.

(b) the set $\{x : g(x) < h(x)\}$ is measurable

(c) $\{x : g(x) = h(x)\}, \{x : g(x) \geq h(x)\},$ $\{x : g(x) > h(x)\}$ are measurable

(d) the set $\{x : g(x) = h(x)\}$ is not measurable

199. Let g_n be a nonegative measurable functions

(a) If $g_n \to g$ and $g_n \le g$ for all n then $\int g_n \to \int g$

(b) If $g_n \to g$ and $g_n \ge g$ for all n then $\int g_n \to \int g$

(c) If $\int g_n \to \int g$ then $g_n \to g$ and $g_n \ge g$

(d) If $\int g_n \to \int g$ then $g_n \to g$ and $g_n \ge 0$ for all n

200. Let S and T be nonempty bounded subsets of R with $S \subset T$. Then

(a) $\inf T \le \inf S \le \sup T \le \sup S$

(b) $\inf T \le \inf S \le \sup S \le \sup T$

(c) $\sup T \le \sup S \le \inf T \le \inf S$

(d) $\sup T \le \sup S \le \inf S \le \inf T$

201. Let $f(x)$ be a periodic function.

(a) If $\lim\limits_{x\to\infty} f(x)$ exists then $f(x)$ is a constant function.

(b) If $\lim\limits_{x\to\infty} f(x)$ does not exists then $f(x)$ is a constant function

(c) $\lim\limits_{x\to\infty} f(x)$ does not exist

(d) If $\lim\limits_{x\to\infty} f(x)$ exists, then $f(x)$ is a non constant function

202. Which of the following function is uniformly continuous

(a) $f(x) = e^x$ on $[0,\infty)$

(b) $f(x) = x\sin x$ on $[0,\infty)$

(c) $f(x) = \text{In}x$ on $(0,1)$

(d) $f(x) = \sqrt{x}$ on $[0,\infty)$

203. Consider a function *f(x)* whose second derivative $f''(x)$ exists and is continuous on $[0,1]$. Assume that $f(0) = f(1) = 0$ and suppose that there exists $A > 0 \ni |f''(x)| \le A$ for $x \in [0,1]$.

(a) $\left|f'\left(\frac{1}{2}\right)\right| \le \frac{A}{4}$ and $|f'(x)| \le \frac{A}{2}$ for $0 < x < 1$

(b) $\left|f\left(\frac{1}{2}\right)\right| \le \frac{A}{4}$ and $|f'(x)| \le \frac{A}{2}$ for $0 < x < 1$

(c) $|f'(x)| \le \frac{A}{4}$ and $|f'(1)| \le A$ for $0 > x > 1$

(d) $|f''(x)| \le \left|f'(x) - f'\left(\frac{1}{2}\right)\right|$

204. $\lim\limits_{x\to 0}(1-\sin x\cos x)^{co\sec 2x} =$

(a) $\frac{1}{\sqrt{e}}$ (b) $\log \frac{1}{2}$

(c) 0 (d) ∞

205. $\lim\limits_{n\to\infty} \frac{1}{n^6}\sum\limits_{k=1}^{n} k^5$

(a) 6 (b) $\frac{1}{6}$

(c) 0 (d) 1

206. The series $\sum\limits_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}, x \in R$ is

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

207. $\sum\limits_{n=1}^{\infty}(-1)^n \frac{n}{n+2}$

(a) convergent
(b) absolutely convergent
(c) conditionally convergent
(d) divergent

208. Let $0 < a < b$ and $f(x) = \begin{cases} 0 & \text{if } x \in [a,b] \cap Q \\ x & \text{if } x \in [a,b] \text{ is irrational} \end{cases}$

Find the upper and lower Riemann integrals of $f(x)$ over $[a,b]$

(a) $L(P,f) = 0$ and $U(P,f) = b-a$

(b) $L(P,f) = a$ and $U(P,f) = b$

(c) $L(P,f)=b$ and $U(P,f)=a-b$

(d) $L(P,f)=0$ and $U(P,f)=a$

209. Let $f:[a,b]\to R$ be a Riemann integrable function. Let $g:[a,b]\to R$ be function $\ni$ $\{x\in[a,b], f(x)\neq g(x)\}$ is finite. Then which of the following is true?

(a) $\int_a^b f(x)dx \neq \int_a^b g(x)dx$

(b) $\int_a^b f(x)dx = \int_a^b g(x)dx$

(c) $\left|\int_a^b f(x)dx\right| \leq \int_a^b g(x)dx$

(d) $\left|\int_a^b f(x)dx\right| \geq \int_a^b g(x)dx$

210. Consider the function $g(x)=\begin{cases}1 & 0\leq x\leq 1\\ 0 & 1<x\leq 2\end{cases}$

$G(x)=\int_0^x g(t)dt$ on $[0,2]$. Then

(a) $G(x)$ is continuous

(b) $G'(x)\neq(x)$ for $x\in(0,2)$ with $x\neq 1$

(c) $G(x)$ is continuous and differentiable expect at 1

(d) $G(x)$ is dicontinuous

211. Find the sum of the series $\sum_{K=1}^{\infty}\text{In}\dfrac{\left(\dfrac{K^{K+1}}{(K+1)^k}\right)}{K(K+1)}$

(a) $S_n=-\dfrac{\text{In}(n+1)}{\text{n+1}}$

(b) $S_n=\dfrac{\text{In}(n-1)}{\text{n-1}}$

(c) $S_n=-\dfrac{\text{In}(n)}{n}$

(d) $S_n=\dfrac{\text{In}(n+1)}{n-1}$

212. Suppose that $\sum x_n$ is a series of positive terms which is convergent. Then

(a) $\sum x_n^2$ and $\sum\sqrt{x_n x_{n+1}}$ are convergent

(b) $\sum x_n^2$ and $\sum\sqrt{x_n x_{n+1}}$ are divergent

(c) $\sum x_n^2$ is convergent and $\sum\sqrt{x_n x_{n+1}}$ is divergent.

(d) $\sum\sqrt{x_n x_{n+1}}$ is convergent $\sum x_n^2$ is divergent

213. Every convergent sequence in a metric space. *(X,d)* is

(a) cauchy sequence

(b) convergent sequence

(c) divergent sequence

(d) bounded sequence

214. For what values of P does the following series converge?

$1-\dfrac{1}{2^p}+\dfrac{1}{3^p}-\dfrac{1}{4^p}+...$

(a) $P<0$ (b) $P=0$

(c) $P>0$ (d) $P<1$

215. The series $\sum_{n=1}^{\infty}(-1)^n\dfrac{\cos n\alpha}{n^2}$ where $\alpha\in R$ is a fixed real number

(a) absolutely convergent

(b) convergent

(c) uniformly convergent

(d) divergent

216. Th series $\sum_{n=1}^{\infty}(-1)^n\dfrac{n\log n}{e^n}$ is

(a) convergent

(b) uniformly convergent

(c) absolutely convergent

(d) divergent

217. Consider the sequence $\{f_n\}$ defined by

$f_n(x)=\dfrac{nx}{e^{nx}}$ for $x\in[0,2]$

(a) The converges is uniform on $[0,2]$

(b) The converges is not uniform on $[0,2]$

(c) $\lim_{n\to\infty} f_n = 1$ for $x \in [0,2]$

(d) The convergence is uniform on $[0,1]$

218. Suppose that $\{g_n\}$ is a sequence of funtions defined on D and $\{M_n\}$ is a sequence of nonnegative numbers such that $|g_n(x)| \le M_n$ for every $x \in D$ for every $n \in N$. Then

(a) if $\sum_{n=0}^{\infty} M_n$ converges then $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on D.

(b) If $\sum_{n=0}^{\infty} g_n(x)$ converges then $\sum_{n=0}^{\infty} M_n$ converges

(c) If $\sum_{n=0}^{\infty} M_n$ diverges then $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on D.

(d) If $\sum_{n=0}^{\infty} g_n(x)$ diverges then $\sum_{n=0}^{\infty} M_n$ converges

219. Which of the following is uniform convergence

(a) $g_n(x) = nx^n(1-x)$ on $[0,1]$

(b) $\sum_{n=1}^{\infty} \frac{\cos^2(nx)}{n^2}$ on R

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ on R

(d) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{\sqrt{n}}$ $[0,2\pi]$

220. Let $g_n : [0,1] \to R$ be continuous $\ni \{g_n\}$ are unfiromly bounded on $[0,1]$ and the derivatives g_n' exist and are uniformly bounded on (0,1). Then

(a) g_n is a divergent subsequence

(b) g_n has a uniformly convergent subsequence

(c) g_n has convergent subsequence

(d) None of the above

221. For each $n \ge 1$, let f_n be a monotonic increasing real valued function on $[0,1]$ $\ni$ the sequence of functions $\{f_n\}$ converges pointwise to the function $f \equiv 0$. Pick out the true statements from the following

(a) If the functions f_n are also non-negative then f_n must be continuous for sufficiently large *n*.

(b) f_n converges to *f* uniformly.

(c) f_n diverges

(d) None of the above

222. The sequence of functions $\{n^2x^2e^{-nx}\}$ over the interval $(0,\infty)$

(a) convergent

(b) uniformly convergent

(c) not uniformly convergent

(d) None of the above

ANSWERS

1. (c)	**2.** (a)	**3.** (a)	**4.** (a)	**5.** (a)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (d)
11. (b)	**12.** (a)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (b)	**18.** (c)	**19.** (a)	**20.** (d)
21. (a)	**22.** (c)	**23.** (b)	**24.** (a)	**25.** (d)	**26.** (b)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (b)
31. (a)	**32.** (b)	**33.** (c)	**34.** (b)	**35.** (a)	**36.** (b)	**37.** (a)	**38.** (a)	**39.** (c)	**40.** (b)
41. (c)	**42.** (b)	**43.** (b)	**44.** (a)	**45.** (c)	**46.** (a)	**47.** (b)	**48.** (a)	**49.** (b)	**50.** (c)
51. (a)	**52.** (a)	**53.** (c)	**54.** (c)	**55.** (b)	**56.** (a)	**57.** (c)	**58.** (d)	**59.** (c)	**60.** (a)
61. (a)	**62.** (d)	**63.** (a)	**64.** (c)	**65.** (b)	**66.** (b)	**67.** (a)	**68.** (d)	**69.** (a)	**70.** (d)
71. (c)	**72.** (b)	**73.** (a)	**74.** (b)	**75.** (b)	**76.** (d)	**77.** (a)	**78.** (b)	**79.** (b)	**80.** (c)
81. (c)	**82.** (a,b,c)	**83.** (a)	**84.** (a)	**85.** (b)	**86.** (a)	**87.** (a)	**88.** (b)	**89.** (d)	**90.** (b)
91. (b)	**92.** (c)	**93.** (a)	**94.** (a,c)	**95.** (a)	**96.** (a)	**97.** (b)	**98.** (c)	**99.** (c)	**100.**(d)
101.(b)	**102.**(c)	**103.**(b)	**104.**(a)	**105.**(a)	**106.**(b)	**107.**(b)	**108.**(a)	**109.**(b)	**110.**(b)
111.(a)	**112.**(c)	**113.**(c)	**114.**(a)	**115.**(c)	**116.**(b)	**117.**(c)	**118.**(b)	**119.**(a)	**120.**(c)
121.(a)	**122.**(a)	**123.**(b)	**124.**(b)	**125.**(b)	**126.**(c)	**127.**(d)	**128.**(d)	**129.**(b)	**130.**(b)
131.(b)	**132.**(b)	**133.**(a)	**134.**(d)	**135.**(b)	**136.**(c)	**137.**(a)	**138.**(a)	**139.**(a)	**140.**(d)
141.(c)	**142.**(a)	**143.**(a)	**144.**(b)	**145.**(c)	**146.**(a)	**147.**(a)	**148.**(b)	**149.**(b)	**150.**(b)
151.(c)	**152.**(c)	**153.**(c)	**154.**(b)	**155.**(b)	**156.**(c)	**157.**(c)	**158.**(a)	**159.**(a)	**160.**(a)
161.(a)	**162.**(a)	**163.**(b)	**164.**(a)	**165.**(a,b)	**166.**(a)	**167.**(b)	**168.**(b)	**169.**(d)	**170.**(d)
171.(c)	**172.**(a)	**173.**(b)	**174.**(a)	**175.**(a)	**176.**(d)	**177.**(a)	**178.**(b)	**179.**(a)	**180.**(b)
181.(b)	**182.**(a)	**183.**(b)	**184.**(a)	**185.**(a)	**186.**(b)	**187.**(b)	**188.**(a)	**189.**(a)	**190.**(a)
191.(a)	**192.**(a,c)	**193.**(d)	**194.**(b)	**195.**(b)	**196.**(a)	**197.**(b)	**198.**(c)	**199.**(a)	**200.**(b)
201.(b)	**202.**(d)	**203.**(a)	**204.**(a)	**205.**(b)	**206.**(b)	**207.**(d)	**208.**(a)	**209.**(a)	**210.**(c)
211.(a)	**212.**(a)	**213.**(a)	**214.**(c)	**215.**(a)	**216.**(c)	**217.**(b)	**218.**(a)	**219.**(b)	**220.**(b)
221.(b)	**222.**(c)								

EXPLANATIONS

1. (c) $f_n = \Sigma_n \dfrac{\sin(nt)}{n^2}$. As $|\sin(nt)| \le 1$, $|a_n| \le \dfrac{1}{n^2}$. $\because \Sigma_n \dfrac{1}{n^2}$ converges. So $\Sigma_n |a_n|$ converges by comparison test $\because f_n = \Sigma_n \dfrac{\sin(nt)}{n^2}$ converges absolutely. $g_n = \sum_n (-1)^n 2^{-n^2}$. For $n \in N, -n^2 \le -n$

$\Rightarrow 0 < 2^{-n^2} \le 2^{-n}$ and so $\sum_{n=1}^{\infty} |(-1)^n 2^{-n^2}|$ converges by comparison with the convergent geometric

series $\sum_{n=1}^{\infty} 2^{-n}$ $\therefore g_n$ converges absolutely.

2. (a) Let $\varepsilon > 0$, choose $\delta = \varepsilon/2$. Given $x, y \in X$ with $d(x,y) < \delta$ we can find $z \in M$ so that $d(x,z) < dist(x,M) + \varepsilon/2$. $\because dist(y,M) \le d(y,z)$.

$dist(y,M) \le d(y,z) \le d(x,y) + d(x,z) < \delta + dist(x,M) + \varepsilon/2 < dist(x,M) + \varepsilon.$

Thus $dist(y,M) - dist(x,M) < \varepsilon$. Reversing the roles of x, y in the above argument gives $dist(x,M) - dist(y,M) < \varepsilon \Rightarrow |dist(y,M) - dist(x,M)| < \varepsilon$, So, the function is continuous.

3. (a) Consider $f(x) = h(x)e^{j(x)}$, $f(a) = f(b) = 0$. So by Rolle's theorem $f'(x) = 0$ for some $x \in (a,b)$.

But $f'(x) = h(x)e^{j(x)} j'(x) + h'(x)e^{j(x)} = e^{j(x)}\left[h(x)j'(x) + h'(x)\right]$, $e^{j(x)} \ne 0$

4. (a) Cosider the function $\dfrac{1}{g}$. Rewrite this as a composite function $h = \dfrac{1}{x}$ and $g = g(x)$ then $h \circ g = 1/g$

h is continuous on every point except 0 and $\because g(x) \ge k > 0$ we know h is continuous on the range $[c,d]$ where $g([a,b]) \subset [c,d]$. $\because$ The composite of a continuous and integrable function is integrable. $\therefore$ $h \circ g$ is integrable on $[a,b]$ which means $1/g$ is integrable on $[a,b]$.

5. (a) The graph g is a sawtooth which is1 at each integer, a narrow triangle near each integer, and 0 otherwise. The sum $\sum g$ diverges; yet each triangle has are 2^{-n}, so $\int_1^{\infty} g(x)dx = \sum_{n=1}^{\infty} \dfrac{1}{2^n} = 1.$

6. (d) The ball is clearly closed and bounded, but not compact, consider the sequence (x_n) in B where in each x_n is a sequence of all zeros, except for a 1 in the n^{th} position. Whenever $n \ne m$ we have $\| x_n - x_m \|_\infty = 1$, so that no subsequence can be cauchy, let alone convergent.

7. (a) $\because x_n \le z_n$ for every n. Then $\liminf z_n \le \limsup z_n \le \liminf x_n \le \liminf z_n$. Hence $\limsup z_n = \liminf z_n$ and so $\lim z_n$ exists.Similarly $\limsup x_n \ge \liminf x_n \ge \limsup z_n \ge \limsup x_n$.

Hence $\limsup x_n = \liminf x_n$ and so $\lim x_n$ exists. Moreover $\lim x_n$ =limsup x_n =limsup z_n =lim z_n

By squeeze theorem we get the result $\lim x_n = \lim y_n = \lim z_n$

8. (b) Fix $\varepsilon > 0$ and find $N \ni n \geq N$. $\Rightarrow na_{2n} = |na_{2n}| < \varepsilon/2$. Now for $n > 2N$ we can find $m \geq N \ni n/3 \leq m \leq n/2$. Hence $|ma_{2m}| < \varepsilon/3$. Since x_K is a decreasing sequence then $|ma_n| \leq |ma_{2m}| < \varepsilon/3$. Hence $|na_n| \leq |3ma_n| < \varepsilon$, $\because n \leq 3m$. Thus $\lim na_n = 0$ Suppose that $\Sigma \frac{1}{n^s}$ converges. Then $0 = \lim n\left(\frac{1}{n^s}\right) = \lim n^{1-s}$. This is false if $0 \leq S \leq 1$. Hence $\Sigma 1/n^s$ diverges.

12. (a) The function g is uniformly continuous on $[-1,1]$ because it is continuous and $[-1,1]$ is a compact set, so it is uniformly continuous. g does not satisfy a Lipschitz condition, suppose for a contradiction that there is some $M \ni |\sqrt{|x|} - \sqrt{|y|}| \leq M|x-y|$ for all $x, y \in [-1,1]$. Take $y = 0$ and $x > 0$, so $\sqrt{x} \leq Mx$ for all $x > 0$. But then $M \geq 1/\sqrt{x}$ for all $x > 0$, which is impossible.

13. (a) $g(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$, If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ then $V(g,P) = \Sigma\left|g(x_i) - g(x_{i-1})\right|$. In particular, if the x_i are taken to be alternately rational and irrational, then $V(g,P) \geq n-2$. For any $\delta > 0$, take a partition P with $|P| < \delta$ and $V(g,P) \geq N$ for any given natural number N. It is clear that no value of L can exist. $\therefore g(x)$ is not a bounded variation.

17. (b) If g is Lipschitz, then we have some $M > 0 \ni$ for all x, y. $|g(x) - g(y)| \leq M|x-y|$. Then we have that $\left|g'(x)\right| = \lim_{y \to x}\left|\frac{g(x) - g(y)}{x - y}\right| \leq M$.

20. (d) Let y be a limit point of $\{x : g(x) = 0\}$. So there is a sequence $\{y_n\} \ni y_n \in \{x : g(x) = 0\}$ for all n and $\lim_{n\to\infty} y_n = y$. Since g is continuous. $\therefore g(y) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} 0 = 0$. Hence, $y \in \{x : g(x) = 0\}$, so $\{x : g(x) = 0\}$ contains all its limit points and is closed subset of R.

21. (a) By Jordan's decomposition theorem, the function g can be represented as the difference of the non-decreasing functions i and h. (i.e) $g = i - h$. Since i and h are non-decreasing, their derivatives exist almost everywhere and are non-negative.

Using the formula $$V_a^b g = V_a^b i + V_a^b h$$

$$\int_a^b \left|g'\right| dx = \int_a^b \left|i' - h'\right| dx \leq \int_a^b \left|i'\right| dx + \int_a^b \left|h'\right| dx$$

$$= \int_a^b i'dx + \int_a^b h'dx = i(b) - i(a) + h(b) - h(a)$$

$$= V_a^b i + V_a^b h$$

$$= V_a^b g$$

22. (c) Fix $y \in R$. For $x \neq y$ $\left|\dfrac{g(x)-g(y)}{x-y}\right| \leq |x-y|$

So, $$-|x-y| \leq \frac{g(x)-g(y)}{x-y} \leq |x-y|$$

$\because \lim\limits_{x\to y} -|x-y| = 0$ and $\lim\limits_{x\to y} |x-y| = 0$. The squeeze theorem for limits implies that

$$\lim_{x\to y} \frac{g(x)-g(y)}{x-y} = 0.$$

Thus, g is differentiable at y and $g'(y) = 0.$ Since this holds for all $y \in R$, g must be constant.

25. (d) For any x, y we have $g(x)^2 - g(y)^2 = (g(x)+g(y))(g(x)-g(y)) \leq 2M(g(x)+g(y))$

Let $P = \{a = x_0 < x_1 < .... < x_n = b\}$ be a partition of $[a,b]$. For any $x, y \in [x_i, x_{i+1}]$

then $g(x)^2 - g(y)^2 \leq 2M(g(x)+g(y)) \leq 2M\left(\sup\limits_{t\in[x_i,x_{i-1}]} g(t) - \inf\limits_{t\in[x_i,x_{i-1}]} g(t)\right)$ denote the number on the right by c. Then for any $x \in [x_i, x_{i+1}]$

$$g(x)^2 - c \leq g(y)^2 \text{ for all } y \in [x_i, x_{i+1}]$$

$$\Rightarrow g(x)^2 - c \leq \inf_{t\in[x_i,x_{i+1}]} g(t)^2 \text{ for all } x \in [x_i, x_{i+1}]$$

$$\Rightarrow g(x)^2 \leq c + \inf_{t\in[x_i,x_{i+1}]} g(t)^2 \text{ for all } x \in [x_i, x_{i+1}]$$

$$\sup_{t\in[x_i,x_{i+1}]} g(t)^2 \leq c + \inf_{t\in[x_i,x_{i+1}]} g(t)^2$$

and thus $$\sup_{t\in[x_i,x_{i+1}]} g(t)^2 - \inf_{t\in[x_i,x_{i+1}]} g(t)^2 \leq 2M\left(\sup_{t\in[x_i,x_{i+1}]} g(t) - \inf_{t\in[x_i,x_{i+1}]} g(t)\right)$$

We conclude that $U(g^2,P) - L(g^2,P) = \Sigma\left(\sup\limits_{t\in[x_i,x_{i+1}]} g(t)^2 - \inf\limits_{t\in[x_i,x_{i+1}]} g(t)^2\right)$ (length of $[x_i, x_{i+1}]$)

$$\leq 2M\left(\sup_{t\in[x_i,x_{i+1}]} g(t) - \inf_{t\in[x_i,x_{i+1}]} g(t)\right) \quad \text{(length of } [x_i, x_{i+1}])$$

$$= 2M[U(g,P) - L(g,P)]$$

28. (c) Let $g:(-1,1)\to R$, $\lim\limits_{x\to 0}\dfrac{g(x)}{x^2}=L\in R$, $\lim\limits_{x\to 0} g(x)=0$. $\therefore$ By L'Hospital rule, $\lim\limits_{x\to 0}\dfrac{g^1(x)}{x}=2L$

$g^1(x)\to 0$ as $x\to 0$, $g'(0)=\lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x}=\lim\limits_{x\to 0}\dfrac{g(x)}{x}$, because $\lim\limits_{x\to 0} g(x)=0$ and g is continuous.

This limit exists since g is differentiable. By L'Hospital rule we get $g'(0)=\lim\limits_{x\to 0}\dfrac{g(x)}{x}=\lim\limits_{x\to 0}\dfrac{g'(x)}{1}=0$

So, $g'(0)=0$. $\therefore g''(0)=\lim\limits_{x\to 0}\dfrac{g'(x)-g'(0)}{x}=\lim\limits_{x\to 0}\dfrac{g'(x)}{x}=2L.$

29. (a) $\sup \displaystyle\int_{|h_n|\geq\alpha} |h_n|dx \leq \int_{|g|\geq\alpha} |g|\,dx$. To show that $\lim\limits_{\alpha\to\infty}\displaystyle\int_{|g|\geq\alpha}|g|\,dx=0.$ This is simply a consequence of the Lebesgue dominated convergence theorem applied to $g_\alpha = 1_{|g|\geq\alpha}|g|$ which is dominated by $|g|$.

Consider $h_n(x)=1_{[n,n+1]}$. This is clearly uniformly integrable. (Choose $\alpha>1$) but if g is a function $\ni g\geq h_n$ for all n then $g\geq 1$ on $[1,\infty)$. Hence g is not integrable.

31. (a) Taylor's theorem gives $g(y)=g(x)+g'(x)(y-x)+\dfrac{g''(c)}{2}(y-x)^2$ for some c between x and y.

Let $y=x+h$, $g(x+h)=g(x)+g'(x)h+\dfrac{g''(c)}{2}h^2$; $-g'(x)h=g(x)-g(x+h)+\dfrac{g''(c)}{2}h^2$

Taking absolute values $|g'(x)h|\leq|g(x)|+|g(x+h)|+\left|\dfrac{g''(c)}{2}h^2\right|$. If $M_k=\sup\{|g'(x)|:x\in(a,\infty)\}$

then $|g^1(x)h|\leq 2M_0+\dfrac{M_2}{2}|h|^2$; $b^2-4ac\leq 0$. $4|g'(x)|^2-16M_0M_2\leq 0$. Thus

$|g'(x)|^2\leq 4M_0M_2$ $\because$ This is true for any $x\in(a,\infty)$ we can take the supremum over x on the left and we get $M_1^2\leq 4M_0M_2$. Hence the result.

36. (b) Let $x=1$. Then $0=K=e^0=c+1$. So, $K=-1$. To find $f(x)$ take the derivative of both sides and use the Fundamental Theorem of calculus $f(x)=2(x-1)e^{(x-1)^2}$

37. (a) By the Mean Value Theorem there are points $a\in(0,1)$ and $b\in(1,3)\ni f'(a)=\dfrac{f(1)-f(0)}{1-0}=-4$

and $f'(b)=\dfrac{f(3)-f(1)}{3-1}=\dfrac{6}{2}=3$. Therefore applying the Mean Value Theorem, f' we find there is a point $c\in(a,b)\ni f''(c)=\dfrac{f'(b)-f'(a)}{b-a}=\dfrac{7}{b-a}>\dfrac{7}{3}>0.$

38. (a) $\phi(y) = y^{2/3}$ is continuous on R, so $f^2 = \phi(f^3)$ is Riemann integrable because it is a continuous function of a Riemann integrable function.

40. (b) Since K is C' then $\exists\ M>0 \ni |K(x)-K(y)| \le M|x-y|$ for all $x, y \in [-1,1]$. Take $M = \sup|K'(x)|$.

Hence for any partition $x_0 < x_1 < \ldots < x_m$. Then we have $\sum |K(x_i) - K(x_{i-1})| \le M \sum |x_i - x_{i-1}| \le 2M$

Hence K is of bounded variation on $[-1,1]$

41. (c) Being by fixing a partition $\Gamma = \{x_i\}_{i=0}^{k}$ of interval $[a,b]$, $V[g_n; a,b] = \sup_{\Gamma} \sum_{i=1}^{k} |g_n(x_i) - g_n(x_{i-1})| \le M$

for all n. Further more $g_k \to g$ pointwise,

$$V[g;a,b] = \sup_{\Gamma} \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| \quad = \sup_{\Gamma} \lim_{n\to\infty} \sum_{i=1}^{k} |g_n(x_i) - g_n(x_{i-1})| \le M$$

45. (c) Let $\varepsilon > 0$. Since (f_n) converges to 0 pointwise $\exists\ N, M \ni |f_n(1)| < \varepsilon$ for $n \ge N$ and $|f_n(3)| < \varepsilon$ for $n \ge M$. Now, since each f_n is decreasing, we have $f_n(1) \ge f_n(x) \ge f_n(3)$ for all n and $x \in [1,3]$. Thus for $n \ge \max\{N, M\}$, we have for any $x \in [1,3]$ $-\varepsilon < f_n(3) \le f_n(x) \le f_n(1) < \varepsilon$. So $|f_n(x)| < \varepsilon$.

$\therefore (f_n)$ converges uniformly to 0.

48. (a) $\lim_{x\to 0} \dfrac{g(x)-g(0)}{x-0}$ exists and is finite, $\lim_{x\to 0} \dfrac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} x^2 \sin\left(\dfrac{1}{x}\right)$,

$-|x| \le x\sin\left(\dfrac{1}{x}\right) \le |x|$. Also $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$. By the squeeze theorem $\lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0$.

$\therefore g$ is differentiable at $x = 0$ and $g'(0) = \lim_{x\to 0} \dfrac{g(x)}{x} = 0$.

49. (b) If $x = 0$; clearly $1 + 0 \le e^0 = 1$ assume $x \ne 0$. Let $g(y) = 1 + y^2$ and $h(y) = e^{y^2}$ for $y \in [0, x]$.

g and h are continuous and differentiable on $[0, x]$, so by Cauchy's mean value thorem, there is some c between 0 and x such that $\dfrac{g(x)-g(0)}{h(x)-h(0)} = \dfrac{g'(x)}{h'(x)}$, thus $\dfrac{1+x^2-1}{e^{x^2}-1} = \dfrac{2c}{e^{c^2}(2c)} = \dfrac{1}{e^{c^2}} \le \dfrac{1}{e^0} = 1$

It is true for any $x \ne 0$. $\therefore x^2 \le e^{x^2} - 1 \Rightarrow 1 + x^2 \le e^{x^2}$ for all $x \in R$.

51. (a) For $x \ne y$, $\dfrac{|g(x)-g(y)|}{|x-y|} \le |x-y|$. Taking limit as $x \to y$ yielding $g'(y) = 0$. This holds for every $y \in R$, so that g must be constant.

53. (c) Since $g(a) < 0$, W is a bounded non-empty subset of $[a,b]$. We have $w = \sup W$ with $x \le w \le b$ for all $x \in W$. To show that $g(w)=0$. Suppose $g(w) \neq 0$ then either $g(w)>0$ or $g(w)<0$

If $g(w)>0$ then $\exists$ a neighborhood $V_\delta(w)=(w-\delta, w+\delta) \cap [a,b]$ $\ni g(x)>0$ for all $x \in V_\delta(w)$.

In this case $x \le w-\delta < w$ if $x \in W$ and thus w is not the least upper bound of W. We get a contradiction. If $g(x)<0$ then by a similar argument, we can find a neighborhood $V_\delta(w)=(w-\delta, w+\delta) \cap [a,b] \ni g(x)>0$ for all $x \in V_\delta(w)$. In this case we have $V_\delta(w) \subseteq W$ and thus w is not an upper bound of W. We also get a contradiction. This shows that $g(w)=0$.

56. (a) Let $X_1=\{x \in N, r>0\}$ and let $x_1=\min X_1$. Let $X_2=\{n \in N; n>n,$ and $r_n > r_{n_1}+1\}$ and let $n_2=\min X_2$. Continue in this way to get $n_3, n_4, \ldots$ so in general $n_s=\min\{n \in N; S>n_{s-1},$ and $r_{n_s} > r_{n_{s-1}}+1\}$. Then (r_{n_s}) is an increasing sequence. Also for $\varepsilon = 1/2$, $\left|r_{n_p}-r_{n_q}\right| > \varepsilon$ for all $p, q \Rightarrow (r_{n_s})$ is not cauchy, so it diverges and $r_{n_s}>0$ for all n_s. Thus, $\lim r_{n_s}=+\infty$.

57. (c) $\because g(x) \neq 0$ on $[a,b]$ the function $1/g$ is defined and continuous on $[a,b]$. Here there is $M>0$ so that

$\dfrac{1}{g(x)} < M$ for all x. $g(x) > \dfrac{1}{M} > 0$ for all x in $[a,b]$. Let $D=\{a,b\}$. Then $L(f,D) \le \int_a^b f$.

However $L(f,D) > \dfrac{1}{M}|b-a| > 0$. $\therefore \int_a^b g(x)dx > 0$.

60. (a) Given $\varepsilon<0$, let N be $\ni n>N$ $\Rightarrow |x_n-0|<\varepsilon^2$ (such an N exists since $\lim S_n=0$). But then

$n>N, \Rightarrow x_n<\varepsilon^2$, Hence $\sqrt{x_n}<\varepsilon$, hence $\left|\sqrt{x_n}-0\right|<\varepsilon$ $\Rightarrow \lim\sqrt{x_n}=0$

61. (a) Given $M>0$, let N_1 be $\ni n>N_1$ $\Rightarrow a_n>b$ such as N_1 exists since $\lim a_n=+\infty$.

Let $N=\max\{N_0, N_1\}$. Then $n>N \Rightarrow M<a_n \le b_n$ so $\lim b_n=+\infty$.

62. (d) Suppose that the sequence is equicontinuous $\exists$ a δ so that for any n and $x, y \in R$ with $|x-y|<\delta$

$|\sin nx - \sin ny|<1$. Let n be so large that $\dfrac{\pi}{2n}<\delta$. Then $\left|\dfrac{\pi}{2n}-0\right|<\delta$, $\left|\sin\left(n.\dfrac{\pi}{2n}\right)-\sin(n.0)\right|=1$

a contradiction. The family is not equicontinuous.

63. (a) Let $x_0 \in (0,1)$, $h_n(x)=h_n(x_0)+\int_{x_0}^{x} h_n'\ (u)du$ for $x \in (0,1)$.

Take the limit as $n \to \infty$ $\quad h(x)-h(x_0)=\lim_{n\to\infty}(h_n(x)-h_n(x_0)) \quad = \lim_{n\to\infty}\int_{x_0}^{x} h_n'(u)du \quad = \int_{x_0}^{x} f(u)du$.

Because uniform convergence allow for interchaning the limit with the integral.

The equation $$h(x)-h(x_0)=\int_{x_0}^{x} f(u)du\,.$$

$$\Rightarrow h'(x)=f(x) \text{ for any } x\in(0,1).$$

67. Let
$$f_-(x)=\lim_{\delta\to+0} f(x-\delta),$$
$$f_+(x)=\lim_{\delta\to+0} f(x+\delta)$$

These one-sided limits exist because *f* is monotone and lence, $f(x-\delta)\le f(x)\le f(x+\delta)$ Moreover, these limits are finite, because $f(x-\delta)\le f_-(x)\le f(x)\le f_+(x)\le f(x+\delta)$. *f* is discontinuous at a point *x* iff $f_-(x)<f_+(x)$. Since rational points are dense in *R*, there is a point $q_x\in\left(f_-(x)+f_+(x)\right)\cap Q$. The mapping $x\to q_x$ is injective; since if $x_1<x_2$ are two different points of discontinuity, then $q_{x_1}<f_-(x_1)\le f\left(x_1+x_2\big/2\right)\le f_+(x_2)<q_{x_2}$ and hence $q_{x_1}\ne q_{x_2}$. Thus we constructed an injective mapping from the set R_f of all discontinuties of *f* to *Q* and $Card\left(R_f\right)\le Card\left(Q\right)$.

$\Rightarrow f$ cannot have more than countable number of discontinuities.

70. (d) We can integrate $\int\frac{\log x}{x}dx=\frac{(\log x)^2}{2}+c$ use comparison test. Notice that for $n\ge 3$,

$$\log(n)>1 \text{ Hence } \frac{\log n}{n}>\frac{1}{n}>0 \text{ for } n\ge 3.$$

Since $$\Sigma\frac{1}{n}\text{ diverges }\Rightarrow\ \Sigma\frac{\log n}{n}\text{ diverges}$$

71. (c) For $\varepsilon_0=1$ and any $\delta>0$ there exist $x,y\in R\ni|x-y|<\delta$ and $|x^3-y^3|\le 1$. To find *x* and *y*, let $y=x+\delta\!\big/\!2$. Then $|x^3-y^3|=\left|x^3-\left(x+\delta\!\big/\!2\right)^3\right|$ $=\frac{3}{2}x^2\delta+\frac{3}{4}x\delta^2+\frac{\delta^2}{8}>\frac{3}{2}x^2\delta$ which is equal to 1 if $x=\sqrt{\frac{2}{3\delta}}$. For any $\delta>0$, let $x=\sqrt{\frac{2}{3\delta}}$ and let $y=x+\frac{\delta}{2}$. Then $|x-y|=\frac{\delta}{2}<\delta$ and $|x^3-y^3|>\frac{3\delta}{2}x^2=1=\xi_0$ so *f* is not uniformly continuous.

73. (a) Claim that $n^3<n!$ for $n\ge 6$. Let $n=6$. Then $6^3=216<720=6!$. Hence this case is true. Assume that $K^3<K!$ for some $K\ge 6$. Then $(K+1)^3=K^3+3K^2+3K+1<K!3K^2+3K+1<K!+K!...+K!$ Where there are *K* + 1summands. Then $(K+1)^3<(K+1)K!=(K+1)!$, $\therefore n^3<n!$ for all $n\ge 6$.

So, $$\left|\frac{n^2}{n!}\right|=\frac{n^2}{n!}<\frac{n^2}{n^3}=\frac{1}{n}.\ \lim_{n\to\infty}\frac{1}{n}=0\Rightarrow\lim_{n\to\infty}\frac{n^2}{n}=0$$

74. (b) $\lim_{n\to\infty}(1+a+...+a^n) = \lim_{n\to\infty}\frac{1-a^{n+1}}{1-a} = \frac{\lim\left(1-a^{n+1}\right)}{\lim(1-a)} = \frac{1-\lim a^{n+1}}{1-a}$ .Since $|a|<1$, $\lim a^n = 0$. So

$\lim_{n\to\infty}(1+a+...+a^n) = \frac{1}{1-a}$

87. (a) To show that for any $\varepsilon>0$ there is $\delta>0$ $\ni$ $\sum|L(G(a_i))-L(G(b_i))|<\varepsilon$ if $\sum|a_i-b_i|<\delta$. Fix $\varepsilon>0$. Since G is absolutely continuous, there is $\delta>0$ $\ni$ $\sum|G(a_i)-G(b_i)|<\varepsilon/M$. $\sum|a_i-b_i|<\delta$. For this choice of δ and $\sum|a_i-b_i|<\delta$ one can estimate

$\sum|L(G(a_i))-L(G(b_i))|<\ \sum M\,|(G(a_i))-G(b_i)|<\varepsilon.$

88. (b) Given $x\in X$, if $\exists\, N\in\mathbb{N}\ni \left|g_n(x)-g_{n+1}(x)\right|<\frac{1}{2^n}$ for all $n\geq N$, then $\{g_n(x)\}$ is cauchy, $\because$ for all $M\in \mathcal{M}$ $\ni M\geq N, n, m\geq M \Rightarrow \left|g_n(x)-g_m(x)\right|<\frac{1}{2^{m-1}}$ .Let $E_n=\{x\in X\colon \left|g_n(x)-g_{n+1}(x)\right|>1/2^n\}$

Then $\sum_{N\in\mathbb{N}}\mu(E_n)=1<\infty$ and so the Borel Cantelli lemma implies that μ almost every $x\in X$ lies in at most a finite number of the E_n's. $\therefore$ for μ almost every $x\in X$ $\exists\, N\ni\left|g_n(x)-g_{n+1}(x)\right|\leq\frac{1}{2^n}$ for all $n\geq N$. $\therefore\{g_n(x)\}$ is cauchy and convergent. The sequence $\{g_n(x)\}$ converges for μ almost all $x\in X$.

89. (d) To find a Cauchy sequence that does not converges with respect to $\|\bullet\|_1$

$$g_n(x)=\begin{cases}0 & x\in[a,(b-a)/2) \\ nx-n(b-a)/2 & x\in[(b-a)/2,(b-a)/2+1/n) \\ 1 & x\in[(b-a)/2+1/n,b)\end{cases}.$$

This converges to the step function $g(x)=\begin{cases}0 & x\in[a,(b-a)/2) \\ 1 & x\in[(b-a)/2,b)\end{cases}$ which is not continuous. The sequence $\|g_n-g\|_1$, then this is a Cauchy sequence but it converges to a function outside of $C([a,b])$. So $([a,b]),\|\bullet\|_1$ is incomplete.

90. (b) Given $\varepsilon>0$, Let $N=\frac{9}{\varepsilon^2}$. Let $x\in R$ and $n>N=\frac{9}{\varepsilon^2}$. Then $\frac{1}{n}<\frac{\varepsilon^2}{9}$, So $\frac{1}{\sqrt{n}}<\varepsilon/3$

$\frac{3}{\sqrt{n}}<\varepsilon$, $\left|g_n(x)-0\right|=\left|\frac{1+2\cos^2(nx)}{\sqrt{n}}-0\right|\leq\frac{3}{\sqrt{n}}<\varepsilon$. Thus $g_n\to g$ uniformly on R.

95. (a) Consider the function $f(x)-g(x)$ which is also differentiable. It derivative $f'(x)-g'(x)=0$. Thus $f(x)-g(x)$ is constant. So $f(x)-g(x)=k$ for some $k\in R$. Thus $f(x)=g(x)+k$

96. (a) Using 2 cases if $\ell = 0$ or if $\ell \neq 0$. If $\ell = 0$ the $\ell.f_n = 0$ is the constant 0 sequence. Thus, it converges to $0 = \ell.a$ as desired. Assume that $l \neq 0$, given $\varepsilon > 0$. since $f_n \to a$ then $\exists\, N \ni$ for all $n > N$ $|f_n - a| < \xi/|\ell|$. So for all $n > N$ $|\ell.f_n - \ell.a| = |\ell(f_n - a)| = |\ell|.|f_n - a| < \xi$. So $\ell.f_n \to \ell.a$

97. (b) To prove that 0 is not an isolated point. Let $\xi > 0$. By the Archimedean principle, there exists an $n \in z_+ \ni 1 < n\varepsilon$ and thus $0 < \frac{1}{n} < \varepsilon$. Thus $\frac{1}{n} \in B(0;\varepsilon) \cap T$. So it is not true that $B(0;\varepsilon) \cap T = \{0\}$. Thus, 0 is not isolated and therefore T is not discrete.

115. (c) The idea is that high p requires functions diverge slowly, but does allowas them to decay slowly as well. For $p > a$, $\int_0^1 x^{-a} dx < \infty$. while $p < a$, $\int_1^\infty x^{-a} dx < \infty$. So, $x^{-p}\chi[0,1]$ is not in L^q for $q > p$ while $x^{-p}\chi[1,\infty)$ is not in L^q for $q < p$. $\therefore L^p \subset L^q$.

116. (b) The function d is bounded on $M \times M$ if $\exists\ k > 0 \ni d(x,y) \leq k$ for all $(x,y) \in M \times M$ (i.e) for all $x, y \in M$. Hence, diam $M = \sup\limits_{x,y \in M} d(x,y) \leq k$

118. (b) Suppose the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges. Then $\exists$ a natural number N with $\sqrt{a_n} < 1$ for all $n > N$. It follows that $a_n < \sqrt{a_n} < 1$ for all $n > N$ and $\therefore \sum_{n=N+1}^{\infty} a_n$ converges. Hence, adding the finitely many terms $a_1, a_2, \ldots, a_N$ to this sum we obtain that $\sum_{n=1}^{\infty} a_n$ converges which is a contradiction.

119. (a) Given any $x \in R$ by the density of rational numbers we can find a sequence of rationals $(r_n) \ni r_n \to x$. Then by continuity of $g, g(x) = \lim g(r_n) = 0$.

138. (a) $\lim\limits_{n\to\infty} \int_{-1}^{1} \sin^n x dx = \int_{-1}^{1} \lim\limits_{n\to\infty} \sin^n x dx = \int_{-1}^{1} 0 dx = 0$

139. (a) The sum of this geometrical series is $= \frac{3}{2}\left(1 - 1/2 + 1/4 - \ldots\right) = \frac{3}{2} \cdot \frac{1}{\frac{3}{2}} = 1$

143. (a) Fix $N \ni n \geq N \Rightarrow |X_n| \leq 1$. Then if $n \geq N$ we have $X_n^2 \leq |X_n|$. Hence $\sum_{n=N}^{\infty} X_n^2$ must converge since $\sum |X_n|$ converge. Hence $\sum_{n=N}^{\infty} X_n^2$ converges as well. Since we are only adding finitely many terms to the series.

145. (c) Every one-point subset $\{y\}$ of a metric space is closed, because the only possible sequence of points in this subset is the constant sequence $y_n = y$ and this has limit y. Every subset X of M is the complement of its complement $X = (X')'$. But since M is finite, X^1 is closed. $\therefore (X')'$ is open

146. (a) $|a_n - b_n| \le 2$ for every n (by the triangle inequality). Hence the n^{th} element in the series is dominated by $2^{-(n-1)}$. Since $\sum 2^{-(n-1)} = 2$ the series converges. $\therefore \sum \frac{|a_n - b_n|}{2^n}$ converges

147. (a) ℓ^2 is a normed vector space over R with norm $\|\{x_n\}\|\ell^2 = \sqrt{\sum z_n^2}$. The Cauchy-Schwarz inequality for any $\{y_n\} \in \ell^2$ $|g(\{y_n\})| \le \|\{x_n\}\|_{\ell^2} \|\{y_n\}\|_{\ell^2}$. Hence, g is bounded by the constant $\|\{x_n\}\|_{\ell^2}$ and so is continuous.

149. (b) Fix $\varepsilon > 0$. Define $G:[0,2p] \to M$ as $G(x) = g(x)$. $\because [0,2p]$ is compact then $\exists\, \delta' > 0 \ni$ if $x, y \in [0,2p]$ and $|x-y| < \delta'$, $d(G(x), G(y)) < \varepsilon$. Let $\delta = \min\{\delta', p\}$. Now fix any $a, b \in R$ and suppose that $|a-b| < \delta$. Then $\exists\ x, y \in [0,2p]$ and $k \in z \ni\ x + kp = a$ and $y + kp = b$.
$\because g$ is periodic then $g(x) = g(a)$ $g(y) = g(b)$. $\because |x-y| = |a-b| < \delta \le \delta'$ then we have that $d(g(a), g(b)) = d(G(x), G(y)) < \varepsilon$. Hence g is uniformly continuous

153. (c) $h_n(x)$ converges to the function $h(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 0 \end{cases}$ pointwise. To prove this convergence is not uniform. Since if the convergence were uniform then h would be continuous. This is not true. Notice that $g_n(x)$ converges to the function $g(x) = 0$ pointwise. If $a \in (0,1)$ then $\lim_{n\to\infty} na^n = 0$. The convergence is not uniform. Let $x_n = 1/(n+1)$ and notice that $g_n(x_n) = (n/(n+1))^{n+1}$. Hence,
$g_n(x_n) = (1 - 1/(n+1))^{n+1} \to 1/e$. Hence, $\limsup \|g_n - g\|_{c[0,1]} \ge 1/e$, where
$\|\cdot\|_{c[0,1]}$ is the sup norm. Since convergence in the sup norm is the same as uniform convergence, then g_n does not converge to g.

154. Let M be $\ni |g'(x)| < M$ for all x. We claim that $|g(x) - g(y)| \le M|x-y|$ for all $x, y \in R$. If it is true then g would be uniformly continuous (since given $\varepsilon > 0$, the corresponding δ would be ε/M) Suppose that $\exists\ x < y \ni\ g(x) - g(y) > M(y-x)$. Then by the Mean value theorem $\exists\ z \in (x,y)$ $\ni\ g'(z) = \dfrac{g(y) - g(x)}{y - x}$. It follows that the $|g'(z)| > M$. This is a contradiction. Similarly $g(y) - g(x) < -M(y-x)$. Hence $|g(x) - g(y)| \le M|x-y|$ for all x, y. $\therefore g$ is uniformly continuous.

160. (a) For any $x \in [-1,1]$, $\left|e^{x^2/n} - 1\right| = e^{x^2/n} - 1 \le e^{1/n} - 1 \to 0$ as $n \to \infty$. Given $\varepsilon > 0$. $\exists\ N \in \mathbb{N} \ni$

$n \ge N \Rightarrow \left|e^{1/n} - 1\right| < \varepsilon$. Hence $n \ge N \Rightarrow \left|e^{x^2/n} - 1\right| \le \left|e^{1/n} - 1\right| < \varepsilon$. For all $x \in [-1,1]$. Since each $e^{x^2/n}$

is continuous on $[-1,1]$. $e^{x^2/n}$ is integrable on $[-1,1]$. $\lim\limits_{n\to\infty} \int_{-1}^{1} e^{x^2/n}\, dx = \int_{-1}^{1} \lim\limits_{n\to\infty} e^{x^2/n}\, dx = \int_{-1}^{1} 1.dx = 2.$

161. (a) For any $x \in [1,3]$, $\left|\dfrac{nx^2+3}{x^3+nx} - x\right| = \left|\dfrac{3-x^4}{x^3+nx}\right| \le \dfrac{3-x^4}{x^3+nx} \le \dfrac{3-3^4}{1^3+n\times 1} = \dfrac{84}{n+1} \to 0$ as $n \to \infty$. Given any

$\varepsilon > 0\ \exists\ n \in \mathbb{N} \ni n \ge N \Rightarrow \left|\dfrac{84}{n+1}\right| < \varepsilon$. Hence $n \ge N \Rightarrow \left|\dfrac{nx^2+3}{x^3+nx} - x\right| \le \dfrac{84}{n+1} < \varepsilon$ for all $x \in [1,3]$.

$\therefore \dfrac{nx^2+3}{x^3+nx} \to x$ uniformly on $[1,3]$. Since each $\dfrac{nx^2+3}{x^3+nx}$ is continuous on $[1,3]$. $\therefore \dfrac{nx^2+3}{x^3+nx}$ is integrable on

$[1,3]$. $\lim\limits_{n\to\infty} \int_{-1}^{3} \dfrac{nx^2+3}{x^3+nx}\, dx = \int_{-1}^{3} \lim\limits_{n\to\infty} \dfrac{nx^2+3}{x^3+nx}\, dx = \int_{1}^{3} x dx = \left[\dfrac{x^2}{2}\right]_1^3 = 4$

167. (b) Since $|h|$ is continuous on $[a,b]$. By extreme value theorem, $\exists\ x_0 \in [a,b] \ni |h(x)| \ge |h(x_0)| > 0$ for all $x \in [a,b]$. Let $c = |h(x_0)|$. Since $h_n \to h$ as $n \to \infty$ uniformly on

$[a,b], \exists\ N_1 \in \mathbb{N} \ni n \ge N \Rightarrow |h_n(x) - h(x)| < c/2$ for all $x \in [a,b]$. So

$|h_n(x)| \ge |h(x)| - c/2 \ge |h(x_0)| - c/2 = c/2 > 0$. for all $n \ge N_1$ and $x \in [a,b]$. Hence $\dfrac{1}{h_n}$ is defined

for large n. From the hypothesis, we know that for any $\varepsilon > 0$, $\exists\ N_2 \in \mathbb{N} \ni n \ge N_2 \Rightarrow | h_n(x) - h(x) | < \dfrac{c^2}{2}\varepsilon$

Let $N = \max\{N_1, N_2\}$. For any $n \ge N$. $\left|\dfrac{1}{h_n(x)} - \dfrac{1}{h(x)}\right| = \left|\dfrac{h_n(x) - h(x)}{h_n(x)h(x)}\right| \le \dfrac{|h_n(x) - h(x)|}{\frac{c^2}{2}} < \varepsilon.$

Hence $\dfrac{1}{h_n(x)}$ converges uniformly to $\dfrac{1}{h(x)}$ as $n \to \infty$ on $[a,b]$. Since g and $\dfrac{1}{h}$ are continuous on

$[a,b]$. By Extreme Value theorem, they are bounded. $\Rightarrow g_n/h_n \to g/h$ uniformly on $[a,b]$.

168. (b) Let $c = \max\{|a|, |b|\}$. $\because [a,b] \subset (-1,1)$, $0 \le c < 1$. $\left|x^k\right| \le c^k$ for all $k \in \mathbb{N}$ and all $x \in [a,b]$

$\because \sum_{k=0}^{\infty} c^k = \frac{1}{1-c} < \infty$. By Weierstrass M-test $\Rightarrow$ The uniformly convergence of $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ on any closed interval $[a,b] \subset (-1,1)$

169. (d) For each $x \neq 0$, $\left|\frac{\sin x \sin y}{x^2+y^2}\right| \leq \frac{|\sin y|}{x^2}$. Thus $\lim_{x\to 0}\lim_{y\to 0}\frac{\sin x \sin y}{x^2+y^2} = 0$ for each $y \neq 0$ $\frac{\sin x \sin y}{x^2+y^2} \leq \frac{|\sin y|}{y^2}$. Thus $\lim_{y\to 0}\lim_{x\to 0}\frac{\sin x \sin y}{x^2+y^2} = 0$. Since $f(x,x) = \frac{\sin x \sin x}{x^2+x^2} = \frac{1}{2}\frac{\sin^2 x}{x^2} \to \frac{1}{2}$ as $x \to 0$, $f(-x,x) = -\frac{\sin x \sin x}{x^2+x^2} = \frac{1}{2}\frac{\sin^2 x}{x^2} \to -\frac{1}{2}$ as $x \to 0$. $\therefore$ The limit does not exist.

170. (d) For each $x \neq 0$, $\frac{x^2+y^4}{x^2+2y^4} \to \frac{x^2}{x^2} = 1$ as $y \to 0$. Thus $\lim_{x\to 0}\lim_{y\to 0}\frac{x^2+y^4}{x^2+2y^4} = 1$. For each $y \neq 0$ $\frac{x^2+y^4}{x^2+2y^4} \to \frac{y^4}{2y^4} = \frac{1}{2}$ as $x \to 0$. Thus $\lim_{y\to 0}\lim_{x\to 0}\frac{x^2+y^4}{x^2+2y^4} = \frac{1}{2}$. Hence the limit does not exist.

171. (c) For any $h \neq 0$ $\frac{g(h,0)-g(0,0)}{h} = \frac{h}{\sin|h|} = sgn(h)\frac{h}{\sin|h|}$

$\because$ The above limit does not exist as $h \to 0$ $g_x(0,0)$ does not exist. Hence g is not differentiable at $(0,0)$.

177.(a) Compute the partial derivatives at $(0,0)$. $g_x(0,0) = \lim_{h\to 0}\frac{g(h,0)-g(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$

and similarly $g_y(0,0) = 0$. Thus, both first partials exist at $(0,0)$. To prove g is differentiable at $(0,0)$. $a = (0,0)$ and $B = \nabla g(a)$,

$$\left|\frac{g(h,k)-g(0,0)-\nabla g(0,0).(h,k)}{\|(h,k)\|}\right| = \frac{|hk|^{\alpha}\left|\log(h^2+k^2)\right|}{\sqrt{h^2+k^2}} \leq \frac{\left(\frac{h^2+k^2}{2}\right)^{\alpha}\left|\log(h^2+k^2)\right|}{\sqrt{h^2+k^2}}$$

$= \frac{1}{2^{\alpha}}\left(h^2+k^2\right)^{\alpha-\frac{1}{2}}\left|\log\left(h^2+k^2\right)\right| \to 0$ as $(h,k) \to (0,0)$. That is f is differentiable at $(0,0)$

178. (b) Compute the partial derivatives at $(0,0)$. $g_x(0,0) = \lim_{h\to 0}\frac{g(h,0)-g(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$

and similarly $g_x(0,0) = 0$. Thus both partials exist at $(0,0)$.

$\frac{g(h,k)-g(0,0)-\nabla g(0,0).(h,k)}{\|(h,k)\|} = \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$. $\because$ The above limit does not exist as $(h,k) \to (0,0)$, g is not differentiable at $(0,0)$

180. (b) By hypothesis on g, $\exists\ M>0$ satisfying $|g(x)|\leq M$ for all $x\in A$. Given $\varepsilon>0\ \exists$

$\delta>0 \ni\ |x-c|<\delta \Rightarrow |f(x)-0|<\varepsilon/M$. Thus $|x-c|<\delta \Rightarrow |f(x)g(x)-0|<(\varepsilon/M)M=\varepsilon$.

181. (b) Let $\varepsilon=1$ and define sequences in $(0,1]$ $\{x_n\}=\left\{\frac{1}{(n^2+1)}\right\}$ and $\{y_n\}=\left\{1/n\right\}$. $\lim|x_n-y_n|=0$ but

$|g(x_n)-g(y_n)|\ =|n^2-n+1|=n(n-1)+1\geq 1=\varepsilon$ for all $n\in\mathbb{N}$. So by the sequential criterion for non-uniform continuity, $g(x)=1/x$ is not uniformly continuous on $(0,1]$.

182. (a) Suppose that $g(c)\neq 0$ for some $c\in(0,a)$. If $g(x)>0$ apply Extreme Value theorem to $x\in[0,a]$

$\Rightarrow$ that g attains its maximum value at some $d\in(0,a)$. $g(d)>0$, $g'(d)=0$ and $g''(d)>0$.

We have $(g''+g'-g)(d)>0$ which contradicts. $g''+g'-g=0$. In conclusion $g(x)=0$ for all $x\in[0,a]$

183. (b) Suppose g has two fixed points $x<y$ on an interval I where $g'(x)\neq 1$. $\because g$ is differentiable on I.

Apply the Mean value theorem to g on $[x,y]$ to conclude that $\dfrac{g(x)-g(y)}{x-y}=g'(c)$ for some

$c\in(x,y)$. However $g(x)=x$ and $g(y)=y$ thus we have $g'(c)=1$. Since this is a contradiction.

$\therefore f$ has no more than one fixed point.

184. (a) By Weierstrass M-test $|f_n(x)|\leq\dfrac{1}{2^n}$ for all $x\in R$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges to 1. Thus $g(x)$ is uniformly convergent on R.

188. (a) Clearly $x_n>0$ assume that $x_n<1+\sqrt{b}$

$$\begin{aligned}x_{n+1}^2&=\left(\sqrt{b+x_n}\right)^2\\&=b+x_n\\&\leq 1+\sqrt{b}+b\\&<1+2\sqrt{b}+b\\&<\left(1+\sqrt{b}\right)^2\end{aligned}$$

Showing that $x_{n+1}<1+\sqrt{b}$.

$\because x_1<1+\sqrt{b}$, it follows from the principle of mathematical induction that $x_n<1+\sqrt{b}$ for all n.

Again using principle of mathematical induction and show that $x_{n+1}<x_n$ for all n.

Clearly $x_2 = \sqrt{b+x_1} > \sqrt{b} = x_1$

Assuming that $x_{n+1} > x_n$

$$x_{n+2} - x_{n+1} = \sqrt{b+x_{n+1}} - \sqrt{b+x_n}$$

$$= \frac{\left(\sqrt{b+x_{n+1}} - \sqrt{a+x_n}\right).\left(\sqrt{a+x_{n+1}} - \sqrt{a+x_n}\right)}{\sqrt{a+x_{n+1}} + \sqrt{a+x_n}}$$

$$= \frac{(b+x_{n+1}) - (b+x_n)}{\sqrt{b+x_{n+1}} + \sqrt{b+x_n}}$$

$$= \frac{(x_{n+1} - x_n)}{\sqrt{b+x_{n+1}} + \sqrt{b+x_n}} > 0$$

Hence, (x_n) is increasing.

189. (a) Let $\varepsilon > 0$. There is $N \in R \ni$ if $m, n > N$ and $x \in [a,b]$ then $|g_m(x) - g_n(x)| < \varepsilon/(b-a)$ so if

$m, n > N$ and $x \in [a,b]$ then $|h_m(x) - h_n(x)| = \left|\int_a^x g_m - \int_a^x g_n\right| = \left|\int_a^x (g_m - g_n)\right| \leq \int_a^x |g_m - g_n|$

$\leq \int_a^b |g_m - g_n| \leq \int_a^b \frac{\varepsilon}{b-a} = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$ So $\langle h_n \rangle$ converges uniformly on $[a,b]$

190. (a) If $0 \leq x < 1$ then $\lim_{n\to\infty} g_n(x) = 1$, $\lim_{n\to\infty} g_n(1) = 1/2$ and if $1 < x$ then $\lim_{n\to\infty} g_n(x) = 0$.

$\therefore \langle g_n \rangle$ converges pointwise to $g(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 1/2 & x = 1 \\ 0 & 1 < x \end{cases}$. But each g_n continuous at 1 and g is not, so

$\langle g_n \rangle$ does not converge uniformly.

191. (a) If $B(a;\varepsilon) \cap A = \{a\}$ then in any sequence of distinct points of A, all but at most one member must be at distance $\geq \varepsilon$ from *a*. Thus no such sequence can converge to A. Conversely, if every ball around *a* meets A in a point other than *a* we can construct a sequence a_n of distinct points of A tending to *a* by the induction argument. Hence the proof.

192. (c) $g : R \to R$ be differentiable $\ni |g'(x)| \leq M$ for all $x \in R$. For any $\varepsilon > 0$ let $\delta = \varepsilon/M$. Then by the

Mean value theorem if $|y - x| < \delta$ we have for some *c* between *x* and *y* $\frac{|g(y) - g(x)|}{|y-x|} = |g'(c)| \leq M$

$\Rightarrow |g(y) - g(x)| \leq |y - x| M < \delta M < \varepsilon$. $\because \delta$ didn't depend on *x* or *y*. $\therefore g$ is uniformly continuous.

198. (c) To show that $\{x : g(x) = h(x)\}$. Coniser the function $f(x) = g(x) - h(x)$. Then $\{x \mid g(x) = h(x)\} = \{x \mid f(x) = 0\}$ $= f^{-1}(0)$. This set is measurable because the function *f* is measurable. To show that $\{x : g(x) \geq h(x)\}$ consider the same definition of the function $f(x)$. This function *f(x)* is measurable; moreover $\{x \mid g(x) \geq h(x)\} = \{x \mid f(x) \geq 0\}$ $= f^{-1}([0, \infty))$. $\therefore$ This set is measurable as well as $\{x : g(x) > h(x)\}$ following similar argument.

199. (a) Considering Fatou's Lemma $\int g \leq \liminf_{i \geq n} \int g_n$. Appling again the Fatou's Lemma to the function $g - g_n \geq 0$, $\int g - g_n \leq \liminf \int g - g_n$. Then $-\int g \leq \liminf \left[-\int g_n\right]$, $\int g \geq \liminf \int g_n$
$\therefore \liminf \int g_n = \int g$.

200. (b) It is always that $\inf S \leq \sup S$ for any bounded nonempty subset of *R*.

To prove $\inf T \leq \inf S$ and $\sup S \leq \sup T$. Let $x \in S$. Then $x \in T$ since $S \subset T$. So $\inf T \leq x$ by definition of $\inf T$. $\Rightarrow \inf T$ is lower bound for *S* because *x* was taken arbitrary in *S*. Since $\inf S$ is the greatest lower bound we get $\inf T \leq \inf S$. Similarly $\sup S \leq \sup T$.

201. (b) Assume that $\lim_{x \to \infty} f(x) = \ell$ exists. To show that $f(x) = \ell$ for any $x \in R$. Let $T > 0$ be period of $f(x)$. Let $a \in R$ and assume that $f(a) \neq \ell$. Take $\varepsilon = \dfrac{|f(a) - \ell|}{2}$ $\because \lim_{x \to \infty} f(x) = \ell$ then $\exists$ $M > 0$ $\ni$ for any $x > M$. We have $|f(x) - \ell| < \varepsilon$. Since *R* is Archimedean, there exists $n \in N \ni n > M - a/T$

Then we have $a + nT > M$. Hence $|f(a + nT) - \ell| < \varepsilon$. $\because f(a + nT) = f(a)$

we get $|f(a) - \ell| < \varepsilon = \dfrac{|f(a) - \ell|}{2}$. Which is a contradiction, $\therefore f(a) = \ell$. $\because$ *a* was arbitrary.

202. (d) $|f(x) - f(c)| = \left|\sqrt{x} - \sqrt{c}\right| \leq \sqrt{|x - c|}$ for $x, c \in [0, \infty)$. $\therefore$ given $\varepsilon > 0$ choose $\delta = \varepsilon^2$ then $|x - c| < \delta$ $\Rightarrow 1\left|\sqrt{x} - \sqrt{c}\right| < \varepsilon$. $\therefore f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

209. (a) Consider the function $h(x) = g(x) - f(x)$. Fix $\varepsilon > 0$. Then the set $A = \{x \in [a, b]; h(x) \neq 0\}$ is finite.

Assume that $A = \{x_1, x_2, ..., x_n\}$ with $a < x_1 < x_2 ... < x_n < b$, set $M = \max\{|h(x_i)| : i = 1, 2, ...n\}$

We have $M > 0$ choose $\delta > 0$ small enough to have $a < x_1 - \delta; x_i + \delta < x_{i+1} - \delta, x_n + \delta < b$ and

$$\int_a^b g(x)dx = \int_a^b f(x)dx + \int_a^b h(x)dx = \int_a^b f(x)dx$$

210. (c) It is clear that $G(x)$ is continuous and differentiable except at 1. (i.e) $F^1(1)$ does not exist.

211. (a) Using the properties of logarithms $\sum_{K=1}^{\infty} \text{In}\dfrac{\left(\dfrac{K^{K+1}}{(K+1)^K}\right)}{K(K+1)} = \sum_{K=1}^{\infty}\dfrac{(K+1)\text{In}K - K\text{In}(K+1)}{K(K+1)}$

$$\therefore S_n = \sum_{K=1}^{n}\frac{\text{In}K}{K} - \frac{\text{In}(K+1)}{(K+1)} = \left(\frac{\text{In}1}{1} - \frac{\text{In}1}{2}\right) + \left(\frac{\text{In}2}{2} - \frac{\text{In}3}{3}\right) + \ldots + \left(\frac{\text{In}n}{n} - \frac{\text{In}(n+1)}{(n+1)}\right) = -\frac{\text{In}(n+1)}{(n+1)}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(-\frac{\text{In}(n+1)}{(n+1)}\right) = 0.$$

217. (b) Let us find the maximum f_n in $[0,2]$ $\because f_n'(x) = \dfrac{ne^{nx} - ne^{nx}nx}{\left(e^{nx}\right)^2} = 0$; $e^{nx}(n - n^2) = 0$; $n^2x = n$

$x = 1/n$. $1/n$ is the maximum with $f_n\left(\dfrac{1}{n}\right) = \dfrac{1}{e}$. Since $\lim_{n\to\infty}\sup\left\{\left|f_n(x) - 0\right| : x \in [0,2]\right\} = \lim_{n\to\infty} f_n\, 1/n$

$= 1/e$. The convergence is not uniform on $[0,2]$

218. (a) Let $S_n(x) = \sum_{k=0}^{n} g_k(x)$ be the nth partial sum. $\because \sum_{n=0}^{\infty} M_n$ converges for all $\varepsilon > 0$ then there exists $N \ni$ if $n \geq m \geq N$, then $M_m + M_{m+1} + \ldots + M_n < \varepsilon$. Thus if $n \geq m \geq N$,

$\left|S_n(x) - S_m(x)\right| = \left|f_{m+1}(x) + \ldots + f_n(x)\right| \leq \left|f_{m+1}{}^{(x)}\right| + \ldots + \left|f_n(x)\right| \leq M_{m+1} + \ldots + M_n < \varepsilon$ for all $x \in D$. $\Rightarrow \{S_n\}$ converges uniformly on *D*. Hence $\sum_{k=0}^{n} g_n$ also converges uniformly on *D*.

219. (b) Let $M_n = 1/n^2$ for $n \geq 1$ then $\left|\dfrac{\cos^2(nx)}{n^2}\right| \leq M_n$

$\because |\cos(nx)| \leq 1$. Because $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ is a p-series with $p = 2 > 1$, it converges. Hence by the Weierstrass M-test, convergence is uniform.

OBJECTIVE TYPE QUESTIONS (PART-C)

1. Which of the following are true?

(a) It is possible to place a linear order relation on the complex numbers C which make $(C,+,X,\leq)$ an ordered field.

(b) It is possible to place an linear order relation $\leq$ on the complex numbers C such that $(C,+,X,\leq)$ is not an ordered field

(c) Both (a) and (b) are true

(d) Both (a) and (b) are false

2. Which of the following statements is true?

(a) Every bounded sequence of real numbers has at least one subsequential limit.

(b) The integral of the limit is equal to the limit of the integrals.

(c) For all sequences of real numbers $\{S_n\}$ we have $\liminf S_n \leq \limsup S_n$

(d) If the radius of convergence of a power series $\Sigma a_k x^k$ is R and if $0 < R < \infty$, then the series $\Sigma a_k x^k$ converges uniformly on $(-R, R)$

3. Which of the following series is convergent

(a) $\sum (-1)^n \frac{n}{100n+1000}$

(b) $\sum \left(\frac{7n}{8n+1}\right)^n$

(c) $\sum \frac{1}{n \log n}$

(d) $\sum \frac{(-1)^n}{\sqrt{n}}$

4. Which of the following is not always true for $x, y, z \in R^k$?

(a) $|x.y| \leq |x||y|$ (b) $|x| = 0$ iff $x = 0$

(c) $|x| \geq 0$ (d) $|x+y| = |x| + |y|$

5. Which of the following statements is / are true?

(a) A metric space X with distinct open sets $\langle U_i : i \in N \rangle \ni \bigcap_{i \in N} U_i$ is open

(b) Every bounded closed subset of a metric space is compact

(c) Suppose $\{s_n\}$ & $\{t_n\}$ are sequences of complex numbers $\ni \lim_{n\to\infty} s_n = s$ and $\ni \lim_{n\to\infty} t_n = t$. Then $\lim_{n\to\infty} s_n t_n = st$.

(d) There exists a bounded sequence of real numbers with no convergent subsequence.

6. Which of the following is true?

(a) Every complete space is compact

(b) Every compact space is complete.

(c) both (a) and (b) are true

(d) both (a) and (b) are false

7. Which of the following are valid intervals of convergence for a power series

(a) $(-\infty, 0]$ (b) $(-\infty, \infty)$

(c) $[2,2] \cup [3,3]$ (d) $[-1,3)$

8. Which of the following statements is / are true?

(a) Suppose $\sum |a_i|$ diverges and $\Sigma a_i = 2$. A rearrangement a_{ik} of the terms $\Sigma a_{ik} = 4$.

(b) Suppose $f : X \to Y$ is continuous and X is compact then f must be uniformly continuous.

(c) Metric spaces X and Y with X closed and bounded and a continuous mapping $f : X \to Y$ such that $f(x)$ is not closed and bounded.

(d) If $f(x), g(x) : R \to R$ are everywhere differentiable then $f(g(x))$ must be everywhere differentiable.

9. Which of the following statements is / are false?

(a) Suppose $f : [a,b]$ is a function and suppose f is a local maximum. Then $f'(x)$ exist and equal to 0.

(b) Suppose $f \in R(\alpha)$.

Then $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$

(c) Suppose $\{f_n\}$ is a sequence of continuous functions on $[a,b]$ such that $(\not\forall\ x \in [a,b])$

$\lim_{n\to\infty} f_n(x) = f(x)$. Then f must be continuous.

(d) Suppose $f(x)$, $g(x)$ are real differentiable functions on (a,b) and $\ni$ $\lim_{x\to c} \frac{f'(x)}{g'(x)} = A$

then $\lim_{x\to 0} \frac{f(x)}{g(x)} = A$

10. Which of the following statements is / are ture?

(a) Every continuous function $f:(0,1)\to(0,1)$ must have a fixed point.

(b) If A is connected then the closure of A is connected.

(c) If $A \subseteq R$ is open and $B = \overline{A}$ closure of A, then interior $(B) = A$

(d) There is a non constant continuous maps from $R \to Q$.

11. Let $S' = \{(x,y)\in R^2 : x^2 + y^2 = 1\}$.

Given $p,q \in S'$ define $d_s(p,q)$ to the shortest distance along S' from p to q. For example if $p=(1,0)$ and $q=(0,1)$ then $d_s(p,q) = \pi/2$. (Note $d_s(p,q)$ is also the angle between the line connecting $(0,0)$ and p and the line connecting $(0,0)$ and q). Then which of the following statements is / are true?

(a) (S', d_s) is a metric

(b) S' is compat

(c) S' is connected

(d) Every continuous function $f: S' \to R$ has a maximum value.

12. Find the coefficient of a_k for $k = 0,1,2,...$ such that $\sum_{k=0}^{\infty} a_k x^k = \frac{x^2+1}{x-1}$

(a) -2 (b) 2

(c) -1 (d) 5

13. Which of the following series is converges

(a) $\sum_{n=0}^{\infty} \frac{(10x)^n}{n^2+1}$ (b) $\sum_{n=0}^{\infty} (2+\sqrt{n})x^n$

(c) $\sum_{n=1}^{\infty} \frac{1}{nx^n}$ (d) $\sum_{n=0}^{\infty} n(x-2)^n$

14. The series

i) $\sum_n \frac{\sin(nt)}{n^2}$

ii) $\sum_n (-1)^n 2^{-n^2}$

(a) (i) is converges absolutely and (ii) is diverges

(b) (ii) is converges absolutely and (i) is diverges

(c) both (i) and (ii) are converges absolutely

(d) both (i) and (ii) are diverges

15. Which of the following series is convergent

i) $S = \sum_{n=1}^{\infty} \frac{1}{f(n)}$ where $f(n)$ is positive and $\leq c_n$ for some constant $c > 0$

ii) $S = \sum_{n=1}^{\infty} \frac{1}{g(n)^n}$ where $g(n)$ is increasing and unbounded

(a) (i) is converge and (ii) does not converge

(b) (ii) is converge and (i) does not converge

(c) both (i) and (ii) are converge

(d) both (i) and (ii) are diverge

16. Which of the following statement is / are true?

(a) Let S and T be non empty bounded subsets of R. If $S \subseteq T$ then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

(b) $\sup(S \cup T) = \max\{\sup S, \sup T\}$

(c) Let I be the set of real numbers that are not rational.If $a < b$ then $\exists\ x \in I \ni a < x < b$.

(d) An ordered field F is Archimedean iff satisfies for every $a \in F$ there exists $n \in N$ when $n > a$.

17. For points $x, y \in R^k$, let

$d_1(x,y) = \max\{|x_j - y_j| : j = 1,2,...,k\}$ and

$d_2(x,y) = \sum_{j=1}^{k} |x_j - y_j|$

i) d_1 and d_2 are matrices for R^k

ii) d_1 and d_2 are complete.

(a) both (i) and (ii) are true
(b) both (i) and (ii) are false
(c) (i) is true but (ii) is false
(d) (ii) is true but (i) is false.

18. The series $\sum \frac{x^n}{1+x^n}$

(a) converges for $x \in [0,1)$

(b) converges uniformly on $[0,a)$ for each a, $0<a<1$.

(c) converges uniformly on $[0,1)$

(d) None of the above

19. Suppose

$$K = \left\{ f \in c([0,1]) : Lip(f) \le 1 \quad \& \int_0^1 f(x)dx = 0 \right\}$$

then

(a) K is equicontinuous in $c([0,1])$

(b) K is closed in $c([0,1])$

(c) K is bounded in $c([0,1])$

(d) K is compact in $c([0,1])$

20. Which of the following statements is/are true

(a) given any two real numbers $a<b$ $\exists$ an irrational number $t \ni$ $a<t<b$

(b) $A = \sup\{r \in Q \mid r < x\} = x$

(c) Let a & b real numbers. If $a \le b + 1/n$ for every $n \in N$ then $a \le b$

(d) None of the above

21. Which of the following series is convergent series

(a) $\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$ (b) $\sum_{n=1}^{\infty} \text{In}\left(1 + 1/n\right)$

(c) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ (d) $\sum_{n=2}^{\infty} \frac{1}{n(\text{In} n)^p}$ if $p>1$

22. Which of the following statements is / are true?

(a) $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ for some integers } m,n \text{ with } n \ne 0 \text{ and reduced to lowest terms} \\ 0 & \text{if } x \notin Q \end{cases}$

is continous only for $x \notin Q$

(b) If f is continuous on $[a,b]$ with $f(x)>0$ for all $x \in [a,b]$ then $1/f$ is bounded on $[a,b]$

(c) $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1]$

(d) None of the above

23. Define $\{f_n(x)\}$ by $f_n(x) = \frac{n}{1+(nx)^2}$ for $x \ne 0$ and $f_n(0)=0$. Fixing $n \in \mathbb{N}$. Compute $\int_{-\infty}^{\infty} f_n(x)dx$. What happens to this integral as $n \to \infty$?

(a) π (b) $1/2$

(c) 0 (d) ∞

24. Compute $x_n = \inf\{a_n, a_{n+1}, \ldots\}$ and $y_n = \sup\{a_n, a_{n+1}, \ldots\}$ and $\liminf a_n$ and $\limsup a_n$ where $a_n = (-1)^n\left(1 + 1/n\right)$

(a) $\liminf a_n = -1$ & $\limsup a_n = 1$

(b) $\liminf a_n = 0$ & $\limsup a_n = 1$

(c) $\liminf a_n = -1$ & $\limsup a_n = 0$

(d) $\liminf a_n = \limsup a_n = 0$

25. i) A contraction mapping on a metric space is continuous

ii) Every closed subset of a compct metric space is compact.

(a) (i) is true but (ii) is not true
(b) (ii) is true but (i) is not true
(c) both (i) and (ii) are true
(d) both (i) and (ii) are not true

26. Find the radius of convergence of the series $\sum \frac{(-1)^n x^n}{n+1}$

(a) 1 (b) 0
(c) ∞ (d) -1

27. Find the power series representation for $f(x) = \frac{1}{(1+x)^2}$

(a) $\sum_{n=1}^{\infty} (-1)^n x^n$ (b) $\sum_{n=1}^{\infty} (-1)^{n+1} nx^{2n}$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1} nx^{n-1}$ (d) $\sum_{n=1}^{\infty} \frac{nx^{2n}}{2}$

28. What is the radius of convergence of $\int \frac{t}{1-t^8} dt$

(a) $R=1$ (b) $R=0$
(c) $R=\infty$ (d) $R=-1$

29. i) Suppose (s_n) and (t_n) are bounded sequences. Then $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$

ii) Suppose (s_n) and (t_n) are bounded sequences. Then $\limsup(s_n t_n) \le \limsup s_n \limsup t_n$

iii) Suppose (s_n) is a bounded but not convergent sequence and $s_n > 0$. If $t_n \to t \ne 0$. Then $(s_n t_n)$ is convergent.

(a) (i) and (ii) are true but (iii) is not true.
(b) (i) and (iii) are true but (ii) is not true.
(c) both (i), (ii) and (iii) are true.
(d) (ii) is true but (i) and (ii) are not true.

30. Find the infimum and supremum of the set $\left\{ \frac{n}{m+n}, \text{with } m, n \in \mathbb{N} \right\}$

(a) $\inf = 0$ and $\sup = 1$
(b) $\inf = -1$ and $\sup = 1$
(c) $\inf = -1$ and $\sup = 0$
(d) $\inf = 0$ and $\sup = 1$

31. Which of the following sets is / are countable?

(a) The set of functions from $\{0,1\}$ to N
(b) The set of finite subsets of N
(c) The set consisting of all sequences of 0^s and 1^s
(d) The set of functions from N to $\{0,1\}$

32. Which of the following statements is / are true?

(a) A cauchy sequence must be monotone
(b) Every monotone sequence is cauchy
(c) An unbounded sequence can contain a cauchy subsequence
(d) A cauchy sequence can have a divergent subsequence.

33. For each $x \in R$, determine whether the sequence $\left(\frac{1}{1+x^n} \right)_{n=1}^{\infty}$ has a limit.

(a) $x=0, \lim = 1$ (b) $|x|<1, \lim = 1$
(c) $x=-1$, no limit (d) $x=0, \lim = 0$

34. Match the following accumulation point

i) The interval $[0,1]$ - The set of all real numbers
ii) The set of natural numbers - $[0,1]$
iii) The set of all irrational numbers - Empty set

(a) (i) $\to$ (iii), (ii) $\to$ (i), (iii) $\to$ (ii)
(b) (i) $\to$ (i), (ii) $\to$ (iii), (iii) $\to$ (ii)
(c) (i) $\to$ (iii), (ii) $\to$ (ii), (iii) $\to$ (i)
(d) (i) $\to$ (ii), (ii) $\to$ (iii), (iii) $\to$ (i)

35. Suppose that the random variables X_n are defined on the same probability space and there is a constant $c \ni X_n$ converges in distribution to the random variable c.

i) X_n converges to c in probability
ii) X_n converges to c a.s

(a) both (i) and (ii) are true
(b) (i) is true but (ii) is not true
(c) (ii) is true but (i) is not true
(d) both (i) and (ii) are not true

36. Suppose f is continuous on $[a,b]$ and $f(a)<0<f(b)$

i) Either $f\left(\frac{(a+b)}{2}\right)=0$ or f has different signs at the end points $\left[a,\frac{(a+b)}{2}\right]$ or f has differnt signs at the end points of $\left[\frac{(a+b)}{2},b\right]$

ii) There is a point x in (a,b) where $f(x)=0$

(a) (i) is true but (ii) is not true
(b) (ii) is true but (i) is not true
(c) both (i) and (ii) are true
(d) both (i) and (ii) are not true

37. Which of the following statements is / are true.

(a) $f(x)=|x|$ is uniformly continuous
(b) $f(x)=x^2$ is uniformly continuous on $[0,3]$
(c) If f is uniformly continuous on a bounded set S, then f is a bounded function on S.
(d) None of the above

38. The series $\sum_{n=1}^{\infty}(-1)^n\frac{n^2-1}{n^3+1}$

(a) converges conditionally
(b) converges
(c) diverges
(d) none of the above

39. The series $\sum_{n=1}^{\infty}\frac{5n^3+2n+\cos^2(e^n)}{\sqrt{n^8+n^5+2}}$

(a) converges
(b) converges conditionally
(c) diverges
(d) none of the above

40. Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(x+1)^n}{n(n+1)}$

(a) $[-2,2]$ (b) $[-2,0]$
(c) $[-1,0]$ (d) $(-2,2)$

41. i) Every σ - finite measure is saturated
ii) The collection τ of locally measurable set is a σ - algebra.

(a) (i) is true but (ii) is not true
(b) (ii) is true but (i) is not true
(c) both (i) and (ii) are true
(d) both (i) and (ii) are not true

42. Which of the following series is convergence

(a) $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ (b) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{1/n}}{n}$

(c) $\sum_{n=1}^{\infty}n^{1/n}$ (d) $\sum_{n=1}^{\infty}n\sin\left(1/n\right)$

43. Define a sequence $\{a_n\}$ by letting $a_1=2$ and $a_{n+1}=4-\frac{1}{a_n}$ for each $n\geq 1$, $2\leq a_n\leq a_{n+1}\leq 4$ for each $n\geq 1$ and find the limit of this sequence.

(a) $L=2+\sqrt{3}$ (b) $L=2-\sqrt{3}$
(c) $L=2$ (d) $L=0$

44. Consider the sequence of partial sums

$$S_n=\sum_{k=1}^{n}\frac{1}{k^2}$$

(a) If $n,m\in z_+$ with $m>n$ then

$$|S_m-S_n|=\sum_{k=n+1}^{n}\frac{1}{k^2}$$

(b) $\frac{1}{k^2}<\frac{1}{k(k-1)}$ for $k\geq 2$

(c) $\sum_{k=n+1}^{m}\frac{1}{k(k-1)}=\frac{1}{n}-\frac{1}{m}$

(d) $|S_m-S_n|<\frac{1}{m}+\frac{1}{n}$

45. Let f,g and h be defined on $[0,1]$ as follows:

$f(x)=g(x)=h(x)=0$ whenever x is irrational

$f(x)=1$ and $g(x)=x$ whenever x is rational

$h(x)=1/n$ if x is the rational number $\frac{m}{n}$, $h(0)=1$

(a) f is continuous in $[0,1]$
(b) g is continuous only at $x=0$

(c) h is continuous only at the irrational points in $[0,1]$

(d) f is not continuous anywhere in $[0,1]$

46. Define $f_n : [0,1] \to [0,1]$ by $f_n(x) = x^n(1-x)$

(a) f_n converges uniformly to 0

(b) f_n does not converges

(c) f_n converges to 1

(d) none of the above

47. Which of the following statement is / are true?

(a) If $\{x_n\}_n$ has an unbouned subsequence, then $\{x_n\}_n$ is unbounded

(b) If $\{x_n\}_n$ is unbounded then some subsequence is unbounded

(c) If $\{x_n\}_n$ is unbounded, then every subsequence is unbounded.

(d) If $\{x_n\}_n$ diverges then its every subsequence diverges.

48. Let $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ be sequence of real numbers. Then

(a) $\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$

(b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

(c) $\liminf_{n\to\infty}(a_n + b_n) \leq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$

(d) $\limsup_{n\to\infty}(a_n + b_n) \geq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

49. Which of the following functions is / are differentiable at 0.

(a) $f(x) = \begin{cases} 0, & x \in Q \\ x, & x \notin Q \end{cases}$

(b) $f(x) = \begin{cases} 0, & x \in Q \\ x^2, & x \notin Q \end{cases}$

(c) $f(x) = \begin{cases} 0, & x = 0 \\ x\sin\left(\frac{1}{x}\right), & x \neq 0 \end{cases}$

(d) $f(x) = x^{1/3}$

50. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients $D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$

(a) If $a_n < 0 < \beta_n$ and $\lim_{n\to\infty} D_n = f'(0)$

(b) If $0 < \alpha_n < \beta_n$ and $\left\{ \beta_n / (\beta_n - \alpha_n) \right\}$ is bounded then $\lim_{n\to\infty} D_n = f'(0)$

(c) If f' is continuous in $(-1,1)$ then $\lim_{n\to\infty} D_n = f'(0)$

(d) none of the above

51. Let $f : [a,b] \to R$ be a function of bounded variation on $[a,b]$ and define a function V on $[a,b]$ by $V(a) = 0$ and $V(x) = V(f,[a,x])$ for all $x \in (a,b]$.

(a) V is increasing on $[a,b]$

(b) If V is continuous at $c \in [a,b]$. Then f is continuous at c.

(c) If f is continuous at $c \in [a,b]$ then V is continuous at c.

(d) $a \leq x < y \geq b$, $V(y) - V(x) = V(f,[x,y]$

52. Which of the following statements is / are ture?

(a) A function that is Riemann integrable on $[a,b]$ must be bounded on $[a,b]$

(b) A continuously differentiable function on a compact interval is Lipschitz on that interval

(c) A Lipschitz function is differentiable

(d) Every continuous function has an anti-derivatie.

53. Which of the following statement is / are true?

(a) Any union of sets of measure zero have measure zero

(b) If $f : [a,b] \to R$ is integrable and $f(x) = \int_a^x f(t)dt$ then $F'(x) = f(x)$

(c) Given any two partitions P and Q of $[a,b]$

and any function $f:[a,b]\to R$ we must have $L(f,P)\le U(f,Q)$

(d) A function $f:[a,b]\to R$ is continuous at $c\in[a,b]$ iff the oscillation of f at c is zero.

54. A function $f:[0,1]\to R$

(a) If f is bounded then f' is bounded

(b) If f' is bounded then f is bounded

(c) If f^2 is Riemann integrable on $[0,1]$ then f is Riemann integrable on $[0,1]$

(d) If f is Riemann integrable on $[0,1]$ then f^2 is Riemann integrable on $[0,1]$.

55. Let X be a compact metric space. Then

(a) X is seperable

(b) X is second countable

(c) $C(X)$ is separable

(d) None of these

56. i) Every nonempty subset of $[0,1]$ is Lebesgue measurable

ii) Suppose H is separable Hilbert space if $T:H\to H$ is a linear mapping $\ni$ $\|I-T\|<1$, where I is the identity map of H to itself then T is invertible.

(a) (i) is true but (ii) is not true

(b) (ii) is true but (i) is not true

(c) (i) and (ii) both are true

(d) (i) and (ii) both are false

57. i) The series $\sum\frac{x^n}{1+x^n}$ converges for $x\in[0,1)$

ii) The series $\sum\frac{x^n}{1+x^n}$ converges uniformly on $[0,1)$

(a) (i) is true but (ii) is not true

(b) (ii) is true but (i) is not true

(c) both(i) and (ii) are true

(d) both(i) and (ii) are not true

58. i) If f is continuous on $[a,b]$ with $f(x)>0$ for all $x\in[a,b]$ then $\frac{1}{f}$ is bounded on $[a,b]$

ii) $f(x)=1/x$ is not uniformly continuous on $(0,1]$.

(a) (i) and (ii) are false

(b) (i) is true but (ii) is not true

(c) (ii) is true but (i) is not true

(d) both (i) and (ii) are true

59. i) The sequence $\{f_n(x)\}$ defined by $f_n(x)=xe^{-nx^2}$ is uniformly convergent on $[0,1]$

ii) $\sum_{n=1}^{\infty}\frac{\cos(2^n x)}{2^n}$ is continuous on R.

(a) (i) is true but (ii) is not true

(b) (ii) is true but (i) is not true

(c) (i) and (ii) both are true

(d) both (i) and (ii) are not true

60. For what values of a and b the series $\sum\left(\frac{a}{n}+\frac{1}{n+1}+\frac{b}{n+2}\right)$ converges?

(a) $a=-1/2,\ 3a+b+2=0$

(b) $a=1,\ b=2$

(c) $a=0,\ b=-1$

(d) $a=2,\ b=3$

61. Which of the following is / are metric?

(a) $d(x,y)=\left|x^2-y^2\right|$ $x\in R^1$ and $y\in R^1$

(b) $d(x,y)=\frac{|x-y|}{1+|x-y|}$ $x\in R^1$ and $y\in R^1$

(c) $d(x,y)=(x,y)^2$

(d) $d(x,y)=|x-2y|$

62. Let X be the set of all $x\in[0,1]$ whose decimal expansion contains only the digits 4 and 7. Then

(a) X is countable

(b) X is dense in $[0,1]$

(c) X is compact

(d) X is perfect.

63. Let A and B be separated subsets of some R^k, suppose $a \in A,\ b \in B$ and define

$$q(t) = (1-t)a + tb$$

for $t \in R^1$. Put $A_0 = q^{-1}(A), B_0 = q^{-1}(B)$.

(a) A_0 and B_0 are separated subsets of R^1

(b) There exists $t_0 \in (0,1) \ni q(t_0) \notin A \cup B$

(c) Every convex subset of R^k is connected

(d) None of the above

64. Define f and g on R^2 by

$$f(0,0) = g(0,0) = 0,\ \ f(x,y) = \frac{xy^2}{x^2 + y^4}$$

$$g(x,y) = \frac{xy^2}{(x^2 + y^6)} \text{ if } (x,y) \neq (0,0)$$

(a) f is bounded on R^2

(b) g is unbounded in every neighborhood of $(0,0)$

(c) f is not continuous at $(0,0)$

(d) restriction of both f and g to every straight line in R^2 are continuous

65. Suppose f is defined in $(-1,1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients $D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$

(a) If $\alpha_n < 0 < \beta_n$ then $\lim D_n \neq f'(0)$

(b) If $0 < \alpha_n < \beta_n$ and $\left\{ \beta_n \big/ (\beta_n - \alpha_n) \right\}$ is bounded then $\lim D_n = f'(0)$

(c) If f' is continuous in $(-1,1)$ then $\lim D_n = f'(0)$

(d) None of the above

66. i) Suppose α increases on $[a,b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$ if $x \neq x_0$. Then $f \in R(\alpha)$ and $\int f d\alpha = 0$.

ii) Suppose $f \geq 0$, is continuous on $[a,b]$ and $\int_a^b f(x)dx = 0$. Then $f(x) = 0$ for all $x \in [a,b]$.

(a) (i) is true but (ii) is not true

(b) (ii) is true but (i) is not

(c) both (i) and (ii) are true

(d) both (i) and (ii) are false

67. Which of the following sequences are bounded

(a) $\dfrac{(-1)^n}{n^2}$ (b) $\sin\left(\dfrac{n\pi}{7}\right)$

(c) $\dfrac{n}{3^n}$ (d) n^5

68. Which of the following statement is / are true?

(a) $|\cos x - \cos y| \le |x - y|$

(b) $ex \le e^x$ for all $x \in R$

(c) $\sin x \le x$ for all $x \ge 0$

(d) $f(x) = \text{In} x$ is uniformly continuous on $[1,\infty)$

69. i) Let f be integrable on $[a,b]$ and suppose that g is a function on $[a,b] \ni g(x) = f(x)$ except for finitely many $x \in [a,b]$. Then g is integrable and $\int_a^b f = \int_a^b g$.

ii) If f is integrable on $[a,b]$ then f is not integrable on every interval $[c,d] \subseteq [a,b]$.

(a) (i) is true but (ii) is not true

(b) (ii) is true but (i) is not true

(c) both (i) and (ii) are true

(d) both (i) and (ii) are false

70. Let $f_n(x) = \dfrac{x}{x+n}$ for $x \ge 0$

(a) $f(x) = \lim f_n(x) = 0$ for all $x \ge 0$

(b) If $t > 0$, the convergence is uniform on $[0,t]$

(c) The convergence is uniform on $[0,\infty)$

(d) None of the above

71. Suppose $f : R \to R$ satisfies $f(x+y) = f(x) + f(y)$ for each $x, y \in R$. Then

(a) $f(nx) = nf(x)$ for all $x \in R,\ n \in N$

(b) f is continuous at a single point iff f is continuous on R

(c) f is continuous iff $f(x) = mx$ for some $m \in R$

(d) None of the above

72. Let $T : (C[0,1], \|.\|) \to (C[0,1], \|.\|)$ be a function defined as $Tf(x) = \int_0^x f(t)dt$ where by $C[0,1]$ mean the vector space of all continuous real valued functions defined on $[0,1]$ and $\|f\| = \sup_{0 \le t \le 1} |f(t)|$ then which of the following is true?

(a) T has a unique fixed point

(b) T is a contraction

(c) T is not a contraction

(d) T^2 is a contraction

73. Which of the following integrals is / are converges?

(a) $\int_1^\infty \frac{\sin x}{x} dx$ (b) $\int_1^\infty \frac{dx}{\sqrt{1+x^3}}$

(c) $\int_0^\infty \frac{x}{1+x^2 \sin^2 x} dx$ (d) None of these

74. Which of the following is / are metric space

(a) $X = R^2$ for $x = (x_1, x_2),\ y = (y_1, y_2) \in X$ the function

$$d(x,y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2 \\ |x_1| + |x_2 - y_2| + |y_1| & \text{if } x_2 \ne y_2 \end{cases}$$

(b) Let X be the set of continuous functions from $[a,b]$ to R. For all $x, y \in X$.

$$d(x,y) = \max\{|x(t) - y(t)| : t \in [a,b]\}$$

(c) Let X be the set of bounded functions from some set A to R. For $x, y \in X$.

$$d(x,y) = \sup\{|x(t) - y(t)| : t \in A\}$$

(d) Let X be the set of continuous functions from $[a,b]$ into R. For all $x, y \in X$.

$$d(x,y) = \int_a^b |x(t) - y(t)|\, dt$$

75. Consider the sequence $\{f_n\}$ defined by

$f_n(x) = \frac{nx}{1+nx}$ for $x \ge 0$. Then

(a) $\lim_{n\to\infty} f_n(x) = 1$, for $x > 0$.

(b) $\{f_n\}$ does not converge uniformly to f on $[0, \infty)$

(c) for $a > 0$, $\{f_n\}$ converges uniformly to f on $[a, \infty)$

(d) $\lim_{n\to\infty} f_n(x) = 0$

76. A sequence of functions $\{g_n\}$ are defined as

$g_n(x) = 2x + \frac{x}{n},\ x \in [0,1]$

(a) $g = \lim_{n\to\infty} g_n = 0$

(b) g is continuous on $[0,1]$

(c) $\left[\lim_{n\to\infty} g_n(x)\right]' = \lim_{n\to\infty} g_n'(x)$ for $x \in [0,1]$

(d) $\int_0^1 \lim_{n\to\infty} g_n(x)dx = \lim_{n\to\infty} \int_0^1 g_n(x)dx$

77. Suppose g is n times continuously differentiable on some interval (a,b) that contains c.

If $g'(c) = g''(c) = \ldots = g^{n-1}(c) = 0$

and $g^n(c) \ne 0$. Then

(a) If n is odd then g has neither a relative minimum nor a relative maximum at c.

(b) If n is even and $g^n(c) > 0$, then g has a relative minimum at c.

(c) If n is even $g^n(c) < 0$ then g has a relative maximum at c.

(d) If n is odd $g^n(c) < 0$ then g has a relative maximum at c.

78. Which of the following statements is / are true?

(a) If (g^n) is a sequence of functions, each one bounded on a set and $A \subseteq R$ and (g^n) converges uniformly to g on A, then g is also bounded

(b) A subset of R is closed iff, it contains all of its boundary points.

(c) The sequence of functions $\left(nxe^{-nx}\right)_{n=1}^{\infty}$ converges uniformly on $[0,\infty)$

(d) $\lim\limits_{x\to 0^+}(\sin(x))^x = 0$

79. Which of the following limits does not exists

(a) $\lim\limits_{(x,y)\to(1,1)} \dfrac{e^x \text{In} y}{x^2+2y^2}$

(b) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2y - z^2}{x^2+y^2+z^2}$

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\sin y}{x^2+2y^2}$

(d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2+2y^2}$

80. Let $g_n(x) = nx^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Then

(a) $\lim g_n(x) = 0$ for $x \in [0,1)$

(b) $\lim\limits_{n\to\infty} \int_0^1 g_n(x)dx = 0$

(c) $\lim\limits_{n\to\infty} \int_0^1 g_n(x)dx = 1$

(d) $\lim g_n(x) = 1$ for $x \in [0,1]$

81. Let E be a set of positive real numbers. We define $\sum\limits_{x\in E} x$ to be $\sup\limits_{F\in\mathcal{F}} S_F$, where $\mathcal{F}$ is the collection of finite subsets of E and S_F is the (finite) sum of the elements of F. Then

(a) $\sum\limits_{x\in E} x < \infty$ only if E is countable

(b) If E is countable and $\langle x_n \rangle$ is onto maping N onto E, then $\sum\limits_{x\in E} x = \sum\limits_{n=0}^{\infty} x_n$

(c) If E is countable and $\langle x_n \rangle$ is a one-to-one mapping of N onto E, then $\sum\limits_{x\in E} x = \sum\limits_{n=1}^{\infty} x_n$

(d) If E is countable then $\sum\limits_{x\in E} x > \infty$.

82. Let f be a real valued function defined on $[a,b]$. We define the lower envelope g of f to be the function g defined by

$$g(y) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(x)$$

and the upper envelope h by

$$h(y) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x)$$

Then which of the following is true?

(a) If f is bounded, the function g is lower semi continuous, while h is upper semi continuous.

(b) If ϕ is any lower semi continuous function $\ni$ $\phi(x) \le f(x)$ for all $x \in [a,b]$ then $\phi(x) \le g(x)$ for all $x \in [a,b]$

(c) For each $x \in [a,b]$, $g(x) \le f(x) \le h(x)$ iff f is lower semi continuous at x, while $g(x) = h(x)$ iff f is continuous at x.

(d) If f is bounded the function g is lower semi continuous while h is not continuous.

83. Let f be a non-negative measurable function. Then

(a) $\int f = 0$ implies $f = 0$ almost everywhere.

(b) $\int f = \sup \int \phi$ over all simple functions $\phi \le f$

(c) There is an increasing sequence $\langle \phi_n \rangle$ of non-negative simple functions each of which vanishes outside a set of finite measure such that $f = \lim \phi_n$

(d) $\int f = 0$ implies $f \ne 0$.

84. Let f be defined on $[a,b]$ and g be a continuous function on $[\alpha,\beta]$ that is differentiable at γ with

$g(\gamma) = c \in [a,b]$

(a) If $g'(\gamma) > 0$ then $D^+(f \circ g)(\gamma) = D^+ f(c).g'(\gamma)$

(b) If $g'(\gamma) < 0$ then $D^+(f \circ g)(\gamma) = D_- f(c).g'(\gamma)$

(c) If $g'(\gamma) = 0$ and all derivatives of f are finite at c then $D^+(f \circ g)(\gamma) = 0$

(d) If $g'(\gamma) < 0$ then $D^+(f \circ g)(\gamma) = D^+ f(c).g(\gamma)$

85. Let E be a subset of a complete metric space

(a) If $\tilde{E}$ is dense and F a closed set contained in E, then F is nowhere dense

(b) If E and $\tilde{E}$ are both dense, then at most one of them is an F_σ

(c) The set of rational numbers in $[0,1]$ is not a G_δ

(d) If E and $\tilde{E}$ are not dense, then F is nowhere dense

86. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space. Then

(a) If $B_n = \bigcup_{i=1}^{n} A_i$ then $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$ for $n = 1, 2, 3, \ldots$

(b) If $B = \bigcap_{i=1}^{\infty} A_i$ then $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$

(c) If $B = \bigcup_{i=1}^{\infty} A_i$ then $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$

(d) If $B_n = \bigcap_{i=1}^{n} A_i$ then $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$ for $n = 1, 2, 3, \ldots$

87. Suppose $x_n > 0$ and $\sum x_n$ converges. Put $r_n = \sum_{m=n}^{\infty} x_m$ then

(a) $\dfrac{x_m}{r_m} + \ldots + \dfrac{x_n}{r_n} > 1 - \dfrac{r_n}{r_m}$

if $m < n$ then $\sum \dfrac{x_n}{r_n}$ diverges

(b) $\dfrac{x_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$ and $\sum \dfrac{x_n}{\sqrt{r_n}}$ converges

(c) $\sum \dfrac{x_n}{r_n}$ is converges

(d) $\sum \dfrac{x_n}{\sqrt{r_n}}$ is diverges

88. Fix a positive number α. Choose $b_1 > \sqrt{\alpha}$ and define $b_2, b_3, b_4, \ldots$ by recursion formula

$b_{n+1} = \dfrac{1}{2}\left(b_n + \dfrac{\alpha}{b_n}\right)$. Then

(a) $\{b_n\}$ decreases monotonically and that limit $b_n = \sqrt{\alpha}$

(b) If $\varepsilon_n = b_n - \sqrt{\alpha}$ then $\varepsilon_{n+1} = \dfrac{\varepsilon_n^2}{2b_n} < \dfrac{\varepsilon_n^2}{2\sqrt{\alpha}}$ so that setting $\beta = 2\sqrt{\alpha}$ then $\varepsilon_{n+1} < \beta\left(\dfrac{\varepsilon_1}{\beta}\right)^{2^n}$

$(n = 1, 2, 3, \ldots)$

(c) If $\alpha = 3$ and $b_1 = 2$ then $\dfrac{\varepsilon_1}{\beta} < \dfrac{1}{10}$ and $\therefore \varepsilon_5 < 4.10^{-16}$, $\varepsilon_6 < 4.10^{-32}$

(d) $\{b_n\}$ increases monotonically and that $\lim b_n = 1/\sqrt{\alpha}$

89. If $\{x_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1} \quad (n = 0, 1, 2, \ldots)$$

Then which of the following imply true?

(a) If $\lim s_n = s$ then $\lim \sigma_n = s$

(b) If $x_n = s_n - s_{n-1}$ for $n \geq 1$. Then

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} kx_k$$

And assume that if $\lim(nx_n) = 0$ and that $\{\sigma_n\}$ converges. Then $\{s_n\}$ is converges.

(c) If $s_n > 0$ for all n then $\limsup s_n = 0$ and $\lim \sigma_n = \infty$

(d) If $M < \infty$ $|nx_n| \le M$ for all n and $\lim \sigma_n = \sigma$. Then $\lim s_n = 0$

90. Suppose $x_n > 0$, $s_n = x_1 + x_2 + ... + x_n$ and $\sum x_n$ diverges. Then

(a) $\sum \frac{x_n}{1+x_n}$ diverges

(b) $\frac{x_{N+1}}{s_{N+1}} + ... + \frac{x_{N+k}}{s_{N+k}} \ge 1 - \frac{s_N}{s_{N+k}}$

and that $\sum \frac{x_n}{s_n}$ diverges.

(c) $\frac{x_n}{s_n^2} \le \frac{1}{s_{n-1}} - \frac{1}{s_n}$ and that $\sum \frac{x_n}{s_n^2}$ converges.

(d) $\sum \frac{x_n}{1+n^2 x_n}$ diverges

91. If E is a non empty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x,z)$$

Then

(a) $\rho_E(x) = 0$ iff $x \in \overline{E}$

(b) If $\rho_E(x)$ is a uniformly continuous function on X then $|\rho_E(x) - \rho_E(y)| \le d(x,y)$ for all $x \in X$, $y \in X$.

(c) $\rho_E(x) = 0$ iff $x \notin \overline{E}$

(d) $\rho_E(x)$ is not a uniformly continuous on X.

92. Suppose g is defined in $(-1,1)$ and $g^1(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients $D_n = \frac{g(\beta_n) - g(\alpha_n)}{\beta_n - \alpha_n}$. Then

(a) If $\alpha_n < 0 < \beta_n$ then $\lim D_n = g^1(0)$

(b) If $0 < \alpha_n < \beta_n$ and $\left\{\frac{\beta_n}{(\beta_n - \alpha_n)}\right\}$ is bounded then $\lim D_n = g^1(0)$

(c) If g^1 is continuous in $(-1,1)$ then $\lim D_n = g^1(0)$

(d) If $\alpha_n > 0 > \beta_n$ then $\lim D_n = g^1(0)$

93. Suppose a and c are real numbers, $c > 0$ and g is defined on $[-1,1]$ by

$$g(x) = \begin{cases} x^a \sin(x^{-c}) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then

(a) g is continuous iff $a > 0$

(b) $g'(0)$ exists iff $a > 1$

(c) g is not continuous iff $a > 0$

(d) $g'(0)$ does not exists.

94. Fix $a > 1$

(a) If m, n, p, q are integers; $n > 0$, $q > 0$ and $r = m/n = p/q$ then $(a^m)^{1/n} = (a^p)^{1/q}$

(b) If x is real, define $B(x)$ to be the set of all numbers a^t, where t is rational and $t \le x$, then $a^r = \sup B(r)$ where r is rational.

(c) $a^{x+y} = a^x + a^y$ for all real x and y

(d) $a^{x+y} = a^x a^y$ for all real x and y

95. Let g be a continuous function from $[0,1]$ to R and $S_n = \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)$ $n = 1,2,3,4,...$

Then

(a) $\{S_n\}_{n=1}^{\infty}$ is a divergent sequence

(b) $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence

(c) Suppose $g(x) \geq 0$ for all $x \in [0,1]$ and $g(x_0) \geq 0$ for some $x_0 \in [0,1]$. Also g is continuous on $[0,1]$. Then $\lim_{n\to\infty} S_n > 0$.

(d) If g is continuous on $[0,1]$ then g is not uniformly continuous on $[0,1]$.

96. If $0 \leq x < 1$ then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ suppose if $x \geq 1$, the series

(a) diverges
(b) converges
(c) uniformly converges
(d) absolutely converges

97. Let E^0 denote the set of all interior points of a set *E*. Then

(a) E^0 is always open

(b) *E* is open iff $E^0 = E$

(c) If *G* is contained in *E* and *G* is open, then *G* is contained in E^0

(d) *E* and $\overline{E}$ always have the same interiors.

98. Let *E* be the set of all $x \in [0,1]$ whose decimal expansion contains only the digit 4 and 7.

(a) *E* is countable

(b) *E* is dence in $[0,1]$

(c) *E* is compact

(d) *E* is perfect

99. Suppose *f* is a real function on $(-\infty, \infty)$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t then f has atmost one fixed point.

(b) The function *f* defined by $f(t) = t + (1+e^t)^{-1}$ has no fixed point, although $0 < f'(t) < 1$ for all real *t*

(c) The function f defined by $f(t) = t + (1+e^t)^{-1}$ has one fixed point although $0 < f'(t) < 1$ for all real *t*.

(d) If f is differentiable and $f'(t) \neq 1$ for every real t then f has two fixed point.

100. Define $g(x) = \int_x^{x+1} \sin(t^2)\,dt$. Then

(a) $|g(x)| < 1/x$ if $x > 0$

(b) $2xg(x) = \cos(x^2) - \cos(x+1)^2 + r(x)$ where $|r(x)| < c/x$ and *c* is a constant

(c) $|g(x)| > 1/x$ if $x > 0$

(d) $2xg(x) = \cos(x^2) - \cos(x+1)^2$

101. Which of the following sets are bounded?

(a) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots.., \frac{1}{n}, \ldots.\right\}$

(b) $C = \{\ldots..-5, -3, -1\}$

(c) $B = \{2, 4, 6, \ldots..\}$

(d) $z = \{\ldots, -2, -1, 0, 1, 2, \ldots.\}$

102. Which of the following is/are true?

(a) An arbitrary union of open sets is open
(b) A finite union of closed sets is closed
(c) An arbitrary intersection of closed set is closed
(d) A finite union of open sets is open

103. A set is said to be nowhere dense if

(a) $\left(\overline{S}\right)^0 = \phi$ (b) $S = S'$

(c) $S \subset S'$ (d) $S \in S'$

104. If *C* is cantour set then

(a) *C* is closed
(b) *C* is nowhere dense
(c) The Lebesgue measure of the set C is zero
(d) *C* is countable

105. 1) Every open cover of a set in *R* has a count able subcover

2) Every bounded infinite set of *R* has a limit point

3) The set $[0,1]$ is countable

(a) 1&2 are true but 3 is false

(b) 1&3 are true but 2 is false
(c) 2&3 are true but 1 is false
(d) 1,2&3 all are true

106. Which of the following is/are incorrect?
(a) Real line is seperable (ie) it has a countable dense subset
(b) Every nonempty perfect set is uncountable
(c) The set $\left\{\frac{r}{2^n}, r = 0,1...,2^n, n \in N\right\}$ is dense in $(0,1)$
(d) Cantour set is compact

107. How many disjoint set can be identified in the universal set displaying Venn-diagram of 5 distinct sets
(a) 5^2 (b) 2^5
(c) 3^5 (d) 5^3

108. Which of the following statement is/are true?
(a) Let $f : D \to R$ and $c \in D' \cap D$ then f has a limit at c iff f is continuous at c
(b) Let $f : N \to R$, then f is continuous on N
(c) Every continuous function on a bounded interval is bounded
(d) Suppose f and g are continuous on $[0,10]$ and differentiable on $(0,10)$. If $f'(x) = g'(x)$ for all $x \in (0,10)$ and $f(\pi) = g(\pi)$ then $f(x) = g(x)$

109. $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) =$
(a) 0 (b) 1
(c) π (d) $n\pi$

110. $\lim\limits_{x\to 1} \frac{\sqrt{x^2+3} - 2\sqrt{x}}{x^2 - 1}$
(a) $-\frac{1}{4}$ (b) -1
(c) 1 (d) 0

111. Let f be continuous on $[a,b]$ and suppose that $f(x) \le 0$ for all $x \in [a,b]$
(a) Then there exists a point $c \in [a,b]$ such that $f(c) = 0$
(b) Then there exists a point $c \in [a,b]$ such that $f(c) < 0$ then $\int_a^b f < 0$
(c) Then there exists a point $c \in (a,b)$ such that $f(c) < 0$ then $\int_a^b f > 0$
(d) Then there exists a point $c \in (a,b)$ such that $f(c) > 0$ then $\int_a^b f < 0$

112. Let $f(x) = \begin{cases} 2x+5 & \text{if } x \le 1 \\ 9x^2 - 2 & \text{if } x > 1 \end{cases}$
(a) $f(x)$ is continuous
(b) $f(x)$ is differentiable at 1
(c) $f(x)$ is continuous but not differentiable at 1
(d) $f(x)$ is discontinuous

113. Let F be a collection of pairwise disjoint open subset of R^n for some fixed $n \in N$.
(a) F is either a finite or a countably infinite
(b) F is uncountable
(c) F is unbounded
(d) None of these

114. Which of the following is/are true?
(a) Every deleted neighbourhood of $x \in R$ is an open set
(b) If a set has a maximum and a minimum, then it is compact
(c) No infinite set is compact
(d) Let $S \subseteq R$ then $\operatorname{int} S \cap bdS = \phi$

115. Then sequence (S_n) deifined by $S_n = \cos(n\pi)$ is

(a) Convergent (b) Divergent
(c) Continuous (d) Not continuous

116. Find the limit of

$$\lim_{n\to\infty}\left(\sqrt{n^2+2n}-\sqrt{n^2-3n}\right)$$

(a) $\frac{5}{2}$ (b) 1
(c) 0 (d) $\frac{1}{2}$

117. Let (S_n) be defined by $S_1=\sqrt{2}$,

$S_{n+1}=\sqrt{2+S_n}$ for all $n\in N$
(a) Monotone
(b) Bounded
(c) Monotone & bounded
(d) Unbounded

118. 1) A sequence is bounded and monotone iff it is a cauchy sequence
2) If for every $\varepsilon>0$ there exists $N\in R$ such that for every $n>N$ we have $S_n<\varepsilon$ then $\lim\limits_{n\to\infty}S_n=0$
(a) 1 is correct but 2 is incorrect
(b) 2 is correct but 1 is incorrect
(c) Both 1 & 2 are correct
(d) Both 1 & 2 are incorrect

119. $\lim\limits_{n\to\infty}\frac{n^2}{1.01^n}=$
(a) 0 (b) 1
(c) ∞ (d) n

120. Find the infimum and supremum of

$$\left\{x\in Q: x=\frac{n}{n+1}, n\in N\right.$$

(a) $\inf=\frac{1}{2}$ & $\sup=1$
(b) $\inf=2$ & $\sup=1$
(c) $\inf=1$ & $\sup=\frac{1}{2}$
(d) $\inf=1$ & $\sup=3$

121. 1)Between any two distinct real numbers there always lies a rational number and therefore infinitely many rational numbers
2)Between any two distinct real numbers. There always lies an irrational number and therefore infinitely many irrational numbers
(a) 1 is true but 2 is not true
(b) 2 is true but 1 is not true
(c) Both 1 & 2 are true
(d) Both 1 & 2 are false

122. If f is defined by $f(x)=|x|+|x^2-1|$ $\forall\ x\in R$ then
(a) f has local minima at $x=\pm\frac{1}{2}$
(b) f has local maxima at $x'=\pm\frac{1}{2}$
(c) f has local maxima at $x'\le 1$
(d) f has no local extrema

123. Which of the following statements is/are true?
(a) The set Z of all integers is not a neighbourhood of any of its points
(b) The set of all rational numbers Q is not a neighbourhood of any of its points
(c) The set of all irrational number is not a neighbourhood of any of its points
(d) None of these

124. If f is integrable on $[a,b]$ and $\int_a^x f(t)\,dt=0$ for all $x\in[a,b]$ then
(a) $f(t)=0$ nowhere in $[a,b]$
(b) $f(t)=0$ almost every where in $[a,b]$
(c) $f(t)\neq 0$ nowhere in $[a,b]$
(d) $f(t)>0$ nowhere in $[a,b]$

125. Which of the following subsets of R are neighbourhoods of 3
(a) $(2,4)$ (b) $[2,4]$
(c) $(2,7)$ (d) $[2,4]-\left\{3\frac{1}{4}\right\}$

126. Find the derived set of the set $S=\{1,3,7,11\}$

(a) ϕ (b) 1

(c) R (d) 3

127. Find limit points of the set

$$S=\left\{\frac{1}{n}:n\in N\right\}=\left\{1,\frac{1}{2},\frac{1}{3}...\frac{1}{n},...\right\}$$

(a) 0 (b) 1

(c) n (d) ∞

128. Find the limit point of the set

$$S=\left\{\frac{3n+2}{2n+1}:n\in N\right\}$$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$

(c) 2 (d) 1

129. Which of the following is/are true?

(a) A closed or semi-closed interval is not an open set

(b) Every non-empty finite set is not a open set

(c) The set of all real number R & the empty set ϕ are open

(d) The set of rational numbers Q is not an open set

130. The set $A=\left\{\frac{1}{2},\frac{2}{3},\frac{3}{4}....\right\}$

(a) Denumerable set (b) Countable

(c) Uncountable set (d) Non-denumerable

131. A set E in R is open if

(a) It contains a neighbourhood of each of its points

(b) It does not contains a neighbourhood

(c) It contains a neighbourhood at a

(d) None of the above

132. The set of all rational numbers is

(a) Countable

(b) Closed

(c) Neither open nor closed

(d) Finite

133. The sequence $\left\{1+\frac{(-1)^n}{n}:n\in N\right\}$ is

(a) Converges to 1

(b) Neither converges nor diverges

(c) Diverges to 1

(d) None of these

134. The sequence $\langle S_n\rangle=\sin n\pi\theta$ and θ is a rational number such that

(a) $0<\theta<1$ is not convergent

(b) $0<\theta<1$ is convergent

(c) $0>\theta>1$ is convergent

(d) $0>\theta>1$ is not convergent

135. The sequence $\left\langle \log\frac{1}{n}\right\rangle$ is

(a) Converges to ∞

(b) Diverges to ∞

(c) Diverges to $-\infty$

(d) Converges to $-\infty$

136. The sequence $e^{n/4}n^{-(n+1)/2}\left(1^1.2^2.....n^n\right)^{1/n}$

(a) Converges to 0 as $n\to\infty$

(b) Converges to 1 as $n\to\infty$

(c) Diverges

(d) None of these

137. Let E be any bounded closed set in the complex plane containing an infinite number of points, and let M_n be the maximum of $|V(x_1,....,x_n)|$ as the points $x_1,....,x_n$ run through the set E where $V(x_1,........,x_n)=\prod_{1\le i<j\le n}(x_i-x_j)$ is the Vandermonde determinant.

(a) $M_n^{2/n(n-1)}$ converges as $n\to\infty$

(b) $M^{2n/(n-1)}$ converges as $n\to\infty$

(c) $M^{n(n-1)}$ converges as $n\to\infty$

(d) M^{2n^2} diverges

138. Let f is differentiable and $f(x)\neq 1$ for any real x then

(a) f is increasing function

(b) f is has at most one fixed point

(c) f has at least two fixed point

(d) f has unique fixed point

139. If x_n is cauchy then

(a) x_n is bounded

(b) $x_{nk} \to x$ then $x_n \to x$

(c) $[a,b]$ is complete

(d) $[0,1)$ is not complete

140. $f(x) = \frac{1}{x}$ on $(0,1]$ and $f(x) = x^2$ on $[1,\infty)$ are

(a) Uniformly continuous
(b) Not uniformly continuous
(c) Continuous
(d) Discontinuous

141. Let $f:[a,b] \to R$ be continuous function then

(a) f is bounded
(b) f is unbounded
(c) f assumes its max and min values
(d) $f(a) < p < f(b) \Rightarrow$ there exists x such that $f(x) = p$

142. $f(x) = x^n$ on $[0,1]$ is

(a) Convergent but not uniformly convergent
(b) Uniformly convergent
(c) Divergent
(d) None of these

143. Which of the following statement is/are correct?

(a) $[a,b]$ is connected
(b) R is connected
(c) Let $f: M \to N$ is continuous and M is connected. Then $f(M)$ is connected
(d) $a,b \in I$ and $a < c < b$, $c \in I$ (ie) I is an interval iff $I \subseteq R$ is connected.

144. The subset $\{(a,b,c,d) \in R^4 \mid ad - bc \neq 0\}$ is

(a) Open and dense in R^n
(b) Compact
(c) Bounded
(d) Deneuramable

145. If a function $f: C \to C$ is entire and the real part of f is bounded function then

(a) f is a constant function
(b) f is a continuous function
(c) f is a onto function
(d) f is bijective function

146. Let $f_n:[0,1] \to R$ be defined for $n \geq 2$

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq 1/n \\ -n^2\left(x - 2/n\right) & 1/n < x < 2/n \\ 0 & 2/n \leq x \leq 1 \end{cases}$$

then choose the correct statement

(a) $\lim\limits_{n\to\infty} \int_0^1 f_n(x) \neq \int_0^1 f(x)\,dx$ where $\lim\limits_{n\to\infty} f_n(x) = f(x)$

(b) $f_n(x)$ is continuous $\forall$ $n \geq 2$ and $x \in [0,1]$

(c) $\langle f_n(x) \rangle$ is uniformly continuous

(d) $\lim\limits_{n\to\infty} f_n(x)$ is continuous function

147. Let f_n be a sequence of continuous real valued functions on $[0,1]$ which converges uniformly to f.

(a) $\lim\limits_{n\to\infty} f_n(x_n) = f\left(1/2\right)$ for any sequence $\{x_n\}$ which converges to $\frac{1}{2}$

(b) $\lim\limits_{n\to\infty} f_n(x_n) = 1$ for any sequence con verges to $\frac{1}{2}$

(c) $\lim\limits_{n\to\infty} f_n(x_n) \neq f(x)$ for any sequence diverges to $f(x)$

(d) $\lim\limits_{n\to\infty} f_n(x_n) = f(0)$

148. Let $f : R \to R$ be differentiable and assume there is no $x \in R$. Such that $f(x) = f'(x) = 0$. Then

(a) $S = \{x \mid 0 \le x \le 1\}$ is continuous

(b) $S = \{x \mid 0 \le x \le 1, \quad f(x) = 0\}$ is finite

(c) $S = \{x \mid 0 \ge x \ge 1, \quad f(x) \ne 0\}$ is infinite

(d) $S = \{f(x) \mid 1 \le x \le \infty\}$ is finite

149. If f is a bounded real valued function on [0,1]. f is Lebesgue measurable iff

(a) $Sup\int \psi\, dm = \inf \int \phi dm$, where m is the Lebesgue measure on [0,1] and $\psi \le f \le \phi$

(b) $Sup\int \psi\, dm > \inf \int \phi dm$, where m is the Lebesgue measure on [0,1] and $\psi \le f \le \phi$

(c) $Sup\int \psi\, dm < \inf \int \phi dm$, where m is the Lebesgue measure on [0,1] and $\psi \le f \le \phi$

(d) $Sup\int \psi\, dm \ne \inf \int \phi dm$, where m is the Lebesgue measure on [0,1] and $\psi \le f \le \phi$

150. Which of the following need not be hold always

(a) f is continuous and bounded on R then f is uniformly continuous on R

(b) f is uniformly continuous on a bounded interval I then f is bounded on I

(c) If $< f_n(x) >$ and $< g_n(x) >$ converges on I. Then $\langle f_n(x) + g_n(x) \rangle$ uniformly convergent

(d) All the above

151. Let $f, g, f_n, \ldots$ be A-measurable function for some fixed σ - algebra A. Then

(a) If $\phi : R^1 \to R^1$ is B_l-measurable then $\phi \circ f$ is A-measurable

(b) If $\phi : R^2 \to R^1$ is continuous then $\phi(f, g)$ is A-measurable

(c) All A-measurable functions form a linear vector space over R

(d) Functions $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$ are A-measurable

152. Let $\| \ \|_1$ and $\| \ \|_2$ be the norms of linear vector space X. Then

(a) $\| \ \|_1$ & $\| \ \|_2$ are equivalent norms

(b) The relative induced distance are topologi cally equivalent

(c) For any $\{x_n\} \subset X, \|x_n\|_1 \to 0$ iff $\|x_n\|_2 \to 0$

(d) None of these

153. Let X & Y be the Banach Spaces with the norm

$\|(x, y)\|_{1 \quad X \times Y} = \|x\|_X + \|y\|_Y$

(a) $\|x, y\|_{P, \quad X \times Y} = \sqrt[P]{\|x\|_X^P + \|y\|_Y^P}, P \ge 1$

(b) $\|x, y\|_{\infty, \quad X \times Y} = \max(\|x\|_X, \|y\|_Y)$

(c) $\|x, y\|_{P, \quad X \times Y} = \sqrt{\|x\| + \|y\|}$

(d) $\|x, y\|_{\infty, \quad X \times Y} = \min(\|x\|_X, \|y\|_Y)$

154. The closed unit ball of a normed linear space X is compact iff

(a) X is finite dimensional

(b) X is closed

(c) X is infinite dimensional

(d) X is bounded

155. If $1 \le p < q < \infty$ then

(a) $\ell_p(R) \subset \ell_q(R) \subset \ell_\infty(R)$

(b) $\ell_p(R)$ is a proper subspace of $\ell_q(R)$

(c) The identity map $Id : \ell_p(R) \to \ell_q(R)$ is continuous

(d) $\ell_1(R)$ is a dense subset of $\ell_q(R)$ with respect to the convergence in $\ell_q(R)$

156. Let (X, d_X) and (Y, d_Y) be two metric space and let $\{f_n\}$ be a sequence of functions $f_n : X \to Y$. For any function $f : X \to Y$ then

(a) $\{f_n\}$ converges uniformly to f

(b) $\{f_n\}$ diverges

(c) $\sup\{d_Y(f_n(x), f(x)) \mid x \in X\} \to 0$ as $n \to \infty$

(d) $\{f_n\} \to 1/2$ as $n \to \infty$

157. The function $f_n : [0, \infty) \to R$ is defined by

$$f_n(x) = e^{-x}\left(\frac{x}{n}\right)^{ne}$$

(a) $\{f_n\}$ converges pointwise

(b) $\{f_n\}$ converges on $\left(\frac{1}{n}, 1\right)$

(c) $\{f_n\}$ converges uniformly but not pointwise converges

(d) $\{f_n\}$ bounded

158. Let $f_n : R \to R$ be given by $f_n(x) = \frac{1}{n}\sin(nx)$

(a) $\{f_n\}$ converges uniformly to zero

(b) $\{f_n'\}$ of derivates converges to zero

(c) $\{f_n'\}$ of derivates does not converges

(d) $\{f_n\}$ diverges

159. Let (X, d_X) and (Y, d_Y) be two metric spaces, and assume that X is compact. Then

(a) $\rho(f, g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}$

(b) $\rho(f, g) = \inf\{d_Y(f(x), g(x)) \mid x \in X\}$

(c) $\rho(f, g) = \|f(x), g(x)\|$

(d) $\rho(f, g) = d(f(x), g(x))$

160. Let E be a normed linear space. E is complete iff

(a) $\sum_1^\infty \|x_n\| < \infty$ then $\sum_1^\infty x_n$ converges to an $s \in E$

(b) $\sum_1^\infty \|x_n\| = \sum_1^\infty x_n$

(c) $\sum_1^\infty \|x_n\| < \infty$ then $\sum_1^\infty x_n$ diverges

(d) $\sum_1^\infty \|x_n\| = 0$ then $\sum_1^\infty x_n$ converges to s where $s \in E$.

161. $\{f_n\}$ is a sequence of Lebesgue measurable function on [0,1] such that $\operatorname{Lim}_{n\to\infty}\int_0^1 |f_n|\,dx = 0$ and there is an integrable function g on [0,1] such that $|f_n|^2 \le g$ for each n. Then

(a) $\lim\limits_{n\to\infty}\int_0^1 f_n\,dx = 0$

(b) $\lim\limits_{n\to\infty}\int_0^1 |f_n|^2\,dx = 0$

(c) $\lim\limits_{n\to\infty}\int_0^1 |f_n|^3\,dx = 0$

(d) $\lim\limits_{n\to\infty}\int_0^\infty |f_n|\,dx = 0$

162. Let S be a Lebesgue measurable subset of R and let $f, g : S \to R$ be measurable function. Then

(a) $f + g$ is measurable

(b) If $\phi \in C(R)$, then $\phi(f)$ is measurable

(c) fg is measurable

(d) $f \circ g$ is measurable

163. The total number of functions from the null set to a finite set *B* having *n* elements is equal to

(a) n (b) 2^n

(c) n^2 (d) 1

164. If $\dfrac{a_n}{a_{n+1}} > \dfrac{b_n}{b_{n+1}}$ $(a_n > 0, \quad b_n > 0)$ then

(a) Σa_n diverges $\Rightarrow \Sigma b_n$ converges

(b) Σb_n diverges $\Rightarrow \Sigma a_n$ converges

(c) Σb_n converges $\Rightarrow \Sigma a_n$ converges

(d) Σa_n & Σb_n diverges

165. Take two arbitrary partitions P_1 and P_2 over [a, b] then

(a) $U(f, P_1) \geq L(f, P_2) \quad \forall\, P_1, P_2$

(b) $U(f, P_1) < L(f, P_2) \quad \forall\, P_1, P_2$

(c) $U(f, P_1) \leq L(f, P_2) \quad \forall\, P_1, P_2$

(d) $U(f, P_1) = L(f, P_2) \quad \forall\, P_1, P_2$

166. For a function $f : R \to R$, if f is monotone then

(a) f is Borel measurable

(b) f is Lebesgue measurable

(c) f is Reimann integral

(d) f is Improper Reimann integral

167. If $< a_n >$ be a sequence such that $\lim\limits_{n \to \infty}(a_{n+1} - a_n) = \ell$ then the sequence $< b_n >$ wher $b_n = \dfrac{a_n}{n}$

(a) Converges to ℓ

(b) Converges to 0

(c) Diverges

(d) Converges to $\dfrac{1}{n}$

168. Let f be a continuous real function on *R* of which it is known that $f'(x)$ exsist for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$ then

(a) $f'(0)$ exists & equal to 3

(b) $f'(0)$ exists & equal to 0

(c) $f'(0)$ exists & equal to ∞

(d) $f'(0)$ need not exists

169. If $f : S \to R^m$ be differentiable at $c \in S \subseteq R^n$ then

(a) f is continuous at c

(b) f is constant at c

(c) f is discontinuous at c

(d) None of these

170. If $f(x)$ is differentiable function for $\forall x \in R$ such that $f(1) = 5$ and $f(x) < 5 \quad \forall\, x \neq 1$. Then

(a) $f(x)$ is monotonically decreasing

(b) $f(x)$ is bounded

(c) $f(x)$ is not monotonic

(d) $f(x)$ is monotonically increasing

171. The function $f(x) = \begin{cases} \cos x & x \geq 0 \\ -\cos x & x < 0 \end{cases}$ has

(a) Continuity at $x = 1$

(b) Removable distcontinuity at $x = 0$

(c) Continuity at $x = 1$

(d) Irremovable discontinuity at $x = 0$

172. For Lebesgue measure which property is not correct?

(a) $\mu(A + x_0) = \mu(A) + \mu(x_0)$

(b) $\mu(A \cup B) = \mu(A) + \mu(B)$

(c) $0 \leq A \leq \mu(B) \leq \infty$

(d) $\mu(I) = l(I)$

173. The series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ converges if

(a) $p<1$ (b) $p>1$

(c) $p\le 0$ (d) $p=1$

174. If f is constant function on [a,b] then total variation of f on [a,b] is

(a) $V_f(a,b)\ge 0$ (b) $V_f(a,b)\le 0$

(c) $V_f(a,b)=0$ (d) $V_f(a,b)\le 1$

175. The space C[a,b] of all continuous functions on [a,b] is a normed linear space if

(a) $\|y(x_1)\| < \underset{a\le x\le b}{Max}|y'(x_1)|$

(b) $\|y(x_1)\| < \underset{a\le x\le b}{Max}|y(x_1)|$

(c) $\|y(x_1)\| < \underset{a\le x\le b}{Max}|y(x_1)| + \underset{a\le x\le b}{Max}|y'(x_1)|$

(d) $\|y(x_1)\| > \underset{a\le x\le b}{Max}|y'(x_1)|$

176. $\int_2^\infty \frac{\cos x}{\log x}dx$ is

(a) Convergent
(b) Conditionally convergent
(c) Divergent
(d) Conditionally divergent

177. If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $z=a$ then it is converges absolutely for

(a) $|z|<|a|$ (b) $|z|>|a|$

(c) $|z|=|a|$ (d) $|z|\ge|a|$

178. Which of the following is/are true?
(a) The set [0,1] is countable
(b) The set [0,1] is not countable
(c) m*[0,1]=0
(d) m*[0,1]=1

179. $\int_0^1 \frac{\sin(1/x)}{\sqrt{x}}dx$

(a) Integral is convergent
(b) Integral is absolutely convergent
(c) Integral is divergent
(d) Integral is not absolutely convergent

180. Which of the following inequality is true for $x>0$

(a) $\log(1+x) < \frac{x}{1+x} < x$

(b) $\frac{x}{1+x} < \log(1+x) < x$

(c) $x < \log(1+x) < \frac{x}{1+x}$

(d) $\frac{1+x}{x} < \log x < x$

181. Which of the following series is absolutely convergent?

(a) $\Sigma\frac{1}{\sqrt{n}}$ (b) $\Sigma\frac{(-1)^n}{n^{3/2}}$

(c) $\Sigma\frac{(-1)^n}{n}$ (d) $\Sigma\frac{1}{\log(n+1)}$

182. Let $f:[0,10)\to[0,10]$ be a continuous mapping then

(a) f has at least 9 fixed points

(b) f has at least one fixed point

(c) f need not to have any fixed point

(d) f has at most one fixed point

183. If $u_n = \sqrt{n+1}-\sqrt{n}$ and $V_n=\sqrt{n^4+1}-n^2$ then

(a) $\sum_{n=1}^{\infty} u_n$ & $\sum_{n=1}^{\infty} V_n$ both converges

(b) $\sum_{n=1}^{\infty} u_n$ converges but $\sum_{n=1}^{\infty} V_n$ diverges

(c) $\sum_{n=1}^{\infty} u_n$ diverges but $\sum_{n=1}^{\infty} V_n$ converges

(d) $\sum_{n=1}^{\infty} u_n$ & $\sum_{n=1}^{\infty} V_n$ both diverges

184. If $f_n(x)=\tan^{-1} nx$ then

(a) $f_n(x)$ is point wise convergent in [0,b], $b<\pi/2$

(b) $f_n(x)$ is uniformly convergent in (0, b).

(c) $f_n(x)$ is uniformly convergent in [0,b], $b > \frac{\pi}{2}$

(d) $f_n(x)$ is uniformly convergent in (0,b), $b > \frac{\pi}{2}$

185. Let $F(x)$ is defined on $[a,b]$ such that $F(x) = x + \frac{1}{20} f(x)$ where *f(x)* is differentiable on $[a,b]$ and and $|f'(x)| < 10$ then from the following statements select the correct statement

(a) $F(x)$ is of bounded variation

(b) $F(x)$ is one -one

(c) $F(x)$ is decreasing

(d) None of these

186. Select the correct statement?

(a) $\text{Lim sup}(a_n)^{1/n} \geq \text{lim sup} \frac{a_{n+1}}{a_n}, \quad a_n > 0$

(b) $\text{Lim inf}(a_n)^{1/n} \geq \text{lim inf} \frac{a_{n+1}}{a_n}, \quad a_n > 0$

(c) $\text{Lim inf}(a_n + b_n) \geq \text{lim inf } a_n + \text{lim inf } b_n$ $< a_n > \& < b_n >$ are bounded

(d) $\text{Lim sup}(a_n + b_n) \leq \text{lim sup } a_n + \text{lim sup } b_n$ $< a_n > \& < b_n >$ are bounded

187. The limit superior & limit inferior of $<(-1)^n>$ is respectively given by

(a) 1,-1 (b) -1,-1

(c) 0,-1 (d) -1,0

188. Which of the following is correct ?

(a) The set of points of discontinuities of a function is countable

(b) The set of pionts of removable discontinuties of a function is countable

(c) There cannot be a function defined *R* which is continuous on *R* but no where differntiable

(d) The set of points of discontinuity of the function defined by $f(x) = \underset{n\to\infty}{\text{Lim}} \; \underset{m\to\infty}{\text{Lim}} |\cos 2^m (n\pi x)|$ is uncountable

189. The series $\sum \left(\frac{1}{n} - \log \frac{n+1}{n} \right)$ is

(a) Divergent

(b) Convergent

(c) Neither convergent nor divergent

(d) None of the above

190. The series $1 + x^2 + x^4 + x^6 + ...$ is a power series with radius of convergence is

(a) 0 (b) 1

(c) 2 (d) 3

191. A monotone function *f* on $[a,b]$, then which of the following imply true?

(a) Any monotone increasing function is the sum of an absolutely continuous function and a singular function.

(b) Let $\langle f_n \rangle$ be a sequence of non-decreasing singular functions on $[a,b] \ni$ the function $f(x) = \sum f_n(x)$ is everywhere finite. Then *f* is also singular.

(c) There is a strictly increasing singular function on $[0,1]$

(d) Any monotone increasing function is the sum of an discontinuous function and a singular function

192. A function *f* satisfy a Lipschitz condition on an interval then

(a) it is absolutely continuous

(b) an absolutely continuous function *f* satisfies a Lipschitz condition iff $|f'|$ is bounded

(c) discontinuous

(d) an absolutely continuous function *f* does not satisfies a Lipschitz condition iff $|f'|$ is bounded.

193. Let $A \subset M$ be connected and let $f : A \to R$ be continuous with $f(x) \neq 0$ for all $x \in A$ then

(a) $f(x) > 0$ for all $x \in A$

(b) $f(x) < 0$ for all $x \in A$

(c) f is constant

(d) $f : A \to R$ be discontiuous at $f(x) > 0$ for all $x \in A$

194. The radius of converges of the power series

$\sum \frac{2^n}{n!} z^n$ is

(a) 0 (b) ∞

(c) 1 (d) e

195. Pick out the sequences which are uniformly convergent?

(a) $f_n(x) = \frac{1}{1+(x-n)^2}$ on $(0, \infty)$

(b) $f_n(x) = \sin^n x$ on $\left[0, \pi/2\right)$

(c) $f_n(x) = \frac{1}{1+(x-n)^2}$ on $(-\infty, 0)$

(d) $f_n(x) = \frac{x^n}{n} + 1$ on $[0,1)$

196. Pick out the functions which are Riemann integrable on the interval $[0,1]$

(a) $f(x) = \begin{cases} 1 & \text{if } x \in \{\alpha_1, \alpha_2, ..., \alpha_n\} \\ 0 & \text{otherwise} \end{cases}$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are fixed, but arbitrarily chosen numbers in $[0,1]$

(b) $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational or if } x = 0 \\ \sin q\pi & \text{if } x = p/q, p \text{ and } q \end{cases}$

are positive coprime integers

(c) $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

(d) None of the above

197. Which of the following improper integrals are convergent

(a) $\int_0^5 \frac{dx}{\sqrt[3]{7x+2x^4}}$

(b) $\int_0^5 \frac{dx}{x^2-5x+6}$

(c) $\int_1^\infty \frac{dx}{\sqrt{x^3+2x+2}}$

(d) None of the above

198. Which of the following series converge uniformly

(a) $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$ over the interval $[-1,1]$

(b) $\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ over the interval $[-\pi, \pi]$ where $\sum_n |a_n| < \infty$ and $\sum_n |b_n| < \infty$

(c) $\sum_{n=0}^{\infty} \overline{e}^{nx} \cos nx$ over the interval $(0, \infty)$

(d) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

199. Let $f \in C[0, \pi]$. Determine the cases where the given condition implies that $f = 0$

(a) $\int_0^\pi f(x) \cos nx dx = 0$ for all integers $n \geq 0$

(b) $\int_0^\pi f(x) \sin nx dx = 0$ for all integers $n \geq 1$

(c) $\int_0^\pi x^n f(x) dx = 0$ for all integers $n \geq 0$

(d) None of the above

200. Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$. Pick out the true statements

(a) The series converges uniformly on R

(b) the series does not converge absolutely for any real value of x
(c) The series converges conditionally
(d) The series is divergent.

201. Let $f: R \to R$ be a continuous function. Which of the following imply that it is uniformly continuous
(a) f is differentiable and its derivative is bounded on R
(b) f is 2π periodic
(c) f is absolutely continuous
(d) None of the above

202. Let f be a continuously differentiable 2π periodic real valued function on the real line. Let $a_n = \int_{-\pi}^{\pi} f(t)\cos ntdt$ where n is a non-negative integer. Pick out the true statements
(a) $a_n \to 0$ as $n \to \infty$
(b) $|a_n| \le c\frac{1}{n}$ for all n, where $c > 0$ is a constant independent of n
(c) The deriative of f is also a 2π periodic function.
(d) $a_n \to n$ as $n \to \infty$

203. Let f_n and f be continuous functions on an interval $[a,b]$ and assume that $f_n \to f$ uniformly on $[a,b]$. Pick out the true statements.
(a) If f_n are all continuously differentiable, then f is continuously differentiable
(b) If f_n are all Riemann integrable, then f is Riemann integrable
(c) If $x_n \to x$ in $[a,b]$, then $f_n(x_n) \to f(x)$
(d) All the above

204. Pick out the convergent series
(a) $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+3/2}}$
(b) $\sum_{n=1}^{\infty} \sqrt{\frac{1+4^n}{1+5^n}}$
(c) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+...$
(d) None of these

205. Which of the following functions are continuous?
(a) $f(x) = \sum_{n=1}^{\infty} n^2x^n, x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$
(b) $f(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{\left(1+x^2\right)^2} + ..., x \in R$
(c) $f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^{3/2}}, x \in [-\pi, \pi]$
(d) All the above

206. Which of the following functions are Riemann integrable on the interval $[0,1]$?
(a) $f(x) = f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$
(b) $f(x) = \lim_{n\to\infty} \cos^{2n}(24\pi x)$
(c) $f(x) = \begin{cases} \cos x, & \text{if } 0 \le x \le \frac{1}{2} \\ \sin x, & \text{if } \frac{1}{2} < x \le 1 \end{cases}$
(d) None of these

207. Consider the polynomial $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ with real coefficients. Pick out the case(s) which ensure that the polynomial $p(.)$ has a root in the interval $[0,1]$
(a) $a_o + \frac{a_1}{2} + ... + \frac{a_n}{n+1} = 0$
(b) $\frac{a_0}{1.2} + \frac{a_1}{2.3} + ... + \frac{a_n}{(n+1)(n+2)} = 0$
(c) $a_0 < 0$ and $a_0 + a_1 + ... + a_n > 0$
(d) None of the above

208. Pick out the true statements
(a) If a continuous function $f: R \to R$ is uniformly continuous, then it maps cauchy sequences into cauchy sequences

(b) The function $f(x)=\dfrac{\sin(x^2)}{\sin^2 x}$ is uniformly continuous on the interval $(0,1)$

(c) A continuous function $f:R\to R$ is uniformly continuous, if it maps cauchy sequences into cauchy sequences

(d) All the above

209. Let $f:[-\pi,\pi]\to R$ be continuous. Pick out the cases which imply that $f\equiv 0$

(a) $\int_{-\pi}^{\pi} f(x)\cos nx dx=0$ for all $n\geq 0$

(b) $\int_{-\pi}^{\pi} f(x)\sin nx dx=0$ for all $n\geq 1$

(c) $\int_{-\pi}^{\pi} x^n f(x)dx=0$ for all $n\geq 0$

(d) None of the above

210. Let $\{a_n\}$ be a sequence of positive terms. Pick out the cases which imply that $\sum a_n$ is convergent

(a) $\dfrac{a_{n+1}}{a_n}<\left(\dfrac{n}{n+1}\right)^2$ for all n

(b) $\sum n^2 a_n{}^2<\infty$

(c) $\lim_{n\to\infty} n^{3/2} a_n = 3/2$

(d) None of thes above

211. Pick out the true statements

(a) $\cos x>1-\dfrac{x^2}{2}$ for all $x>0$

(b) If p is a polynomial in one variable with the real coefficients which has all its roots real, then its derivative p^1 has all its roots real aswell.

(c) The equation $\cos(\sin x)=x$ has exactly one solution in the interval $\left[0,\pi/2\right]$

(d) $\sin x>x^2$ for all $x>0$

212. Let $f:(0,1)\to R$ be continuous. Pick out the statements which imply that f is uniformly continuous

(a) $f(x)=x^{1/2}\sin\dfrac{1}{x^3}$

(b) $f\left(1/n\right)\to 1/2$ and $f\left(1/n^2\right)\to 1/4$

(c) $|f(x)-f(y)|\leq\sqrt{|x-y|}$ for all $x,y\in(0,1)$

(d) All the above

213. Which of the following statements are true?

(a) Let $\{a_{mn}\}, m,n\in N;$ be a double sequence of real numbers $\ni$ $|a_{mn}|\leq\sqrt{m/n}$ for all $m,n\in N$. Then $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{a_{mn}}{m^2 n}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\dfrac{a_{mn}}{m^2 n}$

(b) Let $\{a_{mn}\}, m,n\in N,$ be an arbitrary double sequence of real numbers.

Then $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a^2{}_{mn}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a^2{}_{mn}$

(c) Let $\{a_{mn}\}, m,n\in N$ be an arbitrary double sequence of real numbers.

Then $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a^3{}_{mn}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a^3{}_{mn}$

(d) None of the above

214. Which of the following functions are uniformly continuous?

(a) $f(x)=\sin(x\sin x)$ on $(0,\infty)$

(b) $f(x)=x\sin 1/x$ on $(0,1)$

(c) $f(x)=\sin^2 x$ on $(0,\infty)$

(d) All the above

215. Pick out the functions which are continuous at least one point in the real line

(a) $f(x)=\begin{cases}\sin x & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$

(b) $f(x)=\begin{cases}x & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$

(c) $f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$

(d) None of the above

216. Pick out the functions which are uniformly continuous

(a) $f(x)=\dfrac{\sin x}{x}, x\in(0,1)$

(b) $f(x)=\sin^2 x, x\in R$

(c) $f(x)=\dfrac{1}{x}, x\in(0,1)$

(d) All the above

217. Pick out the uniformly continuous function from the following functions

(a) $f(x)=\sin^2 x, x\in R$

(b) $f(x)=\sqrt{x}, 1\le x\le 2$

(c) $f(x)=x^3, x\in R$

(d) All the above

218. Which of the following functions are differentiable at $x=0$?

(a) $f(x)=\begin{cases}x^2\sin \frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$

(b) $f(x)=\begin{cases}x^2 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$

(c) $f(x)=|x|x$

(d) None of the above

219. Which of the following series are convergent?

(a) $\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\left(\dfrac{1}{n}\right)$

(b) $\sum_{n=1}^{\infty}\sqrt{\dfrac{2n^2+3}{5n^3+7}}$

(c) $\sum_{n=1}^{\infty}\dfrac{(n+1)^n}{n^{n+3/2}}$

(d) None of the above

220. Let $f:R^2\to R$ be a mapping $\ni f(0,0)=0$. Determine which of the following are jointly continuous at $(0,0)$

(a) $f(x,y)=\begin{cases}x\sin\dfrac{1}{y}+y\sin \frac{1}{x} & \text{if } xy\neq 0\\ 0 & \text{otherwise}\end{cases}$

(b) $f(x,y)=\dfrac{x^2y^2}{x^2+y^2}(x,y)\neq(0,0)$

(c) $f(x,y)=\dfrac{xy}{x^2+y^2},(x,y)\neq(0,0)$

(d) None of the above

221. Which of the following functions are uniformly continuous?

(a) $f(x)=x^2, x\in R$

(b) $f(x)=x\sin \frac{1}{x}, x\in(0,1)$

(c) $f(x)=\sin^2 x, x\in R$

(d) $f(x)=x^3, x\in R$

222. Which of the following maps are differentiable everywhere?

(a) $f(x)=x^3\sin\dfrac{1}{\sqrt{|x|}}$ when $x\neq 0$ and $f(0)=0$

(b) $f(x)=|x|^3\ x, x\in R$

(c) $f:R\to R\ni\ |f(x)-f(y)|\le|x-y|^{\sqrt{2}}$ for all x and $y\in R$

(d) None of the above

223. Pick out the true statements

(a) If the series $\sum_n a_n$ convergent; $a_n\ge 0$ for all n, and if the sequence $\{b_n\}$ is bounded, then $\sum_n a_nb_n$ is absolutely convergent

(b) If the series $\sum_n a_n$ is convergent and if $\sum_n b_n$ is absolutely convergent then $\sum_n a_nb_n$ is absolutely convergent.

(c) If the series $\sum_n a_n$ and $\sum_n b_n$ are convergent, then $\sum_n a_nb_n$ is also convergent

(d) None of the above

224. Which of the following functions are continuous?

(a) $f(x) = \lim_{n\to\infty} \frac{1}{n} \log(e^n + x^n); x \geq 0$

(b) $f(x) = [x] + (x-[x])^{[x]}, x \geq \frac{1}{2}$ where $[x]$ denotes the largest integer less than or equal to x.

(c) $f(x) = \lim_{n\to\infty} \frac{x^2 e^{nx} + x}{e^{nx}+1}, x \in R$

(d) None of the above

225. Pick out the uniformly continuous functions over the interval $(0,1)$

(a) $f(x) = \exp\left(-\frac{1}{x^2}\right)$

(b) $f(x) = \sin \frac{1}{x}$

(c) $f(x) = x \sin \frac{1}{\sqrt{x}}$

(d) All the above

226. Pick out the true statements

(a) $|\cos^2 x - \cos^2 y| \leq |x-y|$ for all $x, y \in R$

(b) If $f: R \to R$ satisfies

$|f(x) - f(y)| \leq |x-y|^{\sqrt{2}}$ for all $x, y \in R$ then f must be a constant function.

(c) Let $f: R \to R$ be continuously differentiable and $|f'(x)| \leq \frac{4}{5}$ for all $x \in R$. The $\exists$ a unique $x \in R \ni f(x) = x$

(d) None of the above

227. Pick out the uniformly continuous functions

(a) $f(x) = \sin^2 x, x \in [0, \infty)$

(b) $f(x) = \cos x \cos \frac{\pi}{x}, x \in (0,1)$

(c) $f(x) = \sin x \cos \frac{\pi}{x}, \quad x \in (0,1)$

(d) All the above

228. Pick out the convergent series

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

(b) $\sum_{n=1}^{\infty} \frac{(n+1)n}{n^{n+3/2}}$

(c) $\sum_{n=1}^{\infty} \left((n^3+1)^{1/3} - n\right)$

(d) None of the above

229. Let $\{a_n\}$ be the sequence of consecutive positive solutions of the equation $\tan x = x$ and let $\{b_n\}$ be the sequence of consecutive positive solutions of the equation $\tan \sqrt{x} = x$. Then

(a) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ convergebut $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverges

(b) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges but $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converges

(c) Both $\sum_{n=1}^{\infty} \frac{1}{a_n}$ and $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converge

(d) Both $\sum_{n=1}^{\infty} \frac{1}{a_n}$ and $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverge

230. Let $x_0 = 0$. Define $x_{n+1} = \cos x_n$ for every $n \geq 0$. Then

(a) $\{x_n\}$ is increasing and convergent

(b) $\{x_n\}$ is decreasing and convergent

(c) $\{x_n\}$ is convergent and $x_{2n} < \lim_{m\to\infty} x_m < x_{2n+1}$ for every $n \in \mathbb{N}$

(*d*) $\{x_n\}$ is not convergent

231. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (e^{x+y}, e^{x-y})$. The area of the image of the region $\{(x, y) \in \mathbb{R}^2 : 0 < x, y < 1\}$ under the mapping f is

(*a*) 1 (*b*) $e - 1$

(*c*) e^2 (*d*) $e^2 - 1$

232. Let $f, g : [0, 1] \to \mathbb{R}$ be defined by

and $f(x) = \begin{cases} x & \textit{if } x = \frac{1}{n} \textit{ for } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

Then $g(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \cap [0,1] \\ 0 \text{ otherwise} \end{cases}$

(a) Both f and g are Riemann integrable

(b) f is Riemann integrable and g is Lebesgue integrable

(c) g is Riemann integrable and f is Lebesgue integrable

(d) Neither f nor g is Riemann integrable

233. Consider the following statements :

P : The family of subsets $\left\{ A_n = \left(-\frac{1}{n}, \frac{1}{n} \right), n = 1, 2, \ldots \right\}$ satisfies the finite intersection property.

Q : On an infinite set X, a metric $d : X \times X \to R$ is defined as $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$.

The metric space (X, d) is compact.

R : In a Frechet (T_1) topological space, every finite set is closed.

S : If $f : R \to X$ is continuous, where R is given the usual topology and (X, τ) is a Hausdorff (T_2) space, then f is a one-one function.

Which of the above statements are correct?

(a) P and R (b) P and S

(c) R and S (d) Q and S

234. Consider the statements

P : If X is a normed linear space and $M \subseteq X$ is a subspace, then the closure $\overline{M}$ is also a subspace of X.

Q : If X is a Banach space and $\sum x_n$ is an absolutely convergent scries in X, then $\sum x_n$ is convergent.

R : Let M_1 and M_2 be subspaces of an inner product space such that $M_1 \cap M_2 = \{0\}$. Then

$\forall m_1 \in M_1, m_2 \in M_2; \|m_1 + m_2\|^2 = \|m_1\|^2 + \|m_2\|^2$.

S : Let $f : X \to Y$ be a linear transformation from the Banach Space X into the Banach space Y.

If f is continuous, then the graph of f is always compact.

The correct statements amongst the above are :

(a) P and R only

(b) Q and R only

(c) P and Q only

(d) R and S only

235. The subspace $P = \{(x, y, z) \in R^3 : z = x^2 + y^2 + 1\}$ is

(a) compact and connected

(b) compact but not connected

(c) not compact but connected

(d) neither compact nor connected

236. Let P = (0, 1) Q = [0, 1]; U = (0, 1); S = [0, 1], T = R and A = {P, Q, U, S, T}. The equivalence relation 'homeomorphism' induces which one of the following as the partition of A?

(a) {P, Q, U, S}, {T}

(b) {P, T}, {Q}, {U}, {S}

(c) {P, T}, {Q, U, S}

(d) {P, T}, {Q, U}, {S}

237. Let $x = (x_1, x_2, \ldots) \in l^4$, $x \neq 0$. For which one of the following values of p, the series $\sum_{i=1}^{\infty} x_i y_i$ converges for every $y = (y_1, y_2, \ldots) \in l^p$?

(a) 1 (b) 2

(c) 3 (d) 4

238. For $0 \le x \le 1$, let

$$f_n(x) = \begin{cases} \dfrac{n}{1+n}, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$$

and $f(x) = \lim_{n\to\infty} f_n(x)$. Then, on the interval [0, 1]

(a) f is measurable and Riemann integrable

(b) f is measurable and Lebesgue integrable

(c) f is not measurable

(d) f is not Lebesgue integrable

239. Let $f_n(x) = \dfrac{x}{\{(n-1)x+1\}\{nx+1\}}$

and $s_n(x) = \sum_{j=1}^{n} f_j(x)$ for $x \in [0, 1]$.

The sequence $\{s_n\}$

(a) converges uniformly on $[0, 1]$

(b) converges pointwise on $[0, 1]$ but not uniformly

(c) converges pointwise for $x = 0$ but not for $x \in (0, 1)$

(d) does not converge for $x \in [0, 1]$

240. If $f : [1, 2] \to \mathrm{R}$ is a non-negative Riemann-integrable function such that

$\int_1^2 \dfrac{f(x)}{\sqrt{x}}\,dx = k\int_1^2 f(x)\,dx \neq 0,$ then k belongs to the interval

(a) $\left[0, \dfrac{1}{3}\right]$ (b) $\left[\dfrac{1}{3}, \dfrac{2}{3}\right]$

(c) $\left[\dfrac{2}{3}, 1\right]$ (d) $\left[1, \dfrac{4}{3}\right]$

241. The set $X = \mathbb{R}$ with metric

$d(x, y) = \dfrac{|x-y|}{1+|x-y|}$ is

(a) bounded but not compact

(b) bounded but not complete

(c) complete but not bounded

(d) compact but not complete

242. Let

$$f(x, y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^{5/2}}[1-\cos(x^2+y^2)], & (x,y)\neq(0,0) \\ k, & (x,y)=(0,0) \end{cases}$$

Then the value of k for which $f(x, y)$ is continuous at $(0, 0)$ is

(a) 0 (b) $\dfrac{1}{2}$

(c) 1 (d) $\dfrac{3}{2}$

243. Let A and B be disjoint subsets of $\mathbb{R}$ and let m^* denote the Lebesgue outer measure on $\mathbb{R}$.

Consider the statements:

P : $m^*(A \cup B) = m^*(A) + m^*(B)$

Q : Both A and B are Lebesgue measureable

R : One of A and B is Lebesgue measureable

Which one of the following is correct?

(a) If P is true, then Q is true

(b) If P is NOT true, then R is true

(c) If R is true, then P is NOT true

(d) If R is true, then P is true

244. Let $f : \mathbb{R} \to [0, \infty)$ be a Lebesgue measurable function and E be Lebesgue measurable subset of $\mathbb{R}$ such that then $\int_E f\,dm = 0$, where m is the Lebesgue measure on $\mathbb{R}$. Then

(a) $m(E) = 0$

(b) $\{x \in \mathbb{R} : f(x) = 0\} = E$

(c) $m(\{x \in E : f(x) \neq 0\}) = 0$

(d) $m(\{x \in E : f(x) = 0\}) = 0$

245. Let $\{f_n\}$ be a sequence of real valued differentiable functions on $[a, b]$ such that $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in [a, b]$ and for some Riemann-integrable function $f : [a,b] \to \mathrm{R}$. Consider the statements

P_1 : $\{f_n\}$ converges uniformly

P_2 : $\{f_n'\}$ converges uniformly

P_3 : $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

P_4 : f is differentiable

Then which one of the following need **NOT** be true

(a) P_1 implies P_3 (b) P_2 implies P_1

(c) P_2 implies P_4 (d) P_3 implies P_1

246. Let $f_n(x) = \dfrac{x^n}{1+x}$ and $g_n(x) = \dfrac{x^n}{1+nx}$ for $x \in [0,1]$ and $x \in \mathbb{N}$. Then on the interval [0,1],

(a) both $\{f_n\}$ and $\{g_n\}$ converge uniformly

(b) neither $\{f_n\}$ nor $\{g_n\}$ converges uniformly

(c) $\{f_n\}$ converges uniformly but $\{g_n\}$ does not converge uniformly

(d) $\{g_n\}$ converges uniformly but $\{f_n\}$ does not converge uniformly

247. Consider the power series $\sum_{n=1}^{\infty} \dfrac{x^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$. Then

(a) both converge on (–1,1]

(b) both converge on [–1,1)

(c) exactly one of them converges on (–1,1]

(d) none of them converges on [–1,1)

248. Let X = $\mathbb{N}$ be equipped with the topology generated by the basis consisting of sets $A_n = \{n, n+1, n+2, ...\}$, $n \in \mathbb{N}$. Then X is

(a) Compact and connected

(b) Hausdorff and connected

(c) Hausdorff and compact

(d) Neither compact nor connected

249. Consider the metrics

$$d_2(f, g) = \left(\int_a^b |f(t) - g(t)|^2\,dt\right)^{1/2} \text{ and } d_\infty(f, g) = \sup_{t\in[a,b]} |f(t) - g(t)|$$

on the space X = C[a, b] of all real valued continuous functions on [a, b]. Then which of the following is TRUE?

(a) Both (X, d_2) and (X, d_∞) are complete.

(b) (X, d_2) is complete but (X, d_∞) is NOT complete.

(c) (X, d_∞) is complete but (X, d_2) is NOT complete.

(d) Both (X, d_2) and (X, d_∞) are NOT complete.

250. A function $f : \mathbb{R} \to \mathbb{R}$ need NOT be Lebesgue measurable if

(a) f is monotone

(b) $\{x \in \mathbb{R} : f(x) \geq \alpha\}$ is measurable for each $\alpha \in \mathbb{Q}$

(c) $\{x \in \mathbb{R} : f(x) = \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$

(d) For each open set G in $\mathbb{R}$, f^{-1}(G) is measurable

ANSWERS

1. (b)	**2.** (a,c)	**3.** (b,d)	**4.** (d)	**5.** (a,c)	**6.** (b)
7. (b,d)	**8.** (a,b,c,d)	**9.** (a,b,c,d)	**10.** (b)	**11.** (a,b,c,d)	**12.** (a,c)
13. (a,b,d)	**14.** (c)	**15.** (b)	**16.** (a,b,c,d)	**17.** (a)	**18.** (a,b)
19. (a,b,c,d)	**20.** (a,b,c)	**21.** (c,d)	**22.** (a,b,c)	**23.** (a)	**24.** (a)
25. (c)	**26.** (a)	**27.** (c)	**28.** (a)	**29.** (a)	**30.** (a)
31. (a,b)	**32.** (c)	**33.** (a,b,c)	**34.** (d)	**35.** (b)	**36.** (c)
37. (a,b,c)	**38.** (b)	**39.** (c)	**40.** (b)	**41.** (c)	**42.** (a,b)
43. (a)	**44.** (a,b,c,d)	**45.** (b,c,d)	**46.** (a)	**47.** (a,b)	**48.** (a,b)
49. (b)	**50.** (a,b,c)	**51.** (a,b,c,d)	**52.** (a,b,d)	**53.** (c,d)	**54.** (b,d)
55. (a,b,c)	**56.** (c)	**57.** (a)	**58.** (d)	**59.** (c)	**60.** (a,c)
61. (b)	**62.** (c,d)	**63.** (a,b,c)	**64.** (a,b,c,d)	**65.** (b,c)	**66.** (c)
67. (a,b,c)	**68.** (a,b,c,d)	**69.** (a)	**70.** (a,b)	**71.** (a,b,c)	**72.** (a,c,d)
73. (a,b)	**74.** (a,b,c,d)	**75.** (a,b,c)	**76.** (b,c,d)	**77.** (a,b,c)	**78.** (a,b)
79. (b,c)	**80.** (a,c)	**81.** (a,c)	**82.** (a,b,c)	**83.** (a,b,c)	**84.** (a,b,c)
85. (a,b,c)	**86.** (a,c)	**87.** (a,b)	**88.** (a,b,c)	**89.** (a,b)	**90.** (a,b,c)
91. (a,b)	**92.** (a,b,c)	**93.** (a,b)	**94.** (a,b,d)	**95.** (b,c)	**96.** (a)
97. (a,b,c)	**98.** (c,d)	**99.** (a,b)	**100.** (a,b)	**101.** (a)	**102.** (a,b,c,d)
103. (a)	**104.** (a,b,c)	**105.** (a)	**106.** (c)	**107.** (b)	**18.** (b,d)
109. (a)	**110.** (a)	**111.** (b)	**112.** (a,c)	**113.** (a)	**114.** (a,d)
115. (b)	**116.** (a)	**117.** (a,b,c)	**118.** (d)	**119.** (a)	**120.** (a)
121. (c)	**122.** (b)	**123.** (a,b,c)	**124.** (b)	**125.** (a,b,d)	**126.** (a)

127. (a)	**128.** (a)	**129.** (a,b,c,d)	**130.** (a,b)	**131.** (a)	**132.** (a,c)
133. (a)	**134.** (a)	**135.** (c)	**136.** (b)	**137.** (a)	**138.** (b)
139. (a,b,c,d)	**140.** (b,c)	**141.** (a,c,d)	**142.** (a)	**143.** (a,b,c,d)	**144.** (a)
145. (a,b)	**146.** (a,b,d)	**147.** (a)	**148.** (b)	**149.** (a)	**150.** (a)
151. (a,b,c,d)	**152.** (a,b,c)	**153.** (a,b)	**154.** (a)	**155.** (a,b,c,d)	**156.** (a,c)
157. (a)	**158.** (a,c)	**159.** (a)	**160.** (a)	**161.** (b)	**162.** (a,b,c)
163. (d)	**164.** (c)	**165.** (a)	**166.** (a)	**167.** (a)	**168.** (a)
169. (a)	**170.** (d)	**171.** (a,c,d)	**172.** (a)	**173.** (b)	**174.** (c)
175. (c)	**176.** (a,b)	**177.** (a)	**178.** (b,c)	**179.** (a,b)	**180.** (b)
181. (b)	**182.** (c)	**183.** (c)	**184.** (a)	**185.** (a,b)	**186.** (b,c,d)
187. (a)	**188.** (a,b,d)	**189.** (b)	**190.** (b)	**191.** (a,b,c)	**192.** (a,b)
193. (a,b)	**194.** (b)	**195.** (c,d)	**196.** (a,b)	**197.** (c)	**198.** (b)
199. (a,b,c)	**200.** (b,c)	**201.** (a,b,c)	**202.** (a,b,c)	**203.** (b,c)	**204.** (a,b)
205. (a,c)	**206.** (b,c)	**207.** (a,b,c)	**208.** (a,b)	**209.** (a,b,c)	**210.** (a,b,c)
211. (a,b,c)	**212.** (a,c)	**213.** (a,b)	**214.** (b,c)	**215.** (a,b)	**216.** (a,b)
217. (a,b)	**218.** (a,b,c)	**219.** (a,c)	**220.** (a,b)	**221.** (b,c)	**222.** (a,b,c)
223. (a,b)	**224.** (a,b,c)	**225.** (a,c)	**226.** (a,b,c)	**227.** (a,c)	**228.** (b,c)
229. (b)	**230.** (c)	**231.** (d)	**232.** (b)	**233.** (a)	**234.** (c)
235. (c)	**236.** (d)	**237.** (a)	**238.** (b)	**239.** (b)	**240.** (c)
241. (a)	**242.** (a)	**243.** (c)	**244.** (c)	**245.** (c)	**246.** (d)
247. (b)	**248.** (b)	**249.** (a)	**250.** (a)		

EXPLANATIONS

1. It is possible to place a linear order on C so that we do not get an ordered field, and impossible to place a linear order of C so that we do get an ordered field.

3. (b) Convergent by the root test, $\frac{7n}{8n+1} \to \frac{7}{8} < 1.$ (d)Convergent by alternative series test.

4. $|x+y| = |x| + |y|$ doesn't have to happen. For example; let $y = -x \neq 0.$

5. (a) True: There is a metric space X with open sets. $U_i \ni \bigcap U_i$ is open. For example if X is any set and d is the metric $d(x,x) = 0$ and $d(x,y) = 1$ for $x \neq y$.

(c) True. If $S_n \to S$ and $t_n \to t$ then $S_n t_n \to St$.

7. Intervals of convergence for power series are always either a bounded connected interval or infinite on both sides.

9. (a) False. If $f(x) = -|x|$ then f has a maximum at $x = 0$ but $f^1(0) \neq 0.$ $\because f^1(0)$ doesn't exist

(b) False. It α is not differentiables then $\int_a^b f d\alpha$ cannot equal to $\int_a^b f \alpha^1 dx$

(c) False. Let $[a,b] = [0,1]$ and let $f_n(x) = x^n$. Then each f_n is continuous.

However, $\lim_{n\to\infty} f_n = f(x)$ where $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 1 \end{cases}$ which is clearly not continuous.

(d) False: For an explicit counter example let $f(x) = x+1,\ g(x) = x$ and $c = 1.$

Then $\lim_{x\to c} \frac{f^1(x)}{g^1(x)} = 1$ while $\lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$

12. $\frac{x^2+1}{x-1} = \frac{(x+1)(x-1)+2}{(x-1)} = 1 + x - \frac{2}{1-x} = 1 + x - 2(1 + x + x^2 + ...)$. Thus $a_0 = a_1 = -1$ and $a_k = -2$ for all $k \geq 2$

14. (i) As $|\sin(nt)| \leq 1, |a_n| \leq \frac{1}{n^2}$. Since $\sum_n \frac{1}{n^2}$ converges. So $\sum_n |a_n|$ is also converges by the comparison test. $\therefore \sum_n a_n$ converges absolutely.

(ii) For $n \in N,\ -n^2 \leq -n \cdot\ 0 < 2^{-n^2} \leq 2^{-n}$ and so $\sum_{n=1}^{\infty} |(-1)^n 2^{-n^2}|$ converges by comparison with the covergent geometric series $\sum_{n=1}^{\infty} 2^{-n}$. $\therefore \sum_{n=1}^{\infty} (-1)^n 2^{-n^2}$ converges absolutely.

18. (a) Use the ratio test $\dfrac{\dfrac{x^{n+1}}{1+x^{n+1}}}{\dfrac{x^n}{1+x^n}} = x\dfrac{1+x^n}{1+x^{n+1}} \to x.$ .The series converges for $|x|<1$ and in particular on $[0,1)$.

(b) For $0 \le x \le a < 1$. $\dfrac{x^n}{1+x^n} \le a^n$. $\because a<1$ the geometric series Σa^n converges. By the Weierstrass M--test, the series converges uniformly on $[0,a)$

20. (a) By the density of Q, there is a rational number $r \ni a-\sqrt{2} < r < b-\sqrt{2}.$

Thus $a < r+\sqrt{2} < b.$ $\because r+\sqrt{2}$ is irrational.

(b) x is an upper bound for A. To show that $x = \sup A$. Let $\varepsilon > 0$. Since $x-\varepsilon < x$, $\exists$ a rational number $r \ni x-\varepsilon < r < x$ by the density of 1 rational numbers. $\because r \in A$ and $x-\varepsilon < r.$ $\Rightarrow x = \sup A$

(c) Suppose to the contrary that $a>b$. $\because a-b>0 \; \exists \; n \in N \ni a-b > 1/n$ by the Archimedean property. This is equivalent to $a > b + 1/n$ which is a contradiction.

23. $\because f_n$ has a jump discontinuity at $x=0$ it does not affect the value of the integral. Thus, for any fixed n we have that $\int\limits_{-\infty}^{\infty} f_n(x)dx = \arctan(nx)\Big|_{-\infty}^{\infty} = \pi \Rightarrow \lim\limits_{n\to\infty}\int\limits_{-\infty}^{\infty} f_n(x)dx = \pi$

24. $x_n = \begin{cases} a_{n+1} = -\left(1+\dfrac{1}{n+1}\right) & n \text{ is even} \\ a_n \quad = -\left(1+\dfrac{1}{n}\right) & n \text{ is odd} \end{cases}$, $y_n = \begin{cases} a_n = \left(1+\dfrac{1}{n}\right) & n \text{ is even} \\ a_n \quad = \left(1+\dfrac{1}{n+1}\right) & n \text{ is odd} \end{cases}$ .Hence $-1-\dfrac{1}{n} \le x_n \le -1$

and $1 \le y_n \le 1 + y_n$ .So $x_n \to -1$ and $y_n \to 1$ (i.e.) $\liminf a_n = -1$ and $\limsup a_n = 1$.

26. Here $a_n = \dfrac{(-1)^n x^n}{n+1}$, $\lim\limits_{n\to\infty}\left|\dfrac{a_n+1}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{\left(\dfrac{n+2}{n+1}\right)}$ $=|x|$ by Ratio test the series converges for $|x|<1$ thus $R=1$ the interval of convergence is $(-1,1)$.

27. We have $f(x) = -\dfrac{d}{dx}\left(\dfrac{1}{1+x}\right)$. The power series representation of $\dfrac{1}{1+x}$ is $\sum\limits_{n=0}^{\infty}(-x)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^n$

The representation of $f(x)$ is $-\dfrac{d}{dx}\sum\limits_{n=0}^{\infty}(-1)^n x^n = -\sum\limits_{n=1}^{\infty}(-1)^n nx^{n-1} = \sum\limits_{n=1}^{\infty}(-1)^{n+1} nx^{n+1}$

28. The power series of $\frac{t}{1-t^8}$ is $\sum(t^8)^n = \Sigma t^{8n}$. Hence, the integral as a power series becomes

$$\int \frac{t}{1-t^8}dt = \int \Sigma t^{8n+1}dt = C + \sum \frac{t^{8n+2}}{8n+2}$$

$\therefore$ The radius of convergence is $R = 1$.

31. (a) Let $F = \{f : \{0,1\} \to N\}$. There is a bijection g between F and all pairs $(m,n) \in N^2$, given by $g(f) = (m,n)$ where $m = f(0)$ $n = f(1)$. Hence $F \sim N^2$. But N^2 is countable. $\therefore F$ is also countable.

(b) The set of finite subsets of N is countable, since it can be written as a countable union of countable sets.

32. (c) True. Consider an example $a_n = \left[1+(-1)^n\right]n, \quad n \geq 0$. The odd terms $a_{2n+1} = 0$ for every $n \geq 0$ and the even terms $a_{2n} = 2_n$ for every $n \geq 0$. The sequence is unbounded and it contains the constant subsequence $a_{2n+1} = 0$ which is Cauchy.

33. Put $a_n = \frac{1}{1+x^n}$, x is a fixed real number. If $x = 0$ then $a_n = 1$ for all $n \geq 1$, so $\lim a_n = 1$. If $|x|<1$ then $x^n \to 0$ as $n \to \infty$ so $\lim a_n = 1$. If $|x|>1$ then $x^n \to \pm\infty$ so $\lim a_n = 0$. If $x = 1$, $a_n = \frac{1}{2}$ for all n, so $\lim a_n = \frac{1}{2}$. If If $x = 1$, a_n is not well defined, so no limit.

36. (i) True. If $f\left((a+b)/2\right) \neq 0$ then this number is either > 0 (and f has different signs at the end points of $[a,(a+b)/2]$, or < 0 (and f has different signs at the points of $([(a+b)/2, b]$.

(ii)True. Let x be in each I_n. If $f(x) < 0$ then there is some $\delta > 0 \ni f(y) < 0$ for all y in $[a,b]$ with $|x-y| < \delta.$ Let n be $\ni (b-a)/2^n < \delta.$ Since length $I_n = (b-a)/2^n$ all the points y in I_n satisfy $|x-y| \leq \frac{1}{2^n} < \delta$ hence $f(y) < 0$ for all y in I_n. Contradicting that f has opposite signs on the endpoints of I_n. Similarly we cannot have $f(x) > 0$ thus $f(x) = 0$.

37. (a) True. For $\varepsilon > 0$ let $\delta = \varepsilon$ and for all $|x-y| < \delta$ we have $||x|-|y|| \leq |x-y| < \delta = \varepsilon.$

$\therefore f(x)$ is uniromly continuous (use the triangle inequality to for $||x|-|y|| \leq |x-y|$)

(b) True. $|x+y| \leq 6$ for all $x, y \in [0,3]$ by the triangle inequality. For $\varepsilon > 0$ let $\delta = \varepsilon/6$ so that for all $|x-y| < \delta$ and $x, y \in [0,3]$. we have $|f(x)-f(y)| = |x^2 - y^2| = |x-y|(x+y) \leq 6|x-y| < 6\delta = \varepsilon.$

$\therefore f(x)$ is uniformly continuous $[0,3]$

38. $a_n = \dfrac{n^2-1}{n^3+1}$ is positive and converges to 0 as $n \to \infty$. To show a_n is decreasing. Let $f(x) = \dfrac{x^2-1}{x^3+1}$

$f'(x) = \dfrac{-x^4+3x^2+2x}{\left(x^3+1\right)^2} < 0$ for $x \geq 3$. $\therefore a_n$ is decreasing for $n \geq 3$ and thus the series converges by the alternating series test. $\because \dfrac{n^2-1}{n^3+1} \geq \dfrac{\frac{1}{2}n^2}{2n^3} \geq \dfrac{1}{4n}$ and $\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges by the comparison test, $\sum_{n=1}^{\infty}\dfrac{n^2-1}{n^3+1}$ diverges. $\therefore$ The series converges conditionally.

39. $\dfrac{5n^3+2n+\cos^2(e^n)}{\sqrt{n^8+n^5+2}} \geq \dfrac{n^3}{4\sqrt{n^8}} \geq \dfrac{1}{2}.\dfrac{1}{2}$ and $\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges, the series diverges by the comparison test.

40. Let $a_n = \dfrac{(x+1)^n}{n(n+1)}$; $\left|\dfrac{a_n+1}{a_n}\right| = \left|\dfrac{\dfrac{(x+1)^{n+1}}{(n+1)(n+2)}}{\dfrac{(x+1)^n}{n(n+1)}}\right| = |x+1|\dfrac{n}{n+2} = \dfrac{|x+1|}{1+2/n}$; $\lim_{n\to\infty}\left|\dfrac{a_n+1}{a_n}\right| = |x+1<1$

$(-2,0)$; $x=-2: \sum_{n=1}^{\infty}\dfrac{(x+1)^n}{n(n+1)}\Bigg|_{x=-2} = \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n(n+1)}$ Which implies that convergent by alternation series

$x=0: \sum_{n=1}^{\infty}\dfrac{(x+1)^n}{n(n+1)}\Bigg|_{x=0} = \sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}$. Which implies that convergent by comparison with $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$

$\therefore$ The interval of convergence is $[-2,0]$.

42. (a) Use the comparison test $\because n \geq 1$ we have $1/n \leq 1$; $e^{1/n} \leq e$; $\dfrac{e^{1/n}}{n^2} \leq e/n^2$.

Being smaller than a convergent p-series, the given series must be convergent itself.

(b) by the alternating series test with $a_n = \dfrac{e^{1/n}}{n}$. a_n is certainly non-negative for each $n \geq 1$ and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{e^{1/n}}{n} = \lim_{n\to\infty}\dfrac{e^0}{n} = \lim_{n\to\infty}\dfrac{1}{n} = 0$. Moreover a_n is decreasing for each $n \geq 1$

$\left(\dfrac{e^{1/n}}{n}\right)^1 = \dfrac{e^{1/n}(-n)^{-2}.n - e^{1/n}}{n^2} = \dfrac{-e^{1/n}}{n^2}(n^{-1}+1)$ is negative for each $n \geq 1$. Thus the given series converges by the alternating series test.

43. The sequence $a_{n+1} = 4 - \frac{1}{a_n}$ for each $n \geq 1$ is monotonic and bounded hence also convergent. Let us denote its limit by L.

$$a_{n+1} = 4 - \frac{1}{a_n}; \quad L = 4 - \frac{1}{L}; \; L^2 - 4L + 1 = 0; \; L = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

Since, $a_n \geq 2$ for all n, $L \geq 2$ and this implies $L = 2 + \sqrt{3}$

44. (a) Since all the terms in the sum are positive and $m > n$ then $|S_m - S_n| = S_m - S_n$. The terms is

$S_m - S_n$ are those terms up to m excluding the first n. Thus $|S_m - S_n| = \sum_{k=n+1}^{m} \frac{1}{k^2}$

(b) $k(k-1) = k^2 - k < k^2$. $\because$ All trms are positive $\frac{1}{k^2} < \frac{1}{k(k-1)}$

(c) $\frac{1}{k(k-1)} = \frac{1}{k} - \frac{1}{k-1}$. Then $\sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \sum_{k=n+1}^{m} \frac{1}{k} - \frac{1}{k-1}$ all cancel except for the term $\frac{1}{n}$ and $\frac{-1}{m}$.

Thus $\sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \frac{1}{n} - \frac{1}{m}$

(d) $|S_m - S_n| = \sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} \frac{1}{k(k-1)} < \frac{1}{m} - \frac{1}{n} < \frac{1}{n} + \frac{1}{m}$

46. $f_n(x) - x^n(1-x)$. Note $x^n \geq x^{n+1}$ on $[0,1]$ if follows that $f_{n+1}(x) \leq f_n(x)$. For $0 \leq x < 1$. $x^n \to 0$, also $f_n(x) \to 0$ for $0 \leq x < 1$. For $x = 1$, $f_n(1) = 0$ for all n, $\lim f_n(x) = 0$ for all $x \in [0,1]$.

From Dini's theorem it follows that f_n converges uniformly to 0. $\because$ the limit function is continuous, (f_n) is monotone and $[0,1]$ is compact. $\therefore$ f_n converges uniformly to 0.

47. (a) True. If the sequence is bounded then all of its subsequence are bounded

(b) True. If all subsequences are unbounded, then the sequence must be bounded. In particular, a sequence can be regarded as s subsequence of itself.

49. (d) The function is differentiable at 0 with zero derivative. Its slope function is the fourth function which is continuous at 0.

51. (a) For all $x < y$, $V(y) - V(x) = V(f,[x,y]$ because $V(f,[x,y]$ is the supremum over all possible partitions P and $P = \{x, y\}$ is a partition. So $V(f,[x,y] \geq | f(y) - f(x) | \geq 0$ $V(y) - V(x) \geq 0$

$V(y) \geq V(x)$ for all $y > x$.

$\therefore V$ is increasing on $[a,b]$

(b) $|f(c+)-f(c^-)| \le |f(c+)-f(c)|+|f(c)-f(c^-)| \le |V(c+)-V(c^-)|$

$\because V$ is continuous, $\quad V(c+)-V(c^-)=0$

$|f(c+)-f(c-)| \le 0$

$f(c+)=f(c-)$

So, f is continuous at c.

(c) Given $\xi>0$ by the approximation property of the supremum $P=\{x_0,x_1,...,x_n\}$ a partition of

$[a,b] \ni V(f,[a,b])-\varepsilon/2 < \sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$

Insert $x_{n\in w}$ into the partition $\ni 0<x_{n\in w}-x_0<\delta \;\; |f(x_{n\in w})-f(x_0)|<\varepsilon$

Because f is continuous.

Then $V(f,[c,b])-\varepsilon/2 < |f(x_{n\in w})-f(x_0)|+|f(x_1)-f(x_{n\in w})| + \sum_{i=2}^{n}|f(x_i)-f(x_{i-1})|$

$V(f,[c,b])-\varepsilon/2 < \varepsilon/2+|f(x_1)-f(x_{n\in w})| + \sum_{i=2}^{n}|f(x_i)-f(x_{i-1})|$

$V(f,[c,b])-V(f(x_{n\in w},b))<\varepsilon$

$V(f,[c,x_{n\in w}])<\varepsilon$, for all $\xi>0$ we can find $a\ \delta \ni$ if $x_{n\in w}-x_c<\delta$ then $V(x_{n\in w})-V(c)<\varepsilon$

So, V is continuous.

54. (b) Let $K \ni |f'(x)|<K$ for all $x\in[a,b]$. By mean value theorem , there is a number $c \ni |f(x)-f(0)| = |f'(c)|(x-0) < K.1=K$. So, f is bounded by $f(0)-K$ and $f(0)=K$.

(d) Products of integrable functions are integrable.

57. (i) By the ratio test $\dfrac{\frac{x^{n+1}}{1+x^{n+1}}}{\frac{x^n}{1+x^n}} = x\dfrac{1+x^n}{1+x^{n+1}} \to x$. The series converges for $|x|<1$ and in particular on $[0,1)$

58. (i) By Extreme value function, f attains its minimum m for $x\in[a,b]$. By hypothesis, $f(x)\ge m>0$ for all $x\in[a,b]$. $\quad \therefore 0 \le 1/f(x) \le 1/m$ for all $x\in[a,b] \Rightarrow 1/f$ is bounded on $[a,b]$

(ii) Let $\varepsilon=1$ define sequences in $(0,1]$. $\{x_n\}=\left\{\dfrac{1}{n^2+1}\right\}$ and $\{y_n\}=\left\{1/n\right\}$, $\lim|x_n-y_n|=0$ but $|f(x_n)-f(y_n)|=|n^2-n+1|=n(n-1)+1\ge 1=\varepsilon$ for all $n\in\mathbb{N}$. So by the sequential crieterion for Nonuniform continuity, the claim follows.

59. (i) Fix $n \in N$, $f_n(x) \to f(x) = 0$ pointwise for all x by the Extreme value theorem $f_n^1(x) = \bar{e}^{nx^2}\left(1-2nx^2\right)$

The critical points of f occur at $x = \pm\frac{1}{\sqrt{2n}}$ The extrema of f on $[0,1]$ are $\pm\frac{1}{\sqrt{2en}}$ at $\left(x = \pm\frac{1}{\sqrt{2n}}\right)$.

Thus $M_n = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)| = \frac{1}{\sqrt{2en}}$, $\because \lim\limits_{n\to\infty} M_n = 0$ $\{f_n(x)\}$ is uniformly convergent on $[0,1]$

(ii) Let $f_n(x) = \sum\limits_{n=1}^{\infty}\frac{\cos(2^n(x)}{2^n}$ for all $x \in R$. for any fixed $n \in N$, f_n is continuous on R. $\left|\frac{\cos(2^n(x)}{2^n}\right| \le \frac{1}{2^n}$

for all $x \in R$ and $\sum\limits_{n=1}^{\infty}\frac{1}{2^n}$ converges $\therefore$ By the Weierstrass M test. $\sum\limits_{n=0}^{\infty} f_n(x)$ converges uniformly to $f(x)$ on R. Thus f is continuous on R.

60. $$A_n = \frac{a}{n} + \frac{1}{n+1} + \frac{b}{n+2} = \frac{(a+b+1)n^2 + (3a+b+2)n + 2a}{n(n+1)(n+2)}$$

If $a+b+1 \neq 0$ then $\left|(a+b+1)n^2 + (3a+b+2)n + 2a\right| \ge \frac{|a+b+1|}{2}n^2$

For large n so $|\eta_n| \ge \frac{|a+b+1|}{2} - \frac{n^2}{(n+1)(n+2)} \ge \frac{|a+b+1|}{8}\frac{1}{n}$

$\because (n+1)(n+2) < 4n^2$ for $n \ge 1$.

$\because \Sigma\frac{1}{n}$ diverges, ΣA_n diverges.

Assume that $a+b+1=0$. Then if $3a+b+2 \neq 0$ then $|(3a+b+2)n + 2a| \le |3a+b+2|n$ for large n so that $|A_n| \le \frac{2|3a+b+2|}{(n+1)(n+2)}$ and the series converges since $\Sigma\frac{1}{(n+1)(n+2)}$ converges.

If $3a+b+2=0$, then $2a=-1$, and the series converges.

63. (a) Claim $A_0 \cap \bar{B}_0$ is empty. If not take $x \in: A_0 \cap \bar{B}_0$. $x \in A_0$ and $x \in \overline{B_0}$ $x \in B_0$ or x is a limit point of B_0. $x \in B_0$. $x \in: A_0 \cap B_0$ (i.e) $q(x) \in A \cap B$ a contradiction. $\because A$ and B are separated.

Claim: x is a limit point of $B_0 \Rightarrow q(x)$ is a limit point of B. Take any neighborhood N_r of $q(x)$ and $q(t)$ lies in B for small enought t. More precisely.

$$x - \frac{r}{|b-a|} < t < x + \frac{r}{|b-a|}$$

$\because x$ is a limit point of B_0 and $\left(x-\frac{r}{|b-a|}, x+\frac{r}{|b-a|}\right)$ is a neighborhood N of x, thus N contains a point $y \neq x \ni y \in B_0$, (i.e) $q(y) \in B$. Also $q(y) \in N_r$.

$\therefore q(x)$ is a limit point of B. Hence, $q(x) \in A \cap \overline{B}$ a contradiction. Since A and B are separated.

Hence $A_0 \cap \overline{B}_0$ is empty. (i.e) A_0 and $\overline{B}_0$ are separated subsets of R^1.

(b) Suppose not. For every to $\in (0,1)$ neither $q(t_0) \in A$ nor $q(t_0) \in B$ ($\because A$ and B are separated). Also $q(t_0) \in A \cup B$ for all to $\in (0,1)$. Hence $(0,1) = A_0 \cup B_0$ a contradiction. Since $(0,1)$ is connected.

(c) Let S be a convex subset of R^k. If S is not conncted, then S is a union of two nonempty separated sets A and B. There exists to $\in (0,1) \ni q(t_0) \notin A \cup B$. But S is convex, $q(t_0)$ must lie in $A \cup B$, a contradiction. Hence S is connected.

68. (a) True. Let $x, y \in R$ and assume $x < y$. cosine is a differentiable function on R. So in particular it is continuous on $[x, y]$ and differentiable on $[x, y]$. Thus by the mean value theorem there is some

$c \in [x, y] \ni \cos^1(c) = \dfrac{\cos x - \cos y}{x - y}$. But $\cos^1(c) = \sin c$ and $|\sin c| \leq 1$; $\left|\dfrac{\cos x - \cos y}{x - y}\right| \leq 1$

$\therefore |\cos x - \cos y| \leq |x - y|$.

(b) True. Let $f(x) = e^x - ex$. Then $f(1) = 0$ and $f^1(x) = e^x - e$

For $x < 1; f^1(x) < 0$ so f is a decreasing function on $(-\infty, 1)$. For $x > 1; f^1(x) > 0$ so f is an increasing function on $(1, \infty)$. Thus $f(x) \geq 0$ for all $x \in R. \Rightarrow e^x \geq ex$ for all $x \in R$.

(c) True. Let $g(x) = x - \sin x$ for $x \geq 0$. Then $g(0) = 0$ and $g^1(x) = 1 - \cos x \geq 0$. Thus g is always nondecreasing on $[0, \infty)$. Thus $x - \sin x \geq 0$ and $x \geq \sin x$ for $x \geq 0$.

(d) True $f(x)$ is continuous on $[1, \infty)$ and f is differentiable on $(1, \infty)$. In fact $f^1(x) = \dfrac{1}{x} \leq 1$ for all $x \in (1, \infty)$. $\therefore f$ is uniformly continuous on $[1, \infty)$.

70. (a) For fixed $x \geq 0$ we have $f(x) = \lim_{n\to\infty} f_n(x) = 0$

(b) If $x \in [0, t]$ for some fixed $t > 0$, given $\varepsilon > 0$ let $N = \max\left\{\dfrac{t - \varepsilon t}{\varepsilon}, 1\right\}$. Then if $n > N$ and $x \in [0, t]$

we have $|f_n(x) - f(x)| = \left|\dfrac{x}{x+n}\right| \leq \dfrac{t}{t+n} \leq \dfrac{t}{t+n} = \xi$. $\therefore (f_n)$ converges uniformly to f on $[0, t]$.

73. (a) Use the integration by parts to get $\int_1^A \frac{\sin x}{x}dx = \left[-\frac{\cos x}{x}\right]_{1^`}^A - \int_1^A \frac{\cos x}{x^2}dx$ for any $A > 1$.

$\because \lim_{A\to\infty}\left[-\frac{\cos x}{x}\right]_{1^`}^A - \frac{\cos(1)}{1} = \cos(1)$ and the improper integral $\int_1^\infty \frac{\cos x}{x^2}dx$ is absolutely convergent.

$\therefore \int_1^\infty \frac{\sin x}{x}dx$ is convergent

(b) The inequality $\frac{1}{\sqrt{1+x^3}} \le \frac{1}{\sqrt{x^3}}$ for any $x \ge 1$. $\because$ $\int_1^\infty \frac{dx}{\sqrt{x^3}} = \int_1^\infty \frac{1}{x^{3/2}}dx$ is convergent (because $3/2 > 1$), by the comparison test $\int_1^\infty \frac{1}{\sqrt{1+x^3}}dx$ to be convergent

75. (a) Since $f_n(0) = 0$ for all $n \in N$. $f(0) = 0$ and for $x > 0$ we have $\lim_{n\to\infty}\frac{nx}{1+nx} = \lim_{n\to\infty}\frac{1}{1+\frac{x}{n}} = 1$.

$\Rightarrow f(x) = 1$ for $x > 0$

(b) Let $n \ge 1$, if $0 < x < 1/n$ then $\left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx} > \frac{1}{1+1} = \frac{1}{2}$

$\Rightarrow \{f_n\}$ does not converges uniformly to f on $[0,\infty)$

(c) If $x \ge a$ then $\left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx} \le \frac{1}{1+na}$ and thus $\lim_{n\to\infty}\frac{1}{1+na} = 0$

$\Rightarrow \{f_n\}$ conerges uniformly to f on $[0,\infty)$

76. $\lim_{n\to\infty} = 2x + \frac{x}{n} = 2x$ (b) $2x$ is continuous on $[0,1]$

(c) $\lim_{n\to\infty} g^1_n(x) = \lim_{n\to\infty} 2 + \frac{1}{n} = 2$; $\therefore \left[\lim_{n\to\infty} g_n(x)\right]^1 = g^1(x) = 2.$

(d) $\lim_{n\to\infty}\int_0^1 g_n(x)dx = \lim_{n\to\infty}\int_0^1 2x + \frac{x}{n}dx = \lim_{n\to\infty} 1 + \frac{1}{2n} = 1$

$\int_0^1 g(x)dx = 1$; $\int_0^1 \lim_{n\to\infty} g_n(x)dx = \int_0^1 g(x)dx = 1$

79. (b) Consider the limit along the curve $x = y = 0$. Then $\lim_{z\to 0}\frac{-z^2}{z^2} = -1$. If we consider the limit along the curve $x = z = 0$ then $\lim_{y\to 0}\frac{0}{y^2} = 0$. $\because$ We get two different limits along two different curves. The limit does not exist.

(c) Consider the limit along the curve given by y = x then $\lim_{x\to 0}\frac{x\sin x}{x^2+2x^2} = 1/3$ limit along the curve given by $y=-x$ then $\lim_{x\to 0}\frac{-x\sin x}{x^2+2x^2} = -1/3$ $\therefore$ We get two different limits along two different curves and so the limit does not exist.

80. (a) Fix $x\in[0,1)$ then the sequence (nx^n) consists of non-zero terms and $L = \lim\left|\frac{(n+1)n^{n+1}}{nx^n}\right|$ $=\lim\left(\frac{(n+1)}{n}\right)|x| = 1.|x| = x<1$. Hence $\lim nx^n = 0$ for $x\in[0,1)$ for $x=0$ $g_n(x) = g_n(0) = n.0^n = 0$ for all n. So, $\lim g_n(x) = 0$ for all $x\in[0,1)$

(c) $\int_0^1 g_n(x)dx = \int_0^1 nx^n dx = \left[n\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{n}{n+1}$. So, $\lim_{n\to\infty}\int_0^1 g_n(x)dx = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{\left(1/n\right)+1} = \frac{1}{0+1} = 1$

81. (a) Suppose $\sum_{x\in E} x > \infty$. For each n. Let $E_n = \left\{x\in E : x\geq 1/n\right\}$. Then each E_n is a finite subset of E. Otherwise if E_{n_o} is an infinite set for some n_o, then letting F_k be a subset of E_{n_o} with K_{n_o} elements for each $K\in N$; $S_{F_K}\geq K$. Then $\sum_{x\in E} x\geq S_{F_K}\geq K$ for each K contradiction. Now $E=\bigcup E_n$ so E is countable.

(c) Clearly $\{x_1, x_2, ...x_n\}\in$ F for all n. Thus $\sup S_n \leq \sup_{F\in\mathfrak{F}} S_F$. On the other hand given $F\in$ F

$\exists n \ni F\subset\{x_1, x_2, ..., x_n\}$ so $S_F\leq S_n$ and $\sup_{F\in\mathfrak{F}} S_F\leq \sup S_n$. Hence, $\sum_{x\in E} x = \sup_{F\in\mathfrak{F}} .S_F = \sup S_n = \sum_{n=1}^{\infty} x_n$

85. (a) Let E be a subset of a complete metric space. Suppose E^c is dense and F is a closed set contained in E. Then $F^c\supset E^c$. So F^c is dense and F is nowhere dense.

(b) Suppose E and E^c are both dense in a complete metric space X. Also suppose that E and E^c are both $F_\sigma^1 s$ say $E=\bigcup F_n$ and $E^c = \bigcup F_n^1$ where each F_n and F_n^1 is closed. By part (a) each F_n and F_n^1 is nowhere dense. Then $X = \bigcup F_n \cup \bigcup F_n^1$ so X is of the first category, contradicting the Baire category theorem. Hence at most one of the sets E and E^c is an F_σ

(c) The set of rational numbers in [0,1] is an F_σ. Since the set of rational numbers in [0,1] and the set of irrational numbers in [0,1] are both dense in the complete metric space [0,1] the set of irrational numbers in [0,1] is not an F_σ by part (b). Hence its complement, the set of rational numbers in [0,1] is not a G_δ.

87. (a) $\frac{x_m}{r_m}+...+\frac{x_n}{r_n} > \frac{x_m+...+x_n}{r_m} = \frac{r_m - r_n}{r_m} = 1-\frac{r_n}{r_m}$

If $m<n$. If $\sum \frac{x_n}{r_n}$ converges, for $\varepsilon > 0\ \exists N \ni \frac{x_m}{r_m}+...+\frac{x_n}{r_n} < \varepsilon$ for all *m, n* wherever $n > m \geq N$.

Fix $m = N$. Thus $\frac{x_m}{r_m}+...+\frac{x_n}{r_n} > 1-\frac{r_n}{r_m} = 1-1-\frac{r_n}{r_N}$ for all $n > N$.

But $r_n \to 0$ as $n \to \infty$, thus $\frac{x_m}{r_m}+...+\frac{x_n}{r_n} \to 1$ as $n \to \infty$.

If we take $\varepsilon = \frac{1}{2}$ we will get a contradiction.

(b)

$$r_{n+1} < r_n$$

$$\sqrt{r_{n+1}} < \sqrt{r_n}$$

$$\sqrt{r_n}+\sqrt{r_{n+1}} < 2\sqrt{r_n}$$

$$\frac{\sqrt{r_n}+\sqrt{r_{n+1}}}{\sqrt{r_n}} < 2$$

$$\left(\sqrt{r_n}-\sqrt{r_{n+1}}\right)\left(\frac{\sqrt{r_n}+\sqrt{r_{n+1}}}{\sqrt{r_n}}\right) < \left(\sqrt{r_n}-\sqrt{r_{n+1}}\right)$$

$$\frac{r_n + r_{n+1}}{\sqrt{r_n}} < 2\left(\sqrt{r_n}-\sqrt{r_{n+1}}\right)$$

$$\frac{x_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n}-\sqrt{r_{n+1}}\right)$$

Since for all $x_n > 0$ for all n

Hence $$\sum_{n=1}^{k}\frac{x_n}{\sqrt{r_n}} < \sum_{n=1}^{k} 2\left(\sqrt{r_n}-\sqrt{r_{n+1}}\right) = 2\left(\sqrt{r_1}-\sqrt{r_{k+1}}\right)$$

$r_n \to 0$ as $n \to \infty$. Thus $\sum \frac{x_n}{\sqrt{r_n}}$ is bounded. Hence $\sum \frac{x_n}{\sqrt{r_n}}$ converges.

88. (a) $b_n - b_{n+1} = b_n - \frac{1}{2}\left(b_n + \frac{\alpha}{b_n}\right) = \frac{1}{2}\left(b_n - \frac{\alpha}{b_n}\right) = \frac{1}{2}\left(\frac{b^{2_n} - \alpha}{b_n}\right) > 0$

Since, $b_n > \alpha$: Hence $\{b_n\}$ decreaes monotonically. Also $\{b_n\}$ is bounded by 0, thus $\{b_n\}$ converges.

Let, $\lim b_n = b$ .Hence, $\lim b_{n+1} = \lim \frac{1}{2}\left(b_n + \frac{\alpha}{b_n}\right) = \frac{1}{2}\left(b + \frac{\alpha}{b}\right)$; $b^2 = \alpha$, $b_n > 0$ for all n. Thus

$b = \sqrt{\alpha}. \lim b_n = \sqrt{\alpha}$.

(b) $b_{n+1} = \frac{1}{2}\left(b_n + \frac{\alpha}{b_n}\right)$; $b_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(b_n + \frac{\alpha}{b_n}\right) - \sqrt{\alpha} = \frac{\left(b_n - \sqrt{\alpha}\right)^2}{2b_n}$; $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2b_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$

Hence $\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_n}{\beta}\right)^{2^n}$ where $\beta = 2\sqrt{\alpha}$ by induction.

(c) $\frac{\varepsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{2\sqrt{3}\left(2+\sqrt{3}\right)} = \frac{1}{6+4\sqrt{3}} < \frac{1}{10}$

Thus $\varepsilon_5 < \beta\left(\frac{\varepsilon_1}{\beta}\right)2^4 < 2\sqrt{3}.10^{-16} < 4.10^{-16}$; $\varepsilon_6 < \beta\left(\frac{\varepsilon_1}{\beta}\right)2^5 < 2\sqrt{3}.10^{-32} < 4.10^{-32}$

89. (a) Let $t_n = S_n - S$, $\tau_n = \sigma_n - S$ Then $\tau_n = \frac{t_0 + t_1 + \ldots + t_n}{n+1}$

Choose $M > 0 \ni |t_n| \leq M$ for all n. Gives $\varepsilon > 0$, choose N so that $n > N$ implies $|t_n| < \varepsilon$. Taking

$n > N$ in $\tau_n = \frac{(t_0 + t_1 + \ldots + t_n)}{n+1}$ and then $|\tau_n| \leq \frac{|t_0| + \ldots + |t_N|}{n+1} + \frac{|t_{N+1}| + \ldots + |t_n|}{n+1} < \frac{(N+1)M}{n+1} + \xi$

Hence, $\limsup_{n\to\infty} |\tau_n| < \varepsilon$. since ξ is arbitrary, it follows that $\lim_{n\to\infty} |\tau_n| = 0$ that is $\lim \sigma_n = S$

(b) $\sum_{k=1}^{n} kx_k = \sum_{k=1}^{n} k\left(S_k - S_{k-1}\right) = \sum_{k=1}^{n} ks_k - \sum_{k=1}^{n} S_{k-1} = nS_n \sum_{k=1}^{n-1} S_k - S_0 = (n+1)S_n - \sum_{k=0}^{n} S_k$ = (n + 1)

(i.e) $\left(S_n - \sigma_n\right) = \frac{1}{n+1}\sum_{k=1}^{n} kx_k$. Note that $\{nx_n\}$ is a complex sequence. $\lim_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=1}^{n} kx_k\right) = \lim_{n\to\infty} nx_n = 0$

Also $\lim \sigma_n = \sigma$. Hence $\lim S = \sigma$

93. (a) Assume g is continuous iff for sequence $\{x_n\} \to 0$ with $x_n \neq 0$, $x_n^a \sin x_n^{-c} \to 0$ as $n \to 0$.

$x_n = \left(\frac{1}{\left(2n\pi + \pi/2\right)}\right)^{1/c} > 0$ and thus $x_n^a \to 0$ as $n \to \infty$. Hence $a > 0$. (If not, then $a = 0$ or $a < 0$

when $a = 0$, $x_n^a = 1$ when $a < 0$, $x_n^a = 1/x_n^{-a} \to \infty$ as $n \to \infty$. It contradicts).Conversely assume g

is continuous on $[-1,1] - \{0\}$. Clearly note that $-\left|x^a\right| \leq x^a \sin\left(x^{-c}\right) \leq \left|x^a\right|$ and $\left|x^a\right| \to 0$ as $x \to 0$ since

$a > 0.$ Thus g is continuous at x = 0, Hence g is continuous

(b) $g^1(0)$ exists iff $x^{a-1}\sin\left(x^{-c}\right) \to 0$ as $x \to 0$ $g^1(0)$ exists iff $a-1>0.$ Also $g^1(0)=0.$

96. If $x \neq 1$; $S_n = \sum_{k=0}^{n} x^k = \dfrac{1-x^{n+1}}{1-x}$ .If we let $n \to \infty$. For $x=1$ we get $1+1+1+...$ Which evidently diverges

99. (a) If not, then $\exists$ two distinct fixed points, say x and y of f. Thus $f(x)=x$ and $f(y)=y.$

Since f is differentiable, by applying mean value theorem $f(x)-f(y)=f^1(t)(x-y)$ where t is between x and y. Since $x \neq y$, $f^1(t)=1.$ It contradicts

(b) To show that $0<f^1(t)<1$ for all real t ; $f^1(t)=1+(-1)(1+e^t)^{-2}e^t = 1-\dfrac{e^t}{(1+e^t)^2}$

Since $e^t > 0$; $(1+e^t)^2 = (1+e^t)(1+e^t) > (1+e^t) = (1+e^t) > e^t > 0$

For all real t, thus $(1+e^t)^2 e^t > 0$; $(1+e^t)^{-2}e^t < 1$. For all real t. Hence $0<f^1(t)<1$ for all real t. Since $f(t)-t=(1-e^t)^{-1}$ >0 for all real $t, f(t)$ has no fixed point.

101. (a) Bounded since A is a subset of the closed unit interval $I=[0,1]$

108. (b) True. That is because of all the points in N are isolated points of N.

(d) True. Let $h(x)=f(x)-g(x)$, then h is continuous on $[0,10]$ & differentiable on $(0,10)$ & $h'(x)=0$ for all $x \in (0,10)$. $h(x)=$ constant. Since $h(\pi)=0=$ constant. We have $h(x)=0$ & $\therefore f(x)=g(x)$

109. (a) $f(x)=x\sin\dfrac{1}{x}$ is well defined for all $x \neq 0$. For any $\varepsilon > 0$. There exists $\delta = \varepsilon$ such that whenever

$0<|x|<\delta$ we have $\left|x\sin\left(\dfrac{1}{x}\right)-0\right| \le |x| < \varepsilon$. $\therefore \underset{x\to 0}{Lim}\, x\sin\left(\dfrac{1}{x}\right)=0$

110. (a) $\underset{x\to 1}{\text{Lim}} \dfrac{(x-1)(x-3)}{(x-1)(x+1)\left(\sqrt{x^2+3}+2\sqrt{x}\right)} = -\dfrac{1}{4}$

112. (a), (c) $\underset{x\to 1^-}{\text{Lim}} f(x) = \underset{x\to 1^-}{\text{Lim}}(2x+5)=7$; $\underset{x\to 1^+}{\text{Lim}} f(x) = \underset{x\to 1^+}{\text{Lim}}\left(9x^2-2\right)=7$ .We have $\underset{x\to 1}{Lim} f(x) = f(1)=7$.

$\therefore f(x)$ is continuous at 1. $f'_-(1) = \underset{x\to 1^-}{Lim}\dfrac{f(x)-f(1)}{x-1} = \underset{x\to 1^-}{Lim}\dfrac{2x-2}{x-1}=2$

$f'_+(1) = \underset{x\to 1^+}{Lim}\dfrac{f(x)-f(1)}{x-1} = \underset{x\to 1^+}{Lim}\dfrac{9x^2-9}{x-1} = 18$. We have $f'_-(1) \neq f'_+(1) \Rightarrow f$ is not differentiable at 1

114. (a) True. Since any deleted neighbourhood of x is a union of two open intervals

(d) True. For $x \in \operatorname{int} S$, which means $N(x,\varepsilon_0) \subseteq S$ for some $\varepsilon_0 > 0$, we have $N(x,\varepsilon_0) \cap (R \setminus S) = \phi$ and $x \notin bdS$. $\therefore$ $\operatorname{int} S \cap bdS = \phi$.

115. (b) By choosing $n = 2k$, we have $S_{2k} = 1 \to 1$ as $k \to \infty$ and by choosing $S_{2k-1} = -1 \to -1$ as $k \to \infty$. Therefore $S_n = \cos(n\pi)$ is divergent.

116. (a) $$\lim_{n\to\infty}\left(\sqrt{n^2+2n}-\sqrt{n^2-3n}\right) = \lim_{n\to\infty}\frac{\left(\sqrt{n^2+2n}-\sqrt{n^2-3n}\right)\left(\sqrt{n^2+2n}+\sqrt{n^2-3n}\right)}{\left(\sqrt{n^2+2n}+\sqrt{n^2-3n}\right)}$$

$$= \lim_{n\to\infty}\frac{5n}{\left(\sqrt{n^2+2n}+\sqrt{n^2-3n}\right)} = \frac{5}{2}$$

118. (d) 1) incorrect: Example let $S_n = \frac{(-1)^n}{n}$ for $n \in N$, which is a cauchy sequence (since it is not converges to 0) but it is not monotone

2) Incorrect: Example Let $S_n = -1$ for all $n \in N$, then every $\varepsilon > 0$ there exists $N = 0 \in R$ such that for every $n > N$ we have $S_n < \varepsilon$ but $\lim_{n\to\infty} S_n = -1 \neq 0$

119. (a) $$\frac{S_{n+1}}{S_n} = \frac{\frac{(n+1)^2}{1.01^{n+1}}}{\frac{n^2}{1.01^n}} = \frac{(n+1)^2}{1.01n^2} \to \frac{1}{1.01} < 1$$ as $n \to \infty$. $\therefore$ We have $\lim_{n\to\infty}\frac{n^2}{1.01} = 0$

120. (a) Let $S = \left\{\frac{n}{n+1} : n \in N\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$. The set S is bounded below, $\frac{1}{2}$ being a lower bound for S. $\frac{1}{2} \in S$. glb of S (*ie*) $\inf S = \frac{1}{2}$. Since, $\frac{n}{n+1} < 1 \quad \forall \quad n \in N$, therefore 1 is an upperbound for S. Any real number less than 1 cannot be as upper bound for S. $\therefore$ 1 is as upper bound for S. sup S=1

123. (a) Let $x \in Z$. For any $\varepsilon > 0$, $x-\varepsilon$ & $x+\varepsilon$ are two distinct real numbers and we know that between any two distinct real numbers there lie infinite irrational numbers which are definitely not members of Z.. $(x-\varepsilon, x+\varepsilon) \not\subset Z, \quad \forall\, \varepsilon > 0$. $\therefore$ Z is not a neighbourhood of x

(b) Let $P \in Q$; $(P-\varepsilon, P+\varepsilon) \not\subset Q, \quad \forall\, \varepsilon > 0$. $\therefore$ Q is not a neighbourhood of x.

(c) Let R denote the set of all real numbers and Q denote the set of all rational numbers. Then R-Q is the set of all irrational numbers. Let $x \in (R-Q)$. For any $\varepsilon > 0$, $x-\varepsilon$ and $x+\varepsilon$ are two distinct real number and between any two distinct real numbers there lie infinite rational number which are definitely not members of *R-Q*. Thus there exists no $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset R-Q$

$\therefore$ $R-Q$ is not a neighbourhood of *x*.

125. (a) Since (2,4) is an open interval 3 $\in$ (2,4), (2,4) is neighbourhood of 3
(b) Since there exists an open interval (2,4) such that 3 $\in$ (2,4) $\subset$ [2,4]; $\therefore$ [2,4] is a neighbourhood of 3

(d) [2,4]-$\left\{3\frac{1}{4}\right\}$ is a neighbourhood of 3. Since there exists an open interval $\left(3-\frac{1}{5}, 3+\frac{1}{5}\right)$ such

that $3 \in \left(3-\frac{1}{5}, 3+\frac{1}{5}\right) \subset \left([2,4]-\left\{3\frac{1}{4}\right\}\right)$

126. (a) We shall show that every real number *P* is not a limit point of the set *S* which is a finite set. since the set S is finite, therefore if we take any $\varepsilon > 0$, the open interval $(P-\varepsilon, P+\varepsilon)$ contains at most a finite number of points of the set *S*. Thus $(P-\varepsilon, P+\varepsilon)$ is a neighbourhod of *P* which does not contain infinitely many points of *S* and so *P* is not a limit point of *S*. Thus every real number *P* is not a limit point of *S* and so the derived set of *S* is empty (ie) *D(S)*= ϕ.

130. (a),(b) *A* is denumerable because the mapping $f: N \rightarrow \text{A}$ defined by $f(n) = \frac{n}{n+1} \ \forall \quad n \in N$ is bijective (*ie*) one-one & onto

135. (c) $S_n = \log\frac{1}{n}$. Take any given $K < 0$. Then $S_n < K$ if $\log\frac{1}{n} < K$ (ie) if $-\log n < K$

(ie) if $\log n > -K$ (ie) if $n > e^{-K}$. If we take $m \in \text{N}$ such that $m > e^{-K}$ then $S_n < K$ for all $n \geq m$. Hence $S_n \rightarrow -\infty$ as $n \rightarrow \infty$.

143. (a) Let [*a,b*] = *G* $\cup$ *H*, Such that $G \cup H = \phi$. Let $b \in H$. Then claim $G = \phi$. If not, Let *C* = Sup *G*. Since G is closed $C \in G$. Since *G* is open $B_{\xi}(C) \subseteq G \quad ie \quad [C, C+\xi) \subset G$. That contra dicts C = sup *G*. Thus $G = \phi$

(b) Let $R = \bigcup_{n \in N}[-n, n], \text{O} \in \bigcap[-n, n]$. Therefore, *R* is the union of connected subsets.
$\therefore$ *R* is connected.

(c) Say $f(M) = G \cup H,\ G, H \neq \phi,\ G, H$ open. Then $M = f^{-1}(G) \cup f^{-1}(H)$ where $f^{-1}(G)$ and $f^{-1}(H)$ are both open and nonempty, contradicts. Connectednees of *M*

(d) Assme *I* is a interval *S* = [*a,b*]; *S* = (*a,b*]; $a \geq -\infty$

$S=[a,b)$; $b \le \infty$; S=(a,b), $a \ge -\infty$, $b \le \infty$, $[a,b) = \bigcup_{n \ge n0} \left[a, b-\frac{1}{n}\right], a \in \bigcap [a, b-1/n]$.

Therefore $I \subseteq R$ is connected.

148. (b) Consider $f^{-1}(\{0\})$. Since $\{0\}$ is closed and f continous, $f^{-1}(\{0\})$ is closed. Therefore $S=[0,1] \bigcap f^{-1}(\{0\})$ is a closed and bounded subset of R. Hence S is compact.

179. (a), (b)Let $f(x) = \dfrac{\sin(1/x)}{\sqrt{x}}$. There is no neighbourhood of the point 0, in which $f(x)$ constantly keeps the same sign. Now $\forall \quad x \in (0,1]$ we have $\left|\dfrac{\sin 1/x}{\sqrt{x}}\right| = \dfrac{|\sin 1/x|}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$ also $\int_0^1 \dfrac{1}{\sqrt{x}} dx$ is convergent . $\int_0^1 \left|\dfrac{\sin 1/x}{\sqrt{x}}\right| dx$ is convergent $\Rightarrow \int_0^1 \dfrac{\sin 1/x}{\sqrt{x}} dx$ is absolutely convergent.

192.(a) Suppose f is Lipschitz. $\exists \ M \ni \ |f(x)-f(y)| \le M|x-y|$ for all x, y. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. For any finite collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with $\sum_1^n |x_i' - x_i| < \delta$ we have

$\sum_1^n |f(x_i') - f(x_i) \le M \left|\sum_1^n |x_i' - x_i| < \varepsilon\right.$. Thus f is absolutely continuous.

(b) Let f be absolutely continuous. Suppose f is Lipschitz. Now $f'(x) = \lim_{y \to x} \dfrac{f(y)-f(x)}{y-x}$

So, $|f'(x)| = \lim_{y \to x} \left|\dfrac{f(y)-f(x)}{y-x}\right| \le M$, for all x.Conversely if f is not Lipschitz, then for any M, $\exists \quad x$ and $y \ni \ |f(x)-f(y)| > M\,|x-y|$. Then $|f'(c)| > M$ for some $c \in (x,y)$ by the Mean value theorem. Thus for any $M, \exists \quad c \quad \ni |f'(c)| > M$ so $|f'|$ is unbounded.

193. (a),(b)Suppose $\exists$ points a, b of A for which $f(a) < 0$ and $f(b) > 0$. Then by the Intermediate value theorem $\exists$ a point $c \in (a,b) \quad \ni \quad f(c) = 0$. Contrary to the hypothesis on f. Thus either $f(x) > 0$ for all $x \in A$ or $f(x) < 0$ for all $x \in A$.

194. (b)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1} z^{n+1} n!}{2^n z^n (n+1)!}\right| = \left|\frac{2z}{n+1}\right| = \left|\frac{1}{n+1}\right| 2\,|z|.$$

By taking the limit we get 0, which is always less than 1. Thus, any z. The radius of convergence is infinite.