UNIT 1

METHODS OF MATHEMATICAL PHYSICS

Linear Algebra

MULTIPLICATION OF MATRICES

If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B. If $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{jk}\right]_{n\times p}$ then their product $AB = C = \left[C_{ik}\right]_{m\times p}$, will be matrix of order $m\times p$, where $C_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

PROPERTIES OF MATRIX MULTIPLICATION

If A, B and C are three matrices such that their product is defined, then

(a) $AB \neq BA$ (Generally not commutative)

(b) $(AB)C = A(BC)$ (Associative Law)

(c) $A(B + C) = AB + AC$ (Distributive Law)

(d) If $AB = AC$ it does not mean that $B = C$ (Cancellation /Law is not applicable)

(e) If $AB = 0$. It does not mean that $A = 0$ or $B = 0$

(f) $(AB)^T = B^T A^T$

TRANSPOSE OF A MATRIX

The matrix obtained form a given matrix A by changing its rows into columns or columns into rows is called transpose of matrix A and is denoted by A^T. From the definition it is obvious that if order of A is $m\times n$, then order of A^T is $n\times m$.

PROPERTIES OF TRANSPOSE

(a) $\left(A^T\right)^T = A$

(b) $(A \pm B)^T = A^T \pm B^T$

(c) $(AB)^T = B^T A^T$

(d) $(kA)^T = k(A)^T$

(e) $(A_1 A_2 A_3 \ldots\ldots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \ldots\ldots A_3^T A_2^T A_1^T$

DETERMINANT OF A MATRIX

The determinant of square matrix A is given by $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

PROPERTIES OF THE DETERMINANT OF A MATRIX

(a) $|A|$ exists $\Leftrightarrow$ A is a square matrix

(b) $|AB| = |A||B|$

(c) $|A^T| = |A|$

(d) $|kA| = k^n |A|$, if A is a square matrix of order n.

(e) If A and B are square matrices of same order then $|AB| = |BA|$

(f) If A is a skew symmetric matrix of odd order then $|A| = 0$

(g) If $A = diag(a_1, a_2 \ldots\ldots a_n)$ then $|A| = a_1 a_2 \ldots\ldots a_n$.

(h) $|A|^n = |A^n|, n \in N$.

(i) If $|A| = 0$, then matrix is called singular.

RANK OF A MATRIX

A number is defined as the rank of a $m \times n$ matrix A, if A has at least one minor of order which is not equal to zero and there is no minor of order $(r+1)$ which is not equal to zero. The rank of the matrix A is denoted by $\rho(A)$. Form the definition of the rank of a matrix we concluded that

(a) If a matrix A does not possesses any minor of order $(r+1)$, then $\rho(A) \leq r$

(b) If at least one minor of order of the matrix is not equal to zero, then $\rho(A) \geq r$.

ADJOINT OF A MATRIX

If every element of a square matrix A be replaced by its cofactor in $|A|$, then the transpose of the matrix so obtained is called the Adjoint of matrix A and it is denoted by adj A. Thus if $A = [a_{ij}]$ be a square matrix and be the cofactor of a_{ij} in $|A|$, then adj $A = [F_{ij}]^T$.

PROPERTIES OF ADJOINT MATRIX

If A, B are square matrices of order n and I is corresponding unit matrix, then

(a) $A(adj\ A) = |A| I_n = (adj\ A) A$

(b) $|adj\ A| = |A|^{n-1}$

(c) $adj(adj\ A) = |A|^{n-2} A$

(d) $|adj(adj\ A)| = |A|^{(n-1)^2}$

(e) $adj(A^T) = (adj\ A)^T$

(f) $adj(AB) = (adj\ B)(adj\ A)$

(g) $adj\left(A^{m}\right)=(adj\ A)^{m}$, $m \in N$

(h) $adj(kA)=k^{n-1}(adj\ A), k \in R$

INVERSE OF A MATRIX

If A and B are two matrices such that $AB = I = BA$, then B is called the inverse of A and it is denoted by A^{-1}. Thus $A^{-1} = B \Leftrightarrow AB = I = BA$

To find inverse matrix of a given matrix A we use following formula $A^{-1} = \dfrac{adj\ A}{|A|}$

Thus A^{-1} exists if $|A| \neq 0$ and matrix A is called invertible.

PROPERTIES OF INVERSE MATRIX

Let A and B are two invertible matrices of the same order, then

(a) $\left(A^{T}\right)^{-1}=\left(A^{-1}\right)^{T}$

(b) $(AB)^{-1}=B^{-1}A^{-1}$

(c) $\left(A^{k}\right)^{-1}=\left(A^{-1}\right)^{k}$, $k \in N$

(d) $adj\left(A^{-1}\right)=(adj\ A)^{-1}$

(e) $|A^{-1}|=\dfrac{1}{|A|}=|A|^{-1}$

(f) If $A = diag(a_1, a_2, \ldots\ldots, a_n)$, then $A^{-1} = diag\left(a_1^{-1}, a_2^{-1}, \ldots . a_n^{-1}\right)$

(g) $AB = AC \Rightarrow B = C$ if $|A| \neq 0$,

SYSTEM OF LINEAR EQUATION

Consider the following system of equation

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The system of equation can be written as $Ax = B$

Where $$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, x=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

The above system of equation is said to be consistent if the has at least one solution.

RANK METHOD

The augmented matrix is defined as $[A:B]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix}$

The system of equation as $Ax = B$ is consistent and has solution (unique or infinite) if and only if the matrix A and the augmented matrix $[A:B]$ are of the same rank i.e. $\rho[A:B] = \rho[A]$. If system is of n equation, then we have following conclusion:

1. If $\rho[A:B] = \rho[A] = n$. (The number of unknown) then the system of equation has a unique solution.
2. If $\rho[A:B] = \rho[A] < n$ then $n - r$ unknown may have arbitrary values. Remaining unknowns is determined uniquely. There will be infinite solution for the system.

EIGEN VALUES AND EIGEN VECTOR

Let A be square matrix of order n and let I be a unit matrix of same order, then $|A - \lambda I|$ is called the characteristic matrix, where λ is a scalar.

The determinant of the square matrix i.e. $|A - \lambda I|$ equated to zero gives characteristic equation of the matrix. The root of characteristic equation (i.e. values of λ) are called eigen values.

PROPERTIES OF EIGEN VALUES

(a) Matrix A and A^T have same eigen values.

(b) Determinant of A is equal to the product of the eigen values of A.

(c) If λ is characteristic root of the matrix A then $k + \lambda$ is a characteristic root of the matrix $A + kI$.

(d) Sum of the eigen values of A is equal to the sum of the elements of the principal diagonal of the matrix.

(e) The matrix A and $B^{-1}AB$ have same eigen values.

(f) If A is an invertible matrix, then $A^{-1}B$ and BA^{-1} have the same eigen values.

(g) If $|A| \neq 0$ and λ is the eigen values of A, then $\frac{|A|}{\lambda}$ is the eigen value of A^{-1}.

(h) If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A, then eigen values of

1. A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots\ldots \frac{1}{\lambda_n}$

2. kA are $k\lambda_1, k\lambda_2 \ldots\ldots k\lambda_n$ when k is any scalar.

3. A^m are $\lambda_1^m, \lambda_2^m, \ldots\ldots \lambda_2^m$

CAYLEY-HAMILTON THEOREM

Every square matrix satisfies its characteristic equation, *i,e.*, if A is a square matrix of order n, then its characteristic equation in $\lambda\ |A - \lambda I| = 0$ is satisfied by $\lambda = A$.

Vector Calculus

INTRODUCTION

Vector calculus is a branch of mathematics concerned with differentiation and integration of vector fields, primarily in 3 dimensional Euclidean space.

Vector calculus plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

VECTOR DIFFERENTIATION

Scalar Function : A scalar function $f(x, y, z)$ is a function defined at each point in a certain domain D in space.

Vector Function: A function $\bar{F}(x, y, z) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ where F_1, F_2 and F_3 are functions of x, y and z, defined at each point $P \in D$ is called a vector function.

Position Vector: If a point O is fixed as the origin in space or in plane and P is any point in space then $\overline{OP}$ is called the position vector of a point P with respect to origin O and it is denoted by $\bar{r}$.

$\therefore \overline{OP} = \bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$ and it's magnitude is $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$.

NOTE

In parametric form the position vector $\bar{r}$ is $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$

PARAMETRIC REPRESENTION OF CURVES

The curve 'C' in two dimensional plane can be parameterized by $x = x(t)$, $y = y(t)$ where $a \le t \le b$. Then the position vector of a point P on the curve 'C' is written as $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j}$. Therefore, the position vector of a point on a curve defines a vector of a point on a curve defines a vector function, i.e., $\bar{r} = x\,\bar{i} + y\,\bar{j}$.

Similarly a three-dimensional curve can be parameterized as $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ where $a \le t \le b$.

DIFFERENTIABILITY

A Vector function $\bar{F}(t)$ is said to be differentiable function at a point 't' if $\underset{\delta t \to 0}{Lt} \dfrac{\bar{F}(t + \delta t) - \bar{F}(t)}{\delta t}$ exists and finite.

If $\bar{F}(t) = \bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ is a parametric representation of a curve 'C' then $\dfrac{d\bar{r}}{dt}$ represents the tangent vector to the curve 'C'.

POINT FUNCTION

If the value of the function depends on the position of the point in space but not on any particular coordinate system being used then the function is called point function.

TYPES OF POINT FUNCTIONS

(i) Scalar Point Function: If for each point $P(x, y, z)$ of a region 'R' in space, a unique scalar or a number $\phi(x, y, z)$ or $\phi(P)$ is associated by some function ϕ then the function $\phi(x, y, z)$ is called "scalar point function".

The set of all points of the region R together with the function values $\phi(P)$ is called a scalar field over region R.

Ex: Temperature (T) of a heated body in steady state is different at different points so that T is a scalar point function.

(ii) Vector Point Function: If for every point P (x, y, z) in a region 'R' of space, a unique vector $\bar{f}(x, y, z)$ or $\bar{f}(P)$ is associated by a function $\bar{f}$ then the function $\bar{f}(x, y, z)$ or $\bar{f}(P)$ is called "vector point function" or vector function of position.

The set of all points of the region R together with the function values $\bar{f}(P)$ is called a vector field over region R.

Ex: The velocity of a moving fluid at any time is a vector point function.

LEVEL SURFACE

If $\phi(x, y, z)$ is a scalar point function then the set of all points P (x, y, z) satisfying $\phi(x, y, z) = c$ where 'c' is an arbitrary constant, is called a level surface of ϕ at level c.

NOTE

For different values of c we get different level surfaces and the set of all level surfaces is known as family of level surfaces.

VECTOR DIFFERENTIAL OPERATOR

The vector differential operator is denoted by the symbol ∇ (read as nabla) and defined as

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}.$$

GRADIENT

GRADIENT OF A SCALAR POINT FUNCTION

If $\phi(x, y, z)$ is a scalar point function defined and differentiable at each point in a region of space then the gradient of ϕ is denoted by grad ϕ (or) $\nabla\phi$ and defined as

$$grad\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}.$$

THE PHYSICAL INTERPRETATION OF $\nabla\phi$

The gradient of a scalar function $\phi(x, y, z)$, *i.e.*, $\nabla\phi$ at a point P (x, y, z) is a vector along the normal to the level surface $\phi(x, y, z) = c$ at P and is in increasing direction.

NOTE

- $\nabla\phi$ is always a vector function whose components ϕ_x, ϕ_y, ϕ_z are functions of x, y, z.
- Gradient of constant scalar point function is a zero vector.
- $\frac{\nabla\phi}{|\nabla\phi|}$ is a unit vector normal to the level surface $\phi(x, y, z) = c$

DIRECTIONAL DERIVATIVE (D.D)

If $\phi(x, y, z)$ is a differentiable scalar function then the rate of changed of ϕ at a point P in the direction of a given vector $\bar{a}$ is called directional derivative of ϕ.

It is given by $$(\nabla\phi)_p \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$\therefore D.D = (\nabla\phi)_p \cdot \frac{\bar{a}}{|\bar{a}|}$$

NOTE

- The directional derivative of a scalar function $\phi(x, y, z)$ at a point $P\,(x, y, z)$ in the direction of a unit vector $\bar{e}$ is $(\nabla\phi).\bar{e}$.
- D.D of ϕ in the direction of x-axis is $\frac{\partial\phi}{\partial x} = (\nabla\phi).\bar{i}$
- D.D of ϕ in the direction of y-axis is $\frac{\partial\phi}{\partial y} = (\nabla\phi).\bar{j}$
- D.D of ϕ in the direction of z-axis is $\frac{\partial\phi}{\partial z} = (\nabla\phi).\bar{k}$
- If P is any point of the surface $\phi = c$ then the greatest rate of change of ϕ occurs in the direction of normal to the surface $\phi(x, y, z) = c$ at P.
- The greatest rate of increase (or) maximum value of directional derivative of $\phi(x, y, z)$ at a point P is $|\nabla\phi|$ at P.

ANGLE BETWEEN TWO SURFACES

If $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ are two surfaces and θ is the angle between the two surfaces at their point of intersection P then

$$\theta = \cos^{-1}\left[\frac{(\nabla\phi_1).(\nabla\phi_2)}{|\nabla\phi_1||\nabla\phi_2|}\right].$$

NOTE

- If $\phi(x, y, z) = c$ is level surface and P(x, y, z) is any point on the surface such that $(\nabla\phi)_p = a\,\bar{i} + b\,\bar{j} + c\,\bar{k}$ then

  (i) the equation of tangent plane to the surface is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

  (ii) the equation of normal line to the surface is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.
- If $r = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$ & $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$ then $\nabla[f(r)] = \bar{r}\,\frac{f^1(r)}{r}$.
- If $\bar{F}(t)$ is a vector with constant magnitude then $\bar{F}.\frac{d\bar{F}}{dt} = 0$.
- If $\bar{F}(t)$ is a vector with constant direction then $\bar{F} \times \frac{d\bar{F}}{dt} = \bar{0}$.

Ex: Find a unit normal vector to the given surface $x^2 y + 2xz = 4$ at the point (2, -2, 3).

Sol: Let $\phi(x, y, z) = x^2 y + 2xz - 4$ and $P = (2, -2, 3)$

Then
$$\nabla\phi = (2xy + 2z)\bar{i} + x^2\bar{j} + 2x\bar{k}$$
$$\Rightarrow (\nabla\phi)_p = -2\bar{i} + 4\bar{j} + 4\bar{k} = \bar{a}$$

∴ The unit vector normal to the given surface is
$$\frac{\bar{a}}{|\bar{a}|} = \frac{-2\bar{i} + 4\bar{j} + 4\bar{k}}{\sqrt{4 + 16 + 16}} = \frac{1}{3}\left(-\bar{i} + 2\bar{j} + 2\bar{k}\right)$$

Ex: Find the directional derivative of $f = x^2 - y^2 + 2z$ at P (1, 2, 3) in the direction of the line PQ where $Q = (5, 0, 4)$.

Sol: Given $f = x^2 - y^2 + 2z, P = (1, 2, 3)$ and $Q = (5, 0, 4)$
$$\Rightarrow \overline{PQ} = 4\bar{i} - 2\bar{j} + \bar{k} \text{ and } \nabla f = 2x\,\bar{i} - 2y\,\bar{j} + 4z\,\bar{k} \Rightarrow (\nabla f)_p = 2\bar{i} - 4\bar{j} + 12\bar{k}$$

∴ The directional derivative of f at P along $\overline{PQ}$ is
$$D.D = (\nabla f)_p \cdot \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{8 + 8 + 12}{\sqrt{16 + 4 + 1}} = \frac{28}{\sqrt{21}}$$

Ex: In what direction from the point (– 1, 1, 2) is the directional derivative of $\phi = xy^2 z^3$ a maximum? What is the magnitude of this maximum.

Sol: Given $\phi = xy^2 z^3$ and P = (– 1, 1, 2)

The directional derivative of $\phi(x, y, z)$ is maximum in the direction of normal to ϕ .
$$\Rightarrow \nabla\phi = \left(y^2 z^3\right)\bar{i} + \left(2xyz^3\right)\bar{j} + \left(3xy^2 z^2\right)\bar{k}$$
$$\Rightarrow (\nabla\phi)_p = 8\bar{i} - 16\bar{j} - 12\bar{k}$$

∴ Maximum value of directional derivative is
$$|\nabla\phi| = \sqrt{64 + 256 + 144} = \sqrt{464} .$$

Ex: If $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$ and $r = |\bar{r}|$ then find $\nabla(\cos r)$.

Sol: Given $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$ $\Rightarrow r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

Now
$$\nabla(f(r)) = \bar{r}\,\frac{f^1(r)}{r}$$

∴
$$\nabla(\cos r) = \bar{r}\left(\frac{-\sin r}{r}\right).$$

DIVERGENCE

DIVERGENCE OF A VECTOR FUNCTION

If a vector point function $\bar{F}(x, y, z) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ is defined and differentiable at each point in some region of space then the divergence of $\bar{F}$ is denoted by div $\bar{F}$ or $\nabla.\bar{F}$ and defined as

$$\text{div}\bar{F} = \nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

PHYSICAL INTERPRETATION

Let $\bar{F}(x, y, z)$ be the velocity of a fluid at a point P (x, y, z). Consider a small rectangular box within the fluid. Then div$\bar{F}$ measures the rate per unit volume at which the fluid flows out at any given time, *i.e.*, divergence measures the outward flow (or) expansion of the fluid from their point at any time.

HARMONIC FUNCTION

If $\phi(x, y, z)$ is a scalar point function such that $\nabla^2\phi = 0$ then the function ϕ is called harmonic function and equation $\nabla^2\phi = 0$ is called Laplace's equation where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ is Laplacian operator.}$$

NOTE

- $(\nabla.\bar{F}) \neq (\bar{F}.\nabla)$
- $(\nabla.\bar{F}) = \nabla^2\phi$
- If $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$ & $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$ then $\nabla^2[f(r)] = f''(r) + \frac{2}{r}f'(r)$.

SOLENOIDAL VECTOR

A vector point function $\bar{F}$ is said to be solenoidal vector if $\nabla.\bar{F} = 0$.

CURL

If a vector point function $\bar{F}(x, y, z) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ is defined and differentiable at each point in some region of space then curl of $\bar{F}$ is denoted by curl $\bar{F}$ or $\nabla \times \bar{F}$ and defined as

$$\text{curl } \bar{F} = \nabla \times \bar{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

PHYSICAL INTERPRETATION

If $\bar{\omega}$ is an angular velocity of a rigid body rotating about a fixed axis and $\bar{V}$ is the velocity of any point P (x, y, z) on the body then $\bar{\omega} = \frac{1}{2}\text{curl } \bar{V}$.

NOTE

- $(\nabla \times \bar{F}) \neq (\bar{F} \times \nabla)$
- curl $\bar{F}$ is a vector function
- curl $\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

✧ If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ & $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$ then $\nabla \times \left(\bar{r}f(r)\right) = \bar{0}$ and $\nabla \times (\bar{r}) = \bar{0}$.

IRROTATIONAL VECTOR

A vector point function $\bar{F}$ is said to be an a irrotational vector if curl $\bar{F} = \bar{0}$

SCALAR POTENTIAL FUNCTION

If for a given an irrotational vector $\bar{F}$ there exists a scalar point function $\phi(x, y, z)$ is called scalar potential function of $\bar{F}$.

VECTOR IDENTITIES

If $\bar{A}$ and $\bar{B}$ are differentiable vector functions and *f* and *g* are differentiable scalar functions of position (*x, y, z*) then

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
3. $\nabla.(\bar{A}+\bar{B}) = (\nabla.\bar{A}) + (\nabla.\bar{B})$
4. $\nabla \times (\bar{A}+\bar{B}) = (\nabla \times \bar{A}) + (\nabla \times \bar{B})$
5. $\nabla.(f\bar{A}) = (\nabla f).\bar{A} + f(\nabla.\bar{A})$
6. $\nabla \times (f\bar{A}) = (\nabla f) \times \bar{A} + f(\nabla \times \bar{A})$
7. $\nabla.(\bar{A} \times \bar{B}) = \bar{B}.(\nabla \times \bar{A}) - \bar{A}.(\nabla \times \bar{B})$
8. $\nabla \times (\nabla\phi) = \bar{0}$ (or) curl (grad ϕ)= $\bar{0}$ i.e., grad ϕ is always an irrotational vector.
9. $\nabla.(\nabla \times \mathrm{A}) = 0$ (or) div $(curl\,\bar{A}) = 0$, *i.e.*, curl$\bar{A}$ is always a solenoidal vector.

   (or) curl (curl$\bar{A}$) $= \nabla(\nabla.\bar{A}) - \nabla^2\bar{A}$

Ex: If $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ then find div$\bar{F}$ at (1, -1, 1).

Sol: Given $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$$\text{div}\bar{F} = \nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = y^2 + 2x^2z - 6yz$$

$\therefore$ At (1, -1, 1), div$\bar{F} = 9$

Ex: If $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ then find curl$\bar{F}$ at (1, 01, 1)

Sol: Given $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$$\text{curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix} = \bar{i}\left(-3z^2 - 2x^2y\right) - \bar{j}(0-0) - \bar{k}(4xyz - 2xy)$$

$\therefore$ At (1, -1, 1), curl$\bar{F} = -\bar{i} - 2\bar{k}$

Ex: Show that vector $\overline{F} = (x^2 - yz)\overline{i} + (y^2 - xz)\overline{j} + (z^2 - xy)\overline{k}$ is irrotational and find it's scalar potential.

Sol:
$$\text{curl}\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - xz & z^2 - xy \end{vmatrix} = 0$$

$\therefore \overline{F}$ is irrotational.

Let $\overline{F} = \nabla\phi$ where ϕ is scalar potential.

$$\Rightarrow F_1\overline{i} + F_2\overline{j} + F_3\overline{k} = \overline{i}\frac{\partial\phi}{\partial x} + \overline{j}\frac{\partial\phi}{\partial y} + \overline{k}\frac{\partial\phi}{\partial z} \Rightarrow \frac{\partial\phi}{\partial x} = F_1, \frac{\partial\phi}{\partial y} = F_2, \frac{\partial\phi}{\partial z} = F_3$$

Consider
$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = F_1 dx + F_2 dy + F_3 dz$$

$$\Rightarrow d\phi = (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz$$

$$\Rightarrow d\phi = x^2 dx + y^2 dy - d(xyz)$$

$$\therefore \quad \phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$$

is a scalar potential function.

Ex: If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then find $\nabla^2(\log r)$

Sol: Given $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ and $r = |\overline{r}| = \sqrt{x^2 + y^2 + z^2}$

But $\nabla^2(f(r)) = f^{11}(r) + \frac{2}{r}f^1(r)$ $\quad\therefore\quad \nabla^2(\log r) = \frac{-1}{r^2} + \frac{2}{r}\left(\frac{1}{r}\right) = \frac{1}{r^2}$

Ex: If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then find $\nabla \times (\overline{r}\log r)$

Sol:
$$\nabla \times (\phi\overline{A}) = (\nabla\phi \times \overline{A}) + \phi(\nabla \times \overline{A})$$

where $\phi = \log r$ and $\overline{A} = \overline{r}$

$$\nabla \times (\log r\,\overline{r}) = (\nabla \log r) \times \overline{r} + \log r(\nabla \times \overline{r}) = \frac{1}{r}\overline{r} \times \overline{r} + \overline{0} = \frac{1}{r}(\overline{r} \times \overline{r}) = \overline{0}$$

VECTOR INTEGRATION

LINE INTEGRAL

In general, any integral which is to be evaluated along a curve is called a line integral.

Let $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ be the position vector of any point $P(x, y, z)$ on a curve C joining the points P_1 and P_2. We assume that C is composed of a finite number of curves for each of which $\overline{r}$ has a continuous derivative. Let $\overline{A}(x, y, z) = A_1\overline{i} + A_2\overline{j} + A_3\overline{k}$ be a differentiable vector function. Then the integral of tangential component of $\overline{A}$ along C from P_1 to P_2 is $\int_{P_1}^{P_2} \overline{A}.d\overline{r} = \int_{P_1}^{P_2} A_1 dx + A_2 dy + A_3 dz.$

CIRCULATION

If C is a simple closed curve then the line integral of $\overline{A}$ along a closed curve C is denoted by $\oint_C \overline{A}.d\overline{r}$.

In aerodynamics and fluid mechanics $\oint_C \overline{A}.d\overline{r}$ is called the circulation of $\overline{A}$ around C where $\overline{A}$ represents the fluid velocity.

WORK DONE BY FORCE

If $\overline{A}$ is a force acting on a particle which moves from a point P_1 to a point P_2 along a curve C then the line integral $\int_{P_1}^{P_2} \overline{A}.d\overline{r}$ gives the total work done by force $\overline{A}$.

NOTE

- If $\overline{A} = A_1\overline{i} + A_2\overline{j} + A_3\overline{k}$ and $d\overline{r} = dx\,\overline{i} + dy\,\overline{j} + dz\,\overline{k}$ where $\overline{r} = x\,\overline{i} + y\,\overline{j} + z\,\overline{k}$ then $\int_C \overline{A}.d\overline{r} = \int_C \left(A_1 dx + A_2 dy + A_3 dz\right)$ which is a line integral in Cartesian form.
- The value of the line integral of a vector point function depends (upon) on the path joining A and B (unless the vector function is an irrotational).
- If $\overline{A}$ is a conservative field or an irrotational vector ($i.e.,\nabla \times \overline{A} = 0$) in a region R of space then

  (i) the line integral $\int_{P_1}^{P_2} \overline{A}.d\overline{r}$ is independent of path C joining P_1 and P_2 in R and

  $\int_{P_1}^{P_2} \overline{A}.d\overline{r} = \int_{P_1}^{P_2} \nabla\phi.d\overline{r} = \int_{P_1}^{P_2} d\phi = \phi\left(P_2\right) - \phi\left(P_1\right)$ where $\phi\left(x, y, z\right)$ is a scalar potential function.

  (ii) $\oint_C \overline{A}.d\overline{r} = 0$ around any closed curve C in a region R.

SURFACE INTEGRAL

Suppose S is a piece wise smooth surface and $\overline{F}\left(x, y, z\right)$ is a differentiable vector function over S. Let P be any point on S and let $\overline{n}$ be the unit vector at P in the direction of outward drawn normal to the surface S at P then $\iint_S \left(\overline{F}.\overline{n}\right) dS$ is an example of surface integral.

METHOD OF EVALUATION

- $\iint_S \left(\overline{F}.\overline{n}\right) dS = \iint_{R_1} \left(\overline{F}.\overline{n}\right) \dfrac{dxdy}{\left|\overline{n}.\overline{k}\right|}$
- If R_2 is the projection of 'S' on yz- plane then $\iint_S \left(\overline{F}.\overline{n}\right) dS = \iint_{R_2} \left(\overline{F}.\overline{n}\right) \dfrac{dydz}{\left|\overline{n}.\overline{i}\right|}$

- If R_3 is the projection of 'S' on xz - plane then $\iint_S (\bar{F}.\bar{n})dS = \iint_{R_3} (\bar{F}.\bar{n})\frac{dxdz}{|\bar{n}.\bar{j}|}$

VOLUME INTEGRAL

If S is a closed surface enclosing a volume region V then $\iiint_V \bar{A}\,dV$ and $\iiint_V \phi\, dV$ are examples of volume integrals.

Ex: Find $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = x^2y^2\bar{i} + y\bar{j}$ and C is the curve $y^2 = 4x$ in the XY- plane from (0, 0) to (4, 4)

Sol:
$$\int_C \bar{F}.d\bar{r} = \int_C x^2y^2dx + ydy$$

Given
$$y^2 = 4x$$
$$\Rightarrow 2\,y\,dy = 4\,dx$$
$$\Rightarrow y\,dy = 2\,dx$$
$$\int_C \bar{F}.d\bar{r} = \int_0^4 x^2\,4x\,dx + 2\,dx = \left[4\frac{x^4}{4} + 2x\right]_0^4 = 264$$

Ex: Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight 6 line joining the points (0, 0, 0) and (2, 1, 3).

Sol: Equation of straight line is
$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$
$$\Rightarrow x = 2t,\ y = t,\ z = 3t$$
$$\Rightarrow dx = 2dt,\ dy = dt,\ dz = 3dt$$
$$\text{Work done} = \int_C \bar{F}.d\bar{r}$$
$$= \int_C 3x^2dx + (2xz - y)dy + zdz$$
$$= \int_0^1 \left[3(2t)^2(2dt) + (2(2t)(3t) - t)dt + (3t)3dt\right]$$
$$= \int_0^1 \left[36t^2 + 8t\right]dt = 16$$

Ex: Evaluate $\int_S \bar{F}.\bar{N}ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and 'S' is the part of the surface of the plane $2x + 3y + 6z = 12$ located in the first octant.

Sol: Let the equation of given surface be
$$\phi = 2x + 3y + 6z - 12$$

Then normal to the surface is $\nabla\phi = 2\bar{i} + 3\bar{j} + 6\bar{k} \Rightarrow \bar{N} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\bar{i} + 3\bar{j} + 6\bar{k}}{7}$

$$\therefore \bar{F}.\bar{N} = \frac{6}{7}[6z - 6 + 3y]$$

Let 'r' be projection of 'S' onto XY - plane.

Then $\int_S \bar{F}.\bar{N}\,ds = \int\int_R \bar{F}.\bar{N}\dfrac{dx\,dy}{|\bar{N}.\bar{K}|} = \int\int_R \dfrac{6}{7}(6z - 6 + 3y)\dfrac{dxdy}{6/7} = \int\int_R (6z - 6 + 3y)dxdy$

From the equation of plane, we have

$$6z = 12 - 2x - 3y = \int\int_R (6 - 2x)dxdy = \int_0^6 \int_0^{\frac{12-2x}{3}} (6 - 2x)dx\,dy$$

$$= \int_0^6 (6 - 2x)\frac{(12 - 2x)}{3}dx = \frac{4}{3}\int_0^6 \left(18 - 9x + x^2\right)dx = 24$$

Ex: If $\bar{F} = \left(2x^2 - 3z\right)\bar{i} - 2xy\,\bar{j} - 4x\,\bar{k}$ then evaluate $\int_V \nabla.\bar{F}dV$ where V is the closed region bonded by $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$.

Sol: $\nabla.\bar{F} = 4x - 2x - 0 = 2x$

$$\int_V \nabla.\bar{F}dV = \int_V 2x\,dV$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\;dzdydx = \int_0^2\int_0^{2-x} 2x(4 - 2x - 2y)\,dydx$$

$$= \int_0^2 \left[4\left(2x - x^2\right)y - 4x\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= 4\int_0^2 \left[\left(2x - x^2\right)(2 - x) - \frac{x}{2}(2 - x)^2\right]dx = 2\int_0^2 \left(x^3 - 4x^2 + 4x\right)dx = 8$$

GREEN'S THEOREM

If R is a closed region of the xy- plane bounded by a simple closed curve C and if $M(x, y)$, $N(x,y)$, $\dfrac{\partial N}{\partial x}, \dfrac{\partial M}{\partial y}$ are continuous function of x and y in R then $\oint_C M\,dx + N\,dy = \int\int_R \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)dx\,dy.$

GAUSS DIVERGENCE THEOREM

If V is the volume of the region bounded by a closed surface S and $\bar{A}(x, y, z)$ is a differentiable vector function over S then $\oiint_S \left(\bar{A}.\bar{n}\right)dS = \int\int\int_V \left(\nabla.\bar{A}\right)dV$, where $\bar{n}$ is outward drawn unit normal to the surface S in positive direction.

STOKE'S THEOREM

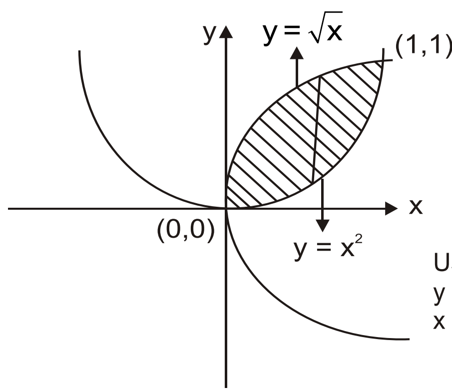
If S is an open, two-sided surface bounded by a simple closed curve C and $\overline{A}(x, y, z)$ is a differentiable vector function then $\oint_C \overline{A}.d\overline{r} = \iint_S \{(\nabla \times \overline{A}).\overline{n}\}dS$, where C is transversed in the positive direction and $\overline{n}$ is outward drawn unit normal to the surface S in positive direction.

Ex: Find $\int_C (2xy - x^2)dx - (x^2 + y^2)dy$ where 'C' is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Sol: Here $M = 2xy - x^2, N = -(x^2 + y^2)$

$$\frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = -2x \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4x$$

$\therefore$ By Green's theorem



$$\oint_C Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]dx\,dy$$

$$\oint_C Mdx + Ndy = \int_0^1 \int_{x^2}^{\sqrt{x}} -4xdydx = \int_0^1 -4x[y]_{x^2}^{\sqrt{x}}\,dx$$

$$= -4\left[\frac{x^{5/2}}{(5/2)} - \frac{x^4}{4}\right]_0^1 = -4\left[\frac{2}{5} - \frac{1}{4}\right] = -\frac{3}{5}$$

Ex: Evaluate $\int_S (x+z)dydz + (y+z)dzdx + (x+y)dxdy$ where 'S' is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

Sol: Since 'S' is the surface of the sphere $x^2 + y^2 + z^2 = 4$, it is a closed surface. So it can be reduced to volume integral using Gauss Divergence theorem.

Here div $\overline{F} = 1 + 1 + 0 = 2$

$$\therefore \int_S F_1 dydz + F_2 dxdy + F_3 dxdy = \int_V div\,\overline{F}\,dV = \int_V 2dV = 2V$$

where V is the volume of the sphere $= 2 \times \frac{4}{3}\pi r^3 = \frac{8}{3}\pi(2)^3 = \frac{64\pi}{3}$

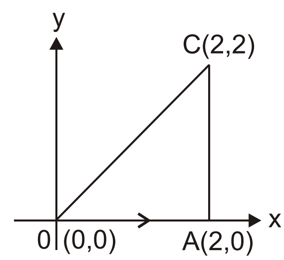
Ex: Evaluate $\int_C \overline{F}.d\overline{r}$ where $\overline{F} = 2y^2\overline{i} + 3x^2\overline{j} - (2x+z)\overline{k}$ and 'C' is the boundary of the triangle whose vertices are (0, 0, 0) and (2, 2, 0)

Sol: Since z - coordinate of each vertex of the triangle is zero, the triangle lies in xy -plane. As 'C' is a closed curve in xy- plane, the line integral can be transformed to a surface integral using Stokes theorem.

$$\therefore Curl\,\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & -2x-z \end{vmatrix} = 2\overline{j} + (6x-4y)\overline{k}$$

Here $\overline{N} = \overline{k}$

$\therefore$By Stokes theorem, we have



$$\oint_C \overline{F}.d\overline{r} = \int_s \left(curl\,\overline{F}.\overline{N}\right)ds = \int_s (6x-4y)ds = \int_0^2\int_0^x (6x-4y)\,dy\,dx = \frac{32}{3}$$

Differential Equations

STANDARD FORM

An equation involving one or more derivatives is called differential standard form for a first order differential equation in the unknown function $y(x)$ is $\frac{dy}{dx} = f(x, y)$.

ORDER & DEGREE

The order of a differential equation is the order of the highest order derivative occurring in it.

The degree of a differential equation is the degree of the highest order derivative occurring in the differential equation, after the equation has been made free of radical signs or fractional powers of the derivatives.

LINEAR AND NON-LINEAR DIFFERENTIAL EQUATION

A differential equation is called linear if the dependent variable and its derivative occurring in it are of the first degree and are not multiplied together.

DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

1. **Variable Separable From:** The equation of this type can be put in the form

$$f(x)dx + g(y)dy = 0$$

The solution is obtained as $\int f(x)dx + \int g(y)dy = c$

2. **Linear differential Equation:** The linear differential equation is of the form $\frac{dy}{dx} + Py = Q$

Where P and Q are the function of only the solution is obtained as

$$y.e^{\int Pdx} = \int Q.e^{\int Pdx}dx + c$$

Where $e^{\int Pdx}$ is called integrating factor.

3. **Bernoulli Equation:** This is of the form $\frac{dy}{dx} + Py = Qy^n$ .... (1)

Where P and Q are function of only dividing this equation by y^n we get

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \quad \text{........... (2)}$$

Now on putting $v = y^{1-n}$ $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

Putting in equation (1) we get, $\frac{dv}{dx} + P(1-n)v = Q(1-n)$ which is a linear equation.

4. **Homogeneous Equation:** A differential equation of the form $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$ is called a homogeneous differential equation, $f_1(x, y)$ if and $f_2(x, y)$ are homogenous function of the same degree.

Let $f_1(x, y) = x^n f_1(y/x)$; $f_2(x, y) = x^n f_2(y/x)$

Thus $$\frac{dy}{dx}=\frac{x^n f_1\left(y/x\right)}{x^n f_2\left(y/x\right)}=f\left(y/x\right) \quad \text{.......... (1)}$$

Put $y=vx$ or $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

From (1) we get $$v+\frac{xdv}{dx}=f(v) \quad \text{......... (2)}$$

Separating the variables, the equation (2) become $\dfrac{dv}{f\left(v\right)-v}=\dfrac{dx}{x}$

Integrating this, we get the required solution.

5. **Reducible to Homogeneous Form:** The equation of this type is
$$\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C} \quad \text{......... (1)}$$

CASE I. When $\dfrac{a}{A}\neq\dfrac{b}{B}$. Put $x=X+h$ and $y=Y+k$
$$\frac{dy}{dx}=\frac{dY}{dX}$$

Thus given equation becomes $$\frac{dY}{dX}=\frac{aX+bY+\left(ah+bk+c\right)}{AX+BY+\left(Ah+Bk+C\right)} \quad \text{......... (2)}$$

Now Choose h and k such that
$$ah+bk+c=0$$
$$Ah+Bk+C=0$$

Thus the equation (2) becomes $$\frac{dY}{dX}=\frac{aX+bY}{AX+BY}$$

Which is homogeneous equation and can be solved by putting $y=xX$

CASE II When $\dfrac{a}{A}=\dfrac{b}{B}$, In this case let $\dfrac{a}{A}=\dfrac{b}{B}=\dfrac{1}{k}$

Thus $A=ak, B=bk$

Equation (1) can be written as $$\frac{dy}{dx}=\frac{ax+by+c}{b\left(ax+by\right)+C} \quad \text{......... (3)}$$

Substituting $ax+by=v$ we get $a+b\dfrac{dy}{dx}=\dfrac{dv}{dx}$

or $$\frac{dy}{dx}=\frac{1}{b}\left(\frac{dv}{dx}-a\right)$$

Thus the equation (3) becomes $$\frac{1}{b}\left(\frac{dv}{dx}-a\right)=\frac{v+c}{kv+C}$$
$$\frac{dv}{dx}=\frac{\left(b+ak\right)v+\left(bc+aC\right)}{kv+C}$$

Which can be solved by variable separation method.

6. Exact Differential Equation: A first order and first degree equation of the form

$$M(x, y)dx + N(x, y)dx = 0 \qquad \text{..........} (1)$$

Is called exact differential equation if it can be obtained by direct differentiation of some function of x and y.

The equation (1) is exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solution of exact differential equation is $\int Mdx + \int Ndy = C$

7. Change of Variable: Let the equation $\dfrac{dy}{dx}$ .... (1)

Substituting $ax + by + c = v$ we get $a + \dfrac{bdy}{dx} = \dfrac{dv}{dx}$

Equation (1) reduced to $\dfrac{1}{b}\left(\dfrac{dv}{dx} - a\right) = f(v)$ or $\dfrac{dv}{bf(v) + a} = dx$

Integrating we find the required solution.

LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

The general form of a linear differential equation of n^{th} order with constant coefficients is

$$\frac{d^n y}{dx^n} P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = X$$

or

$$D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \ldots\ldots + P_n y = X \qquad \text{..........} (1)$$

Where $P_1, P_2 \ldots\ldots P_n$ are function of x and y, $D = \dfrac{d}{dx}$ and X is either a constant or a function of x.

SOLUTION OF THE DIFFERENTIAL EQUATION

If the given differential equation is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2}}{dx^{n-2}} + \ldots + P_n y = 0$$

or

$$\left(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n\right) y = 0$$

or

$$f(D) y = 0$$

Where $f(D) = 0$ is called auxiliary equation.

WORKING RULE FOR SOLUTION

1. Write auxiliary equation A.E. as $f(D) = 0$.

2. Solve the A.E. let the roots of A.E. be $D = m_1, m_2, m_3 \ldots\ldots m_n$

CASE I: When $m_1, m_2, m_3 \ldots\ldots m_n$ are real and distinct then solution of the given equation is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots\ldots + C_n e^{m_n} \times$ where $C_1, C_2, \ldots\ldots C_n$ are arbitrary constant.

CASE II: When $m_1, m_2, m_3 \ldots\ldots m_n$ are real and say $m_1 = m_2$ and others are distinct. Then solution of the given equation is $y = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x}$

CASE III: When $m_1, m_2, m_3, m_4 .....m_n$ are all real and $m_1 = m_2 = m_3$ and others are different, then solution of the given equation is $y = \left(C_1 + C_2 x + C_3 x^2\right)e^{m_1 x} + C_4 e^{m_4 x} + ...+ C_n e^{m_n x}$

CASE IV: When $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ (complex roots) and $m_3, m_4 .....m_n$ are real and different. $y = e^{ax}\left(C_1 \cos\beta x + C_2 \sin\beta x\right) + C_3 e^{m_3 x} + C_4 e^{m_4 x} ....C_n e^{m_n} x$

CASE V: When $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha + i\beta$ (complex repeated roots) where $m_3 ....m_n$ are real and different,

$$y = e^{ax}\left[\left(C_1 + C_2 x\right)\cos\beta x + \left(C_3 + C_4 x\right)\sin\beta x\right] + C_5 e^{m_5 x} + ... + C_x e^{m_n x}$$

SOLUTION OF THE EQUATION

The general solution of the differential equation

$$f(D)y = X \qquad ... (1)$$

Consist of two parts. The solution of equation on (1) corresponding to the LHS is called the complementary Function (C.F.) and the one corresponding to the RHS called the particular Integral (P.I).

The complete solution is $y = C.F + P.I$

PARTICULAR INTEGRAL

Particular Integral is defined as $P.I = \dfrac{1}{f(D)} X$

CASE I: When $X = e^{ax+b}$ $P.I = \dfrac{1}{f(\alpha)} e^{ax+b}$, if $f(\alpha) \neq 0$

If $f(\alpha) = 0$ then $P.I = x^2 \dfrac{1}{f''(\alpha)} e^{ax+b}$, if $f''(\alpha) \neq 0$

CASE II: When $X = x^n$ $P.I = \dfrac{1}{f(D)} x^n = \left[f(D)\right]^{-1} x^n$

Expand $\left[f(D)\right]^{-1}$ by binomial theorem in ascending powers of D.

CASE III: When $X = \sin(ax + b)$

$$P.I = \frac{1}{f\left(D^2\right)}\sin\left(ax + b\right) = \frac{1}{f\left(-a^2\right)}\sin\left(ax + b\right) \qquad \text{if } f\left(-a^2\right) \neq 0$$

If $f\left(-a^2\right) = 0$ then $P.I = \dfrac{1}{\dfrac{d}{dD} f\left(D^2\right)}\sin\left(ax + b\right)$

Similarly we can find the P.I. of $x = \cos ax$

CASE IV: When $X = e^{\alpha x}V(x)$ $P.I = \dfrac{1}{f(D)} e^{\alpha x}V(x) = e^{\alpha x}\dfrac{1}{f\left(D + \alpha\right)}V(x)$

CASE V: When $X = xV(x)$

$$P.I = \frac{1}{f(D)} xV(x) = \left[x - \frac{f'\left(D\right)}{f\left(D\right)}\right]\frac{1}{f\left(D\right)}V(x) = x\frac{1}{f\left(D\right)}V - \frac{f'(D)V(x)}{\left[f(D)\right]^2}$$

CASE VI: Where x is any function of x other than the function given in previous cases

Let $$f(D) = (D - m_1)(D - m_2)......(D - m_n)$$

Then $$P.I = \frac{1}{f(D)} x$$

Resolve denominator into partial fraction and use $\frac{1}{D - m} x = e^{mx} \int e^{-mx} x dx$

METHOD OF VARIATION OF PARAMETER

Consider the differential equation $\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R$ .... (1)

Assume that y_1 and y_2 are a complementary function of equation (1) , then particular solution to this equation is $y = -y_1 \int \frac{y_2 R}{y_1 y_2' - y_1' y_2} dx + y_2 \int \frac{y_1 R}{y_1 y_2' - y_1' y_2}$

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

The linear differential equation

$$\frac{x^n d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2}}{dx^{n-2}} +.....+ P_{n-1} x \frac{dy}{dx} + P_n y = X \quad .... (1)$$

is called Homogeneous Linear Differential Equation or Cauchy's Differential Equation. $P_1, P_2, P_3 .....P_{n-1}, P_n$ are constants and X is a function of x or constant. This equation can be reduced to a linear differential equation with constant coefficients by putting or and using following results:

$$x\frac{dy}{dx} = \frac{dy}{dz} = D_y \quad \text{where} \quad D = \frac{d}{dz}$$

$$\frac{x^2 d^2 y}{dx^2} = D(D-1) y$$

$$\frac{x^3 d^3 y}{dx^3} = D(D-1)(D-2) y$$

$$\frac{x^n d^n y}{dx^n} = D(D-1)(D-2).....(D-n+1) y$$

Substituting these in equation (1) we get a linear differential equation with constant coefficient, which can be solved by method, discussed earlier.

EULER EQUATION

The following type of equation are called Euler Equation $x^2 + y'' + by' + cy = 0$

The auxiliary equation is $aD(D-1) + bD + C = 0$

This equations is quadratic so we will have following three case:

CASE I: If roots and distinct, then solution is $y = C_1 x^{m_1} + C_2 x^{m_2}$

CASE II: If roots are equal and real, than solution is $y = C_1 x^{m_1} + C_2 x^{m_2} Inx = x^{m_1} (C_1 + C_2 Inx)$

CASE III: If root are complex then solution is $y = x^{\alpha} [C_1 \cos(\beta Inx) + C_2 \sin(\beta Inx)]$

Complex Analysis

I. COMPLEX NUMBER

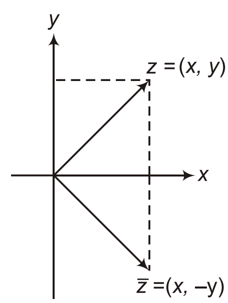
An order of real numbers $Z = (x_1, y_1)$ which are subjected to following operations.

- $x \in IR$ and $y \in -IR$ (Imaginary) I_c $R_e(Z) = x$ and $I_m(Z) = y$
- $Z_1 = Z_2 \Rightarrow x_1 = x_2,\ y_1 = y_2$
- $Z_1 + Z_2 = (x_1 + x_2, y_1 + y_2)$

II. ALGEBRA OF COMPLEX NUMBERS

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$

- $Z_1 \pm Z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$
- $Z_1 Z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$
- $\dfrac{Z_1}{Z_2} = \left(\dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\dfrac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)$



III. ARGAND PLANE (COMPLEX PLANE)

- $|z| = \sqrt{x^2 + y^2}$
- $z + \bar{z} = 2x$ & $z - \bar{z} = 2iy \Rightarrow x = \frac{1}{2}(z + \bar{z})$ & $y = \frac{1}{2i}(z - \bar{z})$
- $z\bar{z} = x^2 + y^2 = (|z|)^2$
- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
- $\overline{Z_1 Z_2} = \bar{Z}_1\ \bar{Z}_2$
- $\overline{\left(\dfrac{Z_1}{Z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$
- Triangle Inequality: $(|z_1 + z_2|) \subseteq (|Z_1|) + (|Z_2|)$

IV. POLAR REPRESENTATION OF A COMPLEX NUMBER

$$x = r\cos\theta,\ y = r\sin\theta,\ z = re^{i\theta}$$

- $(|Z|) = r = \sqrt{x^2 + y^2}$
- Argument of z, $\theta = \tan^{-1}(y/x)$
- **Demoivre's Theorem**

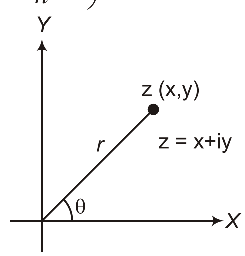
If $z = re^{i\theta} \Rightarrow Z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, n \in z$

This theorem is used to find the roots of complex numbers.

- ***n^{th}* roots of *z***

Let $z = re^{i\theta} \Rightarrow z^{1/n} = \text{Re}^{i\phi}$

where $R = z^{1/n}$ & $\phi = \left(\frac{\theta + 2k\pi}{n}\right)$ and $(k = 0, \pm 1, \pm 2, \ldots., n)$



- Cube roots of unity $\sqrt[3]{1} = \left(1, \frac{-1 \pm \sqrt{3}i}{2}\right)$
- Fourth roots of unity $\sqrt[4]{1} = (\pm 1, \pm i)$

V. ANALYTIC FUNCTION (REGULAR FUNCTION)

A function $f(z)$, which is single valued and possess a unique derivative w.r.t. z at all points of a region R is said to be analytic function. *i.e.* $f(z)$ is said to be Analytic in a Domain 'D' if $f(z)$ is well defined and differentiable at all points of D.

VI.CAUCHY-RIEMANN EQUATION (CONDITIONS FOR DIFFERENTIABILITY)

$$f(z) = u(x, y) + i\,v(x, y)$$

$f(z)$ is analytic if u and v satisfy C-R equation $\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \ and \ \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$

- C-R Equation in polar form.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

- Harmonic function:

  If $f(z) = u(x, y) + iv(x, y)$ is analytic. Thus $\nabla^2 u = 0$ and $\nabla^2 v = 0$.

  i.e. u and v are called harmonic function x, which satisfy Laplacian equation.
  $u(x, y)$ and $v(x, y)$ are called conjugate function $f(z)$.
  when $u(x, y)$ is given,

$$V = \int \frac{-\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy + c$$

  when $v(x, y)$ is given,

$$u = \int \frac{\partial v}{\partial y} dx - \int \frac{\partial v}{\partial x} dy + c$$

VII. SINGULARITIES

- Singular point: A point at which a function $f(z)$ is not analytic is called singular point.

  Eq: $f(z) = \dfrac{1}{z - a}$ has a singular point at $z = a$

- Laurent's Expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$

- **Types of Singularities**

  (i) If principal part of $f(z)$ contains infinite no. of terms at $z = a$.
  $z = a$ is called essential singular point.
  Ex: $f(z) = e^{1/z}$, $z = 0$ is essential singular point.

  (ii) If principal part of $f(z)$ contains finite number of terms at $z = a$.
  $z = a$ is called pole of order n.

$$f(z) = \frac{z^2}{(z-1)^2 (z+2)}, \quad z = 1 \text{ is pole of order 2.}$$

  (iii) If principal part of $f(z)$ has no principal terms of $(z - a)$, $z = a$ is called removable singular point.

$$\text{Eg. } f(z) = \frac{\sin z}{z}$$

VIII. RESIDUE THEOREM

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

- If $f(z)$ has a pole of order n at $z = a$

$$\operatorname{Res}(f,a)=\frac{1}{(n-1)!}\lim_{z\to a}\left[\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z)\right]$$

- If $f(z)$ has essential singularity at $z=a$

  $\operatorname{Res}(f,a)=$ coefficient b_1 in the Laurent's expansion.

- Residue of $f(z)$ at $z=\infty$ is negative coefficient of $\frac{1}{z}$ in $f(z)\big|_{z=0}$.

IX CAUCHY INTEGRAL THEOREM

- Cauchy integral theorem 1, if no poles are inside the contour,

$$\int_C f(z)dz=0$$

- Cauchy integral theorem 2, if there is a pole inside the contour,

$$f(a)=\frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

- Derivative of analytic function

$$f^n(a)=\frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}dz$$

CONSEQUENCES OF CAUCHY'S THEOREM

Let $f(z)$ be analytic function in a simply connected region R.

- If a and z are any two points in R, then $\int_a^z f(z)dz$ is independent of the path is R joining a and z.

- If a and z are any two points in R and $g(z)=\int_a^z f(z)\,dz$ then $g(z)$ is analytic in R and $g(z)=f(z)$.

- If a and b are any two points in R and $F'(z)=f(z)$ then,

$$\int_a^b f(z)dz=F(b)-F(a)$$

- Let $f(z)$ be analytic in a region bounded by two simple closed curves C and C_1 where C_1 lies inside C and on the curves, then

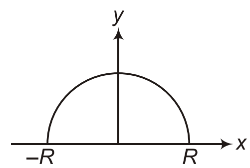
$$\oint_C f(z)\,dz=\oint_{C_1} f(z)\,dz$$

X. EVALUATION OF DEFINITE INTEGRAL

The evaluation of definite integral is often achieved by using two residue theorem together with a suitable function $f(z)$ and a suitable path or contour C.

✧ $\int_{-\infty}^{\infty} f(x)dx$, where $f(x)$ is a rational function.

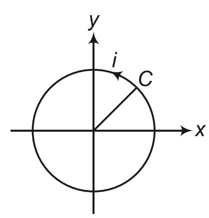
Consider $\oint_c f(z)dz$ along a contour C consisting of the line along the x - axis from $-R$ *to* R and the semi-circle Γ above the x-axis having this line as a diameter as shown in figure. Then, let $R \to \infty$.



✧ $\int_{o}^{2\pi} f(\sin\theta, \cos\theta)$, where $f(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.

Let $z = e^{i\theta}$ then $\sin\theta = \frac{z - z^{-1}}{(2i)}$, $\cos\theta = \frac{z + z^{-1}}{(2)}$ and $dz = ie^{i\theta} d\theta$ or $d\theta = \frac{dz}{iz}$.

The given integral is equivalent to $\oint_C f(z)dz$ where C is the unit circle with centre at the origin as shown in the figure.



✧ $\int_{-\infty}^{\infty} f(x)\cos mx\, dx$ or $\int_{-\infty}^{\infty} f(x)\sin mx\, dx$, where $f(x)$ is a rational function.

Here, we consider $\oint_c f(z)\, e^{imz}\, dz$ where C is the same contour as that in type (i).

XI. TAYLOR'S SERIES

Let $f(x)$ be analytic inside and on a simple closed curve C. Let a and b are two points inside C.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ...$$

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a)....$$

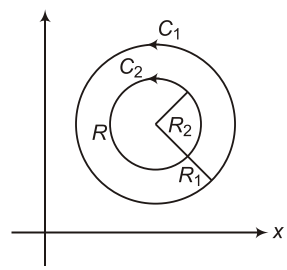
NOTE :

- The region of convergence of the series is $(|z-a|) < R$.
- For $(|z-a|) = R_1$ the series may or may not converge.
- For $(|z-a|) > R_1$ the series diverge.

where R is the distance from a to the nearest singularity of the function $f(z)$.

XII. LAURENT'S SERIES

Let C_1 and C_2 be concentric circles of radii R_1 and R_2 respectively as shown in the figure. Let $(a+b)$ be any point in R, then



$$f(a+h) = a_0 + a_1 h + a_2 h^2 + ...+ \frac{a-1}{h} + \frac{a-2}{h^2} + ...$$

where $$a_n = \frac{1}{2\pi i}\oint_{c_1} \frac{f(z)}{(z-a)^{n+1}} dz \; ; \; n = 0,1,2,3...$$

$$a_n = \frac{1}{2\pi i}\oint_{c_2} (z-a)^{n-1} f(z) dz \; ; n = 0,1,2,3....$$

Fourier Series

You have seen from Maclaurin's and Taylor's series that an infinitely differentiable function can be expressed in the form of an infinite series in x. Fourier series on the other hand, enables us to represent a *Periodic Function* as an infinite trigonometrical series in sine and cosine terms. We can use Fourier series to represent a function containing discontinuities unlike Maclaurin's and Taylo's series.

PERIODIC FUNCTION

A function $f(t)$ is periodic if $f(t) = f(t + nT), n = 0, \pm 1, \pm 2.....$
T is called the period. For sine and cosine the period $T = 2\pi$ so that

$$\sin t = \sin(t + 2\pi n) \text{ and } \cos t = \cos(t + 2\pi n)$$

ANALYTICAL DESCRIPTION OF A PERIODIC FUNCTION

Many periodic functions are non-sinusoidal

Ex: **1.** $f(t) = 3 \quad 0 < t < 4$

$f(t) = 0 \quad 4 < t < 6$

$f(t) = f(t+6)$ i.e. the period is 6

2. $f(t) = \frac{5}{8}t \quad 0 < t < 8$

$f(t) = f(t+8)$

Sketch the following periodic functions

1. $f(t) = 4 \quad 0 < t < 5$

$f(t) = 0 \quad 5 < t < 8$

$f(t) = f(t+8)$

2. $f(t) = 3t - t^2 \quad 0 < t < 3$

$f(t) = f(t+3)$

FOURIER SERIES OF FUNCTIONS OF PERIOD 2π

Any periodic function $f(x) = f(x + 2\pi n)$ can be written in Fourier series as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + ...+ b_1 \sin x + b_2 \sin 2x + ...$$

(where $a_0, a_n, b_n, n = 1,2,3......$ are Fourier coefficients) or as

$$f(x) = \frac{1}{2}a_0 + c_1 \sin(x + \alpha_1) + c_2 \sin(2x + \alpha_2) + ...$$

where $c_i = \sqrt{a_i^2 + b_i^2}$ *and* $\alpha_i = arc\tan\left(\frac{b_i}{a_i}\right)$.

$c_1 \sin(x+\alpha_1)$ is the first harmonic or fundamental $c_2 \sin(2x+\alpha_2)$ is the second harmonic $c_n \sin(nx+\alpha_n)$ *is the* n^{th} harmonic.

For the Fourier series to accurately represent $f(x)$ it should be such that if we put $x = x_1$ in the series the answer should be approximately equal to the value of $f(x_1)$ i.e. the value should converge to $f(x_1)$ as more and more terms of the series are evaluated. For this to happen $f(x)$ must satisfy the following:

DIRICHLET CONDITIONS

(a) $f(x)$ must be defined and single-valued.

(b) $f(x)$ must be continuous or have a finite number of discontinuities within a periodic interval.

(c) $f(x)$ and $f'(x)$ must be piecewise continuous in the periodic interval.

If these conditions are met the series converges fairly quickly to $f(x_1)$ if $x = x_1$, and approximation of the function $f(x)$

FOURIER COEFFICIENTS

The Fourier coefficients above are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

ODD AND EVEN FUNCTIONS

(a) *Even Functions*: A function $f(x)$ is said to be even if $f(-x) = f(x)$. The graph of an even function is, therefore, *symmetrical about the y-axis*. e.g. $f(x) = x^2$ $f(x) = \cos x$

(b) *Odd Functions*: A function $f(x)$ is said to be odd if $f(-x) = -f(x)$; the graph of an odd function is thus asymmetrical about the origin. e.g. $f(x) = x^3$; $f(x) = \sin x$

PRODUCTS OF ODD AND EVEN FUNCTIONS

(even) × (even) = even

(odd) × (odd) = even

(neither) × (odd) = neither

(neither) × (even) = neither

Theorem 1: If $f(x)$ is defined over the internal $-\pi < x < \pi$ and $f(x)$ is even, then the Fourier series for $f(x)$ contains cosine terms only. Here a_0 is included.

Ex:

$$f(x)=0 \quad -\pi< x< -\frac{\pi}{2}$$
$$f(x)=4 \quad -\frac{\pi}{2}< x< \frac{\pi}{2}$$
$$f(x)=0 \quad \frac{\pi}{2}< x< \pi$$
$$f(x)=f(x+2\pi)$$

The waveform is symmetrical about the y-axis, therefore, it is even.

$$\Rightarrow f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

(a) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}4dx=\frac{2}{\pi}[4x]_0^{\frac{\pi}{2}}=4$

(b) $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}4\cos nxdx=\frac{8}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\frac{\pi}{2}}=\frac{8}{n\pi}\sin\frac{n\pi}{2}$

But $\sin\frac{n\pi}{2}=\begin{cases}0 \text{ for } n \text{ even}\\ 1 \text{ for } n=1, 5, 9...\\ -1 \text{ for } n=3, 7, 11.....\end{cases}$

$$f(x)=2+\frac{8}{\pi}\left[\cos x-\frac{1}{3}\cos x+\frac{1}{5}\cos 5x-\frac{1}{7}\cos 7x+......\right]$$

Theorem 2: If $f(x)$ is an odd function defined over the interval $-\pi< x< \pi$, then the Fourier series for $f(x)$ contains sine terms only. Here $a_0=a_n=0$.

Ex:

$$f(x)=-6 \quad -\pi< x< 0$$
$$f(x)=6 \quad 0< x< \pi$$
$$f(x)=f(x+2\pi)$$

This is an odd function so $f(x)$ contains only the sine terms

i.e. $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$

and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$

$f(x)\sin nx$ is even since it is a product of two odd functions.

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nxdx=\frac{2}{\pi}\int_0^{\pi}6\sin nxdx=\frac{12}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}=\frac{12}{\pi n}(1-\cos n\pi)$$

$$f(x)=\frac{24}{\pi}\left\{\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+......\right\}$$

If $f(x)$ is neither even nor odd we must obtain expressions for a_0, a_n and b_n in full

Ex Determine the Fourier series of the function shown.

$$f(x) = \frac{2x}{\pi} \quad 0 < x < \pi$$
$$f(x) = 2 \quad \pi < x < 2\pi$$
$$f(x) = f(x + 2\pi)$$

This is neither odd nor even,

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

(a) $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\{\int_0^{\pi} \frac{2}{\pi} xdx + \int_{\pi}^{2\pi} 2dx\}$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{\pi}\right]_0^{\pi} + [2x]_{\pi}^{2\pi}\right\} = \frac{1}{\pi}\{\pi + 4\pi - 2\pi\} = 3$$

$$\Rightarrow a_0 = 3$$

(b) $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx$

$$= \frac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2x}{\pi}\right)\cos nxdx + \int_{\pi}^{2\pi} 2\cos nxdx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{\pi n}\int_0^{\pi}\sin nxdx + \int_{\pi}^{2\pi}\cos nxdx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi n}(\pi \sin n\pi x) + \frac{1}{\pi n}\left[\frac{\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\sin n\pi x + \frac{1}{\pi n^2}(\cos nx - 1) + \frac{1}{n}(\sin 2\pi nx - \sin \pi x)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi n^2}(\cos \pi nx - 1) + \frac{1}{n}\sin 2n\pi x\right\}$$

$a_n = 0$ (n even); $a_n = \frac{-4}{\pi^2 n^2}$ (n odd)

(c) $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx$

$$= \frac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2x}{\pi}\right)\sin nxdx + \int_{\pi}^{2\pi} 2\sin nxdx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{x\cos nx}{n}\right]_0^{\pi} - \frac{1}{\pi n}\int_0^{\pi}\cos nxdx + \int_{\pi}^{2\pi}\sin nxdx\right\}$$

$$=\frac{2}{\pi}\left\{\frac{1}{\pi n}\left(-\pi\cos n\pi x\right)+\frac{1}{\pi n}\left[\frac{\sin nx}{n}\right]_0^{\pi}+\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{2}{\pi}\left\{-\frac{1}{n}\cos n\pi x+\left(0-0\right)-\frac{1}{n}\left(\cos 2\pi nx-\cos n\pi x\right)\right\}$$

$$=\frac{2}{\pi}\left\{-\frac{1}{n}\cos 2n\pi x\right\}=-\frac{2}{\pi n}\cos 2n\pi x$$

But $\cos 2n\pi=1 \Rightarrow b_n=-\frac{2}{\pi n}$

$$f\left(x\right)=\frac{3}{2}-\frac{4}{\pi^2}\left\{\cos x+\frac{1}{9}\cos 3x+\frac{1}{25}\cos 5x+\ldots\ldots\right\}-\frac{2}{\pi}\left\{\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\frac{1}{4}\sin 4x\ldots\ldots\right\}$$

HALF-RANGE SERIES

Sometimes a function of period 2π is defined over the range 0 to π instead of the normal $-\pi$ *ot* π, *or* 0 *to* 2π. In this case one can choose to obtain a half range cosine series by assuming that the function is part of an even function or a sine series by assuming that the function is part of an odd function.

Ex:

$$f\left(x\right)=2x \quad 0<x<\pi$$

$$f\left(x\right)=f\left(x+2\pi\right)$$

To obtain a half-range cosine series we assume an even function

$$a_0=\frac{2}{\pi}\int_0^{\pi}f\left(x\right)dx=\frac{2}{\pi}\int_0^{\pi}2x\,dx=\frac{2}{\pi}\left[x^2\right]_0^{\pi}=2\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}2x\cos nx dx=\frac{4}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi}-\frac{1}{n}\int_0^{\pi}\sin nx dx\right\}$$

Simplifying, $a_0=0$ for n even and $a_n=\frac{-8}{\pi n^2}$ for n odd. In this $b_0=0$ and so

$$f\left(x\right)=\pi-\frac{8}{\pi}\left\{\cos x+\frac{1}{9}\cos 3x+\frac{1}{25}\cos 5x+\ldots\ldots\right\}$$

Obtain a half-range sine series for $f\left(x\right)$.

FUNCTIONS WITH ARBITRARY PERIOD *T*

$f\left(t\right)=f\left(t+T\right)$, frequency $f=\frac{1}{T}$ and angular frequency $\omega=2\pi f$

$\Rightarrow \omega=\frac{2\pi}{T}$ *and* $T=\frac{2\pi}{\omega}$. *The angle* $x=\omega t$ and the Fourier series is

$$f\left(t\right)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left\{a_n\cos n\omega t+b_n\sin n\omega t\right\}$$

$$=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n\cos\frac{2\pi nt}{T}+b_n\sin\frac{2\pi nt}{T}\}$$

where $a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)dt$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\cos n\omega t dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\sin n\omega t dt$$

Ex: Determine the Fourier series for a periodic function defined by

$$f(t) = 2(1+t) \quad -1 < t < 0$$

$$f(t) = 0 \qquad 0 < t < 1$$

$$f(t) = f(t+2) \quad 0 < t < 1$$

Answer: $f(t) = \frac{1}{2} + \frac{4}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t.....\right\}$

$$-\frac{2}{\omega}\left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{4}\sin 4\omega t\right\}$$

SUM OF A FOURIER SERIES AT A POINT OF FINITE DISCONTINUITY

At $x = x_1$ the series converges to the value $f(x_1)$ as the number of terms included increases to infinity. But if there is a "jump" at x_1 $f(x_1 - 0) = y_1$ (approaching x_1 from below) $f(x_1 + 0) = y_2$ (approaching x_1 from above).

If we sustitute $x = x_1$ in the Fourier series for $f(x)$, it can be shown that the series converges to the value $\frac{1}{2}\{f(x_1 - 0) + f(x_1 + 0)\}$ *i.e.* $\frac{1}{2}(y_1 + y_2)$, the average of y_1 *and* y_2.

FOURIER INTEGRALS

THE FOURIER INTEGRAL

While Fourier series is for periodic functions Fourier integral is for non-periodic function. If a non-periodic $f(x)$ (i) satisfies the Dirichlet conditions in every finite interval $(-a, a)$ and (ii) is absolutely integrable in $(-\infty, \infty)$, i.e. $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, then $f(x)$ can be represented by a Fourier's integral as follows:

$$f(x) = \int_0^{\infty} \{A(k)\cos kx + B(k)\sin kx\}dx \qquad ...(1)$$

where $A(k) = \int_{-\infty}^{\infty} f(x)\cos kx dx$ ...(2)

$$B(k) = \int_{-\infty}^{\infty} f(x)\sin kx dx \qquad ...(3)$$

If x is a point of discontinuity, then $f(x)$ must be replaced by $\left(\dfrac{f(x+0)+f(x-0)}{2}\right)$ as in the case of Fourier series. This can, in other words, be expressed by the following theorem.

Theorem 1: If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists, then $f(x)$ can be represented by a Fourier integral. At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point.

Ex: Find the Fourier integral representation of the function in below

$$f(x)=\begin{cases}1 & \text{if } |x|<1\\ 0 & \text{if } |x|>1\end{cases}$$

Solution: From (2) and (3) we have

$$A(k)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos kx dx=\frac{1}{\pi}\int_{-1}^{1}\cos kx dx=\left.\frac{\sin kx}{\pi k}\right|_{-1}^{1}=\frac{2\sin k}{\pi k}$$

$$B(k)=\frac{1}{\pi}\int_{-1}^{1}\sin kx dx=0$$

and (1) gives the answer

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\frac{\cos kx\sin k}{k}dx \qquad ...(4)$$

The average of the left- right-hand limits of $f(x)$ *at* $x=1$ is equal to (1+0)/2, that is, 1/2. Furthermore, from (4) and Theorem 1 we obtain

$$\int_{0}^{\infty}\frac{\cos kx\sin k}{k}=\frac{\pi}{2}f(x)=\begin{cases}\dfrac{\pi}{2} & \text{if } 0\le x\le 1\\ \dfrac{\pi}{4} & \text{if } x=1\\ 0 & \text{if } x>1\end{cases}$$

This integral is called *Dirichlet's discontinuous factor*. If $x = 0$, then $\int_{0}^{\infty}\frac{\sin kx}{k}dx=\frac{\pi}{2}$.

This integral is the limit of the so-called ***sine integral*** $Si(u)=\int_{0}^{u}\frac{\sin k}{k}dx$ as $u\to\infty$

In the case of a Fourier series the graphs of the partial sums are approximation curves of the periodic function represented by the series. Similarly, in the case of the Fourier integral, approximations are obtained by replacing ∞ by numbers a. Hence the integral $\int_{0}^{a}\frac{\cos kx\sin k}{k}dx$ approximates the integral in (4) and therefore $f(x)$.

Laplace Transform

LT is used in solving ordinary differential equations (ode). It has the following advantages:

- Solution of the ode is obtained by algebraic processes.
- The initial conditions are involved from the early stages so that the determination of the particular solution is considerably shortened.
- The method enables us to deal with situations where the function is discontinuous.

The LT of a function $f(t)$ is denoted by $L\{f(t)\}$ *or* $F(s)$ and is defined by the integral

$$\int_0^\infty f(t)e^{-st}\,dt$$

i.e.

$$L\{f(t)\} \text{ or } F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where s is a positive constant such that $f(t)e^{-st}$ converges as $t \to \infty$

Ex: **1.** To find the LT of a constant function $f(t) = a$

$$L\{a\} = \int_0^\infty ae^{-st}\,dt = a\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{a}{s}\left[e^{-st}\right]_0^\infty = -\frac{a}{s}[0-1] = \frac{a}{s}$$

$$\Rightarrow \quad L\{a\} = \frac{a}{s} \qquad ...(1)$$

e.g. for $a = 1$, $L\{1\} = \frac{1}{s}$

2. If $f(t) = e^{at}$

$$L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,dt\int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}[0-1] = \frac{1}{s-a}$$

$$\Rightarrow \quad L\{e^{at}\} = \frac{1}{s-a} \qquad ...(2)$$

Similarly $L\{e^{-at}\} = \frac{1}{s+a}$ ...(3)

3. If $f(t) = \sin at$

$$L\{\sin at\} = \int_0^\infty \sin(at)e^{-st}\,dt = \int_0^\infty \left(\frac{e^{iat} - e^{-iat}}{2i}\right)e^{-st}\,dt$$

$$= \frac{1}{2i}\left(\int_0^\infty e^{-(s-ia)t}\,dt - \int_0^\infty e^{-(s+ia)t}\,dt\right) = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a+is} + \frac{1}{a-is}\right) = \frac{1}{2}\frac{a-is+a+is}{(a+is)(a-is)} = \frac{a}{s^2+a^2}$$

$$\Rightarrow L\{\sin at\} = \frac{a}{s^2+a^2} \qquad \text{...(4)}$$

e.g. $L\{\sin 2t\} = \dfrac{2}{s^2+4}$ Similarly (Show that):

4. If $f(t) = \cos at \quad \Rightarrow L\{\cos at\} = \dfrac{s}{s^2+a^2}$ ...(5)

e.g. $L\{\cos 4t\} = \dfrac{s}{s^2+16}$

5. If $f(t) = t^n \quad \Rightarrow L\{t^n\} = \dfrac{n!}{s^{n+1}}$ ...(6)

e.g. $L\{t^3\} = \dfrac{3!}{s^{3+1}} = \dfrac{6}{s^4}$.

6. If $f(t) = \sinh at$

$$L\{\sinh at\} = \int_0^\infty \sinh(at)e^{-st}\,dt = \int_0^\infty \left(\frac{e^{at}-e^{-at}}{2}\right)e^{-st}\,dt$$

$$= \frac{1}{2}\left(\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2}$$

$$\Rightarrow L\{\sinh at\} = \frac{a}{s^2-a^2} \qquad \text{...(7)}$$

e.g. $L\{\sinh 2t\} = \dfrac{2}{s^2-4}$ Similarly (show that)

7. If $f(t) = \cosh at = \dfrac{1}{2}\left(e^{at}+e^{-at}\right) \Rightarrow L\{\cosh at\} = \dfrac{s}{s^2-a^2}$ ...(8)

e.g. $L\{4\cosh 3t\} = 4\dfrac{s}{s^2-3^2} = \dfrac{4s}{s^2-9}$

EXISTENCE THEOREM FOR LAPLACE TRANSFORMS

Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies $|f(t)| \leq Me^{-kt}$ for all $t \geq 0$ and for some constants k and M. Then the LT of $f(t)$ exists for all $s > k$. A function $f(t)$ is said to be *piecewise continuous* in an interval (a, b) if

(i) the interval can be divided into a finite number of subintervals in each of which $f(t)$ is continuous.

(ii) the limits of $f(t)$, in other words a piecewise continuous function is one that has a finite number of finite discontinuities.

INVERSE TRANSFORM

Given a LT, $F(s)$ one can find the function $f(t)$ by inverse transform

$f(t) = L^{-1}\{F(s)\}$ where L^{-1} indicates inverse transform.

e.g.

$$L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$$

$$L^{-1}\left\{\frac{4}{s}\right\} = 4$$

$$L^{-1}\left\{\frac{s}{s^2+25}\right\} = \cos 5t$$

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\} = L^{-1}\left\{\frac{1}{s+2}+\frac{2}{s-3}\right\} \text{ (by partial fractions)}$$

$$= L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{2}{s-3}\right\}$$

(Note: L, L^{-1} are linear operators. Prove it)

$$= e^{-2t} + 2e^{3t}$$

RULES OF PARTIAL FRACTIONS

1. The numerator must be of lower degree than the denominator. If it is not then we first divide out

2. Factorize the denominator into its prime factors. These determine the shapes of the partial fractions.

3. A linear factor ($s+a$) gives a partial fraction $\frac{A}{s+a}$ where A is a constant to be determined.

4. A repeated factor $(s+a)^2$ give $\frac{A}{s+a}+\frac{B}{(s+a)^2}$

5. Similarly $(s+a)^3$ give $\frac{A}{s+a}+\frac{B}{(s+a)^2}+\frac{C}{(s+a)^3}$

6. Quadratic factor (s^2+ps+q) gives $\frac{As+B}{s^2+ps+q}$

7. repeated quadratic factor $(s^2+ps+q)^2$ give $\frac{As+B}{s^2+ps+q}+\frac{Cs+D}{(s^2+ps+q)^2}$

Ex: **1.** $$\frac{s^2-15s+41}{(s+2)(s-3)^2} = \frac{3}{s+2}-\frac{2}{s-3}+\frac{1}{(s-3)^2}$$

2. $$L^{-1}\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$$

but $$\frac{4s^2-5s+6}{(s+1)(s^2+4)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}=\frac{3}{s+1}+\frac{s-6}{s^2+4}=\frac{3}{s+1}+\frac{s}{s^2+4}-\frac{6}{s^2+4}$$

$$\Rightarrow f(t)=L^{-1}\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}=L^{-1}\left\{\frac{3}{s+1}+\frac{s}{s^2+4}-\frac{6}{s^2+4}\right\}$$

$$\Rightarrow f(t)=3e^{-t}+\cos 2t-3\sin 2t$$

PROPERTIES OF LAPLACE TRANSFORM

Linearity: $L\{af(t)+bg(t)\}=aL\{f(t)\}+bL\{g(t)\}$ (Prove!)

1. THE FIRST SHIFT THEOREM (OR S-SHIFTING)

It states that if $L\{f(t)\}=F(s)$ *then*

$$\Rightarrow L\{e^{-at}f(t)\}=F(s+a) \quad ...(9)$$

i.e. $L\{e^{-at}f(t)\}$ is the same as $L\{f(t)\}$ with s replaced by (s + a)

Ex: **1.** If $L\{\sin 2t\}=\dfrac{2}{s^2+4}$ then $L\{e^{-3t}\sin 2t\}=\dfrac{2}{(s+3)^2+4}=\dfrac{2}{s^2+6s+13}$

2. If $L\{t^2\}=\dfrac{2}{s^3}$ then $L\{t^2e^{4t}\}=\dfrac{2}{(s-4)^3}$

2. THEOREM 2: MULTIPLYING BY *T* (OR DERIVATIVE OF LT)

If $L\{f(t)\}=F(s)$ then $\Rightarrow L\{tf(t)\}=-\dfrac{d}{ds}F(s)$ ...(10)

e.g. if $L\{\sin 2t\}=\dfrac{2}{s^2+4}\Rightarrow L\{t\sin 2t\}=-\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right)=\dfrac{4s}{(s^2+4)^2}$

3. THEOREM 3: DIVIDING BY *T*

If $L\{f(t)\}=F(s)$ then $L\left\{\dfrac{f(t)}{t}\right\}=\int_s^\infty F(s)ds$ ...(11)

If limit of $\dfrac{f(t)}{t}$ as $t\to 0$ exists, we use L' Hospital's rule to find out if it does

e.g. $L\left\{\dfrac{\sin at}{t}\right\}$; here $\lim\limits_{t\to 0}\left\{\dfrac{\sin at}{t}\right\}=\dfrac{0}{0}$ (undefined).

By L'Hospital's rule, we differentiate top and bottom separately and substitute $t=0$ in the result to ascertain the limit of the new function.

$$\lim_{t\to 0}\left\{\frac{\sin at}{t}\right\}=\lim_{t\to 0}\left\{\frac{a\cos at}{1}\right\}=a,$$

i.e. the limit exists. The theorem can, therefore, be applied.

Since $L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \left[\arctan\left(\frac{s}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right) = \arctan\left(\frac{a}{s}\right)$

4. TRANSFORM OF DERIVATIVE

Let $\frac{df(t)}{dt} = f'(t)$ *and* $\frac{d^2 f(t)}{dt^2} = f''(t)$

Then $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$

Integrating by parts, $L\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\left\{-se^{-st}\right\}dt$

i.e. $L\{f'(t)\} = -f(0) + sL\{f(t)\}$

$$\Rightarrow L\{f'(t)\} = sF(s) - f(0) \quad ...(12)$$

Similarly $L\{f''(t)\} = -f'(0) + sL\{f'(t)\} = -f'(0) + s\left[-f(0) + sL\{f(t)\}\right]$

$$\Rightarrow L\{f''(t))\} = s^2 F(s) - sf(0) - f'(0) \quad ...(13)$$

Similarly $\Rightarrow L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$ ...(14)

ALTERNATIVE NOTATION

Let $x = f(t), f(0) = x_0, f'(0) = x_1, f''(0) = x_2, ..., f^{(n)}(0) = x_n$ and

$\bar{x} = L\{x\} = L\{f(t)\} = F(s)$

we now have

$$L\{x\} = \bar{x}$$

$$L\{\dot{x}\} = s\bar{x} - x_0$$

$$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$$

$$L\{\dddot{x}\} = s^3\bar{x} - s^2 x_0 - sx_1 - x_2$$

SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORM

PROCEDURE

(a) Rewrite the equation in terms of LT.
(b) Insert the given initial conditions.
(c) Rearrange the equation algebraically to give the transform of the solution.
(d) Determine the inverse transform to obtain the particular solution

SOLUTION OF FIRST ORDER DIFFERENTIAL EQUATIONS

Ex: Solve the equation $\frac{dx}{dt} - 2x = 4$, given that at $t = 0$, $x = 1$.

We go through the four stages as follows:

(a) $L\{x\} = \bar{x},\ L\{\dot{x}\} = s\bar{x} - x_0,\ L\{4\} = \frac{4}{s}$

Then the equation becomes $\left(s\bar{x} - x_0\right) - 2\bar{x} = \frac{4}{s}$

(b) Insert the initial condition that at $t = 0,\ x = 1,\ i.e.,\ x_0 = 1 \quad \Rightarrow s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$

(c) Now we rearrange this to give an expression for $\bar{x}$: i.e. $\bar{x} = \frac{s+4}{s(s-2)}$

(d) Finally, we take inverse transform to obtain x:

$$\frac{s+4}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \quad \Rightarrow s + 4 = A(s-2) + Bs$$

(i) Put $(s-2) = 0,\ i.e.,\ s = 2 \Rightarrow 6 = 2B$ or $B = 3$

(ii) Put $s = 0 \quad \Rightarrow \quad s = -2A$ or $A = -2$

$$\bar{x} = \frac{s+4}{s(s-2)} = \frac{3}{s-2} - \frac{2}{s} \quad \Rightarrow x = 3e^{2t} - 2$$

Solve the following equations:

1. $\frac{dx}{dt} + 2x = 10e^{3t}$, given that at $t = 0,\ x = 6$

2. $\frac{dx}{dt} - 4x = 2e^{2t} + e^{4t}$, given that at $t = 0,\ x = 6$

Ex: Solve the equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t}$, given that at $t = 0,\ x = 5$ and $\frac{dx}{dt} = 7$

$L\{x\} = \bar{x}$

(a) $L\{\dot{x}\} = s\bar{x} - x_0$

$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$

The equation becomes $(s^2\bar{x} - sx_0 - x_1) - 3\,(s\bar{x} - x_0) + 2\bar{x} = \frac{2}{s-3}$

(b) Insert the initial conditions. In this case $x_0 = 5$ and $x_1 = 7$

$$(s^2\bar{x} - 5s - 7) - 3\,(s\bar{x} - 5) + 2\bar{x} = \frac{2}{s-3}$$

(c) Rearrange to obtain $\bar{x}$ *as* $\bar{x} = \frac{5s^2 - 23s + 26}{(s-1)\,(s-2)\,(s-3)}$

(d) Now for partial fractions $\frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

$\Rightarrow\ 5s^2 - 23s + 26 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$

$\Rightarrow\ A = 4,\ B = 0,\ C = 1$

$$\bar{x} = \frac{4}{s-1} + \frac{1}{s-3}$$

$\Rightarrow\ x = 4e^{t} + e^{3t}$

Solve $\frac{d^2x}{dt^2} - 4x = 24\cos 2t$, given that at $t = 0$, $x = 3$ and $\frac{dx}{dt} = 4$

SOLUTION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

Ex: Solve the pair of simultaneous equations

$$\dot{y} - x = e^t$$

$$\dot{x} + y = e^{-t}$$

given that at $t = 0$, $x = 0$ and $y = 0$

(a) $\left(s\bar{y} - y_0\right) - \bar{x} = \frac{1}{s-1}$; $\left(s\bar{x} - x_0\right) + \bar{y} = \frac{1}{s+1}$

(b) Insert the initial conditions $x_0 = 0$ and $y_0 = 0$

$$s\bar{y} - \bar{x} = \frac{1}{s-1} ; \qquad s\bar{x} + \bar{y} = \frac{1}{s+1}$$

(c) Eliminating = $\bar{y}$ we have

$$s\bar{y} - \bar{x} = \frac{1}{s-1} ; \qquad s\bar{y} + s^2\bar{x} = \frac{s}{s+1}$$

$$\Rightarrow \bar{x} = \frac{s^2 - 2s - 1}{(s-1)(s+1)\left(s^2+1\right)} = -\frac{1}{2}.\frac{1}{s-1} - \frac{1}{2}.\frac{1}{s+1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

(d) $x = \frac{1}{2}e^t - \frac{1}{2}e^{-t} + \cos t + \sin t = \sin t + \cos t - \cosh t$

Eliminating $\bar{x}$ in (b) we have

$$\Rightarrow \bar{y} = \frac{s^2 + 2s - 1}{(s-1)(s+1)\left(s^2+1\right)} = \frac{1}{2}.\frac{1}{s-1} + \frac{1}{2}.\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\Rightarrow y = \frac{1}{2}e^t + \frac{1}{2}e^{-t} - \cos t + \sin t = \sin t - \cos t + \cosh t$$

So the results are:

$$x = \sin t + \cos t - \cosh t$$

$$y = \sin t - \cos t + \cosh t$$

THE DIRAC DELTA FUNCTION (THE IMPULSE FUNCTION)

It represents an extremely large force acting for a minutely small interval of time. Consider a single rectangular pulse of width b and height $\frac{1}{b}$ ocurring at $t = a$. If we reduce the width of the pulse to $\frac{1}{b}$ and keep the area of the pulse constant (1 unit) the height of the pulse will be $\frac{2}{b}$. If we continue reducing the width of the pulse while maintaining an area of unity, then as $b \to 0$, the height $\frac{1}{b} \to \infty$ and we have the Dirac delta function. It is denoted by $\delta\left(t - a\right)$.

Graphically it is represented by a rectangular pulse of zero width and infinite height.

If the Dirac delta function is at the origin, a = 0 and so it is denoted by $\delta(t)$

INTEGRATION INVOLVING THE IMPULSE FUNCTION

From the definition of $\delta(t-a)$.

$$\int_p^q \delta(t-a)dt = 1 \text{ for } \begin{cases} (i) \;\; p < t < a, \;\; \delta(t-a) = 0 \\ (ii) \;\; t = a \;\; \text{area of pulse} = 1 \\ (iii) \;\; a < t < q, \;\; \delta(t-a) = 0 \end{cases}$$

Now consider $\int_p^q f(t)\delta(t-a)dt$ since $f(t)\,\delta(t-a)$ is zero for all values of t within the interval $[p, q]$ except at the point $t = a$, $f(t)$ may be regarded as a constant $f(a)$ so that

$$\int_p^q f(t)\delta(t-a)dt = f(a)\int_p^q \delta(t-a)dt = f(a)$$

Ex: Evaluate $\int_1^3 \left(t^2+4\right).\delta\left(t-2\right)dt$. *Here* $a = 2$ $f(t) = t^2 + 4 \Rightarrow f(a) = f(2) = 2^2 + 4 = 8$

Evaluate

1. $\int_0^6 5.\delta\left(t-3\right)dt$

2. $\int_2^5 e^{-2t}.\delta\left(t-4\right)dt$

LAPLACE TRANSFORM OF $\delta(t-a)$

Recall that $\int_p^q f(t).\delta(t-a)dt = f(a), \quad p < a < q$

$\Rightarrow$ If $p = 0$ and $q = \infty$ then $\int_0^\infty f(t).\delta(t-a)dt = f(a)$

Hence, if $f(t) = e^{-st}$, this becomes $\int_0^\infty e^{-st}.\delta(t-a)dt = L\{\delta\;(t-a)\} = e^{-as}$

Similarly $L\{f(t).\delta(t-a)\} = \int_0^\infty e^{-st}.f(t).\delta\;(t-a)\;dt = f(a)e^{-as}$

DIFFERENTIAL EQUATIONS INVOLVING THE IMPULSE FUNCTION

Ex: Solve the equation $\ddot{x} + 4\dot{x} + 13x = 2\delta(t)$ where, at $t = 0$, $x = 2$ and $\dot{x} = 0$

(a) Expressing in LT, we have $\left(s^2\bar{x} - sx_0 - x_1\right) + 4\left(s\bar{x} - x_0\right) + 13\bar{x} = 2\times 1$

(b) Inserting the initial conditions and simplifying we have $\bar{x} = \dfrac{2s+10}{s^2+4s+13}$

(c) Rearranging the denominator by completing the square, this can be written as

$$\bar{x} = \frac{2(s+2)}{(s+2)^2+9} + \frac{6}{\left(s+2\right)^2+9}$$

(d) The inverse LT is $x = 2e^{-2t}\cos 3t + 2e^{-2t}\sin 3t = 2e^{-2t}\left(\cos 3t + \sin 3t\right)$

Special Functions

THE GAMMA AND BETA FUNCTIONS

THE GAMMA FUNCTION Γ

The gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad ...(1)$$

and is convergent for $x > 0$. It follows from equation (1) that

$$\Gamma(x) = \int_0^\infty t^{x} e^{-t}\, dt$$

Integrating by parts $\qquad \Gamma(x+1) = \left[t^x \left(\frac{e^{-t}}{-1} \right) \right]_0^\infty + x\int_0^\infty t^{x-1} e^{-t}\, dt$

$$\Gamma(x+1) = x\Gamma(x) \qquad ...(2)$$

This is a fundamental recurrence relation for gamma functions. It can also be written as $\Gamma(x) = (x-1)\Gamma(x-1)$.

A number of other results can be derived from this as follows:

If $x = n$, a positive integer, i.e. if $n \geq 1$, then

$$\Gamma(n+1) = n\Gamma(n).$$

$$= n(n-1)\Gamma(n-1) \text{ since } \Gamma(n) = (n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2) \text{ since } \Gamma(n-1) = (n-2)\Gamma(n-2)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= n(n-1)(n-2)\Gamma(n-3)\ldots\ldots\Gamma(1)$$

$$= n!\,\Gamma(1)$$

But $\qquad \Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = \left[-e^{-t} \right]_0^\infty = 1 \ldots\ldots (3)$

$$\Rightarrow \Gamma(n+1) = n!$$

Ex: $\Gamma(7) = 6! = 720,\ \Gamma(8) = 7! = 5040,\ \Gamma(9) = 40320$

We can also use the recurrence relation in reverse $\Gamma(x+1) = x\Gamma(x) \Rightarrow \Gamma(x) = \frac{\Gamma(x+1)}{x}$

Ex: If $x = \frac{1}{2}$ it can be shown that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Using the recurrence relation $\Gamma(x+1) = x\Gamma(x)$ we can obtain the following:

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\left(\sqrt{\pi}\right) \Rightarrow \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right) \Rightarrow \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

NEGATIVE VALUES OF X

Since $\Gamma(x)=\dfrac{\Gamma(x+1)}{x}$,then as $x\to 0$, $\Gamma(x)\to\infty \Rightarrow \Gamma(0)=\infty$

The same result occurs for all negative integral values of x

Ex: At $x=-1$, $\Gamma(-1)=\dfrac{\Gamma(0)}{-1}=\infty$

At $x=-2$, $\Gamma(-2)=\dfrac{\Gamma(-1)}{-2}=\infty$ etc.

Also at $x=\dfrac{-1}{2}$, $\Gamma\left(-\dfrac{1}{2}\right)=\dfrac{\Gamma\left(\dfrac{1}{2}\right)}{-\dfrac{1}{2}}=-2\sqrt{\pi}$ and at $x=-\dfrac{3}{2}$, $\Gamma\left(-\dfrac{3}{2}\right)=\dfrac{\Gamma\left(-\dfrac{1}{2}\right)}{-\dfrac{3}{2}}=\dfrac{4}{3}\sqrt{\pi}$

Ex: 1. Evaluate $\int_0^\infty x^7 e^{-x}dx$

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$$

Let $\Gamma(v)=\int_0^\infty x^{v-1}e^{-x}dx \Rightarrow v=8$

i.e., $\int_0^\infty x^7 e^{-x}dx=\Gamma(8)=7!=5040$

2. Evaluate $\int_0^\infty x^3 e^{-4x}dx$

Since $\Gamma(v)=\int_0^\infty x^{v-1}e^{-x}dx$ we use the substitution $y=4x \Rightarrow dy=4dx$

$$\Rightarrow I=\frac{1}{4^4}\int_0^\infty y^3 e^{-y}dy=\frac{1}{4^4}\Gamma(v) \text{ where } v=4 \Rightarrow I=\frac{3}{128}$$

3. Evaluate $\int_0^\infty x^{\frac{1}{2}} e^{-x^2}dx$

Use $y=x^2$ therefore $dy=2xdx$. Limits $x=0,\ y=0\ x=\infty,\ y=\infty$

$$x=y^{\frac{1}{2}} \Rightarrow x^{\frac{1}{2}}=y^{\frac{1}{4}}$$

$$I=\int_0^\infty \frac{y^{\frac{1}{4}}e^{-y}}{2x}dy=\int_0^\infty \frac{y^{\frac{1}{4}}e^{-y}}{2y^{\frac{1}{2}}}dy=\frac{1}{2}\int_0^\infty y^{-\frac{1}{4}}e^{-y}dy=\frac{1}{2}\int_0^\infty y^{v-1}e^{-y}dy$$

where, $v=\dfrac{3}{4} \Rightarrow I=\dfrac{1}{2}\Gamma\left(\dfrac{3}{4}\right)$

From tables, $\Gamma(0.75)=1.2254 \Rightarrow I=0.163$

THE BETA FUNCTION, β

The beta function $\beta(m,n)$ is defined by $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

It can be shown that the beta function and the gamma function are related

as $$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\frac{(m-1)!(n-1)!}{(m+n-1)!}$$

APPLICATION OF GAMMA AND BETA FUNCTIONS

Ex: 1. Evaluate $I=\int_0^1 x^5(1-x)^4\,dx$

Comparing this with $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

then $m-1=5 \Rightarrow m=6$ and $n-1=4 \Rightarrow n=5$

$$I=\beta(6,5)=\frac{5!4!}{10!}=\frac{1}{1260}$$

2.Evaluate $I=\int_0^1 x^4\sqrt{1-x^2}\,dx$

Comparing this with $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

we see that we have x^2 in the root, instead of a single x. Therefore, put

$$x^2=y \Rightarrow x=y^{\frac{1}{2}} \; and \; dx=\frac{1}{2}y^{-\frac{1}{2}}dy$$

The limits remain unchanged.

$$I=\int_0^1 y^2(1-y)^{\frac{1}{2}}\frac{1}{2}y^{-\frac{1}{2}}dy=\frac{1}{2}\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}}dy$$

$$m-1=\frac{3}{2} \Rightarrow m=\frac{5}{2} \text{ and } n-1=\frac{1}{2} \Rightarrow n=\frac{3}{2}$$

Therefore, $$I=\frac{1}{2}\beta\left(\frac{5}{2},\frac{3}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)}$$

$$=\frac{1}{2}\frac{\left(\frac{3}{4}\sqrt{\pi}\right)\left(\frac{1}{2}\sqrt{\pi}\right)}{3!}=\frac{\pi}{32}$$

3. Evaluate $I=\int_0^3 \frac{x^3\,dx}{\sqrt{3-x}}$

BESSEL'S FUNCTIONS

Bessel's functions are solutions of the Bessel's differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-v^2\right)y=0 \qquad ...(1)$$

where v is a real constant.

By the Frobenius method we assume a series solution of the form

$$y = x^c\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_rx^r + \ldots\right) \text{ or } y = x^c\sum_{r=0}^{\infty} a_rx^r$$

i.e. $$y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \ldots\ldots + a_rx^{c+r} + \ldots\ldots \text{ or } y = \sum_{r=0}^{\infty} a_rx^{c+r} \qquad \ldots(2)$$

where $c, a_0, a_1, a_2, \ldots\ldots, a_r$ are constants. a_0 is the first non-zero coefficient. c is called the indicial constant.

$$\frac{dy}{dx} = a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \ldots\ldots + a_r(c+r)x^{c+r-1} + \qquad \ldots(3)$$

$$\frac{d^2y}{dx^2} = a_0c(c-1)x^{c-2} + a_1c(c+1)x^{c-1} + a_2(c+1)(c+2)x^c + \ldots\ldots + a_r(c+r-1)(c+r)x^{c+r-2} + \ldots \qquad (4)$$

Substituting eqs.(2),(3) and (4) into (1) and equating coefficients of equal powers of x, we have $c = \pm v \text{ and } a_1 = 0.$

The recurrence relation is

$$a_r = \frac{a_{r-2}}{v^2 - (c+r)^2} \quad for\ r \geq 2.$$

It follows that $$a_1 = a_3 = a_5 = a_7 = \ldots\ldots = 0$$
so that when $c = +v$

$$a_2 = \frac{-a_0}{2^2(v+1)}$$

$$a_4 = \frac{a_0}{2^4 \times 2!(v+1)(v+2)}$$

$$a_6 = \frac{-a_0}{2^6 \times 3!(v+1)(v+2)(v+3)}$$

$$a_r = \frac{(-1)^{\frac{r}{2}} a_0}{2^r \times \frac{r}{2}!(v+1)(v+2)\ldots\ldots\left(v + \frac{r}{2}\right)}$$

for r even. The resulting solution is

$$y_1 = a_0x^v\left\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!(v+1)(v+2)} - \frac{x^6}{2^6 \times 3!(v+1)(v+2)(v+3)} + \ldots\right\}$$

This is valid provided v is not a negative integer. Similarly, when $c = .v$

$$y_2 = a_0x^{-v}\left\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!(v-1)(v-2)} + \frac{x^6}{2^6 \times 3!(v-1)(v-2)(v-3)} + \ldots\right\}$$

This is valid provided v is not a positive integer.

The complete solution is

$$y = Ay_1 + By_2$$

with the two arbitrary constants A and B.

SOLUTIONS OF BESSSEL'S FUNCTIONS

Let $a_0 = \dfrac{1}{2^v \Gamma(v+1)}$ then the solution y_1 gives for $c = v = n$ (where n is a positive integer) Bessel's functions of the first kind of order n denoted by $J_n(x)$ where

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)} - \frac{x^2}{2^2(1!)\Gamma(n+2)} + \frac{x^4}{2^4(2!)\Gamma(n+3)} - \ldots\ldots. \right\}$$

$$= \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(n+k+1)}$$

$$= \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(n+k)!}$$

Similarly for $c = .v = .n$ (a negative integer)

$$J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \left\{ \frac{1}{\Gamma(1-n)} - \frac{x^2}{2^2(1!)\Gamma(2-n)} + \frac{x^4}{2^4(2!)\Gamma(3-n)} - \ldots\ldots. \right\}$$

$$= \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-n+1)}$$

$$= (-1)^n \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)(n+k)!}$$

$$= (-1)^n J_n(x)$$

$\Rightarrow$ The two solutions $J_n(x)$ and $J_{-n}(x)$ dependent on each other. Further more the series for $J_n(x)$ is

$$J_n(x) = \left(\frac{x}{2}\right)^{-n} \left\{ \frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \ldots\ldots. \right\}$$

From this we obtain two commonly used functions

$$J_1(x) = \frac{x}{2}\left\{ 1 - \frac{1}{(1!)(2!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^4 + \ldots\ldots. \right\}$$

Remark: Note that $J_0(x)$ and $J_1(x)$ are similar to cos x and sin x respectively.

The generating function for $J_n(x)$ is $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n$

RECURRENCE FORMULA

$J_n(x)$ can also be obtained from the recurrence formula $\rightarrow J_{n+1}(x) = \frac{2n}{x}\left[J_n(x) - J_{n-1}(x)\right]$

For $(0 < x < 1)$ $J_n(x)$ are orthogonal.

LEGENDRE'S POLYNOMIALS

These are solutions of the Legendre's differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + k\left(k+1\right)y = 0$$

where k is a real constant. Solving it by the Frobenius method as before we obtain $c = 0$ and $c = 1$ and the corresponding solutions are

(a) $c = 1 : y = a_0\left\{1 - \frac{k\left(k+1\right)}{2!}x^2 + \frac{k\left(k-2\right)\left(k+1\right)\left(k+3\right)}{4!}x^4 - \ldots\ldots\right\}$

(b) $c = 0 : y = a_1\left\{x - \frac{\left(k-1\right)\left(k-2\right)}{3!}x^3 + \frac{\left(k-1\right)\left(k-3\right)\left(k+2\right)\left(k+4\right)}{5!}x^5 - \ldots\ldots\right\}$

where a_0 and a_1 are the usual arbitrary constants. When k is an integer n, one of the solution series terminates after a finite number of terms. The resulting polynomial in x denoted by $P_n(x)$ is called Legendre polynomial with a_0 and a_1 being chosen so that the polynomial has unit value when $x = 1$.

$(-1 < x < 1)$ orthogonality

e.g. $P_0(x) = a_0\{1 - 0 + 0 - \ldots\} = a_0$.

We choose $a_0 = 1$ so that $P_0(x) = 1$

$$P_1(x) = a_1\{x - 0 + 0 - \ldots\} = a_1 x$$

a_1 is then chosen to make $P_1(x) = 1$ when $x = 1 \Rightarrow a_1 = 1 \Rightarrow P_1(x) = x$

$$p_2(x) = a_0\left\{1 - \frac{2x^3}{2!}x^2 + 0 + 0 + \ldots\right\} = a_0\{1 - 3x^2\}$$

If $P_2(x) = 1$ when $x = 1$ then $a_0 = \frac{-1}{2} \Rightarrow P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$

Using the same procedure obtain:

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) \text{ etc.}$$

Legendre polynomials can also be expressed by Rodrigue's formula given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

(Use this formula to obtain $P_0(x),\ P_1(x),\ P_2(x),\ P_3(x),\ etc$)

The generating function is $\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$

To show this, start from the binomial expansion of $\dfrac{1}{\sqrt{1-v}}$ where $v = 2xt - t^2$, multiply the powers of $2xt - t^2$ out, collect all the terms involving t^n and verify that the sum of these terms is $P_n(x)t^n$.

The recurrence formula for Legendre polynomials is

$$P_{n+1}(x) = \frac{2n+1}{n+1} xP_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

This means that if we know $P_{n-1}(x)$ and $P_n(x)$ we can calculate $P_{n+1}(x)$.

e.g. given that $P_0(x) = 1$ *and* $P_1(x) = x$ we can calculate $P_2(x)$ using the recurrence formula by taking $P_{n-1} = P_0,\ P_n = P_1$ *and* $P_{n+1} = P_2 \Rightarrow n = 1.$

Substituting these in the formula,

$$P_2(x) = \frac{2\times 1+1}{1+1} xP_1(x) - \frac{1}{1+1} P_0(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

Similarly to find $P_3(x)$ we set $P_{n-1} = P_1,\ P_n = P_2$ *and* $P_{n+1} = P_3$ *where* n = 2. Substituting these in the formula we have

$$P_3(x) = \frac{2\times 2+1}{2+1} xP_2(x) - \frac{2}{2+1} P_1(x)$$

$$= \frac{5}{3} x \times \frac{1}{2}\left(3x^2 - 1\right) - \frac{2}{3} x$$

$$= \frac{1}{2}\left(5x^3 - 3x\right)$$

(Using the recurrence formula obtain $P_4(x)$ and $P_5(x)$)

HERMITE POLYNOMIALS

They are solutions of the Hermite differential equation

$$\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2vy = 0 \qquad ...(1)$$

where v is a parameter. Using the Frobenius method the solution is

$$y = \sum_{r=0}^{\infty} a_r x^{c+r}, \text{ where } a_0 \neq 0$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (c+r)x^{c+r-1} \text{ and}$$

$$\frac{d^2 y}{dx^2} = \sum_{r=0}^{\infty} a_r (c+r)(c+r-1)x^{c+r-2}$$

Substituting these in eq.(1) and equating coefficients of like terms we have

$a_0 c(c-1) = 0 \Rightarrow c = 0,\ or\ c = 1$ and c = 0

$$y_1 = a_0\left\{1 - \frac{2v}{2!}x^2 + \frac{2^2 v(v-2)}{4!}x^4 - \ldots + \frac{(-1)^r 2^r v(v-2)\ldots(v-2r+2)}{2!}x^{2r} + \ldots\right\}$$

(where $a_1 = 0$). When $c = 1$.

$$y_2 = a_0 x\left\{1 - \frac{2(v-1)}{2!}x^2 + \frac{2^2 (v-1)(v-3)}{4!}x^4 - \ldots + \frac{(-1)^r 2^r (v-1)(v-3)\ldots(v-2r+1)}{(2r+1)!} + \ldots\right\}$$

The complete solution of eq.(1) is then given by $y = Ay_1 + By_2$ i.e.where A and B are arbitrary constants. When $v = n$, an integer, the series terminates after a few terms. The resulting polynomials $H_n(x)$ are called Hermite polynomials. The first 5 of them are:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 1$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

They can also be given by a corresponding Rodrigue□fs formula

$$H_n(x) = e^{x^2}(-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

The generating function is given by

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

This can be proved using the formula for the coefficients of a Maclaurin series and noting that

$$tx - \frac{1}{2}t^2 = \frac{1}{2}x^2 - \frac{1}{2}(x-t)^2$$

Hermite polynomials satisfy the recursion formula

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

(Given that $H_0 = 1$ and $H_1 = 2x$ use this formula to obtain H_2, H_3, H_4 and H_5).

LAGUERRE POLYNOMIALS

They are solutions of the Laguerre differential equation

$$x\frac{d^2 y}{dx^2} + (1-x)\frac{dy}{dx} + vy = 0 \qquad ...(1)$$

Using the Frobenius method again we have

$$y=\sum_{r=0}^{\infty} a_r x^{c+r}, \text{ where } a_0 \neq 0$$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r (c+r)x^{c+r-1} \text{ and } \frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r (c+r)(c+r-1)x^{c+r-2}$$

Substituting these in eq.(*1) and equating coefficients of like terms we have $c^2=0 \Rightarrow c=0$ and

$$a_{r+1}=\frac{c+r-v}{(c+r+1)^2}a_r=\frac{r-v}{(r+1)^2}a_r$$

$$y_1=a_0\left\{1-vx+\frac{v(v-1)}{(2!)^2}x^2-......+\frac{(-1)^r v(v-1).....(v-r+1)}{(r!)^2}x^r+......\right\} \quad ...(2)$$

In case $v=n$ (a positive integer) and $a_0=n!$ the solution eq.(2) is said tobe the Laguerre polynomial of degree n and is denoted by $L_n(x)$ i.e.

$$L_n(x)=(-1)^n\left\{x^n-\frac{n^2}{1!}x^{n-1}+\frac{n^2(n-1)^2}{2!}x^{n-2}+.....+(-1)^n n!\right\} \quad ...(3)$$

Then the solution of Laguerre equation for v to be a positive integer is $y=AL_n(x)$

From eq.(*3) it is easy to show that

$$L_0(x)=1$$

$$L_1(x)=1-x$$

$$L_2(x)=x^2-4x+2$$

$$L_3(x)=-x^3+9x^2-18x+6$$

$$L_4(x)=x^4-16x^3+72x^2-96x+48$$

They can also be given by the Rodrigue's formula

$$L_n(x)=e^x\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

Their generating function is

$$e^{-\frac{xt}{(1-t)}}=\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}t^n$$

They satisfy the recursion formula

$$L_{n+1}=(2n+1-x)L_n(x)-n^2L_{n-1}(x)$$

They are orthogonal for $0< x< \infty$

Probability

1. EVENTS

Two or more events are said to be **Mutually Exclusive** if the occurrence of one prevents the occurrence of the others. In other words they cannot occur together.

Two or more events are said to be **Independent** if happening of one does not affect other events.

These events have the following differences.

(a) Independent events are always taken from different experiments, while mutually exclusive events are from only one experiment.

(b) Independent events can happen together but in mutually exclusive events one event may happen at one time.

(c) Independent events are represented by the word "and" but mutually exclusive events are represented by the word "or".

2. MATHEMATICAL DEFINITION OF PROBABILITY

Let there are exhaustive, mutually exclusive and equally likely cases for an event A and m of those are favorable to it, then probability of happening of the event A is defined by the ratio m/n which is denoted by $P(A)$.

Thus
$$P(A) = \frac{m}{n} = \frac{\text{No. of favourable cases to } A}{\text{No. of exhaustive cases to } A}$$

It is obvious that $0 \le m \le n.$ If an event A is certain to happen, then $m = n$. Thus, if A is impossible to happen then $m = 0$ and so $P(A) = 0.$ If $\overline{A}$ denotes negative of, *i.,e.*, event that A doesn't happen, then for above cases we shall have

$$P(\overline{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

3. ODDS FOR AN EVENT

If an event A happens in number of cases out of total number of cases then

Odds in favour of A
$$\frac{P(A)}{P(\overline{A})} = \frac{m/n}{(n-m)/n} = \frac{m}{n-m}$$

Odds in against of A
$$\frac{P(\overline{A})}{P(A)} = \frac{(n-m)/n}{m/n} = \frac{n-m}{m}$$

4. JOINT PROBABILITY

Given two events A and B, the compound event "A or B or both" is denoted as and the compound event "A and B" is denoted as the probability $P(A \cap B)$ is called the joint probability. If A and B are not necessarily mutually exclusive, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Equivalently

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

If A and B are mutually exclusive events then $P(A \cap B) = 0$. *i.,e.*, the probability of happening of any one of several mutually exclusive events is equal to the sum of their probabilities.

5. CONDITIONAL PROBABILITY

Let $P(A/B)$ denote the probability of A occurring given that B occurred and $P(A/B)$ denote the probability of B given A, these probabilities are defined, respectively as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

and

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

A special case of Bayes rules results by putting these two definitions then

$$P(B/A) = \frac{P(A/B)P(B)}{P(A)}$$

The event is called **Statistically independent** if the probability of occurrence of one event is not affected by the occurrence of the other event. For statistically independent event

$$P(A/B) = P(A),$$

$$P(B/A) = P(B)$$

and

$$P(A \cap B) = P(A)P(B)$$

6. BINOMIAL DISTRIBUTION FOR REPEATED TRIALS

Let an experiment is repeated n times and probability of happening of any event called success is p and not happening the event called failure is $q = p - 1$ then by binomial theorem.

$$(p+q)^n = q^n + {}^nC_1 q^{n-1} p + \ldots + nC_r q^{n-r} p^r + \ldots + p^n$$

Now probability of

(a) Occurrence of the event exactly r times $= {}^nC_r q^{n-r} p^r$

(b) Occurrence of the event at the most r times $= q^n + {}^nC_1 q^{n-1} p + \ldots + {}^nC_r q^{n-r} p^r$

Numerical Analysis

INTRODUCTION

Numerical methods are extremely powerful problem solving tools. They are capable of handling large systems of equations, non linearities, and complicated geometries which are often impossible to solve analytically.

NUMERICAL SOLUTIONS OF ALGEBRAIC EQUATIONS

BISECTION METHOD

Consider the equation $f(x) = 0$ ...(1)

Using trail and error methods, we have to find two numbers a and b such that $f(a)$ and $f(b)$ are of opposite signs.

By intermediate value theorem, a root of the equation lies in the interval (a, b).

As a first approximation to the root take

$$x_1 = \frac{a+b}{2}$$

For second approximation, compute $f(x_1)$

Case 1: If $f(x_1) < 0$ then a root of the equation lies in (x_1, b). In this case, choose $x_2 = \frac{x_1 + b}{2}$.

Case 2: If $f(x_1) = 0$, then x_1 is a root of the equation and the problem is solved

Case 3: If $f(x_1) > 0$ then the root lies in (a, x_1). Choose $x_2 = \frac{a + x_1}{2}$ similarly, we compute x_3, x_4.......

We have to continue this process till we get the required accuracy, *i.e.*, the absolute difference between two consecutive values is less than a pre assigned number $\in$.

Ex: Find a real root of the equation $f(x) = x^3 - x - 1 = 0$

Sol: Since $f(1)$ and $f(2)$ are of opposite signs, a root of the equation lies in the interval (1, 2)

We take $$x_1 = \frac{1+2}{2} = 1.5$$

then $$f(x_1) = \frac{15}{2} > 0$$

Hence, the root lies in the interval (1, 1.5) and we obtain $x_2 = \frac{1+1.5}{2} = 1.25$

Now, $$f(x_2) = f(1.25) = -\frac{19}{64} < 0$$

$\therefore$ The root lies in the interval (1.25, 1.5) take $x_3 = \frac{1.25+1.5}{2} = 1.375$

The procedure is repeated and the successive approximations are $x_3 = 1.3125, x_4 = 1.34375$, $x_5 = 1.328125$ etc.

✧ Bisection method is a simplest iterative method and convergence to the root is guaranteed.

- The Bisection method is considered as slow and steady. Rate of convergence is only one bit per iteration.
- The Bisection method cannot be applied to find the complex roots of an equation.

NEWTON - RAPHSON METHOD

Consider the equation $f(x)=0$ ...(1)

Let x_0 be the initial approximation to the root.

Let $x_1=x_0+h$ ...(2)

Substituting in (1), we have

$$f(x_0+h)=0$$

Expanding by Taylor's series, and neglecting h^2, h^3, etc

We have, $f(x_0)+h.f^1(x_0)=0$

$$h=-\frac{f(x_0)}{f^1(x_0)}$$

Substituting in (2), we have

$$x_1=x_0-\frac{f(x_0)}{f^1(x_0)}$$

Which is first approximation to the root similarly,

$$x_2=x_1-\frac{f(x_1)}{f^1(x_1)}$$

$$x_3=x_2-\frac{f(x_2)}{f^1(x_2)}$$

and so on

The iteration formula is $x_{n+1}=x_n-\dfrac{f(x_n)}{f^1(x_n)}$

- Geometrically, the method consists in replacing the part of the curve between the point $(x_0, f(x_0))$ and the x - axis by means of a tangent to the curve at the point. The point where this tangent intersect x - axis is taken as first approximation to the root.
- The method is sensitive to the initial guess x_0. If x_0 is not sufficiently close to the root, then convergence to the root is not guaranteed
- Newton - Raphson method has quadratic convergence.

  i.e. order of convergence = 2

  i.e. the error at each iteration is proportional to the square of the error at previous iteration
- Newton-Raphson method converges more rapidly than the other methods
- Since two function evaluations are required for each iteration, Newton Raphson method required more computing time.

- The method can be used for solving algebraic and transcendental equations and it can also be used when the roots are complex.
- Newton - Raphson method is useful in cases of large values of $f^1(x)$, *i.e.*, when the graph of the function $y = f(x)$ while crossing x - axis is nearly vertical.
- Newton - Raphson method is generally used to improve the result obtained by other methods

Ex: Using the Newton - Raphson method find a real root of the equation $x^3 - 2x - 5 = 0$

Sol: Let
$$f(x) = x^3 - 2x - 5 = 0$$
$$f^1(x) = 3x^2 - 2$$
$$f(0) = -5 < 0$$
$$f(1) = -6 < 0$$
$$f(2) = -1 < 0$$
$$f(3) = 16 > 0$$

Root lies in the interval (2, 3) . $f(2)$ is nearer to zero than $f(3)$

Let us choose $x_0 = 2$
$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 2 - \left(\frac{-1}{10}\right) = 2.1$$
$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)} = 2.1 - \frac{0.061}{11.23} = 2.0904568$$

Ex: Newton - Raphson iteration formula to evaluate a real root of the equation $x^3 - a = 0$ is __ .

Sol: Let
$$f(x) = x^3 - a = 0$$
$$f^1(x) = 3x^2$$

The Newton - Raphson iteration formula is
$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} = x_n - \frac{(x_n^3 - a)}{3x_n^2} \Rightarrow x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right)$$

SECANT METHOD (MODIFIED VERSION OF REGULA FALSI OR INTERPOLATION METHOD)

- Newton Raphson method is very powerful, but the evaluation of derivative involved may some times be difficult or computationally expensive.
- This suggests the idea of replacing $f^1(x_n)$ by the difference quotient.

$$f^1(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

in the N-R iteration formula
$$x_{n+1} = x_n - f(x_n)\left\{\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right\}$$

- Instead of choosing two values of x such that the function has opposite signs at these values, we choose two values nearest the root for each iteration
- This method is applicable only if one is sure that there is a root in the vicinity of x_1 the starting value.
- In this method, we approximate the graph of the function $y = f(x)$ in the neighborhood of the root by a straight line (secant) passing through the points $(x_i - 1, f_i - 1)$ and (x_i, f_i)
- The order of convergence is 1.62. Converges faster than false position method.
- No guarantee of convergence if not near root. The method fails if $f(x_i) = f(x_{i-1})$
- It may be considered the most economical method giving reasonably rapid convergence at a low cost.
- The amount of computational effort is one function evaluation
- On the average secant method is superior to Newton - Raphson method

Ex: Find first and second approximations to a real root of the equations $x^3 - 2x - 5 = 0$ by secant method between 2 and 3

Sol: Let $$f(x) = x^3 - 2x - 5 = 0$$

$a = 2$ and $b = 3$

$$f(a) = -1$$
$$f(b) = 16$$

First approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2(16) - 3(-1)}{16 - (-1)} = 2.058$$

$$f(x_1) = f(2.058) = -0.38 < 0$$

The two nearest values to the root are 2.058 and 2.

For second approximation, take $a = 2$ and $b = 2.058$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2(-0.38) - 2.058(-1)}{(-0.38) - (-1)} = 2.09 \text{ (approx)}$$

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

SINGLE AND MULTI - STEP METHODS

- A one step method is a method that, in each step uses only values obtained in a single step, viz. in the preceding step.
- Some of the one - step methods are: Euler's method, Heun's method, Runge's method, Runge - Kutta method and Taylor series method.
- A multi - step method is a method that in each step uses values from more than one of the preceding steps. The reason for using the additional information is to increase the accuracy.
- Some of the multi - step methods are: Milne's method, Simpson's method, Adams - Bash forth - Moulton methods.

EULER'S METHOD

Consider $dy/dx = f(x, y)$ with initial condition $y(x_0) = y_0$. We want to find out the value of y at $x = l$.

Divide (x_0, l) into 'n' equal parts of width 'h'. Let $x_1, x_2, ...x_{n-1}, (x_n = l)$ be the intermediate points. Now in (x_0, x_1) we have

$$y_1 = y_0 + h\,f(x_0, y_0)$$
$$y_2 = y_1 + h\,f(x_1, y_1)$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})$$

- The method is stable if $\left|1 + h(\partial f / \partial y)\right| < 1$ then the errors will be damp down with successive iterations. Otherwise, the errors increase in successive iterations and the procedure will be unstable.
- It is based on the linear term in the Taylor's expansion of $f(x, y)$.
- In practice, the error build up in using the method is substantial and the method is rarely used.
- This method is also called Runge - Kutta first order method.

Ex: Given $y^1 = -y$ with initial condition $y(0) = 1$. Find y (0.04) by using Euler's method with $h = 0.01$.

Sol: Here, $x_0 = 0,\ y_0 = 1,\ f(x, y) = -y$

$$y_1 = y_0 + h.f(x_0, y_0) = 1 + (0.01)(-1) = 0.99$$
$$y_2 = y_1 + h.f(x_1, y_1) = 0.99 + (0.01).(-0.99) = 0.9801$$
$$y_3 = y_2 + h.f(x_2, y_2) = 0.9801 + (0.01).(-0.9801) = 0.9701$$
$$y_4 = y_3 + h.f(x_3, y_3) = 0.9701 + (0.01).(0.9701) = 0.9606$$

The exact solution is $y = e^{-x}$

$$y(0.04) = e^{-0.04} = 0.9608$$

NOTE

By direct integration the solution is $y(0.04) = 0.9606$

$\therefore$ Error due to Euler's method $= 0.9608 - 0.9606$
$= 0.0002$

HEUN'S METHOD (MODIFIED EULER METHOD OF RUNGE-KUTTA SECOND ORDER METHOD)

Consider $dy/dx = f(x, y)$ with the initial condition $y(x_0) = y_0$. Initially y_1 is computed by Euler's formula.

$$y_1^p = y_0 + h\,f(x_0, y_0) \qquad \text{[where } p \text{ indicates predictor]}$$

Then modified value of y_1 is given by

$$y_1^c = y_0 + (h/2)\left[f(x_0, y_0) + f\left(x_1, y_1^p\right)\right] \text{[where indicates corrector]}$$

Similarly, we can find $y_2, y_3, ... y_n$

This method is also called modified Euler method/ second order Runge - Kutta method.

Ex: Given that $\frac{dy}{dx} = x^2 + y$ with initial condition $y(0) = 1$. Find y (0.05) and y(0, 1) using Runge - Kutta second order method with $h = 0.05$.

Sol: Here $x_0 = 0, y_0 = 1, f(x, y) = x^2 + y$

By Euler's formula $y_1^p = y_0 + h\,f(x_0, y_0) = 1 + (0.05).(0+1) = 1.05$

The modified value of y_1 is $y_1^c = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f\left(x_1, y_1^p\right)\right\} = 1.0513$

$\therefore y(0.05) = 1.0513$

To Compute $y(0.1)$ $x_1 = 0.05, y_1 = 1.0513$

By Euler's formula $y_2^p = y_1 = h.\,f(x_1, y_1) = 1.0513 + (0.05)\left\{(0.05)^2 + 1.05\right\} = 1.1040$

The modified value of y_2 is $y_2^c = y_1 + \frac{h}{2}\left\{f(x_1, y_1) + f\left(x_2, y_2^p\right)\right\} = 1.1055$

RUNGE'S METHOD (R-K THIRD ORDER METHOD)

To solve $(dy/dx) = x + y$ with $y(x_0) = y_0$

Calculate successively

$$k_1 = h\,f(x_0, y_0)$$
$$k_2 = h\,f(x_0 + h/2,\ y_0 + k_1/2)$$
$$k^1 = h\,f(x_0 + h, y_0 + k_1)$$
$$k_3 = h\,f\left(x_0 + h, y_0 + k^1\right)$$

Finally, compute $k = 1/6(k_1 + 4k_2 + k_3)$ and the solution is $y_1 = y_0 + k$.

RUNGE - KUTTA METHOD (R-K FOURTH ORDER METHOD)

To solve $(dy/dx) = f(x, y)$ with the condition $y(x_0) = y_0$. Let 'h' denotes the interval between equidistant values of x.

If the initial values are (x_0, y_0) then the first increment in y is computed from the formula given by

$$k_1 = h\,f(x_0, y_0)$$
$$k_2 = h\,f(x_0 + h/2,\ y_0 + k_1/2)$$
$$k_3 = h\,f(x_0 + h/2, y_0 + k_2/2)$$
$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$
$$\Delta y = 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_1 = y_0 + \Delta y$$

Ex: Given that $\frac{dy}{dx} = 1 + y^2$ where $y(0) = 0$ find $y(0.2)$ using Runge - Kutta fourth order method with $h = 0.2$.

Sol: We take $x_0 = 0, y_0 = 0$ $k_1 = h, f(x_0, y_0) = 0.2$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.202$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.20204$$

$$k_4 = hf\left(x_0 + h,\ y_0 + k_4\right) = 0.20816$$
$$y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.2027$$

PROPERTIES

1. R - K methods do not require prior computations of the higher derivatives of $y(x)$ as the taylor method does.
2. The R - K formulae involve the computation of $f(x, y)$ at various positions and this function occurs in the given equation.
3. To evaluate y_{n+1}, we need information only at y_n. Information at y_{n-1}, y_{n-2} etc not directly required. Thus R - K methods are one step methods.
4. These methods agree with Taylor's series solution upto the terms of h^r, where 'r' differs from method to method and is known as the order of R - K method.

TAYLOR'S SERIES METHOD

Consider $\left(dy/dx\right) = f\left(x, y\right), y\left(x_0\right) = y_0$. If the solution curve y(x) is expanded in a Taylor series around $x = x_0$. We obtain $y(x) = y_0 + \left(x - x_0\right)y_0^1 + \frac{\left(x - x_0\right)^2}{2!} y_0^{11} + \frac{\left(x - x_0\right)^3}{3!} y_0^{111} + \ldots$ ...(1)

- In equation (1), if we take only the first two terms then it corresponds to the Euler method of extrapolation. Thus the errors due to the truncation of the series would be of the order of *h*2.
- An improvement of the Euler method would thus be to include the *h*2 term in the above expansion. Then the truncation error in the above formula would of the order of *h*3.
- This method is not applicable in general, because the partial derivatives *fx, fy* are not always easy to obtain and considerable computation effort is involved.
- If $f(x, y)$ is given in a tabular form then this method is not applicable.
- R - K methods are equivalent to Taylor series method but will use only the values of $f(x, y)$ at specified values of x and y and will not require the derivative to be evaluated.
- R - K methods agree with Taylor's series solution upto the term in h^r, where r differs from method to method and is called the order of that method.

Ex: Find by Taylor's series method, the values of y at $x = 0.1$ from $dy/dx = x^2 y - 1,\ y(0) = 1$. (Consider the Taylor's series expansion upto h^3 terms). (with $h = 0.1$)

Sol: Here $y(0) = 1,\ y^1 = x^2 y - 1,\ y^1(0) = -1$

∴Differentiating successively and substituting, we get

$$y^{11} = 2xy + x^2 y^1 \qquad y^{11}(0) = 0$$
$$y^{111} = 2y + 4xy^1 + x^2 y \qquad y^{111}(0) = 2$$

Putting these values in the Taylor's series

$$y\left(x\right) = y_0 + \left(x - x_0\right)y_0^1 + \frac{\left(x - x_0\right)^2}{2!} y_0^{11} + \frac{\left(x - x_0\right)^3}{3!} y_0^{111} +$$

Choosing $x = 0.1$ and $x_0 = 0$; $y\left(0.1\right) = -1 + \left(0.1\right) + \frac{\left(0.01\right)^2}{2}.0 + \frac{\left(0.01\right)^3}{6}.2 = 0.9003.$

TOPICWISE SOLVED PROBLEMS

VECTORS, LAPLACE TRANSFORM

Ex. 1. Find $\vec{\nabla}.\vec{F}$ and $\vec{\nabla}\times\vec{F}$ where the vector field $F=(yz,xz,xy)$ **(Ans:** $\vec{\nabla}.\vec{F}=0$ $\vec{\nabla}\times\vec{F}=(0,0,0)$**)**

Sol : $\vec{F}=yz\hat{i}+xz\hat{j}+xy\hat{k}$; $\vec{\nabla}.\vec{F}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left(yz\hat{i}+xz\hat{j}+xy\hat{k}\right)=0+0+0+0=0$

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & xz & xy\end{vmatrix}=\hat{i}\left[x-x\right]-\hat{j}\left[y-y\right]+\hat{k}\left[z-z\right]=0\hat{i}+0\hat{j}=0\hat{k}$$

Ex. 2. Find the value of the following integral $\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\left(t_3-t_2\right)^2\delta(t_1+t_3-t-t_2)$ **(Ans: t^4/12)**

Sol : $\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\left(t_3-t_2\right)^2\delta(t_1+t_3-t-t_2)=\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}(t_3-t_2)^2\delta(t_3-(t+t_2-t_1))$

From the properties of dirac delta function $\int_a^b f(x)\delta(x-m)dx=f(m)\ \ if\ \ a<m<b$.

Given integral $\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}(t_3-t_2)^2\delta(t_3-(t+t_2-t_1))=\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3 f(t_3)dt_3\delta(t_3-m)$

$$=\int_0^t dt_1\int_0^{t_1}dt_2.(t+t_2-t_1-t_2)^2=\int_0^t dt_1\int_0^{t_1}(t-t_1)^2dt_2=\int_0^t dt_1\int_0^{t_1}(t^2-2tt_1+t_1^2)dt_2$$

$$=\int_0^t\left.(t^2t_2-2tt_1t_2+t_1^2t_2)\right|_0^{t_1}dt_1=\int_0^t(t^2t_1-2tt_1^2+t_1^3)dt_1=\left.\frac{t^2t_1^2}{2}-\frac{2tt_1^3}{3}+\frac{t_1^4}{4}\right|_0^t=\frac{t^4}{2}-\frac{2t^4}{3}+\frac{t^4}{4}$$

$$\frac{6t^4-8t^4+3t^4}{12}=\frac{t^4}{12}$$

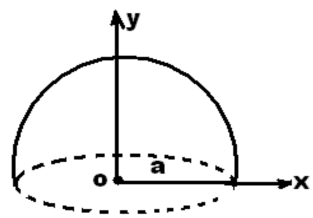
Ex. 3. What is the Laplace transform of $\begin{aligned}f(t)&=t^2-1 \quad for\ t>1\\ &=0 \quad\quad for\ 0<t<1\end{aligned}$ **(Ans:** $2e^{-s}\left(s+1\right)/s^3$**)**

Sol : $L\left[f(t)\right]=\int_0^\infty f(t)e^{-st}dt=\int_0^1 0dt+\int_1^\infty\left(t^2-1\right)e^{-st}dt=\int_1^\infty\left(t^2-1\right)e^{-st}dt=\int_1^\infty t^2e^{-st}dt-\int_1^\infty e^{-st}dt;$

$$\int_1^\infty t^2e^{-st}dt=\frac{e^{-s}}{s}+\frac{2e^{-s}}{s^2}+\frac{2e^{-s}}{s^3},\ \int_1^\infty e^{-st}dt=\left.\frac{e^{-st}}{-s}\right|_1^\infty=0+\frac{e^{-s}}{s};$$

$$L[f(t)] = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} = 2e^{-s}\left[\frac{1}{s^3} + \frac{1}{s^2}\right] = 2e^{-s}\left[\frac{s^2+s^3}{s^2s^3}\right] = 2e^{-s}\frac{s^2(1+s)}{s^2s^3} = \frac{2e^{-s}(s+1)}{s^3}$$

Ex. 4. If $\vec{F} = r^2\hat{r}$, S is the surface of a hemisphere including the circular plane base radius 'a', centered at the origin, $\hat{r}$ represents the unit vector in the radial direction at any point and $\hat{n}$ represents the unit outward normal at each point of the closed surface S. Find the value of $\int_s \vec{F}.\hat{n}ds$ **(Ans: $2\pi a^4$)**



Sol : $\int_s \vec{F}.\hat{n}ds = \int_v (\vec{\nabla}.\vec{F})\,d\tau; \nabla.A\big|_{\text{spherical}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(A_1h_2h_3) + \frac{\partial}{\partial q_2}(A_2h_1h_3) + \frac{\partial}{\partial q_3}(A_3h_1h_3)\right]$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \times r^2\sin\theta\right)\right] = \frac{4r^3}{r^2} = 4r$$

$$\Rightarrow \int_0^r\int_0^\pi\int_0^\pi 4rr^2dr\sin\theta d\theta d\phi = \frac{4r^4}{4}\Bigg|_0^a (-\cos\theta)_0^\pi \times \pi = 2\pi a^4$$

Ex. 5. If $f(s) = \int_0^\infty F(t)e^{-st}dt$ then, find the value of $\int_0^\infty tF(t)e^{-st}dt$ **(Ans:** $-\frac{df}{ds}$**)**

Sol : $\int_0^\infty tF(t)e^{-st}dt = (-1)^1\frac{df(s)}{ds} = \frac{-df(s)}{ds}.$

Ex. 6. Find the unit tangent vectors to any point on the curve $x = t^2 - t, y = 4t - 3, z = 2t^2 - 8t$ at $t = 2$?

(Ans: $\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$**)**

Sol : Tangent to the curve is

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(t^2-t)\hat{i} + (4t-3)\hat{j} + (2t^2-8t)\hat{k}\right] = (2t-1)\hat{i} + 4\hat{j} + (4t-8)\hat{k}\Big|_{t=2} = 3\hat{i} + 4\hat{j};$$

Unit tangent $= \frac{3\hat{i}+4\hat{j}}{\sqrt{25}} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$

Ex. 7. Find an equation of the tangent plane to the surface $x^2 + 2xy^2 - 3z^3 = 6$ at the point $p(1, 2, 1)$ **(Ans:** $10x + 8y - 9z = 17$**)**

Sol : $\nabla\phi = \hat{i}\left[2x+2y^2\right]+\hat{j}\left[4xy\right]+\hat{k}\left[-9z^2\right];\ \nabla\phi\big|_{121} = (2+2\times 4)\hat{i}+8\hat{j}+-9\hat{k}\ = 10\hat{i}+8\hat{j}-9\hat{k}.$

This is the normal to the surface at p equation of the plane with normal. $N = a\hat{i}+b\hat{j}+c\hat{k}$ has the form. $ax+by+cz=k; i.e., 10x+8y-9z=k;$ 10+16-9 = k; k = 17; substitute (1, 2, 1) $\therefore$ equation of tangent plane $10x+8y-9z=17.$

Ex. 8. Find $\nabla\phi\ if\ \phi = \ln|\vec{r}|$ and $\phi = \frac{1}{r}$ **(Ans:** $\frac{\vec{r}}{r^2}, \frac{-\vec{r}}{r^3}$ **)**

Sol : $\ln|\vec{r}| = \ln\sqrt{x^2+y^2+z^2};\ = \ln(x^2+y^2+z^2)^{1/2}$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}; \frac{\partial\phi}{\partial x} = \frac{1}{\left(x^2+y^2+z^2\right)^{1/2}}\times\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}\times 2x\ = \frac{1}{r^2}x,$$

$$lly\frac{\partial\phi}{\partial y} = \frac{y}{r^2}, \frac{\partial\phi}{\partial z} = \frac{z}{r^2}; \nabla\phi = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^2}; \nabla\phi = \frac{\vec{r}}{r^2}$$

$$\nabla r^n\ = nr^{n-2}\vec{r}, \nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$$

Ex. 9. Find the angle between the surfaces $\phi_1 = x^2+y^2-z$ and $\phi_2 = \left(x-\frac{\sqrt{6}}{6}\right)^2+\left(y-\frac{\sqrt{6}}{6}\right)^2-z$ at the point $p = \left(\frac{\sqrt{6}}{12}, \frac{\sqrt{6}}{12}, \frac{1}{12}\right)$ **(Ans: 60°)**

Sol : Angle between the surfaces equal to angle between their normal.

$\nabla\phi_1 = \hat{i}2x+\hat{j}2y-\hat{k};$

$$\nabla\phi_1\big|_p = \frac{2\sqrt{6}}{12}\hat{i}+\frac{2\sqrt{6}}{12}\hat{j}-\hat{k} = \frac{\hat{i}}{\sqrt{6}}+\frac{\hat{j}}{\sqrt{6}}-\hat{k}; |\nabla\phi_1| = \sqrt{\frac{1}{6}+\frac{1}{6}+1}$$

$$\nabla\phi_2 = 2\left(x-\frac{1}{\sqrt{6}}\right)i+2\left(y-\frac{1}{\sqrt{6}}\right)j\Big|_p -\hat{k}$$

$$= 2\left(\frac{\sqrt{6}}{12}-\frac{1}{\sqrt{6}}\right)\hat{i}+2\left(\frac{\sqrt{6}}{12}-\frac{1}{\sqrt{6}}\right)\hat{j}-\hat{k}\ = 2\left(\frac{-6}{12\sqrt{6}}\right)\hat{i}+2\left(\frac{-6}{12\sqrt{6}}\right)\hat{j}-\hat{k} = \frac{-1}{\sqrt{6}}\hat{i}-\frac{1}{\sqrt{6}}\hat{j}-\hat{k}$$

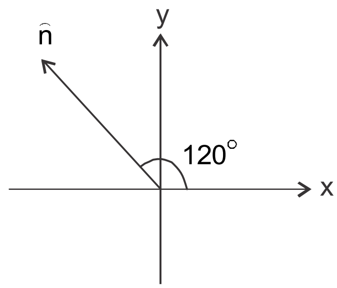
$$\Rightarrow |\nabla\phi_2| = \sqrt{\frac{1}{6}+\frac{1}{6}+1}$$

Angle between the normals

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{-\frac{1}{6}-\frac{1}{6}+1}{\sqrt{\frac{1}{6}+\frac{1}{6}+1}\sqrt{\frac{1}{6}+\frac{1}{6}+1}} = \frac{-\frac{1}{3}+1}{\frac{1}{6}+\frac{1}{6}+1}$$

$$\frac{2/3}{4/3} = \frac{1}{2}\cos\theta = 1/2 \Rightarrow \theta = \cos^{-1}(1/2) = 60^{o}$$

Ex. 10. A unit vector $\hat{n}$ on the xy plane is at an angle of 120° with respect to $\hat{i}$. The angle between the vector, $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60° if **(Ans:** $b = \frac{a}{2}$ **)**



Sol : The angle between $\bar{u}$ and $\bar{v}$, *i.e.*, $\vec{u}\cdot\vec{v} = |u||v|\cos 60^\circ$

$$a^2\,\hat{i}.\hat{n} + b^2\hat{n}.\hat{i} + ab + ba = |u||v|\cos 60^\circ$$

Ex. 11. Find the inverse Laplace transform of $t-1+e^{-t}$ is **(Ans:** $t-1+e^{-t}$**)**

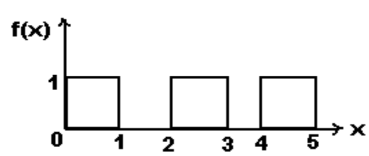
Sol : $L\left[t-1+e^{-t}\right] = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} = \frac{s+1-s(s+1)+s^2}{s^2(s+1)} = \frac{s+1-s^2-s+s^2}{s^2(s+1)} = \frac{1}{s^2(s+1)}$

$$\Rightarrow a^2\cos 120^o + 2ab + b^2\cos 120^o = \frac{|u||v|}{2}$$

$$\Rightarrow -\left(\frac{a^2+b^2}{2}\right) + 2ab = \frac{1}{2}\left(a^2+b^2+2ab\cos 120^\circ\right)$$

$$\Rightarrow -\left(a^2+b^2\right) + 4ab = a^2+b^2-ab \Rightarrow 2\left(a^2+b^2\right) = 5ab \Rightarrow b = a/2$$

Ex. 12. The graph of the function $f(x) = \begin{cases} 1, & for\ 2n \le x \le 2n+1 \\ 0, & for\ 2n+1 \le x \le 2n+2 \end{cases}$ where $(n = 0,1,2,\ldots..)$ is shown below. Its Laplace transform $\tilde{f}(s)$ is? **(Ans:** $\frac{1}{s\left(1+e^{-s}\right)}$**)**



Sol : $L[f(t)] = \dfrac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$ where $f(t)$ is a periodic function of period T. Here $T = 2$, $f(t) = 1$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left\{\int_0^1 e^{-st}dt + \int_1^2 0dt\right\} = \left[\frac{1}{1-e^{-2s}}\right]\frac{e^{-st}}{-s}\bigg|_0^1 = \left(\frac{1}{1-e^{-2s}}\right)\left(\frac{e^{-s}-1}{-s}\right)$$

$$= \left(\frac{1-e^{-s}}{s}\right)\frac{1}{(1-e^{-2s})} = \frac{(1-e^{-s})}{s(1+e^{-s})(1-e^{-s})} = \frac{1}{s\left(1+e^{-s}\right)}$$

Ex. 13. Find $L^{-1}\left[\dfrac{1}{s^3\left(s^2+1\right)}\right]$ **(Ans:** $\dfrac{t^2}{2} + \cos t - 1$ **)**

Sol : $L^{-1}\left[\bar{f}(s)\right] = f(t)$ *then* $L^{-1}\left[\dfrac{\bar{f}(s)}{s}\right] = \int_0^t f(t)dt; L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t;\ L^{-1}\left[\dfrac{1}{s^3\left(s^2+1\right)}\right] = \int_0^t\int_0^t\int_0^t \sin t dt$;

$$= \int_0^t\int_0^t (-\cos t)_0^t = \int_0^t\int_0^t (1-\cos t)\,dt = \int_0^t t - \sin t\big|_0^t = \int_0^t (t-\sin t)\,dt = \frac{t^2}{2} + \cos t\big|_0^t = \frac{t^2}{2} + \cos t - 1$$

Ex. 14. Find the inverse Laplace transform of $L^{-1}\left[\dfrac{1}{\left(s^2+a^2\right)^2}\right]$ **(Ans:** $\dfrac{1}{2a^2}\left[\dfrac{2\sin at}{2a} - t\cos(at)\right]$ **)**

Sol : $L^{-1}\left(\dfrac{1}{\left(s^2+a^2\right)^2}\right) = L^{-1}\left[\dfrac{1}{\left(s^2+a^2\right)\left(s^2+a^2\right)}\right]; f(t) = \dfrac{1}{a}\sin(at), g(t) = \dfrac{1}{a}\sin at;$

$$L^{-1}\left[\frac{1}{\left(s^2+a^2\right)^2}\right] = \frac{1}{a}\sin(at) * \frac{1}{a}\sin(at) = \int_0^t \frac{1}{a}\sin(au)\frac{1}{a}\sin a(t-u)du = \frac{1}{a^2}\int_0^t \sin(au)\sin a(t-u)du\ ;$$

$$= \frac{1}{2a^2}\int_0^t 2\sin(au)\sin a(t-u)du = \frac{1}{2a^2}\int_0^t \cos\left(au - at + au\right) - \cos(at)du$$

$$= \frac{1}{2a^2}\int_0^t [(\cos(2au - at) - \cos(at)]du = \frac{1}{2a^2}\left\{\frac{\sin(2au-at)}{2a} - u\cos(at)\right\}_0^t$$

$$= \frac{1}{2a^2}\left\{\frac{\sin(at)}{2a} + \frac{\sin(att)}{2a} - t\cos(at)\right\} = \frac{1}{2a^2}\left\{\frac{2\sin(at)}{2a} - t\cos(at)\right\}$$

Ex. 15. Find $L^{-1}\left[\dfrac{1}{\sqrt{2s-1}}\right]$ **(Ans:** $\dfrac{e^{t/2}}{\sqrt{2\pi t}}$ **)**

Sol : $L^{-1}\left[\frac{1}{\sqrt{2s-1}}\right]=L^{-1}\left[\frac{1}{\left(2(s-1/2)\right)^{1/2}}\right]=\frac{1}{2^{1/2}}L^{-1}\left[\frac{1}{(s-1/2)^{1/2}}\right]=\frac{e^{1/2t}}{\sqrt{2}}L^{-1}\left[\frac{\sqrt{1/2}}{\sqrt{1/2}s^{1/2}}\right];$

$$L\left[t^n\right]=\frac{n!}{s^{n+1}}=\frac{\sqrt{n+1}}{s^{n+1}};L^{-1}\left[\frac{1}{\sqrt{2s-1}}\right]=\frac{e^{\frac{1}{2}t}t^{-1/2}}{\sqrt{2\pi}}=\frac{e^{t/2}}{\sqrt{2\pi t}}$$

Ex. 16. Find $L\left[\int_0^t \frac{e^t \sin t}{t}dt\right]$ **(Ans:** $\frac{1}{s}\cot^{-1}(s-1)$**)**

Sol : $L[\sin t]=\frac{1}{s^2+1};L\left[e^t \sin t\right]=\frac{1}{(s-1)^2+1};L\left[\frac{e^t\sin t}{t}\right]=\int_s^\infty \frac{1}{1+(s-1)^2}ds=\tan^{-1}(s-1)\Big|_s^\infty$

$$=\frac{\pi}{2}-\tan^{-1}(s-1)=\cot^{-1}(s-1)\ ;\ L\left[\int_0^t \frac{e^t\sin t}{t}dt\right]=\frac{1}{s}\cot^{-1}(s-1)$$

Ex. 17. If $f(t)=|t-1|+|t+1|$ where $t\geq 0$ then. Find $L[f(t)]$ is? **(Ans:** $\frac{-2}{s}\left[e^{-s}-1\right]+2\left[\frac{e^{-s}}{s}+\frac{e^{-s}}{s^2}\right]$**)**

Sol : $|t-1|=\begin{cases}-(t-1) & t<1\\ t-1 & t>1\end{cases}$; $|t+1|=\begin{cases}-(t+1) & t<-1\\ t+1 & t>-1\end{cases}$

Laplace is defined in the interval 0 to ∞ $f(t)=\begin{cases}-(t-1)+t+1 & 0<t<1\\ t-1+t+1 & 1<t<\infty\end{cases}$

$$L\left[f(t)\right]=\int_0^\infty f(t)e^{-st}dt=\int_0^1\left[(t+1)-(t-1)\right]e^{-st}dt+\int_0^\infty 2te^{-st}dt=\int_0^1 2e^{-st}dt+\int_1^\infty 2te^{-st}dt;$$

$$\left.\frac{-te^{-st}}{s}-\frac{e^{-st}}{s^2}\right|_1^a=0+\frac{2e^{-s}}{s}+\frac{2e^{-s}}{s^2}=L\left[f(t)\right]=\left.\frac{2e^{-st}}{-s}\right|_0^1+\frac{2e^{-s}}{s}+\frac{2e^{-s}}{s^2}=\frac{-2}{s}\left[e^{-s}-1\right]+2\left[\frac{e^{-s}}{s}+\frac{e^{-s}}{s^2}\right]$$

Ex. 18. Find an equation for the tangent plane to the surface $x^2yz-4xyz^2=-6$ at point p(1,2,1) **(Ans:** $4x+3y+14z=24$**)**

Sol : $\phi=x^2yz-4xyz^2+6;\nabla\phi=\hat{i}\left[2xyz-4yz^2\right]+\hat{j}\left[x^2z-4xz^2\right]+\hat{k}\left[x^2y-8xy\right]$

$\nabla\phi\big|_{(1,2,1)}=\hat{i}\left[4-8\right]+\hat{j}\left[1-4\right]+\hat{k}\left[2-16\right]=-4\hat{i}-3\hat{j}-14\hat{k}\ \therefore\ 4\hat{i}+3\hat{j}+14\hat{k}$ is normal to the surface at p.

An equation of the plane with normal. $N=a\hat{i}+b\hat{j}+c\hat{k}$ has the form

$ax+by+cz=k$ i.e, $4x+3y+14z=k$ put $(xyz)=(1,2,1)=k=24$

$\therefore$ Equation of tangent plane is $4x+3y+14z=24$

Ex. 19. Find the total work done in moving a particle in the force field given by $\vec{F} = z\hat{i} + z\hat{j} + x\hat{k}$ along the helix c given by $x = \cos t, y = \sin t, z = t$ from $t = 0$ to $t = \pi/2$. **(Ans:** $\frac{\pi}{2} - 1$**)**

Sol : $\vec{F} = z\hat{i} + z\hat{j} + x\hat{k}$; for helix $x = \cos t$, $y = \sin t$, $z = t$ from $t = 0$ to $\frac{\pi}{2}$; $W = \int \vec{F}.d\vec{r}$

$$= \int \left(z\hat{i} + z\hat{j} + x\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = \int \left(zdx + zdy + xdz\right) = \int_0^{\pi/2} \left(-t\sin t + t\cos t + \cos t\right) dt$$

$$= \int_0^{\pi/2} \left(\cos t(t+1) - t\sin t\right) dt = \int_0^{\pi/2} (t+1)\cos t dt - \int_0^{\pi/2} t\sin t dt$$

$$\int_0^{\pi/2} (t+1)\cos t dt = (t+1)\sin t\Big|_0^{\pi/2} + \cos t\Big|_0^{\pi/2} = \left(\frac{\pi}{2} + 1\right) - 0 + 0 - 1; \frac{\pi}{2} + 1 - 1 = \frac{\pi}{2}$$

$$\int_0^{\pi/2} t\sin t\, dt = -t\cos t\Big|_0^{\pi/2} + \sin t\Big|_0^{\pi/2} = 0 + 1; \text{ Work done } = \frac{\pi}{2} - 1$$

Ex. 20. Evaluate $\iint_s \vec{F}.\hat{n}ds$ where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and 's' is the surface of the cube bounded by $x = 0$, $x = 2; y = 0, y = 2, z = 0, z = 2$. **(Ans: 24)**

Sol : $$\iint_0 \vec{F}.\hat{n}ds = \int_V (\nabla.F)\, d\tau, \nabla.\vec{F} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right].\left[4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right] = 4z - 2y + y = (4z - y);$$

$$\int_v \left(\vec{\nabla}.\vec{F}\right) d\tau = \iiint (4z - y)dxdydz = \int_0^2\int_0^2 4xz - xy\Big|_0^2 dydz = \int_0^2\int_0^2 \left(8z - 2y\right) dydz = \int_0^2 8yz - \frac{2y^2}{2}\Big|_0^2 dz;$$

$$\int_v \left(\vec{\nabla}.\vec{F}\right) d\tau = \int_0^2 \left(16z - 4\right) dz = \frac{16z^2}{2} - 4z\Big|_0^2 = 8z^2 - 4z\Big|_0^2 = 32 - 8 = 24$$

Ex. 21. Let $\vec{F} = \frac{-y\hat{i}}{\left(x^2 + y^2\right)} + \frac{x\hat{j}}{\left(x^2 + y^2\right)}$. Calculate $\vec{\nabla} \times \vec{F} = ?$ **(Ans: zero)**

Sol : $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} & 0 \end{vmatrix} = \hat{i}[0 - 0] - \hat{j}[0] + \hat{k}\left[\frac{\left(x^2 + y^2\right) - x(2x)}{\left(x^2 + y^2\right)^2} + \frac{\left(x^2 + y^2\right) - y(2y)}{\left(x^2 + y^2\right)^2}\right]$$

$$= \hat{k}\left[\frac{y^2 - x^2 + x^2 - y^2}{\left(x^2 + y^2\right)^2}\right] = 0$$

Ex. 22. In the above problem Find $\oint_c \vec{F}.d\vec{r}$ **(Ans: 2π)**

Sol : $$\oint_c \vec{F}.d\vec{r} = \oint\left[\left(\frac{-y}{x^2+y^2}\right)\hat{i} + \left(\frac{x}{x^2+y^2}\right)\hat{j}\right].\left[dx\hat{i}+dy\hat{j}\right] = \oint \frac{-ydx+xdy}{\left(x^2+y^2\right)};$$

$$x = \rho\cos\phi;\ y = \rho\sin\phi;\ \rho^2 = x^2+y^2;\ dx = -\rho\sin\phi d\phi + \cos\phi d\rho;\ dy = \rho\cos\phi d\phi + \sin\phi d\rho$$

$$\frac{-ydx+xdy}{\left(x^2+y^2\right)} = \frac{-\rho\sin\phi\left[-\sin\phi\rho d\phi+\cos\phi d\rho\right]+\rho\cos\phi\left[\rho\cos\phi d\phi+\sin\phi d\rho\right]}{\rho^2}$$

$$= \frac{\rho^2\sin^2\phi d\phi - \rho\sin\phi\cos\phi d\rho.d\phi + \rho^2\cos^2\phi d\phi + \rho\sin\phi\cos\phi d\phi d\rho}{\rho^2} = d\phi;$$

$$\oint \vec{F}.d\vec{s} = \oint d\phi = 2\pi$$

FOURIER SERIES, FOURIER TRANSFORM, PROBABILITY DISTRIBUTIONS

Ex. 23. What is the Fourier transform of $\exp(-|x|)$ is **(Ans: $\frac{2}{1+k^2}$)**

Sol : Fourier transform of a function f(x) is

$$F\left[f(x)\right] = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_{-\infty}^{\infty} e^{-|x|}e^{-ikx}dx$$

$$= \int_{-\infty}^{0} e^{x}e^{-ikx}dx + \int_{0}^{\infty} e^{-x}.e^{-ikx}dx = \int_{-\infty}^{0} e^{x(1-ik)}dx + \int_{0}^{\infty} e^{-x(1+ik)}dx = \frac{1}{1-ik}+\frac{1}{1+ik} = \frac{2}{1+k^2}$$

Ex. 24. Two cards are selected from a well shuffled usual pack of 52 cards. What is the probability that the selected cards are a king and a queen. **(Ans: 0.01207)**

Sol : $$p = \frac{{}^4C_1 \times {}^4C_1}{{}^{52}C_2} = \frac{\frac{4!}{3!1!}\times\frac{4!}{3!1!}}{\frac{52!}{50!2!}} = \frac{4\times 4}{\frac{51\times 52}{2}} = \frac{16}{51\times 26} = 0.01207$$

Ex. 25. You are given two pairs of shoes (is a total of four shoes) such that one pair is brown and the other black. You pick out two shoes randomly. What is the probability that you have picked a matching pair of same colour? **(Ans: 1/2)**

Sol : $p = 1/2$

Ex. 26. Which of the following is a valid probability distribution (x and y are random variables).

(Ans: II and III)

(I) $\frac{1}{\sqrt{\pi}}e^{-xy}\quad -\infty < x < \infty, -\infty < y < \infty$

(II) $\frac{1}{\pi\left(1+x^2\right)}\quad -\infty < x < \infty$

(III) $4\sin^2(\pi x)\cos^2(\pi y)\ 0<x<1,\ 0<y<1$

(IV) $\exp\{-[a(x^2+y^2)-2bxy]\}\quad -\infty<x<\infty,\ -\infty<y<\infty$

Sol : A valid probability distribution satisfies the condition.

$$\int_{-\infty}^{\infty} f(x)dx = 1;$$

$$\text{(b)} \int_{-\infty}^{\infty}\frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1+x^2)}dx = \frac{1}{\pi}\tan^{-1}x\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left[\tan^{-1}\infty-\tan^{-1}(-\infty)\right] = \frac{1}{\pi}\left[\frac{\pi}{2}-\frac{-\pi}{2}\right]=1$$

$$\text{(c)} \int_0^1 4\sin^2(\pi x)\cos^2(\pi y)dxdy = 4\int_0^1\sin^2\pi x dx\int_0^1\cos^2(\pi y)dy$$

$$=4\int_0^1\left(\frac{1-\cos 2\pi x}{2}\right)dx\int_0^1\left(\frac{1-\cos 2\pi y}{2}\right)dy = \left[x-\frac{\sin 2\pi x}{2\pi}\right]_0^1\left[y-\frac{\sin 2\pi y}{2\pi}\right]_0^1 = 1$$

Ex. 27. Find the Fourier sine siries transform of the function $f(t)=e^{-at}$ is **(Ans:** $\frac{\omega}{\omega^2+a^2}$**)**

Sol : $F_s[f(t)] = \int_0^\infty f(t)\sin\omega t dt = \int_0^\infty e^{-at}\sin(\omega t)dt. = \int_0^\infty \sin(\omega t)e^{-at}dt;$

$$I = \sin(\omega t)\frac{e^{-at}}{-a}\Bigg|_0^\infty - \int_0^\infty \cos(\omega t)\omega\frac{e^{-at}}{-a}dt = +\int_0^\infty\frac{\omega}{a}\cos(\omega t)e^{-at}dt$$

$$=\frac{\omega}{a}\left[\cos\omega t\frac{e^{-at}}{-a}\right]_0^\infty - \int_0^\infty -\sin(\omega t)\times\frac{\omega e^{-at}}{-a}$$

$$=+\frac{\omega}{a}\left[\frac{1}{a}-\frac{\omega^2}{a^2}I\right]\quad I=+\frac{\omega}{a^2}-\frac{\omega^2}{a^2}I;\quad I\left(1+\frac{\omega^2}{a^2}\right)=+\frac{\omega}{a^2};\quad I\left(\frac{a^2+\omega^2}{a^2}\right)=\frac{\omega}{a^2}\Rightarrow I=\frac{\omega}{a^2+\omega^2}$$

Ex. 28. In a series of five cricket matches, one of the captains calls “Heads” every time when the toss is taken. The probability that he will win 3 times and lose 2 times? **(Ans: 5/16)**

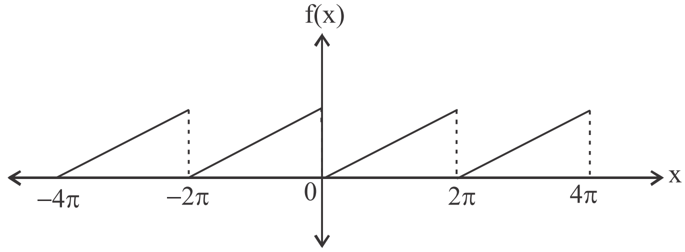
Sol : $p(x) = {}^nC_x p^x q^{n-x}; n=5, x=3; p(x=3) = {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5!}{2!3!}\frac{1}{2^3}\frac{1}{2^2}=\frac{5}{16}$

Ex. 29. An unbiased dice is thrown three times successively. Find the probability that the number of dots on the uppermost surface add up to 16 is **(Ans: 1/36)**

Sol : Sample space = 6^3 = 216, favourable events (5, 5, 6), (5,6,5), (6,5,5), (4,6,6), (6,4,6), (6,6,4);

$$p=\frac{6}{216}=\frac{1}{36}$$

Ex. 30. Find the Fourier series representing **(Ans:** $f(x)=\pi-2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+....\right]$**)**



Sol : $f(x)=x;\theta<x<2\pi;f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right);\ a_0=\frac{1}{\pi}\int_c^{c+2L}f(x)dx$;

$$a_n=\frac{1}{\pi}\int_c^{c+2L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx;\ b_n=\frac{1}{\pi}\int_c^{c+2L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx;$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}xdx=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}=\frac{1}{\pi}\times\frac{4\pi^2}{2}=2\pi;$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x\cos\left(\frac{n\pi x}{\pi}\right)dx=\frac{1}{\pi}\int_0^{2\pi}x\cos(nx)dx;$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x\cos(nx)dx=\frac{x\sin(nx)}{n}\Bigg|_0^{2\pi}+\frac{\cos nx}{n^2}\Bigg|_0^{2\pi}=\frac{1}{n^2}[1-1]=0\qquad\therefore a_n=0;$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x\sin(nx)dx=-\frac{x\cos nx}{n}\Bigg|_0^{2\pi}+\frac{\sin nx}{n^2}\Bigg|_0^{2\pi}\qquad\therefore b_n=\frac{1}{\pi}-\frac{2\pi}{n}=\frac{-2}{n}$$

$$f(x)=\sum_{n=1}^{\infty}\frac{-2}{n}\sin(nx);\ f(x)=\pi-2\left[\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+....\right]$$

Ex. 31. Find the Fourier series of the function defined as $f(x)=\begin{cases}x+\pi & \text{for } 0<x\le\pi\\ -x-\pi & \text{for } -\pi\le x<0\end{cases}$ and $f(x+2x)=f(x)$

(Ans: $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+...\right]+4\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+...\right]$**)**

Sol : $f(x)=\begin{cases}x+\pi & for\ \ 0\le x\le\pi\\ -x-\pi & -\pi\le x<0\end{cases};\ f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right);$

$$a_0=\frac{1}{\pi}\int_c^{c+2L}f(x)dx;=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left\{\int_{-\pi}^{0}(-x-\pi)dx+\int_0^{\pi}(x+\pi)dx\right\}$$

$$=\frac{1}{\pi}\left[-\int_{-\pi}^{0}(x+\pi)dx+\int_0^{\pi}(x+\pi)dx\right]=\frac{1}{\pi}\left[-\left(\frac{x^2}{2}+\pi x\right)_{-\pi}^{0}+\left(\frac{x^2}{2}+\pi x\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}-\pi^2+\frac{\pi^2}{2}+\pi^2\right]=\frac{\pi^2}{\pi}=\pi; a_n=\frac{1}{\pi}\int_c^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx;$$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-(x+\pi)\cos(nx)dx+\int_0^{\pi}(x+\pi)\cos(nx)dx\right]$$

$$\int_{-\pi}^{0}-(x+\pi)\cos(nx)dx=-\left[0+\frac{1}{n^2}\left[1-(-1)^n\right]\right]=\frac{-1}{n^2}+\frac{(-1)^n}{n^2};$$

$$\int_0^{\pi}(x+\pi)\cos(nx)dx=\frac{(-1)^n}{n^2}-\frac{1}{n^2}\quad\therefore a_n=\frac{1}{\pi}\left[\frac{-1}{n^2}+\frac{(-1)^n}{n^2}\right]+\frac{1}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right];$$

$$a_n=\frac{2}{n^2\pi}\left[(-1)^n-1\right];\therefore a_n=0 \text{ if } n \text{ is even}; a_n=\frac{-4}{n^2\pi} \text{ if } n \text{ is odd};$$

$$b_n=\frac{1}{\pi}\int_c^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx;$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-(x+\pi)\sin(nx)dx+\int_0^{\pi}(x+\pi)\sin(nx)dx\right]=\frac{-2\pi(-1)^n}{n}+\frac{\pi}{n};$$

$$b_n=\frac{1}{\pi}\left[\frac{\pi}{n}-\frac{2\pi(-1)^n}{n}+\frac{\pi}{n}\right]=\frac{1}{n}\left[2-2\pi(-1)^n\right]=\frac{2}{n}\left[1-(-1)^n\right]=\frac{4}{n} \text{ if } n \text{ is odd}; =0 \text{ if } n \text{ is even};$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$=\frac{\pi}{2}+\frac{-4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+...\right]+4\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}=...\right]$$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+...\right]+4\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+...\right]$$

Ex. 32. Find the Fourier sine series for the function $f(x)=e^{ax}$ for $0<x<\pi$; where a is a constant?

(Ans: $e^{ax}=\frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x+\frac{2(1-e^{a\pi})}{a^2+2^2}\sin 2x+...\right]$**)**

Sol : Fourier sine series contains only sine terms

$$e^{ax}=\sum_{n=1}^{n}b_n\sin\left(\frac{n\pi x}{L}\right); b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin\left(\frac{n\pi x}{L}\right); L=\pi, b_n=\frac{2}{\pi}\int_0^{\pi}e^{ax}\sin(nx)dx;$$

$$I=\left[\sin(nx)\frac{e^{ax}}{a}-\int_0^{\pi}\cos(nx)\frac{n\times e^{ax}}{a}dx\right]=\left[-\frac{n}{a}\int_0^{\pi}\cos(nx)e^{ax}dx\right]$$

$$=\frac{-2n}{a\pi}\left[\frac{\cos nxe^{ax}}{a}-\int_0^{\pi}-\sin(nx)\frac{ne^{ax}}{a}dx\right]=\frac{-n}{a}\left[\frac{(-1)^n e^{a\pi}}{a}-\frac{1}{a}+\frac{n}{a}\int_0^{\pi}\sin(nx)e^{ax}dx\right];$$

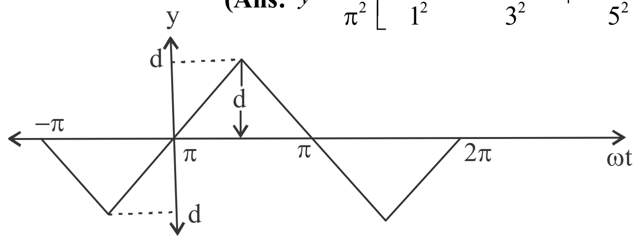
$$I=\frac{-n}{a}\frac{(-1)^n e^{a\pi}}{a}+\frac{n}{a^2}-\frac{n^2}{a^2}I;\quad I\left(1+\frac{n^2}{a^2}\right)=\frac{2n-n(-1)^n e^{a\pi}}{a^2}$$

$$\frac{I(a^2+n^2)}{a^2}=\frac{n\left[1-(-1)^n e^{a\pi}\right]}{a^2};\quad I=\frac{n\left[1-(-1)^n e^{a\pi}\right]}{(a^2+n^2)}$$

$$b_n=\frac{2n}{\left(a^2+n^2\right)\pi}\left[1-\left(-1\right)^n e^{a\pi}\right];f(x)=\frac{2}{\pi}\left[\frac{1+e^{a\pi}}{\left(a^2+1^2\right)}\sin x+\frac{2(1-e^{a\pi})}{\left(a^2+2^2\right)}\sin 2x+....\right]$$

Ex. 33. Determine the first four terms of the Fourier series for the triangular wave form in the following figure.

(Ans: $y=\frac{8d}{\pi^2}\left[\frac{\sin\omega t}{1^2}-\frac{\sin 3\omega t}{3^2}+\frac{\sin 5\omega t}{5^2}-\frac{\sin 7\omega t}{7^2}+......\right]$**)**



Sol : The function is odd. Hence the Fourier series contains only sine terms.

$$y=f(\omega t);=\frac{d}{(\pi/2)}\omega t=\frac{2d}{\pi}(\omega t)$$

$$y=\frac{2d}{\pi}\omega t\ for\ 0\le\omega t\le\pi/2=\frac{-2d\omega t}{\pi}+2d\ for\ \pi/2\le\omega t\le 3\pi/2=\frac{2d\omega t}{\pi}-4d\text{ for }\frac{3\pi}{2}\le\omega t\le 2\pi.;$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(\omega t)\sin\frac{n\pi(\omega t)}{\pi}d(\omega t)=\frac{1}{\pi}\int_0^{2\pi}f(\omega)\sin n(\omega t)d(\omega t);$$

$$I=\frac{1}{\pi}\left\{\int_0^{\pi/2}\frac{2d\omega t}{\pi}\sin n(\omega t)d(\omega t)+\int_{\pi/2}^{3\pi/2}\left(\frac{-2d\omega t}{\pi}+2d\right)\sin n(\omega t)d(\omega t)\right\}+\int_{3\pi/2}^{2\pi}\left(\frac{2d}{\pi}\omega t-4d\right)\sin n(\omega t)d(\omega t);$$

$$I=I_1+I_2+I_3;\ I_1=\frac{2d}{\pi^2}\int_0^{\pi/2}(\omega t)\sin n\omega td(\omega t)=\frac{2d}{\pi^2}\left[\frac{-\pi}{2}\cos\frac{n(\pi/2)}{n}+\frac{\sin n(\pi/2)}{n^2}\right];$$

$$=-\frac{\omega t\cos n(\omega t)}{n}\bigg|_0^{\frac{\pi}{2}}+\frac{\sin n(\omega t)}{n^2}\bigg|_0^{\frac{\pi}{2}};\ -\frac{\pi}{2}\frac{\cos n\left(\frac{\pi}{2}\right)}{n}+\frac{\sin n\left(\frac{\pi}{2}\right)}{n^2};$$

$$I_2=\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left[-\frac{2d(\omega t)}{\pi}+2d\right]\sin n(\omega t)d(\omega t);\ -\left[\frac{-2d(\omega t)}{\pi}+2d\right]\frac{\cos n(\omega t)}{n}\bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}-\frac{2d}{\pi}\frac{\sin n(\omega t)}{n^2}\bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$\therefore I_2 = \frac{1}{\pi}\left[-\left(-3d+2d\right)\frac{\cos\left(\frac{3n\pi}{2}\right)}{n} - \frac{2d}{\pi}\frac{\sin\left(\frac{3n\pi}{2}\right)}{n^2} + (-d+2d)\frac{\cos\left(\frac{n\pi}{2}\right)}{n} + \frac{2d}{\pi}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right];$$

$$I_3 = \frac{1}{\pi}\int_{\frac{3\pi}{2}}^{2\pi}\left[\frac{2d}{\pi}(\omega t) - 4d\right]\sin n(\omega t)d(\omega t); \quad -\left[\frac{2d}{\pi}(\omega t) - 4d\right]\frac{\cos n(\omega t)}{n}\Bigg|_{\frac{3\pi}{2}}^{2\pi} + \frac{2d}{\pi}\frac{\sin n(\omega t)}{n^2}\Bigg|_{\frac{3\pi}{2}}^{2\pi};$$

$$I_3 = \frac{1}{\pi}\left[-(4d-4d)\frac{\cos(2n\pi)}{n} + \frac{2d}{\pi}\frac{\sin(2n\pi)}{n^2} + (3d-4d)\frac{\cos\left(\frac{3n\pi}{2}\right)}{n} - \frac{2d}{\pi}\frac{\sin\left(\frac{3n\pi}{2}\right)}{n^2}\right];$$

$$b_n = \frac{d}{\pi}\left[\begin{array}{c}\frac{-1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n}\cos\left(\frac{3n\pi}{2}\right) - \frac{2}{n^2\pi}\sin\left(\frac{3n\pi}{2}\right) + \frac{1}{n}\cos\left(\frac{n\pi}{2}\right) \\ + \frac{2}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n}\cos\left(\frac{3n\pi}{2}\right) - \frac{2}{n^2\pi}\sin\left(\frac{3n\pi}{2}\right)\end{array}\right]$$

$$= \frac{d}{\pi}\left[\frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi}\sin\left(\frac{3n\pi}{2}\right)\right] = \frac{4d}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right]$$

$$\frac{4d}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right] = \frac{8d}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

The Fourier expansion for $y = f(\omega t)$; $f(\omega t) = \frac{8d}{\pi^2}\left[\frac{\sin(\omega t)}{1^2} - \frac{\sin 3(\omega t)}{3^2} + \frac{\sin 5(\omega t)}{5^2} + ...\right]$

Ex. 34. A bag contains many balls, each with a number painted on it. There are exactly n balls which have the number n. [one ball with 1, 2 balls with 2 and so on N balls with N on them]. An experiment consists of choosing a ball at random and noting the number on it and returning it to the box. Find the probability of getting the number 36? **(Ans :** $\frac{72}{n^2+n}$**)**

Sol : $p = \frac{36 \times 2}{n(n+1)} = \frac{72}{n^2+n}$

Ex. 35. The probability of hitting a target is 2/5. A person fires at the target 10 times. What is the probability that he hits the target 6 times. **(Ans: 0.11)**

Sol : $p = 2/5; q = 1 - 2/5 = 3/5; p = {}^{10}C_6\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^4 = \frac{10!}{6!4!}\frac{2^6}{5^6}\frac{3^4}{5^4} = \frac{1088640}{15625 \times 625} = 0.11$

Ex. 36. In a box there are 10 alphabet cards with the letters 3A's 4Ms, 3Ns. One draws 3 cards one after another and places three cards on the table? find the probability that the word MAN appears? **(Ans: 1/20)**

Sol : $MAN = \frac{4}{10} \times \frac{3}{9} \times \frac{3}{8} = \frac{36}{10 \times 9 \times 8} = \frac{1}{20}$

Ex. 37. Compute the probability of obtaining at least two "Six" in rolling a fair die 4 times. **(Ans: 0.132)**

Sol : $p(A) = 1/6, q = 1 - 1/6 = 5/6$

$$= {}^4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 + {}^4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^1 + {}^4C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^0$$

$$= \frac{1}{6^4}\left[\frac{4!}{2!2!}5^2 + \frac{4!}{3!1!}5 + 1\right] = \frac{1}{6^4}[6 \times 25 + 20 + 1] = 0.132$$

Ex. 38. If the probability of producing a defective screw is $\rho = 0.01$, what is the probability that a lot of 100 screws will contain more than 2 defectives? **(Ans: 8.03%)**

Sol : Poisson distribution $\mu = np = 100 \times 0.01 = 1; p(X = x) = \frac{e^{-\mu}\mu^x}{x!};$

$$p(A^c) = p(x=0) + p(x=1) + p(x=2) = \left(1 + 1 + \frac{1}{2}\right)e^{-1} = 0.9197 = 91.97\%; p(A) = 8.03\%$$

Ex. 39. If the probability that on any working day a garage will get 10 – 20, 21 – 30, 31 – 40, over 40 cars to service is 0.20, 0.35, 0.25, 0.12 respectively, what is the probability that on a given working day the garage gets at least 21 cars to service? **(Ans: 0.72)**

Sol : Since there are mutually exclusive event. $p = 0.35 + 0.25 + 0.12 = 0.72$

Ex. 40. In tossing a fair die, what is probability of getting an odd number or a number less than 4? **(Ans: 4/6)**

Sol : A be the event of getting odd number. B be the event of getting number less than 4.

$$p(A \cup B) = p(A) + p(B) - p(A \cap B); 1\ 2\ 3\ 4\ 5\ 6; p(A) = 3/6; p(B) = 3/6; p(A \cap B) = 2/6;$$

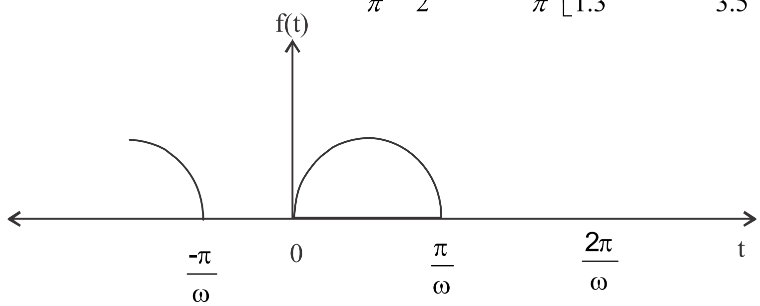
$$p = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6}$$

Ex. 41. Five coins are tossed simultaneously. Find the probability of the event A: At least one head turns up. Assume that the coins are fair? **(Ans: 31/32)**

Sol : Sample space contains $2^5 = 32$ outcomes. Since the coins are fair, we may assign same probability 1/32 to each outcome. The event A^c (No heads up) $= 1/32; p(A) = 1 - p(A^c) = 1 - 1/32 = 31/32$

Ex. 42. A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half wave rectifier that dips the negative position of the wave. Find the Fourier series of the resulting periodic function.

(Ans: $f(t) = \frac{E}{\pi} + \frac{E}{2}\sin \omega t - \frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + ...\right]$**)**



Sol : $f(t)=\begin{cases}0 & if \quad -L<t<0\\ E\sin\omega t & if \quad \theta<t<L\end{cases}$

$$a_0=\frac{1}{L}\int_c^{c+2L} f(t)dt=\frac{\omega}{\pi}\int_0^{\pi}\sin\omega t dt \;=\frac{\omega E}{\pi}\frac{-\cos(\omega t)}{\omega}\Bigg|_0^{\frac{\pi}{\omega}}$$

$$=\frac{\omega E}{\pi}\left[1+1\right]=\frac{2E}{\pi};a_n=\frac{1}{L}\int_c^{c+2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$=\frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin(\omega t)\cos\frac{n\pi t\omega}{\pi}dt\;;$$

$$\frac{E\omega}{\pi}\frac{1}{2}\int_0^{\frac{\pi}{\omega}}\left[\sin(n+1)\omega t+\sin(1-n)\omega t\right]dt=\frac{\omega E}{2\pi}\left[\frac{-\cos(1+n)\omega t}{(1+n)\omega}-\frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega};$$

if $n=1$ the integral on the right is zero. If $n=2, 3, ..$ we have;

$$a_n=\frac{\omega E}{2\pi}\left[\frac{-\cos(1+n)\omega t}{(1+n)\omega}-\frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$=\frac{\omega E}{2\pi}\left[\frac{-\cos(1+n)\pi}{(1+n)\omega}-\frac{\cos(1-n)\pi}{(`1-n)\omega}+\frac{1}{(1+n)\omega}+\frac{1}{(1-n)\omega}\right]$$

$$=\frac{E}{2\pi}\left[\frac{-\cos(1+n)\pi+1}{(1+n)}+\frac{-\cos(1-n)\pi+1}{(1-n)}\right];\text{ If } n \text{ is odd. } a_n=\frac{E}{2\pi}\left[\frac{-1+1}{(1+n)}+\frac{-1+1}{(1-n)}\right]=0;$$

if n is even

$$a_n=\frac{E}{2\pi}\left[\frac{2}{1+n}+\frac{2}{1-n}\right]=\frac{E}{2\pi}\left[\frac{2(1-n)+2(1+n)}{(n+1)(1-n)}\right]=-\frac{E}{2\pi}\left[\frac{2-2n+2+2n}{(n+1)(n-1)}\right]=-\frac{2E}{(n+1)(n-1)\pi}$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega t)+b_n\sin(n\omega t);b_1=\frac{E}{2};b_n=0\text{ for } n=2,3,...$$

$$\therefore f(t)=\frac{E}{\pi}+\frac{E}{2}\sin\omega t-\frac{2E}{\pi}\left(\frac{1}{1.3}\cos 2\omega t+\frac{1}{3.5}\cos(4\omega t)+....\right)$$

Ex. 43. Two drunkards start out together at the origin, each having equal probability of making a step simultaneously to the left or right along the x-axis. Find the probability that they meet after n steps is

(Ans: $\frac{1}{4^n}\frac{2n!}{n!^2}$**)**

Sol : N_1 steps right; N_2 steps left $N=N_1+N_2;N_1-N_2=0;N_1=N_2=n;p=1/2;q=1/2;$

$$p={}^N C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n={}^{2n}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n=\frac{2n!}{n!n!}\frac{1}{2^n2^n}=\frac{2n!}{n!^2}\frac{1}{2^{2n}}=\frac{1}{4^n}\frac{2n!}{n!^2}$$

COMPLEX ANALYSIS

Ex. 44. If $\alpha, \beta, x, y \in R$ and $\sin(\alpha + i\beta) = x + iy$ then. Find $\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = ?$ **(Ans: 1)**

Sol : $\sin(\alpha + i\beta) = \sin\alpha\cos(i\beta) + \cos\alpha\sin(i\beta); x + iy = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta$

$$\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = \frac{\sin^2\alpha\cosh^2\beta}{\cosh^2\beta} + \frac{\cos^2\alpha\sinh^2\beta}{\sinh^2\beta} = \sin^2\alpha + \cos^2\alpha = 1$$

Ex. 45. If $\bar{z}$ is the complex conjugate of z, then find the value of the integral $\int_c \bar{z}dz$ from z = 0 to $z = 4 + 2i$ along the curve c given by $z = t^2 + it$ is **(Ans:** $10 - \frac{8i}{3}$ **)**

Sol : $z = t^2 + it; dz = 2tdt + idt = (2t + i)dt; \bar{z}dz = \left(t^2 - it\right)\left(2t + i\right)dt.$

$$\int \bar{z}dz = \int\left(2t^3 + it^2 - 2it^2 + t\right)dt = \int_0^2\left(2t^3 - it^2 + t\right)dt = \left.\frac{2t^4}{4}\right|_0^2 - \left.\frac{it^3}{3}\right|_0^2 + \left.\frac{t^2}{2}\right|_0^2 = \frac{2\times 2}{4} - \frac{i2^3}{3} + \frac{2^2}{2} = 8 - \frac{i8}{3} + 2$$

$$\Rightarrow = 10 - \frac{8i}{3}$$

Ex. 46. Find the value of the integral $\oint \frac{e^z \sin z}{z^2}\frac{dz}{2\pi}$ around the unit circle in complex plane is **(Ans: i)**

Sol : $\oint \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(a); \frac{1}{2\pi}\oint \frac{e^z \sin z}{(z-0)^2} = \frac{2\pi i}{1!2\pi}\left[e^z \cos z + \sin z e^z\right]_{z=0} = i$

Ex. 47. Find the value of $\oint \frac{dz}{(z-a)^n}$ where n is a positive integer and $z = a$ lies inside the simple closed curve c. **(Ans: $2\pi i$ for $n = 1$ and 0 for all other value of n)**

Sol : if $n = 1$ $\oint \frac{dz}{(z-a)} = 2\pi i \sum R; R = \lim_{z\to a}(z-a)\frac{1}{(z-a)} = 1; \oint \frac{dz}{(z-a)} = 2\pi i$;

if $n = 2$ $\oint \frac{dz}{(z-a)^2} = 2\pi i \sum R = 0; R = \frac{1}{1!}\lim_{z\to a}\frac{d}{dz}(1) = 0$

Ex. 48. Evaluate $\int_c \left(x^2 - iy\right)\left(dx + idy\right)$ along $y = x^2$; $x = y^2$. **(Ans:** $\frac{19i}{30}$ **)**

Sol : I: $\int_0^1 \left(x^2 - iy\right)\left(dx + idy\right)$ along $y = x^2; dy = 2xdx;$

$$= \int_0^1 \left(x^2 - ix^2\right)(dx + i2xdx) = (1-i)\int_0^1 (x^2 + 2ix)dx$$

$$= (1-i)\int_0^1 x^2(1+2ix)\,dx = (1-i)\left[\frac{x^3}{3}+\frac{2ix^4}{4}\right]_0^1 = \frac{5}{6}+\frac{i}{6} \text{......(1)}$$

II: $\int_0^1 (x^2-iy)(dx+idy) = \int_1^0 (y^4-iy)(2ydy+idy) = \int_0^0 (y^4-iy)(2y+i)dy$

$$= \int_1^0 (2y^5+iy^4-2iy^2+y)\,dy = \frac{2y^6}{6}+\frac{iy^5}{5}-\frac{2iy^3}{3}+\frac{y^2}{2}\Bigg|_1^0 = \frac{-1}{3}-\frac{i}{5}+\frac{2i}{3}-\frac{1}{2} = \frac{-5}{6}+\frac{7i}{15} \text{......(2)}$$

From (1) and (2) we have

$$\int_c (x^2-iy)(dx+idy) = \frac{5}{6}+\frac{i}{6}-\frac{5}{6}+\frac{7i}{15} = \frac{(15+42)i}{6\times 15} == \left(\frac{57}{6\times 15}\right)i = \left(\frac{19}{30}\right)i$$

Ex. 49. Branch points of multiple valued functions are non isolated singular points. Find the branch points of the function $f(z) = \ln(z^2+z-2)$ **(Ans:** $z = 1, z = -2$ **)**

Sol : Branch points are at $z^2+z-2=0 \Rightarrow (z-1)(z+2)=0; z=1,-2$

Ex. 50. If the potential function is $\log\sqrt{x^2+y^2}$, find the complex potential function.

[Ans: $\omega = \log z + c$ (where c is a complex constant) **]**

Sol : Let u be the potential function, v be the flux function ω be the complex potential function.

$$\omega = u+iv; u = \log\sqrt{x^2+y^2}; u = \frac{1}{2}\log(x^2+y^2); \frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{(x^2+y^2)} = \frac{x}{x^2+y^2};$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\omega^1 = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \left(\frac{x}{x^2+y^2}\right)+i\left(\frac{-y}{x^2+y^2}\right) \Rightarrow \omega^1 = \left(\frac{1}{z}\right) \Rightarrow \omega = \log z + c \quad \text{'}c\text{' is complex constant.}$$

Ex. 51. Find the values of constants *a, b, c* and *d* such that the function $f(z) = x^2+axy+by^2+i(cx^2+dxy+y^2)$ is analytic. **(Ans: *a* = 2, *b* = -1, *c* = –1, *d* = 2)**

Sol : $f(z) = x^2+axy+by^2+i(cx^2+dxy+y^2)$; $u = x^2+axy+by^2; v = cx^2+dxy+y^2$

$$\frac{\partial u}{\partial x} = 2x+ay, \frac{\partial u}{\partial y} = ax+2by; \frac{\partial v}{\partial y} = dx+2y, \frac{\partial v}{\partial x} = 2cx+dy; \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$2x+ay = dx+2y----(1), ax+2by = -2cx-dy---(2)$

From (1) $(2-d)x+(a-2)y=0$,

from (2) $(a+2c)x+(2b+d)y=0$; $2-d=0, d=2$; $a-2=0, a=2$ $a+2c=0$; $c=-1$; $2b+d=0$; $b=-1$

Ex. 52. Locate and name the singularity for the function $f(z) = \sec(1/z)$ in the finite z plane. Determine whether they are isolated singularity or not?

[Ans: $z = \dfrac{2}{\pi(2n+1)}$ **is a pole of order one and $z = 0$ essential singularity (not isolated)]**

Sol : $f(z) = \sec\left(\dfrac{1}{z}\right); f(z) = \dfrac{1}{\cos\left(\dfrac{1}{z}\right)}$; poles are at $\cos\left(\dfrac{1}{z}\right) = 0; \dfrac{1}{z} = (2n+1)\dfrac{\pi}{2} \Rightarrow z = \dfrac{2}{\pi(2n+1)}$; Also $f(z)$ is not defined at $z = 0$, it is also a singularity. $z = \dfrac{2}{\pi(2n+1)}$ is a pole of order one. These poles are located on the real axis at $z = \pm\dfrac{2}{\pi}, \pm\dfrac{2}{3\pi}, \ldots$ These are isolated singularities. At $z = 0$ we cannot define any +ve integer 'n' such that $\lim\limits_{z\to 0}(z-0)^n f(x) = A \neq 0$.

$\therefore$ $z = 0$ is an essential singularity which is not isolated.

Ex. 53. A particle moves along a curve $z = e^{-t}(2\sin t + i\cos t)$. Determine the magnitude of velocity at $t = 0$?

(Ans: $\sqrt{5}$ **)**

Sol : $z = e^{-t}(2\sin t + i\cos t)$; velocity = dz/dt; $v = e^{-t}[2\cos t - i\sin t] + [2\sin t + i\cos t]e^{-t} \times -1$ at $t = 0$

$v = 2 - i; \sqrt{4+1} = \sqrt{5}$.

Ex. 54. If $f(\xi) = \int_c \dfrac{4z^2 + z + 5}{z - \xi} dz$ where c is the ellipse $\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 1$. Find the value of $f(3.5)$? **(Ans: zero)**

Sol : $f(3.5) = \int_c \dfrac{4z^2 + z + 5}{z - 3.5} dz \Rightarrow$ 3.5 is the singular point. $\therefore f(3.5) = 0$

Ex. 55. Find the value of the integral $\int_c e^{\frac{1}{z}} dz$ where c: $|z| = 1$ and z is a complex variable is: **(Ans:** $2\pi i$ **)**

Sol : $e^{\frac{1}{z}} = 1 + \dfrac{1}{z} + \left(\dfrac{1}{z}\right)^2 \dfrac{1}{2!} + \ldots\ldots\ldots$ principle part contains infinite number of terms essential singularity.

Residue = 1. $\int_c e^{\frac{1}{z}} dz = 2\pi i$.

Ex. 56. Find the value of the integral $\int \dfrac{e^{iz}}{z + 3i} dz$, where c is the circle $|z + 3i| = 1$, traversed the counter clockwise is: **(Ans:** $2\pi i e^3$ **)**

Sol : $\int \frac{f(z)}{z-a}dz = 2\pi i f(a); \int \frac{e^{iz}}{z-3i} = 2\pi i f(-3i) = 2\pi i e^{i(-3i)} = 2\pi i e^{3}$

Ex. 57. Find the residue at infinity $f(z) = \frac{z^3}{z^2-1}$ **(Ans: -1)**

Sol : $f(z) = \frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)} = z\left(1-\frac{1}{z^2}\right)^{-1} = z\left[1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+....\right] = z+\frac{1}{z}+\frac{1}{z^3}+...$ Residue at infinity is -1

Ex. 58. Evaluate the integral using Cauchy integral theorem $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$ **(Ans: $4\pi i$)**

Sol : $\oint \frac{f(z)}{(z-a)} = 2\pi i f(a) \quad I = \oint \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}}{(z-2)}dz + \oint \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}}{(z-1)}dz = 2\pi i\left[1+\frac{-1}{-1}\right] = 4\pi i$

Ex. 59. What is the real part of principal value of 4^{4-i} is **[Ans: 256 cos (ln 4)]**

Sol : Let $z = 4^{4-i}; x+iy = 4^{4-i}$

$\ln(x+iy) = \ln(4^{4-i}) \Rightarrow \ln(x+iy) = (4-i)\ln 4;$

$x+iy = e^{(4-i)\ln 4};\ x+iy = e^{4\ln 4}e^{-i\ln 4} = e^{\ln 4^4} \times e^{-i\ln 4};\ x+iy = 4^4 \times e^{-i\ln 4};$

$x+iy = 4^4\left[\cos(\ln 4) - i\sin(\ln 4)\right] \Rightarrow x = 256\cos(\ln 4)$

Ex. 60. Find the value of the contour integral $\left|\int_c \vec{r} \times d\vec{\theta}\right|$, for a circle '*c*' of radius *r* with center at origin is.

(Ans: $2\pi r$)

Sol : $z = re^{i\theta}; \frac{dz}{iz} = d\theta; \int_c \vec{r} \times d\overline{\theta} = \int_c \frac{rdz}{iz} = \frac{r}{i}\int \frac{dz}{z} = \frac{r}{i} \times 2\pi i = 2\pi r$

Ex. 61. Evaluate $\oint_c \frac{1-2z}{(z^2+2z+5)}dz \ for \ |z+1+i| = 2$ **[Ans: $\frac{-\pi}{2}(3+4i)$]**

Sol : $z^2+2z=5=0; z = \frac{-2\pm\sqrt{4-20}}{2} = \frac{2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2} = \frac{-2+4i}{2}, \frac{-2-4i}{2} = -1+2i, -1-2i;$

$(z^2+2z+5) = [z-(-1+2i)]\ [z-(-1-2i)];\ |z+1+i| = |-1+2i+1+i| = |3i|$

is outside the contour. $|z+1+i| = |-1-2i+1+i| = |-i|$ is within the contour. –1 – 2i is inside the contour

$$\oint \frac{\frac{1-2z}{z-(-1+2i)}}{[z-(-1-2i)]} dz = 2\pi i f(a),\ a = -1-2i;$$

$$= \frac{2\pi i[1-2(-1-2i)]}{-1-2i+1-2i} = \frac{2\pi i[1+2+4i]}{-4i} = \frac{2\pi(3+4i)}{-4i} = -\frac{\pi}{2}(3+4i)$$

CONTOUR INTEGRATION TAYLOR'S & LAURENT'S SERIES VECTOR CALCULUS

Ex. 62. Find Taylor's expansion of $f(z) = \frac{1}{(z+1)^2}$ about the point $z = -i$

(Ans: $\frac{i}{2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n (n+1)(z+i)^n}{(1-i)^n}\right]$**)**

Sol : $f(z) = \frac{1}{(z+1)^2}$ put $t = z + i$; $f(z) = \frac{1}{(t-i+1)^2} = \frac{1}{(1-i+t)^2} = \frac{1}{(1-i)^2\left[1+\frac{t}{1-i}\right]^2}$

$$= \frac{1}{(1-i)^2}\left[1+\frac{t}{(1-i)}\right]^{-2} = \frac{1}{1-2i-1}\left[1+\frac{t}{(1-i)}\right]^{-2};\ f(z) = \frac{i}{2}\left[1+\frac{t}{(1-i)}\right]^{-2}$$

$$= \frac{i}{2}\left[1-\frac{2t}{(1-i)}+\frac{3t^2}{(1-i)^2}-\frac{4t^3}{(1-i)^3}+.....\right];\ f(z) = \frac{i}{2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n (n+1)t^n}{(1-i)^n}\right];$$

$$f(z) = \frac{i}{2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n (n+1)(z+i)^n}{(1-i)^n}\right]$$

Ex. 63. Find the Laurent's expansion of $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in the region $1 < z + 1 < 3$.

(Ans: $\frac{-2}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+.....-\frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+........\right]$**)**

Sol : $f(z) = \frac{7z-2}{(z+1)z(z-2)}$; put $u = z+1$; region is $f(z) = \frac{7(u-1)-2}{u(u-1)(u-3)} = \frac{7u-9}{u(u-1)(u-3)}$ partial fractions

$$\frac{7u-9}{u(u-1)(u-3)} = \frac{A}{u}+\frac{B}{u-1}+\frac{C}{u-3};\ 7u\text{-}9 = A(u-1)(u-3)+Bu(u-3)+Cu(u-1)$$

put $u = 1;\ 7-9 = B(-2) \Rightarrow B = 1;$ put $u = 3;\ 21-9 = C\times 6; 12 = 6C; C = 2;$

put u = 0 $\Rightarrow -9 = -A(-3) \Rightarrow -9 = 3A; A = -3$;

$$f(z) = \frac{-3}{u} + \frac{1}{u-1} + \frac{2}{u-3}; 1 < u < 3; \frac{1}{u} < 1; \frac{u}{3} < 1 \quad = \frac{-3}{u} + \frac{1}{u(1-\frac{1}{u})} - \frac{2}{3(1-\frac{u}{3})}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} = -\frac{3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1};$$

$$= \frac{-3}{u} + \frac{1}{u}\left[1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + ...\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \left(\frac{u}{3}\right)^3 + ...\right]$$

$$= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + ... - \frac{2}{3}\left[1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \left(\frac{u}{3}\right)^3 + ...\right]$$

$$f(z) = \left(\frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \frac{1}{(z+1)^4} + ...\right) + \frac{-2}{3}\left[1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \left(\frac{z+1}{3}\right)^3 + ...\right]$$

Ex. 64. Consider a right-handed orthonormal system of co-ordinates $(\hat{u}, \hat{v}, \hat{\omega})$. If the infinitesimal displacement in these co-ordinates is $\frac{1}{f} du\hat{u} + \frac{1}{g} dv\hat{v} + \frac{1}{h} d\omega\hat{\omega}$. Then find the gradient of a scalar field T in these co-ordinates is: **(Ans:** $f\left(\frac{\partial T}{\partial u}\right)\hat{u} + g\left(\frac{\partial T}{\partial v}\right)\hat{v} + h\left(\frac{\partial T}{\partial \omega}\right)\hat{\omega}$ **)**

Sol : For any system of co-ordinates q_1, q_2, q_3 with length elements $h_1 dq_1\hat{q} + h_2 dq_2\hat{q} + h_3 dq_3\hat{q}$

$$\nabla\phi = \frac{\hat{q}_1}{h_1}\frac{\partial\phi}{\partial q_1} + \frac{\hat{q}_2}{h_2}\frac{\partial\phi}{\partial q_2} + \frac{\hat{q}_3}{h_3}\frac{\partial\phi}{\partial q_3}$$

In given system of co-ordinates

$$h_1 = \frac{1}{f}; h_2 = \frac{1}{g}; h_3 = \frac{1}{h} \quad \nabla T = \hat{u}f\frac{\partial T}{\partial u} + \hat{v}g\frac{\partial T}{\partial v} + \hat{\omega}h\frac{\partial T}{\partial \omega}; \nabla T = f\frac{\partial T}{\partial u}\hat{u} + g\frac{\partial T}{\partial v}\hat{v} + h\frac{\partial T}{\partial \omega}\hat{\omega}$$

Ex. 65. Find the value of $\nabla\cdot(r^n\hat{r})$, $\hat{r}$ being the unit vector where in 'd' dimensions, is **(Ans:** $(n+d)r^{n-1}$**)**

Sol : For 3-D space $\nabla\cdot(r^n\vec{r}) = (n+3)r^n$ *or* $\nabla\cdot(r^n\hat{r}) = (n+3)r^{n-1}$ for 3-D space for 'd' dimensional space $\nabla\cdot(r^n\hat{r}) = (n+d)r^{n-1}$.

Ex. 66. If $\vec{F} = (x^2+y^2+z^2)^n(x\hat{i} + y\hat{j} + z\hat{k})$ and if $\vec{F} = -\nabla V$ then. V is equal to? **(Ans:** $\frac{-(x^2+y^2+z^2)^{n+1}}{2(n+1)}$ **)**

Sol : Let $V = \frac{-(x^2+y^2+z^2)^{n+1}}{2(n+1)}$; $\nabla V = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$; $\frac{\partial V}{\partial x} = \frac{-(n+1)(x^2+y^2+z^2)^n}{2(n+1)} \times 2x$;

$$\frac{\partial V}{\partial x} = -x\left(x^2+y^2+z^2\right)^n \Rightarrow -\frac{\partial V}{\partial x} = x(x^2+y^2+z^2)^n,\ -\frac{\partial V}{\partial y} = y\left(x^2+y^2+z^2\right)^n;$$

$$-\frac{\partial V}{\partial z} = z\left(x^2+y^2+z^2\right)^n \Rightarrow -\nabla V = \left(x^2+y^2+z^2\right)^n\left[x\hat{i}+y\hat{j}+z\hat{k}\right] \text{ given function.}$$

Ex. 67. Find the value of the integral $\int_c \frac{z^3 dz}{z^2-5z+6}$ where c is a closed contour defined by the equation $2|z|$ -5 =0 traversed in the anticlockwise direction? **(Ans:** $-16\pi i$ **)**

Sol : $\int_c \frac{z^3 dz}{z^2-5z+6} dz = 2\pi i \sum$ Residues

$z^2-5z+6=0 \Rightarrow (z-2)(z-3)=0 \Rightarrow z=2,3$ are the poles. $Z=2$ lies inside the contour.

Residue $= \lim_{z\to 2}(z-2)\frac{z^3}{(z-2)(z-3)} = \frac{8}{-1} = -8 \therefore \int_c \frac{z^3 dz}{z^2-5z+6} = 2\pi i(-8) = -16\pi i$

Ex. 68. What is the directional derivative of xyz^2+xz at $(1,1,1)$ in the direction of normal to the surface $3xy^2+y=z$ at $(0,1,1)$? **(Ans:** $\frac{4}{\sqrt{11}}$ **)**

Sol : $\nabla\phi_1 = \hat{i}\left(yz^2+z\right)+\hat{j}\left(xz^2\right)+\hat{k}\left(2xyz+x\right)\Big|_{(1,1,1)} = 2\hat{i}+\hat{j}+3\hat{k}$.

To find the normal to the surface

$3xy^2+y-z=0$; $\nabla\phi_2 = \hat{i}\left(3y^2\right)+\hat{j}\left(6xy+1\right)+\hat{k}\left(-1\right)\Big|_{(0,1,1)} = 3\hat{i}+\hat{j}-\hat{k}$

The directional derivative in the direction of normal to the surface is

$$\nabla\phi_1 . \frac{\nabla\phi_2}{|\nabla\phi_2|} = \left(2\hat{i}+\hat{j}+3\hat{k}\right).\frac{\left(3\hat{i}+\hat{j}-\hat{k}\right)}{\sqrt{9+1+1}} = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

Ex. 69. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2}$ **(Ans:** $\frac{2\pi a^2}{1-a^2}$ **)**

Sol : $\cos\theta = \frac{1}{2}(z+\frac{1}{z}); \cos 2\theta = \frac{1}{2}(z^2+\frac{1}{z^2});$

$$\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} = \int_0^{2\pi} \frac{1/2(z^2+1/z^2)}{1-\frac{2a}{2}(z+\frac{1}{z})+a^2}\left(\frac{dz}{iz}\right)$$

$$= \frac{1}{2i}\int_c \frac{(z^2+1/z^2)dz}{z\left[1-a\left(z+1/z\right)+a^2\right]} = \frac{1}{2i}\int_c \frac{z^4+1}{z^3[1-a\frac{(z^2+1)}{z}+a^2]} = \frac{1}{2i}\int_c \frac{z^4+1}{z^3-az^4-az^2+a^2z^3}$$

$$= \frac{-1}{2i}\int_c \frac{(z^4+1)dz}{az^4-z^3-a^2z^3+az^2} = \frac{-1}{2i}\int_c \frac{(z^4+1)dz}{z^2[az^2-z-a^2z+a]} = \frac{1}{2i}\int_c \frac{(z^4+1)}{z^2[z(1-az)+a(az-1)]}dz$$

$f(z)$ has a second order pole at $z = 0$ and simple poles at $z = a$; $1 - az = 0$; $1 = az$; $z = 1/a$ outside the unit circle.

At $z = 0$;

$$R = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[\frac{z^2(z^4+1)}{z^2(1-az)(z-a)}\right]$$

$$= \frac{(1-az)(z-a)4z^3-(z^4+1)[(1-az)+(z-a)-a]}{[(1-az)(z-a)^2} = \frac{1(-a)\times 0-[1+a^2]}{a^2} = \frac{-(1+a^2)}{a^2}$$

At $z = a$;

$$R = \lim_{z\to a}\frac{(z-a)(z^4+1)}{z^2(1-az)(z-a)} \Rightarrow R = \frac{a^4+1}{a^2(1-a^2)};$$

$$\sum R = \frac{-(1+a^2)}{a^2} + \frac{a^4+1}{a^2(1-a^2)} = \frac{1}{a^2}\left[-1-a^2+\frac{a^4+1}{(1-a^2)}\right]$$

$$= \frac{1}{a^2}\left[\frac{(1-a^2)(-1-a^2)+a^4+1}{(1-a^2)}\right] = \frac{1}{a^2}\left[\frac{-1-a^2+a^2+a^4+a^4+1}{(1-a^2)}\right]$$

$$= \frac{2a^4}{a^2(1-a^2)};\ \sum R = \frac{2a^2}{1-a^2}$$

Substitute in (1) $I = \frac{1}{2i}\left(\frac{2a^2}{1-a^2}\right)2\pi i$; $I = \frac{2\pi a^2}{1-a^2}$

Ex. 70. Evaluate $\int_c \frac{z\sec z}{\left(1-z\right)^2}dz \quad C: |z-1| = 1$ **[Ans:** $2\pi i\sec(1)\left(1+\tan(1)\right)$ **]**

Sol : $f(z) = \frac{z\sec z}{(1-z)^2} \Rightarrow$ pole of order '2' at $z = 1$; $R = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(1-z)^2\frac{z\sec z}{(1-z)^2}$;

$R = z\sec z\tan z + \sec z = \sec(1)\tan(1) + \sec(1) = \sec(1)\left[1+\tan(1)\right]$;

$$\int_c \frac{z\sec z}{(1-z)^2}dz = 2\pi i\sec(1)\left[1+\tan(1)\right]$$

Ex. 71. Find the Laurent series about $z = 1$ for the function $f(z) = \dfrac{e^{2z}}{(z-1)^3}$. Give the region of convergence

(Ans: $f(z) = \dfrac{e^2}{(z-1)^3} + \dfrac{2e^2}{(z-1)^2} + \dfrac{2e^2}{(z-1)} + \dfrac{4e^2}{3} + \dfrac{2e^2}{3}(z-1) + ...$ **series converges for all values of** $z \neq 1$**.)**

Sol : $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$; put $t = z-1; z = t+1; f(z) = \dfrac{e^{2(t+1)}}{(t+1)^3} = \dfrac{e^{2t}e^2}{t^3}$

$$= \frac{e^2}{t^3}\left[1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + ...\right]$$

$$= \frac{e^2}{(z-1)^3}\left[1 + 2(z-1) + \frac{4(z-1)^2}{2} + \frac{8(z-1)^3}{6} + \frac{2^4(z-1)^4}{2.3.4} + ...\right]$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{(z-1)} + \frac{4e^2}{3} + \frac{2}{3}e^2(z-1) + ...$$

Ex. 72. Find the residue of $f(z) = \dfrac{\cot z \coth z}{z^3}$ $at\ z = 0$? **(Ans: -7/45)**

Sol : $f(z) = \dfrac{\cos z \cosh z}{z^3 \sin z \sinh z} = \dfrac{1}{z^3}\dfrac{\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...\right]\left[1 + \frac{z^2}{2!} + \frac{z^4}{4!} + ...\right]}{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...\right]\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...\right]}$

$$\frac{1}{z^3}\frac{\left[1 + \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^2}{2!} - \frac{z^4}{2!2!} - \frac{z^6}{2!4!} + \frac{z^4}{4!} + \frac{z^6}{2!4!}\right]}{\left[z^2 + \frac{z^4}{3!} + \frac{z^6}{5!} - \frac{z^6}{5!} - \frac{z^4}{3!} - \frac{z^6}{3!3!} + \frac{z^8}{3!5!} + \frac{z^6}{5!}\right]}$$

$$= \frac{1}{z^5}\left[\frac{1 + z^4\left[\frac{1}{12} - \frac{1}{4}\right]}{1 + z^4\left(\frac{1}{120} - \frac{1}{36} + \frac{1}{120}\right)}\right] = \frac{1}{z^5}\left[\frac{1 - z^4/6}{1 + z^4\left[\frac{1}{60} - \frac{1}{36}\right]}\right]$$

$$= \frac{1}{z^5}\left[\frac{1 - z^4/6}{1 - \frac{24 \times z^4}{60 \times 36}}\right] = \frac{1}{z^5}\left[\frac{1 - z^4/6}{1 - z^4/90}\right]$$

$$= \frac{1}{z^5}\left[\left(1-\frac{z^4}{6}\right)\left(1-\frac{z^4}{90}\right)^{-1}\right] = \frac{1}{z^5}\left[\left(1-\frac{z^4}{6}\right)\left(1+\frac{z^4}{90}\right)\right]$$

$$= \frac{1}{z^5}\left[1+\frac{z^4}{90}-\frac{z^4}{6}-\frac{z^8}{6\times 90}\right] = \frac{1}{z^5}+\frac{1}{z}\left[\frac{1}{90}-\frac{1}{6}\right]-\frac{z^3}{6\times 90}$$

Coefficient of ½ is $\frac{(6-90)}{6\times 90} = \frac{84}{6\times 90} = \frac{-7}{45}$

Ex. 73. Evaluate $\int \frac{1-2z}{z(z-1)(z-2)}$ where $|z| = 3/2$ **(Ans: $3\pi i$)**

Sol : $f(z) = \frac{1-2z}{z(z-1)(z-2)}$; $z = 0$ and $z = 1$ are poles inside the contour.

$$\underset{z=0}{R} = \lim_{z\to 0}\frac{z(1-2z)}{z(z-1)(z-2)} = \frac{1}{2}$$

$$\underset{z=1}{R} = \lim_{z\to 1}\frac{(z-1)(1-2z)}{z(z-1)(z-2)} = \frac{-1}{1(-1)} = 1;$$

$$I = 2\pi i\sum R = 2\pi i\times\frac{3}{2} = 3\pi i$$

LINEAR ALGEBRA, MATRICES

Ex. 74. Consider the matrix $\begin{bmatrix} 7 & 0 & 0 \\ 4 & 1 & 0 \\ 6 & 2 & -7 \end{bmatrix}$. *If* $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of this matrix, then find the value of

$(\lambda_1+\lambda_2+\lambda_3)^2 + (\lambda_1^2+\lambda_2^2+\lambda_3^2)$ **(Ans: 100)**

Sol : $(\lambda_1+\lambda_2+\lambda_3) = 1\, and\, \lambda_1\lambda_2\lambda_3 = -49 \Rightarrow \lambda_1 = 7; \lambda_2 = -7; \lambda_3 = 1$

$(\lambda_1+\lambda_2+\lambda_3)^2 + (\lambda_1^2+\lambda_2^2+\lambda_3^2)$; $1+49+49+1 = 100$

Ex. 75. Find the eigen values of the matrix $\begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix}$? **(Ans: 3 and 9)**

Sol : Trace $= 12 \Rightarrow \lambda_1+\lambda_2 = 12\, and\, |A| = 27 \Rightarrow \lambda_1\lambda_2 = 27\, which\, gives\, \lambda_1 = 3\, and\, \lambda_2 = 9$

Ex. 76. Find the eigen vectors v_1 and v_2 of the matrix $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$? **[Ans: $v_1 = (2,-1)^T$ $v_2 = (2,1)^T$]**

Sol : $\begin{vmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0; (2-\lambda)^2 - 4 = 0 \Rightarrow (2-\lambda)^2 = 4 \Rightarrow 2-\lambda = \pm 2 \Rightarrow \lambda = 0, 4$

$$\lambda = 0; \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; 2x + 4y = 0; x + 2y = 0; x = -2y; X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\lambda = 4; \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}; 2x + 4y = 4x; 2x = 4y; x = 2y; X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Ex. 77. Consider the following 4x4 matrix S. $S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ if λ_i where $i = 1, 2, 3, 4$ are the eigen values of S, then find the value of $\sum_{i=1}^{4} (\lambda_i)^4$ **(Ans: 4)**

Sol : $S^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$; $S^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \sum_{i=1}^{4} \lambda_i^4 = \text{Trace}(S^4)$$

Ex. 78. Find the eigen values of the antisymmetric matrix.

$$A = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

where n_1, n_2 and n_3 are the components of a unit vector? **(Ans: 0, *i*,-*i*)**

Sol : $A = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}; |A| = n_3(-n_1 n_2) + n_2(n_1 n_2); \begin{vmatrix} 0-\lambda & -n_3 & n_2 \\ n_3 & 0-\lambda & -n_1 \\ -n_2 & n_1 & 0-\lambda \end{vmatrix} = 0$

$$-\lambda[\lambda^2 + n_1^2] + n_3[-\lambda n_3 - n_1 n_2] + n_2[n_1 n_3 - n_2 \lambda] = 0$$

$$-\lambda^3 - n_1^2\lambda - n_3^2\lambda - n_1 n_2 n_3 + n_1 n_2 n_3 - n_2^2\lambda = 0; -\lambda^3 - \lambda(n_1^2 + n_2^2 + n_3^2) = 0;$$

Since n_1, n_2 and n_3 are the components of a unit vector, $n_1^2 + n_2^2 + n_3^2 = 1$;

$$-\lambda^3 - \lambda = 0; \lambda^3 + \lambda = 0 \Rightarrow \lambda(\lambda^2 + 1) = 0; \lambda = 0, \lambda = \pm i$$

Ex. 79. Find the trace of the real 4x4 matrix $U = \exp(A)$ where $A = \begin{bmatrix} 0 & 0 & 0 & \frac{-\pi}{4} \\ 0 & 0 & \frac{-\pi}{4} & 0 \\ 0 & \frac{-\pi}{4} & 0 & 0 \\ \frac{-\pi}{4} & 0 & 0 & 0 \end{bmatrix}$ **(Ans:** $4\cosh\frac{\pi}{4}$**)**

Sol : Put $x = \pi/4$.

$$A = \begin{bmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & -x & 0 \\ 0 & -x & 0 & 0 \\ -x & 0 & 0 & 0 \end{bmatrix};$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & -x & 0 \\ 0 & -x & 0 & 0 \\ -x & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & -x & 0 \\ 0 & -x & 0 & 0 \\ -x & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x^2 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^2 \end{bmatrix};$$

$$A^3 = \begin{bmatrix} -x^3 & 0 & 0 & 0 \\ 0 & -x^3 & 0 & 0 \\ 0 & 0 & -x^3 & 0 \\ 0 & 0 & 0 & -x^3 \end{bmatrix}; \; A^4 = \begin{bmatrix} x^4 & 0 & 0 & 0 \\ 0 & x^4 & 0 & 0 \\ 0 & 0 & x^4 & 0 \\ 0 & 0 & 0 & x^4 \end{bmatrix}$$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots\ldots$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -x \\ 0 & 0 & -x & 0 \\ 0 & -x & 0 & 0 \\ -x & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} x^2 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^2 \end{bmatrix} - \frac{1}{3!}\begin{bmatrix} 0 & 0 & 0 & -x^3 \\ 0 & 0 & -x^3 & 0 \\ 0 & -x^3 & 0 & 0 \\ x^{-3} & 0 & 0 & 0 \end{bmatrix}$$

$$+\frac{1}{4!}\begin{bmatrix} x^4 & 0 & 0 & 0 \\ 0 & x^4 & 0 & 0 \\ 0 & 0 & x^4 & 0 \\ 0 & 0 & 0 & x^4 \end{bmatrix} + \ldots\ldots\ldots$$

$$\text{Trace of } e^A = 4\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right] = 4\cosh x = 4\cosh\frac{\pi}{4} \qquad \therefore x = \pi/4$$

Ex. 80. Matrix $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is an orthogonal matrix. Find the value of $|B|$? **(Ans: 1)**

Sol : $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ For orthogonal matrix $MM^T = I$;

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} A & C \\ B & 0 \end{bmatrix} = I; \begin{bmatrix} A^2+B^2 & AC \\ AC & C^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$C^2 = 1; C = \pm 1\, and\ AC = 0 \Rightarrow A = 0\, and\ A^2 + B^2 = 1 \Rightarrow B^2 = 1\, hence\ |B| = 1$$

NUMERICAL ANALYSIS, SPECIAL FUNCTIONS

Ex. 81. Find the root of $x^3 - x - 1 = 0$ after two approximations by using bisection method is **(Ans: 1.375)**

Sol : $f(1) = 1-1-1 = -\text{ve}; f(-1) = -1+1-1 = -\text{ve}; f(2) = 8-2-1 = 5 + \text{ve}$ root lies in between 1 and 2.

$x_1 = \dfrac{a+b}{2} = 1.5; f(x_1) = (1.5)^2 - 1.5 - 1 = 0.875 + ve;$ Root lies between 1 and 1.5 = 1.25 = x_2;

$f(x_2) = (1.25)^3 - 1.25 - 1 = -0.26 - ve$; roots lies between 1.5 and 1.25 $\Rightarrow x_3 = 1.375$

Ex. 82. Given $a > 0$, the reciprocal value of a is $1/a$ by Newton Rephson method for $f(x) = 0$. What is the Newton Raphson Algorithm is **(Ans: $x_{i+1} = 2x_i - ax_i^2$)**

Sol : $\dfrac{1}{x} = a; f(x) = \dfrac{1}{x} - a; f'(x) = \dfrac{-1}{x^2}; x_{n+1} = x_n - \dfrac{(1/x_n - a)}{-1/x_n^2};\ x_{n+1} = x_n + x_n^2\left(\dfrac{1}{x_n} - a\right); x_{n+1} = 2x_n - ax_n^2$

Ex. 83. In the above problem for a = 7 and starting x_0 = 0.2, then find the first two approximations are **(Ans: 0.12, 0.1392)**

Sol : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2x_0 - ax_0^2; x_1 = 2\times 0.2 - 7\times(0.2)^2 = 0.12$

$x_2 = 2\times 0.12 - 7\times(0.12)^2 = 0.24 - 7(0.0144) = 0.24 - 0.1008 = 0.1392.$

Ex. 84. The recurrence relation to solve $x = e^{-x}$ using Newton Raphson method is

(Ans: $x_{i+1} = \dfrac{(1+x_i)e^{-x_i}}{1+e^{-x_i}}$)

Sol : $f(x) = x - e^{-x}; f'(x) = 1 + e^{-x}; x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{(x_n - e^{-x_n})}{1+e^{-x_n}}$

$$= \frac{x_n(1+e^{-x_n}) - x_n + e^{-x_n}}{(1+e^{-x_n})} = \frac{x_n + x_n e^{-x_n} - x_n + e^{-x_n}}{(1+e^{-x_n})} = \frac{e^{-x_n}(x_n+1)}{1+e^{-x_n}}$$

Ex. 85. If $f(0) = 1; f(\pi/6) = 1.6487; f(\pi/3) = 2.3632; f(\pi/2) = 2.7182,\ \int_0^{\pi/2} f(x)dx$ is. Use Trapezoidal rule.

(Ans: 3.072)

Sol :

x	0	π/6	π/3	π/2
f(x)	1	1.6487	2.3632	2.7182

Here we have three sub intervals.

$$\int_0^{\pi/2} f(x)dx = \frac{h}{2}\left[y_0 + y_n + 2\left(y_1 + ... + y_{n-1}\right)\right] = \frac{\pi}{2\times 6}\left[1 + 2.7182 + 2\left(1.6487 + 2.3632\right)\right]$$

$$= \frac{\pi}{12}\left[3.7182 + 2(4.0119)\right] = \frac{\pi}{12}\left[3.7182 + 8.0238\right] = 3.072.$$

Ex. 86. $\frac{dy}{dx} = f(x)\ y(0) = 0.$ Find the solution at $x = h$ is **(Ans:** $y = \frac{h}{6}\left[f(x_0) + 4f(h/2) + f(h)\right]$**)**

Sol : $k_1 = hf(x_0); k_2 = hf\left[\left(x_0 + h/2\right)\left(y_0 + k_1/2\right)\right];$

$$k_2 = hf\left(h/2\right); k_4 = hf(h); y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$y = \frac{h}{6}\left[f(x_0) + 4f(h/2) + f(h)\right]$$

Ex. 87. The Backward difference is defined as $\nabla y_r = y_r - y_{r-1}$ and shifting operator E is define as $Ey_r = y_{r+1}$.

Find the relation between ∇ and E? **(Ans:** $\nabla^n = \left(1 + E^{-1}\right)^n$**)**

Sol : $\nabla y_1 = y_1 - y_0$; $\nabla y_1 = y_1 - E^{-1}(y_1)$; $\nabla y_1 = \left[1 - E^{-1}\right]y_1; \nabla = 1 - E^{-1}; \nabla^n = \left(1 - E^{-1}\right)^n$

Ex. 88. Find a relation between Δ and ∇ and E **(Ans:** $\Delta\nabla = \left(E^{1/2} - E^{-1/2}\right)^2$**)**

Sol : $\Delta\nabla = \left(E - 1\right)\left(1 - E^{-1}\right) = E - EE^{-1} - 1 + E^{-1} = E - 2 + E^{-1} = \left[E^{1/2} - E^{-1/2}\right]^2$

Ex. 89. Find the value of $\Delta\tan^{-1}\left(\frac{n-1}{n}\right)$ **(Ans:** $\tan^{-1}\left(\frac{1}{2n^2}\right)$**)**

Sol : $\Delta\tan^{-1}\left(\frac{n-1}{n}\right) = \Delta\tan^{-1}\left(1 - \frac{1}{n}\right); \Delta\tan^{-1}\left(1 - \frac{1}{n}\right); \Delta\tan^{-1}\left(\frac{n-1}{n}\right) = \tan^{-1}\left[1 - \frac{1}{n+1}\right] - \tan^{-1}\left[1 - \frac{1}{n}\right];$

$$\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right); \Delta\tan^{-1}\left(\frac{n-1}{n}\right) = \tan^{-1}\left[\frac{1 - \frac{1}{n+1} - 1 + \frac{1}{n}}{1 + \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{1}{n}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{1}{n} - \frac{1}{n+1}}{1 + \frac{(n+1-1)}{n+1}\frac{(n-1)}{n}}\right] = \tan^{-1}\left[\frac{\frac{n+1-n}{n(n+1)}}{1 + \frac{n(n-1)}{n(n+1)}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{1}{n(n+1)}}{\frac{n(n+1)+n(n-1)}{n(n+1)}}\right] = \tan^{-1}\left[\frac{1}{n^2+n+n^2-n}\right] = \tan^{-1}\left[\frac{1}{2n^2}\right]$$

Ex. 90. Find a relation between average operator and central difference operator **(Ans:** $\mu^2 = \frac{1}{4}(\delta^2+4)$**)**

Sol : $\mu y_r = \frac{1}{2}\left[y_{r+1/2} + y_{r-1/2}\right] = \frac{1}{2}\left\{E^{1/2}[y_r] + E^{-1/2}[y_r]\right\}$

$\mu y_r = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]y_r; \mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

$\mu^2 = \frac{1}{4}\left[E^{1/2} + E^{-1/2}\right]^2; \ \mu^2 = \frac{1}{4}\left[\left(E^{1/2} - E^{-1/2}\right)^2 + 4\right]$........(1);

$\delta y_r = y_{r+1/2} - y_{r-1/2} = E^{1/2}(y_r) - E^{-1/2}(y_r) = \left[E^{1/2} - E^{-1/2}\right]y_r \ \delta = E^{1/2} - E^{-1/2}$

$\therefore$ Equation (1) become ; $\mu^2 = \frac{1}{4}\left[\delta^2 + 4\right]$

Ex. 91. Given that $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2tx}$ what is the value of $H_4(0)$ is **(Ans: 12)**

Sol : $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2tx}$; $H_0(x)\frac{t^0}{0!} + H_1(x)\frac{t^1}{1!} + H_2(x)\frac{t^2}{2!} + H_3(x)\frac{t^3}{3!} + H_4(x)\frac{t^4}{4!} + ...$

$= 1 - t^2 + \frac{(t^2)^2}{2!} - \frac{(t^2)^3}{3!} + \frac{(t^2)^4}{4!} + ...\ 1 - t^2 + \frac{t^4}{2!}; \ \frac{H_4(x)}{4!} = \frac{1}{2!}; H_4(x) = \frac{4!}{2!} = 3\times 4 = 12$

Ex. 92. Express cos*x* in terms of Bessel function **(Ans:** $J_0 - 2J_2 + 2J_4 - .....$**)**

Sol : $e^{\frac{1}{2}x(t-\frac{1}{t})} = J_0(x) + (t-\frac{1}{t})J_1(x) + (t^2+\frac{1}{t^2})J_2(x) + (t^3-\frac{1}{t^3})J_3(x) + ...$ put $t = \cos\theta + i\sin\theta$

$t^p = \cos p\theta + i\sin p\theta; \ \frac{1}{t} = \cos\theta - i\sin\theta; \frac{1}{t^p} = \cos p\theta - i\sin p\theta \Rightarrow t^p + \frac{1}{t^p} = 2\cos p\theta;$

$t^p - \frac{1}{t^p} = 2i\sin np\theta; \ e^{\frac{1}{2}n2i\sin\theta} = J_0(x) + 2i\sin\theta J_1(x) + 2\cos 2\theta J_2(x) + 2i\sin 3\theta J_3(x) + ...$

$e^{xi\sin\theta} = J_0(x) = 2\cos 2\theta J_2(x) + 2\cos 4\theta J_4(x) + ...\ +2i\sin\theta J_1(x) + 2i\sin 3\theta J_3^{(x)} + ...$

equate real and imaginary part; $\cos(x\sin\theta) = J_0(x) + J_2(x)2\cos 2\theta + 2J_4(x)\cos 4\theta + ...$

$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta = ...$ *put* $\theta = \frac{\pi}{2}$; $\cos x = J_0(x) - 2J_2(x) + 2J_4(x) + ...$

Ex. 93. The generating function $F(x,t)=\sum_{n=0}^{\infty} p_n(x)t^n$. For the Legendre polynomial $F(x,t)=\left(1-2xt+t^2\right)^{-1/2}$ then find the value of $p_3(-1)$? **(Ans: –1)**

Sol : $\left(1-2xt+t^2\right)^{-1/2}=\sum_{n=0}^{\infty} p_n(x)t^n$; $\left(1+2t+t^2\right)^{-1/2}=\sum_{n=0}^{\infty} p_n(-1)t^n$; $\left[\left(1+t\right)^2\right]^{-1/2}=\sum_{n=0}^{\infty} p_n(-1)t^n$;

$\left(1+t\right)^{-1}=1-t+t^2-t^3+t^4+.......\sum^n p_n(-1)=1-t+t^2-t^3+......(-1)^n t^n=\sum -^n p_n(-1);$

$p_n(-1)=(-1)^n=-1$

Ex. 94. Find $\int_{-1}^{1} xp_n(x)p_{n-1}(x)dx$ **(Ans: $\frac{2n}{4n^2-1}$)**

Sol : We have the Recurrence formula $\left(2n+1\right)xp_n=\left(n+1\right)p_{n+1}+np_{n-1}; n\to n-1$

$\left[2(n-1)+1\right]xp_{n-1}=np_n+(n-1)p_{n-2}$;$(2n-1)xp_{n-1}=np_n+\left(n-1\right)p_{n-2}$;

multiplying by p_n we get $\left(2n-1\right)xp_np_{n-1}=np_n+\left(n-1\right)p_{n-2}$ $xp_np_{n-1}=\frac{n}{2n-1}p_n^2+\frac{n-1}{2n-1}p_{n-2}.p_n$;

$$xp_np_{n-1}=\frac{1}{2n-1}\left[np_n^2+(n-1)p_{n-2}p_n\right];$$

Integrating both sides with x

$$\int_{-1}^{1} xp_np_{n-1}dx=\frac{n}{2n-1}\int_{-1}^{1} p_n^2(x)dx+\frac{n-1}{2n-1}\int_{-1}^{1} p_{n-2}p_ndx=\left(\frac{n}{2n-1}\right)\frac{2}{2n+1}=\frac{2n}{4n^2-1}$$

Ex. 95. Find $\int_{-1}^{1} x^2 p_{n-1}p_{n+1}dx$ **(Ans: $\frac{2n\left(n+1\right)}{\left(2n-1\right)\left(2n+1\right)\left(2n+3\right)}$)**

Sol : $\int_{-1}^{1} x^2 p_{n-1}p_{n+1}dx$ we have $\left(2n+1\right)xp_n=\left(n+1\right)p_{n+1}+np_{n-1}; xp_{n-1}\ n\to n-1;$

$\left[2(n-1)+1\right]xp_{n-1}=np_n+(n-1)p_{n-2}$;

$$xp_{(n-1)}=\frac{1}{\left(2n-1\right)}\left[np_n+\left(n-1\right)p_{n-2}\right]........(1),\ xp_{n+1}=\frac{1}{2n+3}\left[\left(n+2\right)p_{n+2}+(n+1)p_n\right].....(2)$$

$$(1)\times(2)\Rightarrow x^2p_{n-1}p_{n+1}=\frac{1}{(2n-1)(2n+3)}\begin{bmatrix} n(n+2)p_np_{n+2}+n(n+1)p_n^2 \\ (n-1)(n+2)p_{n-2}p_{n+2}+(n^2-1)p_np_{n-2}\end{bmatrix}$$

Integrate both sides $\int_{-1}^{1} x^2p_{n-1}p_{n+1}dx=\frac{n(n+1)}{(2n-1)(2n+3)}\int_{-1}^{1} p_n^2dx=\frac{2n(n+1)}{\left(2n-1\right)\left(2n+3\right)\left(2n+1\right)}$

Ex. 96. Find $\int xJ_0^2(x)dx$ **(Ans:** $\frac{J_0^2x^2}{2}+\frac{(xJ_1)^2}{2}$ **)**

Sol : $\int xJ_0^2(x)dx=\int J_0^2 xdx$;

$$=J_0^2\frac{x^2}{2}-\int 2J_0J_0^1\frac{x^2}{2}=\frac{J_0^2x^2}{2}-\int J_0J_0^1x^2;=\frac{J_0^2x^2}{2}+\int J_0J_1x^2$$

$$=\frac{J_0^2x^2}{2}+\int xJ_1xJ_0$$

$$\int xJ_0^2(x)dx=\frac{J_0^2x^2}{2}+\int xJ_1\frac{d}{dx}(xJ_1)......(1)$$

$$\int xJ_1\frac{d}{dx}(xJ_1)=[xJ_1xJ_0]-\int xJ_0xJ_1 \quad \int xJ_1xJ_0=\frac{(xJ_1)^2}{2};$$

$$\therefore \int xJ_0^2(x)dx=\frac{J_0^2x^2}{2}+\frac{(xJ_1)^2}{2}$$

Ex. 97. Find $J_0^{11}=?$ **(Ans:** $\frac{1}{4}[J_{n-2}(x)-2J_n(x)+J_{n+2}(x)]$**)**

Sol : $J_n^1(x)=\frac{1}{2}[J_{n-1}(x)-J_{n+1}(x)];J_n^{11}(x)=\frac{1}{2}[J_{n-1}^1(x)-J_{n+1}^1(x)];$

$$J_{n-1}^1(x)=\frac{1}{2}[J_{n-2}(x)-J_n(x)];$$

$$J_{n+1}^1(x)=\frac{1}{2}[J_n(x)-J_{n+2}(x)];J_n^{11}(x)$$

$$=\frac{1}{4}[J_{n-2}(x)-2J_n(x)+J_{n+2}(x)]$$

OBJECTIVE TYPE QUESTIONS

VECTOR ALGEBRA

1. The vectors $(x, 1, 0)$, $(1, x, 1)$ and $(0, 1, x)$ in R^3 are linearly dependent when the scalar x is

(a) 0 (b) ± 1

(c) $\pm\sqrt{2}$ (d) $\pm\sqrt{3}$

2. The Vectors $\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{k}$, $2\hat{i}+4\hat{j}-\hat{k}$ are

(a) linearly independent

(b) linearly dependent

(c) orthogonal to each other

(d) parallel

3. The maximum value of the function $f(x) = xe^{-x^2}$ is

(a) $\sqrt{2}$ (b) $\frac{1}{2}e^{-1/2}$

(c) $\frac{1}{\sqrt{2}}e^{-1}$ (d) $\frac{1}{\sqrt{2}}e^{-1/2}$

4. A function $f(x) = x(x\text{-}1)$ has

(a) a minimum at $x = 1$

(b) a maximum at $x = 1$

(c) a minimum at $x = 1/2$

(d) zero at $x = \text{-}1$

5. Let f, g, h be vectors

$$f = \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}, g = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, h = \begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix}$$

in three dimensional complex vector space. Then f is

(a) orthogonal to both g and h

(b) orthogonal to g but not h

(c) orthogonal to h but not g

(d) not orthogonal to g and not orthogonal to h.

6. If $\vec{v}$ is a differentiable vector function and f is sufficiently differentiable scalar function then $curl\left(f\,\vec{v}\right)$

(a) 0

(b) $\left(grad\ f\right)\times\vec{v}+\left(f\ curl\ \vec{v}\right)$

(c) $f\,curl\ \vec{v}$

(d) $\left(grad\ f\right)\times\vec{v}$

7. The value of λ for which the vectors $(1,-2,\lambda)$, $(2,-1,5)$ and $(3,-5,7\lambda)$ are linearly dependent.

(a) 5 (b) 3/5

(c) 5/7 (d) 5/14

8. Let $\sum u_n$ be a series of positive terms. Given that $\sum u_n$ is convergent and also $\underset{n\to\infty}{Lt}\frac{u_n+1}{u_n}$ exists, then the limit is

(a) necessarily equal to 1

(b) necessarily greater than 1

(c) may be equal to 1 or less than 1

(d) necessarily less than 1.

9. A linear transformation T, defined as $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ x_2-x_3 \end{pmatrix}$, transforms a vector for a three dimensional real space to a two-dimensional real space. The transformation matrix T is

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

VECTOR CALCULUS

10. Consider a vector field $F = (yz,xz,xy)$ then

(a) $\nabla.F = 0, \nabla\times F = (0,0,0)$

(b) $\nabla.F = 1, \nabla\times F = (0,0,0)$

(c) $\nabla.F = 0, \nabla\times F = (y,-z,0)$

(d) $\nabla.F = 1, \nabla\times F = (0,z,-x)$

11. The value of the integral $\int_{(0,0)}^{(\pi/2,\pi/2)}\left(y\cos xdx+\sin xdy\right)$ is

(a) 0 (b) π/2
(c) π (d) 2π

12. Find the values of the constants a, b, c for which the vector

$\vec{V}=(x+y+az)\hat{i}+(bx+3y-z)\hat{j}+(3x+cy+z)\hat{k}$

is irrotational

(a) $a=1, b=1, c=-1$
(b) $a=1, b=-1, c=1$
(c) $a=3, b=1, c=-1$
(d) $a=3, b=-1, c=1$

13. Which of the following statement is INCORRECT?

(a) The vectors $\bar{a}\times(\bar{b}\times\bar{c})$, $\bar{b}\times(\bar{c}\times\bar{a})$ and $\bar{c}\times(\bar{a}\times\bar{b})$ are coplanar

(b) If $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$ are coplanar, then $[\bar{a}\bar{b}\bar{c}]=[\bar{a}\bar{b}\bar{d}]+[\bar{a}\bar{d}\bar{c}]+[\bar{d}\bar{b}\bar{c}]$

(c) $(\bar{a}\times\bar{b}).\{(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})\}=\{\bar{a}.(\bar{b}\times\bar{c})\}$

(d) If $\bar{a},\bar{b},\bar{c}$ represents the sides of a tetrahedron, then the volume of tetrahedron is $\frac{1}{6}[\bar{a}\bar{b}\bar{c}]$

14. The work done when a force $\vec{F}=(x^2-y^2+x)\hat{i}-(2xy+y)\hat{j}$ moves a particle from origin to (1, 1) along a parabola $y^2=x$

(a) 2/3 (b) 4/3
(c) 5/3 (d) 7/3

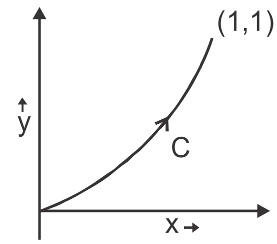
15. If for a scalar function $\phi, \vec{\nabla}\phi=\frac{\vec{r}}{r^2}$ then ϕ is equal to

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$
(c) $\ln(|r|)$ (d) $\ln(r)$

16. The divergence of vectors $u=\vec{i}$ and $v=x\vec{i}$ is respectively

(a) 0, 0 (b) 0, 1
(c) 1, 0 (d) 1, 1

17. The value of the line integral $\int_c \vec{F}.d\vec{l}$ where $\vec{F}=x\hat{i}+y\hat{j}$ and c is a shown, is



(a) 0 (b) 1
(c) 1/2 (d) 2

18. Consider two vectors $\vec{A}=(x+y+3z)\hat{i}+(x+3y-z)\hat{j}+(3x-y+z)\hat{k}$ and $\vec{B}=(e^x\sin y)\hat{i}+(e^x\cos y)\hat{j}$ which of the following is correct

(a) $\vec{A}$ is irrotational and $\vec{B}$ is irrotational
(b) $\vec{A}$ is irrotational, $\vec{B}$ is solenoidal
(c) $\vec{A}$ is solenoidal, $\vec{B}$ is irrotational
(d) $\vec{A}$ is solenoidal and $\vec{B}$ is solenoidal

19. If F is a solenoidal vector, then the value of curl curl curl curl F is

(a) F (b) zero
(c) $\nabla^4 F$ (d) $\nabla^2 F$

20. Find the turning points of $f(x)=x^2e^{-x^2}$, where the function become maximum.

(a) $x=0$ (b) $x=+1$ only
(c) $x=-1$ only (d) $x=\pm 1$

21. Consider the vector $V=r/r^3$

(A) The surface integral of this vector over the surface of a cube of side a and centered at the origin

(a) 0 (b) 2π
(c) $2\pi a^3$ (d) 4π

(B) Which one of the following is NOT correct?

(a) Value of the line integral of this vector around any closed curve is zero
(b) This vector can be written as the gradient of some scalar function
(c) The line integral of this vector from point P to point Q is independent of the path taken
(d) This vector can represent the magnetic field of some current distribution

22. The directional derivative of $f(x,y) = 2x^2 + 2y^2 + z^2$ at point $p(2,1,3)$ in the direction of the vector $\vec{a} = \hat{i} - 2\hat{k}$ is ?

(a) $4/\sqrt{5}$ (b) $-4/\sqrt{5}$

(c) $\sqrt{5}/4$ (d) $-\sqrt{5}/4$

23. The curl of a vector field $\vec{F}$ is $4\hat{x}$. Identify the appropriate vector field $\vec{F}$ from the choices given below.

(a) $\vec{F} = 2z\hat{x} + 3z\hat{y} + 5y\hat{z}$

(b) $\vec{F} = z\hat{y} + 5y\hat{z}$

(c) $\vec{F} = 3x\hat{y} + 5y\hat{z}$

(d) $\vec{F} = 2\hat{x} + 5y\hat{z}$

24. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector, then the value of $\nabla(\log r)$ is

(a) $\frac{\vec{r}}{r}$ (b) $\frac{\vec{r}}{r^2}$

(c) $\frac{-\vec{r}}{r^3}$ (d) $\frac{\vec{r}}{r^3}$

25. If $\vec{F}(t)$ has a constant magnitude then

(a) $\frac{d}{dt}\vec{F}(t) = 0$ (b) $\vec{F}(t).\frac{d\vec{F}(t)}{dt} = 0$

(c) $\vec{F}(t)\times\frac{d\vec{F}(t)}{dt} = 0$ (d) $\vec{F}(t) - \frac{d\vec{F}(t)}{dt} = 0$

26. If all the surfaces are closed in a region containing volume V then the following theorem is applicable.

(a) Stoke's theorem
(b) Green's theorem
(c) Gauss divergence theorem
(d) None the above

27. The curl of a vector $\vec{A} = e^{xyz}(\hat{i} + \hat{j} + \hat{k})$ at the point (1,2,3) is

(a) $e^6(\hat{i} - 4\hat{j} + 3\hat{k})$ (b) $(\hat{i} - 4\hat{j} + 3\hat{k})$

(c) $e^6(\hat{i} + \hat{j} + \hat{k})$ (d) zero

28. Consider the following statements and identify the CORRECT ONE.

I. If $\vec{A}$ and $\vec{B}$ are irrotational, $\vec{A}\times\vec{B}$ is also irrotational

II. If $\vec{A}$ and $\vec{B}$ are irrotational, $\vec{A}\times\vec{B}$ is solenoidal

III. If $\vec{A}$ and $\vec{B}$ are irrotational, $\vec{A}\times\vec{B} = 0$

(a) II & III only (b) II only
(c) III only (d) I & III only

29. If S is the sphere of radius R and $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$, then $\int_s \vec{A}.ds =$

(a) $\frac{12}{5}\pi R^5$ (b) $4\pi R^2$

(c) $\frac{4}{3}\pi R^2$ (d) $\frac{4}{5}\pi R^5$

30. The angle between the surfaces $x^2 + y^2 + z^2 = 1$ and $z = x^2 + y^2 - 1$ at the point (1, 1, -1)

(a) $\cos^{-1}\left(\frac{10}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{10}{6\sqrt{3}}\right)$

(c) $\cos^{-1}\left(\frac{5}{6\sqrt{3}}\right)$ (d) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

31. A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$, which is irrotational. The velocity potential is given by

(a) $xy - yz + zx$ (b) $xy + yz - zx$

(c) $xy + yz + zx$ (d) $yz - xy + zx$

32. If $f(r)\,\vec{r}$ is a solenoidal, which of the following represents the correct function for $f(r)$? Where c is constant.

(a) $\frac{c}{r^2}$ (b) $\frac{c}{r}$

(c) $\frac{c}{r^3}$ (d) $\frac{c}{r^4}$

33. A vector field F is said to be conservative if and only if

(a) F is a curl of some vector r

(b) F can be represented as a gradient of scalar function ϕ

(c) $\nabla \bullet F = 0$

(d) $\nabla \times F = F$

34. If $u = e^{xyz}$, then $\frac{\partial^3 u}{\partial x \partial y \partial z}$ is equal to

(a) $e^{xyz}\left[1 + xyz + 3x^2y^2z^2\right]$

(b) $e^{xyz}\left[1 + xyz + x^3y^3z^3\right]$

(c) $e^{xyz}\left[1 + 3xyz + x^2y^2z^2\right]$

(d) $e^{xyz}\left[1 + 3xyz + x^3y^3z^3\right]$

35. If $r = x\hat{i} + y\hat{j} + z\hat{k}$ then $(r.\nabla)r^2$ is equal to

(a) $2r^2$ (b) $3r^2$

(c) $4r^2$ (d) 0

36. The four vertices of a regular tetrahedron are located at $O(0,0,0)$, $A(0,1,0)$, $B(0.5\sqrt{3},0.5,0)$ and $C\left(\frac{0.5}{\sqrt{3}},0.5,\frac{\sqrt{2}}{3}\right)$. The unit vector perpendicular (outward) to the face ABC is

(a) $0.41\hat{x} + 0.71\hat{y} + 0.29\hat{z}$

(b) $0.47\hat{x} + 0.82\hat{y} + 0.33\hat{z}$

(c) $-0.47\hat{x} + 0.82\hat{y} + 0.33\hat{z}$

(d) $-0.41\hat{x} - 0.71\hat{y} - 0.29\hat{z}$

37. If A is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, then $\iint_A y\,dx\,dy$ is equal to

(a) $\frac{48}{5}$ (b) $\frac{36}{5}$

(c) $\frac{32}{5}$ (d) None of these

38. The unit vector to the surface $x^2 + y^2 - z = 1$ at the point $P(1,1,1)$ is

(a) $\frac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$ (b) $\frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}}$

(c) $\frac{\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{6}}$ (d) $\frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$

LINEAR ALGEBRA

MATRICES

39. The matrix representing the linear operator $\frac{\partial^2}{\partial x^2}$ in the basis $u_1 = e^{\alpha x}$, $u_2 = e^{-\alpha x}$ is

(a) $\begin{bmatrix} 0 & \alpha^2 \\ \alpha^2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & \alpha^2 \\ -\alpha^2 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} \alpha^2 & 0 \\ 0 & -\alpha^2 \end{bmatrix}$ (d) $\begin{bmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{bmatrix}$

40. The matrix representing the linear operator $\frac{d}{dx}$ in the basis $u_1 = e^{\alpha x}, u_2 = e^{-\alpha x}$ is

(a) $\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$

(c) $\begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$ (d) $\begin{bmatrix} \alpha & 0 \\ \alpha & -\alpha \end{bmatrix}$

41. Which of the following matrix rotates the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ through 30° about x - axis

(a) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

(c) $\begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

42. Which of the following is INCORRECT regarding characteristic roots/ vectors?

(a) The characteristic roots of a unitary matrix must be uni modular

(b) Characteristic vectors corresponding to a system of distinct characteristic roots of any square matrix are linearly dependent

(c) Characteristic roots of a Hermitian matrix are real

(d) Characteristic vectors of a unitary matrix are orthogonal

43. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

(a) 2 (b) 3

(c) 1 (d) 4

44. The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ is similar to the matrix

(a) $\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

45. Given two matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} A' = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ whose elements are real numbers other than zero. If $A.A' = I$ where I is the identity matrix and D is the determinant of A then D is equal to

(a) *e/d* (b) *d/e*

(c) *bc/ad* (d) none of these

46. If $A_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, then consider the following statements:

I) $A_\alpha . A_\beta = A_{\alpha\beta}$

II) $A_\alpha . A_\beta = A_{\alpha+\beta}$

III) $(A_\alpha)^n = \begin{pmatrix} \cos^n\alpha & \sin^n\alpha \\ -\sin^n\alpha & \cos^n\alpha \end{pmatrix}$

IV) $(A_\alpha)^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$

which of the above statements are true.

(a) I and II (b) I and IV

(c) II and III (d) II and IV

47. Let $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then A^n is given by, where n is any positive integer

(a) $\begin{bmatrix} 1+3n & 1-4n \\ 1+n & -1+n \end{bmatrix}$ (b) $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$

48. The condition for a Matrix

$\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ to be a unitary matrix

(a) $\alpha^2 + \beta^2 = \gamma^2 + \delta^2$

(b) $\alpha^2 + \beta^2 = -(\gamma^2 + \delta^2)$

(c) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$

(d) $\alpha^2 + \beta^2 = 2(\gamma^2 + \delta^2)$

49. If $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, then $(I-A)(I+A)^{-1}$ is

(a) $\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -1-2i \\ 1-2i & -2 \end{bmatrix}$

(c) $\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ -2+4i & -4 \end{bmatrix}$

(d) $\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$

50. The rank of the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix}$ is

(a) 0 (b) 1

(c) 2 (d) 3

51. The value of the determinant

$\begin{bmatrix} 1989 & 1990 & 1991 \\ 1992 & 1993 & 1994 \\ 1995 & 1996 & 1997 \end{bmatrix}$ is

(a) 0 (b) 1

(c) – 1 (d) none of these

52. If product of matrices

$A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and

$B=\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is a null matrix,

then θ and ϕ differ by

(a) An odd multiple of π

(b) An even multiple of π

(c) An odd multiple of $\frac{\pi}{2}$

(d) An even multiple $\frac{\pi}{2}$

53. If A is a square matrix of order 2 and $|A|=2$ then A (adj A) is equal to

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) None of these

EIGEN VALUES AND EIGEN VECTORS

54. Consider the matrix $A=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ the eigen values of this matrix are

(a) 1 and -1 (b) 2 and -2

(c) $\sqrt{2}$ and $-\sqrt{2}$ (d) 1/2 and -1/2

55. Let A, B and C be three matrices Tr(ABC) is same as

(a) Tr(BAC) (b) Tr(CBA)

(c) Tr(BCA) (d) Tr(ACB)

56. The matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ has three eigen values λ_i defined by $Av_i = \lambda_i v_i$. Which of the following statements are NOT true?

(a) $\lambda_1 + \lambda_2 + \lambda_3 = 0$

(b) λ_1, λ_2 and λ_3 are all real numbers.

(c) $\lambda_1\lambda_2 = +1$ for some pair of roots.

(d) $\lambda_i^3 = +1, i = 1,2,3$

57. The eigen values of the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ are}$$

(a) $e^{\pm i\theta}$ (b) $e^{\pm 2i\theta}$

(c) $e^{\pm 3i\theta}$ (d) $e^{\pm i\theta/2}$

58. If $\lambda_1, \lambda_2, ...\lambda_n$ are the eigen values of A, then the eigen values of A^m are

(a) $m\lambda_1, m\lambda_2, ...m\lambda_n$ (b) $\lambda_1^m, \lambda_2^m, ...\lambda_n^m$

(c) $\lambda_1, \lambda_2, ...\lambda_n$ (d) $\frac{\lambda_1}{m}, \frac{\lambda_2}{m}, ... \frac{\lambda_n}{m}$

59. It is given that the eigenvalues of the matrix $\begin{pmatrix} 3 & 2 & 1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ are integers then the eigenvalues are,

(a) 5, -2, 3 (b) 2, 2, 2

(c) -1, 2, -3 (d) 1, 2, 3

60. The eigen values of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ are

(a) 1, 0, 0 (b) 1, 2, 0

(c) 1, 1, 1 (d) 3, 0, 0

61. Given a matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $Tr\, A^2$ is

(a) 10 (b) 9

(c) 2 (d) zero

62. For arbitrary matrices E, F, G and H, if $EF - FE = 0$ then Trace ($EFGH$) is equal to

(a) Trace ($HGFE$)

(b) Trace (E) Trace (F) Trace (G) Trace (H)

(c) Trace ($GFEH$)

(d) Trace ($EGHF$)

63. The eigen values of the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

(a) 0, 0, 1, 5 (b) 0, 0, 6, 0

(c) 0, 0, 2, 4 (d) 0, 0, -7, 1

64. The eigen values of the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ are}$$

(a) $\frac{1}{2}(\sqrt{3} \pm i)$ when $\theta = 45^\circ$

(b) $\frac{1}{2}(\sqrt{3} \pm i)$ when $\theta = 30^\circ$

(c) $\pm$ since the matrix is unitary

(d) $\frac{1}{\sqrt{2}}(1 \pm i)$ when $\theta = 30^\circ$

65. State which of the following statements are TRUE regarding eigen values and eigen vectors of a matrix A

I) Determinant of A is equal to the product of eigen value of A

II) If A is Real its eigen values are must be Real.

III) Eigen values of A^{-1} are multiplicative reciprocals of A.

(a) I, II, III (b) I, III only

(c) I, II only (d) II & III only

66. One of the eigenvalues of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is 5:

The other two eigenvalues are

(a) 0 and 0 (b) 1 and 1

(c) 1 and-1 (d) -1 and -1

67. Eigen values of the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 0 & 2i & 0 \end{pmatrix}$ are

(a) -2,-1,1,2 (b) -1,1,0,2

(c) 1,0,2,3 (d) -1,1,0,3

68. A matrix whose determinants is zero is called

(a) singular matrix (b) unit matrix

(c) null matrix (d) orthogonal matrix

69. The eigen values of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ are

(a) 1, 1, 2 (b) 0, 1, 2

(c) 0, 2, 3 (d) 1, 2, 2

70. If A and B are unitary matrices of the same order, which of the following combinations is unitary?

(a) $A^{-1}B$ (b) AB^{-1}

(c) AB (d) $A+B$

71. The rank of the matrix, $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 7 & \lambda \\ 1 & 4 & 5 \end{pmatrix}$ is 2, then the value of λ is

(a) -13 (b) 13

(c) 3 (d) 6

72. Let A and B be non – singular square matrices of the same order. Consider the following statements.

I) $(AB)^T = A^T B^T$

II) $(AB)^{-1} = A^{-1}B^{-1}$

III) $adj(AB) = (adj\ A)(adj\ B)$

IV) Rank(AB) = Rank (A) Rank(B)

V) $|AB| = |A|.|B|$

(a) I, III and IV (b) IV and V

(c) I and II (d) I, II and V

73. The system equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=12$ is inconsistent, if λ is

(a) 3 (b) -3

(c) 0 (d) 2

74. The eigen values of $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ are

(a) 0, 3, -15 (b) 0, -3, -15

(c) 0, 3, 15 (d) 0, -3, -15

75. If A is a 3 –rowed square matrix, then $|adj(adj(A))|$ is equal to

(a) $|A|^6$ (b) $|A|^3$

(c) $|A|^4$ (d) $|A|^2$

76. If $A = \begin{pmatrix} t^2 & \cos t \\ e^t & \sin t \end{pmatrix}$, then $\frac{dA}{dt}$ will be

(a) $\begin{pmatrix} t^2 & \sin t \\ e^t & \sin t \end{pmatrix}$ (b) $\begin{pmatrix} 2t & \cos t \\ e^t & \sin t \end{pmatrix}$

(c) $\begin{pmatrix} 2t & -\sin t \\ e^t & \cos t \end{pmatrix}$ (d) $\begin{pmatrix} t^2 & \sin t \\ e^t & \cos t \end{pmatrix}$

77. If $A \in R_{n\times n}$, det $A \neq 0$, then

(a) A is non singular and the rows and columns of A are linearly independent.

(b) A is non singular and the rows of A are linearly dependent.

(c) A is non singular and the A has one zero rows.

(d) A is singular

78. The matrix shown transforms the components of a vector in one coordinate frame S to the components of the same vector in a second coordinate frame S^1. This matrix represents a rotation of the reference frame S by

$$\begin{pmatrix} a_x^1 \\ a_y^1 \\ a_z^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & i\frac{\sqrt{3}}{2} & 0 \\ i\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

(a) 30° counterclockwise about the z-axis

(b) 45° clockwise about the z-axis

(c) 60° clockwise about the y-axis

(d) 60° counterclockwise about the x-axis

CAYLEY– HAMILTON THEOREM

79. The characteristic equation $|A - \lambda I| = 0$ is $\lambda^3 + 2\lambda^2 - 3\lambda + 4 = 0$. The characteristic equation for A^{-1} is

(a) $\lambda^3 - 2\lambda^2 + 3\lambda - 4 = 0$

(b) $\lambda^3 + \frac{1}{2}\lambda^2 - \frac{1}{3}\lambda + \frac{1}{4} = 0$

(c) $\lambda^3 - \lambda^2 - \frac{3}{2}\lambda - 2 = 0$

(d) $\lambda^3 - \frac{3}{4}\lambda^2 + \frac{1}{2}\lambda + \frac{1}{4} = 0$

80. The rank of the matrix $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \end{bmatrix}$ is

(a) 3 (b) 2

(c) 1 (d) 4

81. The rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1

(c) 2 (d) 3

82. The characteristic equation of a square matrix

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

(a) $A^3 + A(a + b + c) = 0$

(b) $A^3 + A(a^2 + b^2 - c^2) = 0$

(c) $A^3 + A(a^2 + b^2 + c^2) = 0$

(d) $A^3 = A(a^2 + b^2 + c^2)$

LINEAR ORDINARY DIFFERENTIAL EQUATIONS

FIRST ORDER

83. The solution of the equation $d^2x/dt^2 + 2dx/dt + 5x = 0$ subject to the initial condition $x = 5, dx/dt = -3$ at $t = 0$.

(a) $e^{-t}(5\cos 2t + \sin 2t)$

(b) $e^{+t}(5\sin 2t + \cos 2t)$

(c) $(5\sin 2t + \cos 2t)$

(d) $e^{-t}(5\sin 2t)$

84. The solution of the differential equation $y\, dx = (3x + y^4)\, dy$

(a) $\frac{x}{y^3} + y = c$ (b) $\frac{x}{y^2} + y = c$

(c) $\frac{x}{y} + y^2 = c$ (d) $\frac{x}{y^3} = y + c$

85. Introducing a parameter $p = y' = \frac{dy}{dx}$, the equations $y = y'^2 + xy' - x$ can be reduced to the equation

(a) $\frac{dp}{dx} = 2p + x$ (b) $\frac{dx}{dp} = 2p + x$

(c) $\frac{dp}{dx} = p + x$ (d) $\frac{dx}{dp} = p + x$

Common Data (Q.86 - 87)

Consider the differential equation

$$y'' + P(x)y' + q(x)y(x) = 0\,.$$

86. If $x\ p(x)$ and $x^2\ q(x)$ have the Taylor series expansion $x\,p(x) = 4 + x + x^2 + \ldots$ and $x^2 q(x) = 2 + 3\ x + 5\ x^2 + ..$ then the roots of the indicial equation are

(a) -1, 0 (b) -1, -2

(c) -1, 1 (d) -1, 2

87. If $p(x) = 0$ with the Wronskin at $x = 0$ as $W(x = 0) = 1$ and one of the solutions is x, then the other linearly independent solution which vanishes at $x = 1/2$ is

(a) 1 (b) $1 - 4x^2$

(c) x (d) $-1 + 2x$

88. To change $\left(x^3y^3 + xy\right)dx = dy$ to linear form the substitution is?

(a) $x^3 = v$ (b) $y^3 = v$

(c) $\frac{1}{x^2} = v$ (d) $\frac{1}{y^2} = v$

89. $x\,dx + ydy + zdz = 0$ is the first order differential equation of

(a) Sphere (b) Right circular cone

(c) Cylinder (d) Ellipsoid

90. Match the following and find the correct alternative

Group - A

I) Cauchy's equation

II) Bernouli's equation

III) Method of variation of parameters

Group - B

p) $(x+2)^2 \frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} + y = 5$

q) $x^2 \frac{d^3y}{dx^3} - x\frac{d^2y}{dx^2} = e^x$

r) $\left(D^2 + a\right)y = \tan x$

s) $\frac{dy}{dx} + xy = x^2y^2$

(a) I (p), II (q), III (r) (b) I (s), II (q), III (r)

(c) I (s) II (p), III (q) (d) I (q), II (s), III (r)

91. What are the order and degree, respectively of the differential equation of the family of curves $y^2 = 2c(x + \sqrt{c})$

(a) 1.1 (b) 1,2

(c) 1,3 (d) 2,1

92. The differential equation ydx - $2x\ dy$ = 0 represents a family of

(a) straight lines (b) parabolas

(c) circles (d) catenaries

93. The solution of $(dy/dx) + 2xy = 2xy^2$, is

(a) $y = (cx)/(1 + e^{-x^2})$

(b) $y = 1/(1 - ce^x)$

(c) $y = 1/(1 + ce^{x^2})$

(d) $y = (cx)/(1 + e^{x^2})$

SECOND ORDER

94. The most general solution of the differential equation $x^2 \frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 15y = 0$ is given by

(a) $y = c_1e^{3x} + c_2e^{5x}$

(b) $y = c_1e^{-3x} + c_2e^{-5x}$

(c) $y = c_1x^3 + c_2x^5$

(d) $y = c_1x^3 + c_2x^3 \log x$

95. A moving body is opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. The velocity of the particle interms of x, if it starts from rest.

(a) $V^2 = \frac{cx}{b} - \frac{c}{2b^2} + \frac{c}{2b^2}e^{-2bx}$

(b) $V^2 = -\frac{c}{2b^2} + \frac{c}{2b^2}e^{-2bx}$

(c) $V^2 = \frac{-cx}{b} + \frac{c}{2b^2} - \frac{c}{2b^2}e^{-2bx}$

(d) $V^2 = \frac{cx}{2b^2} + \frac{c}{b^2}x - \frac{c}{2b^2}e^{-2bx}$

96. The family of conic represented by the solution by the solution of the DE $(4x+3y+1)dx+(3x+2y+1)dy=0$ is

(a) Circles (b) Parabolas

(c) Hyperbolas (d) Ellipses

97. The points, where the series solution of the Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\frac{3}{2}(\frac{3}{2}+1)y=0$$

will diverge, are located at

(a) 0 and 1 (b) 0 and -1

(c) -1 and 1 (d) $\frac{3}{2}$ and $\frac{5}{2}$

98. Solution of the differential equation

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{5x}$$

(a) $c_1e^{2x}+c_2e^{3x}+e^{5x}$

(b) $c_1e^{2x}+c_2e^{3x}$

(c) $c_1e^{2x}+c_2e^{3x}+\frac{e^{5x}}{6}$

(d) $c_1e^{2x}+c_2e^{3x}+\frac{e^{5x}}{5}$

99. The particular Integral of differential equation

$$\frac{d^4x}{dt^4}+4x=\cos ht$$

(a) $\frac{1}{5}\cos ht$ (b) $\frac{1}{5}\sin ht$

(c) $\frac{1}{5}\cos\ t$ (d) $\frac{1}{5}\sin\ t$

100. The particular integral of the differential equation $(D^3-D)\,y=e^x+e^{-x}, D=\frac{d}{dx}$ is

(a) $(e^x+e^{-x})/2$ (b) $x(e^x+e^{-x})/2$

(c) $x^2(e^x+e^{-x})/2$ (d) $e^x(e^x-e^{-x})/2$

101. The integrating factor of the equation $y'-\frac{\tan y}{1+x}=(1-x)e^x\sec y$ is

(a) $\frac{1}{1+x}$ (b) $\frac{-1}{1+x}$

(c) $\frac{e^x}{1+x}$ (d) e^{x+1}

102. The general solution of the differential equation $x^2(y-px)=yp^2$ is given by

(a) $y^2=c^2+x^2$ (b) $y=cx+1$

(c) $y=(c+x)^2$ (d) $y^2=cx^2+c^2$

103. If $y=x$ is a solution of $x^2y''+xy'-y=0$, then the second linearly independent solution of the above equation is

(a) $1/x$ (b) x^2

(c) x^{-2} (d) x^n

104. Two linearly independent solutions of the differential equation

$4(d^2y/dx^2)+4(dy/dx)+5y=0$ are

(a) $e^{-x/2}\cos x, e^{-x/2}\sin x$

(b) $e^{x/2}\cos x, e^{x/2}\sin x$

(c) $e^{x/2}\cos x, e^{-x/2}\sin x$

(d) $e^{-x/2}\cos x, e^{x/2}\sin x$

105. The solution of the differential equation. $(D^2+4D+4)y=2\sinh 2x$ is

(a) $(C_1+C_2x)e^{2x}-\frac{1}{16}e^{2x}+\frac{x^2}{2}e^{-2x}$

(b) $(C_1x+C_2x^2)e^{2x}+\frac{1}{16}e^{-2x}-\frac{x^2}{2}e^{2x}$

(c) $(C_1+C_2x)e^{2x}+\frac{1}{16}e^{2x}-\frac{x^2}{2}e^{-2x}$

(d) $(C_1+C_2x^2)e^{2x}+\frac{1}{8}e^{2x}-\frac{x^2}{6}e^{-2x}$

106. Integrating factor of the differential equation

$$x(1-x^2)dy+(2x^2y-y-ax^3)dx=0$$

is

(a) $\dfrac{1}{x(x^2+1)^{1/2}}$ (b) $\dfrac{1}{(x^2+1)^{1/2}}$

(c) $\dfrac{1}{x(1-x^2)^{1/2}}$ (d) $\dfrac{1}{2x(1-x^2)^{1/2}}$

107. The solution of the differential equation $y''+(3i-1)y'-3iy=0$ is

(a) $y=c_1e^x+c_2e^{3ix}$

(b) $y=c_1e^{-x}+c_2\cos 3x-c_3\sin 3x$

(c) $y=c_1e^x+c_2e^{-3ix}$

(d) $y=c_1e^{-x}+c_2\cos 3x-c_3\sin 3x$

108. The solution of $(d^2y/dx^2)+y=0$ satisfying the condition $y(0)=1, y(\pi/2)=2$, is

(a) $\cos x+2\sin x$ (b) $\cos x+\sin x$
(c) $2\cos x+\sin x$ (d) $2(\cos x+\sin x)$

109. The set of linearly independent solutions of the differential equation $(dy^4/dx^4)-(dy^2/dx^2)=0$ is

(a) $\{1,x,e^x,e^{-x}\}$ (b) $\{1,x,e^{-x},xe^{-x}\}$

(c) $\{1,x,e^x,xe^x\}$ (d) $\{1,x,e^x,xe^{-x}\}$

110. Consider the following statements: The equation $(2x/y^3)dx+[(y^2-3x^2)/y^4]dy=0$ is

1) exact
2) homogeneous
3) linear

(a) 1 and 2 are correct(b) 1 and 3 are correct
(c) 2 and 3 are correct(d) 1,2 and 3 are correct

111. Consider the following statements in respect of the differential equation

$$2xy(dy/dx)=y^2-x^2.$$

1) The differential equation is a homogeneous equation
2) The curve represented by the differential equation is a family of circles
3) The differential equation of the orthogonal trajectories is $dy/dx=(2xy)(x^2-y^2)$, which one of the following is correct one.

(a) 1 and 2 only (b) 1 and 3 only
(c) 2 and 3 only (d) 1,2 and 3

112. If the population of a country doubles in 50 years, in how many years will it treble under the assumption that the rate of increase is proportional to the number of inhabitants.

(a) $25\dfrac{\log 3}{\log 2}$ (b) $50\dfrac{\log 3}{\log 2}$

(c) $20\dfrac{\log 3}{\log 2}$ (d) $25\dfrac{\log 2}{\log 3}$

113. The solution of the differential equation $(D^3-D)y=2\cosh x$ is

(a) $C_1+C_2e^x+C_3e^{-x}+\dfrac{x}{2}(e^x-e^{-x})$

(b) $C_1+C_2e^x+C_3e^{-x}+\dfrac{x}{2}(e^x+e^{-x})$

(c) $C_1x+C_2e^x+C_3e^{-x}-\dfrac{x}{2}(e^x-e^{-x})$

(d) $C_1+C_2e^{-x}+C_3e^x+\dfrac{x}{2}(e^x-e^{-x})$

114. The solution of $d^2s/dt^2=g$. (g is a constant, s = 0 and ds/dt =u when t = 0) is

(a) $s=gt$ (b) $s=ut+(1/2)gt^2$

(c) $s=(1/2)gt^2$ (d) $s=ut+gt^2$

Common Data (Q.115 - 116)

115. Which one of the following equations has the same order and degree

(a) $d^4y/dx^4+8(dy/dx)^4+5y=e^x$

(b) $5(d^3y/dx^3)^4+8(dy/dx+1)^2+5y=x^3$

(c) $\{1+(dy/dx)^3\}^{2/3}=4(d^3y/dx^3)$

(d) $y=x^2(dy/dx)+\{(dy/dx)^2+1\}^{1/2}$

116. The number of arbitrary constants in the differential equation of the form $\phi(x,y,dy/dx,d^3y/dx^3)=0$ is

(a) 1 (b) 2
(c) 3 (d) 4

Common Data (Q.117 - 118)

117. A particular integral of the differential equation $(D^2 + 4)y = x$ is

(a) xe^{2x} (b) $x\cos 2x$

(c) $x\sin 2x$ (d) $x/4$

118. The homogeneous differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ can be reduced to a differential equation, in which the variables are separated, by the substitution of

(a) $y = vx$ (b) $xy = v$

(c) $x+y = v$ (d) $x - y = v$

Common Data (Q.119 - 120)

119. $e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + c_3 e^{2x}$ is the general solution of

(a) $(d^3y/dx^3) + 4y = 0$

(b) $(d^3y/dx^3) + 8y = 0$

(c) $(d^3y/dx^3) - 8y = 0$

(d) $(d^3y/dx^3) - 2(d^2y/dx^2) + (dy/dx) - 2 = 0$

120. Which of the following represents the solution of the differential equation $\frac{d^2i}{dt^2} + \left(\frac{R}{L}\right)\frac{di}{dt} + \left(\frac{1}{LC}\right)i = 0$, where $R^2C = 4L$ and R, C, L are constants (where C_1 and C_2 are arbitrary constants)

(a) $(C_1 + C_2 t)e^{-\frac{tR}{L}}$

(b) $(C_1 t + C_2 t^2)e^{-\frac{tR}{2L}}$

(c) $(C_1 \cos t + C_2 \sin t)e^{-\frac{tR}{2L}}$

(d) $(C_1 + C_2 t)e^{-\frac{tR}{2L}}$

Common Data (Q.121 - 122)

121. For non exact differential equation $(1 + xy)\,ydx + (1 - xy)\,xdy = 0$, what is the integrating factor.

(a) $1/(2x^2y^2)$ (b) $1/(2xy)$

(c) $1/(2xy^2)$ (d) $1/(2x^2y)$

122. The cooling law " The rate at which a hot body cools is proportional to the difference in temperature between the body and the surrounding medium",

(a) Ohm's law (b) Kepler's law

(c) Newton's law (d) Kelvin's law

Common Data (Q.123 - 124)

123. The differentiale quation $dy/dx + x\sin 2y = x^3\cos^2 y$ is reduced to the linear form $dv/dx + Pv = Q$, where P and Q are functions of 'x' alone, by changing the variable as

(a) $\sin y = v$ (b) $\cos y = v$

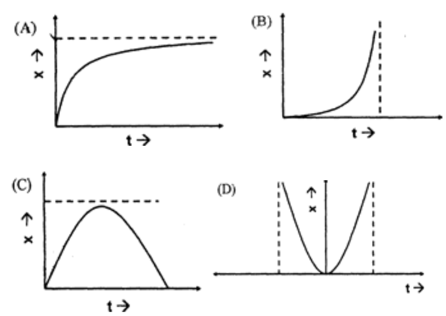
(c) $\tan y = v$ (d) $\sin 2y = v$

124. Solution of $d^2y/dx^2 + dy/dx - 2y = 0$, $y(0) = 0$, $y^1(0) = 3$ is

(a) $e^x + e^{-2x}$ (b) $e^x - e^{-2x}$

(c) $e^x - e^{2x}$ (d) $e^x + e^{2x}$

125. Which one of the following curves gives the solution of the differential equation $k_1\frac{dx}{dt} + k_2 x = k_3$, where k_1, k_2 and k_3 are positive constants with initial conditions $x = 0$ at $t = 0$.



126. Consider the differential equation $dx^2/dt^2 + 2dx/dt + x = 0$. At time $t = 0$, it is given that $x = 1$ and $dx/dt = 0$. At $t = 1$, the value of x is given by

(a) $1/e$ (b) $2/e$

(c) 1 (d) $3/e$

127. Consider the following three functions $e^{m_1x}, e^{m_2x}, e^{m_3x}$. These functions linearly independent if

(a) $m_1 = m_2 \neq m_3$ (b) $m_2 = m_3 \neq m_1$

(c) $m_1 = m_2 = m_3$ (d) $m_1 \neq m_2 \neq m_3$

128. Consider the Assertion (A) and Reason (R) given below.]

Assertion (A) : The curves $y = ax^3$ and $x^2 + 3y^2 = c^2$ form orthogonal trajectories.

Reason (R) : The differential equation of the second curve is obtained from the differential equation of the first by replacement of $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. The correct answer is

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true and R is not a correct explanation of A.
(c) A is true but R is false
(d) A is false but R is true.

129. The differential equation of the family of circles of radius r whose centre lies on the x – axis is

(a) $y\frac{dy}{dx} + y^2 = r^2$ (b) $y\left(\frac{dy}{dx} + 1\right) = r^2$

(c) $y^2\left(\frac{dy}{dx} + 1\right) = r^2$ (d) $y^2\left(\left(\frac{dy}{dx}\right)^2 + 1\right) = r^2$

130. Let $(y - c)^2 = cx$ be the primitive of the differential equation

$$4x\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0$$

The number of integral curves which will pass through (1,2) is

(a) one (b) two
(c) three (d) four

131. Solution curves of $ydx - xdy = 0$ form a family of

(a) circles (b) straight lines
(c) Hyperbolas (d) parabolas

132. The solution of the differential equation is $\frac{d^2y}{dx^2} + y = 0$ satisfying the condition $y(0) = 1, y\left(\frac{\pi}{2}\right) = 1$

(a) $\cos x + 2\sin x$ (b) $\cos x + \sin x$
(c) $2\cos x + \sin x$ (d) $2(\cos x + \sin x)$

133. The solution of the differential equation $(D^4 + 8D^2 + 16)y = 0$ is given by

(a) $c_1e^{2x} + c_2e^{-2x} + c_3e^x + c_4e^{-x}$

(b) $(c_1 + c_2x)e^{2x} + (c_3 + c_4x)e^{-2x}$

(c) $(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$

(d) $(c_1 + c_2x)\cosh 2x + (c_3 + c_4x)\sinh 2x$

134. If I_1, I_2 are integrating factors of the equations $xy^1 + 2y = 1$ and $xy^1 - 2y = 1$ then

(a) $I_1 = -I_2$ (b) $I_1I_2 = x^2$

(c) $I_1 = x^2I_2$ (d) $I_1I_2 = 1$

Common Data (Q.135 - 136)

For the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

135. One of the solutions is

(a) e^x (b) $\ln x$
(c) e^{-x^2} (d) e^{x^2}

136. The second linearly independent solution is

(a) e^{-x} (b) xe^x
(c) x^2e^x (d) x^2e^{-x}

Common Data (Q.137 - 138)

137. Ordinary points of the differential equation $(1-x^2)y^{11} - 2xy^1 + 2y = 0$ are

(a) ± 1 (b) except ± 1
(c) all (d) no points

138. The most general solution of the differential equation $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$ is

(a) $y(x) = (A + Bx)e^{x/2}$

(b) $y(x) = (A + Bx)e^{-x/2}$

(c) $y(x) = Ae^{x/2} + Be^{-x/2}$

(d) $y(x) = A\cos h\,(x/2) + B\sin h\,(x/2)$

Common Data (Q.139 - 140)

139. The solution of the differential equation for

$y(t): \frac{\partial^2 y}{\partial t^2}$-y=2cosh$(t)$, subject to the initial conditions $y(0)=0$ and $\frac{dy}{dt}|_{t=0}=0$, is

(a) $\frac{1}{2}\cosh(t)+t\sinh(t)$

(b) $-\sinh(t)+t\cosh(t)$

(c) $t\cosh(t)$

(d) $t\sinh(t)$

140. Solution of the differential equation $x\frac{dy}{dx}+y=x^4$, with the boundary condition that $y=1$, at $x=1$, is

(a) $y=5x^4-4$ (b) $y=\frac{x^4}{5}+\frac{4x}{5}$

(c) $y=\frac{4x^4}{5}+\frac{1}{5x}$ (d) $y=\frac{x^4}{5}+\frac{4}{5x}$

FOURIER SERIES

141. Consider a function defined as

$$f(t)=\begin{cases}0 & -2<t<-1\\ k & -1<t<1\\ 0 & 1<t<2\end{cases}.$$

The Fourier coefficients a_0, b_n will be.

(a) 0, 0 (b) $k, \frac{2k}{n\pi}$

(c) $k, 0$ (d) $k, \frac{k}{n\pi}$

142. Given $f(x)=\begin{cases}0 & 0<x<c\\ 1 & c<x<2c\end{cases}$ which of the following represents the expansion of $f(x)$ in a Fourier series of period $2c$.

(a) $f(x)=\frac{1}{2}-\frac{2}{\pi}\left(\sin\frac{\pi x}{c}+\sin\frac{3\pi x}{c}+...\right)$

(b) $f(x)=\frac{2}{\pi}\left(\sin\frac{\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}+...\right)$

(c) $f(x)=\frac{1}{2}-\frac{2}{\pi}\left(\sin\frac{\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}+...\right)$

(d) $f(x)=\frac{1}{2}+\frac{2}{\pi}\left(\sin\frac{\pi x}{c}+\sin\frac{3\pi x}{c}+...\right)$

143. Which of the following statements is wrong

(a) Fourier series has sine terms when $f(x)$ is an odd function

(b) Fourier series has cosine terms when $f(x)$ is an even function

(c) Fourier series has sine as well as cosine terms when $f(x)$ is neither even nor odd function

(d) Fourier series has neither sine nor cosine terms when $f(x)$ is neither even nor odd function

144. Fourier series for the function $f(x)$ is given by

$f(x)=1+2x/\pi, -\pi\le x\le 0;$

$;1-2x/\pi,\ 0\le x\le\pi$ is

$f(x)=\frac{8}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+=........\right]$ by using the above result deduce $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......$

(a) $\frac{8}{\pi^2}$ (b) $\frac{4}{\pi^2}$

(c) $\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{8}$

FOURIER & LAPLACE TRANSFORMS

145. The Fourier sine transform of the function $f(t)=e^{-at}$ is

(a) $\sqrt{\frac{2}{\pi}}\frac{a}{\omega}$ (b) $\sqrt{\frac{2}{\pi}}\frac{\omega}{a}$

(c) $\sqrt{\frac{2}{\pi}}\frac{a}{\omega^2+a^2}$ (d) $\sqrt{\frac{2}{\pi}}\frac{\omega}{\omega^2+a^2}$

146. The inverse Laplace transform of

$$\frac{s+4}{s(s-1)(s^2+4)}$$

(a) $e^t - \frac{1}{2}\sin 2t$ (b) $e^t - \frac{1}{2}\sin 2t - 1$

(c) $1 - e^t + \frac{1}{2}\sin 2t$ (d) $e^t + \frac{1}{2}\sin 2t$

147. The Fourier transform to k-space of a Gaussian function in x, $f(x) = e^{-x^2}$ is

(a) a Gaussian in $1/k$, like e^{-1/k^2}
(b) a power law in k, like $1/k^2$
(c) a sinusoidal function like sin k or cos k
(d) a Gaussian in k like e^{-k^2}

148. If $f(s) = \int_0^\infty F(t)e^{-st}dt$ then $\int_0^\infty t\,F(t)e^{-st}dt$ is

(a) $-\frac{df}{ds}$ (b) $\frac{df}{ds}$

(c) 0 (d) – 1

149. The Laplace transform of a function $f(t)$ exist if
(a) It is uniformly continuous
(b) It is piecewise continuous
(c) It is uniformly continuous and of exponential
(d) It is piecewise continuous of exponential order

150. The Laplace inverse transform of $\sqrt{t}$ is

(a) $\sqrt{\frac{\pi}{s}}$ (b) $\frac{1}{2}\sqrt{\frac{\pi}{s}}$

(c) $\frac{\sqrt{\pi}}{2s^{3/2}}$ (d) $\sqrt{\frac{\pi}{s^3}}$

151. Laplace transform of $f(t) = te^{at}\sin(at), t > 0$.

(a) $\frac{2a(s-a)}{\left[(s-a)^2 + a^2\right]^2}$ (b) $\frac{a(s-a)}{(s-a)^2 + a^2}$

(c) $\frac{s-a}{(s-a)^2 - a^2}$ (d) $\frac{(s-a)^2}{(s-a)^2 + a^2}$

152. The Laplace transform of $f(t) = \sin \pi t$ is $F(s) = \frac{\pi}{(s^2 + \pi^2)}, s > 0.$ Therefore, the Laplace transform of $t \sin \pi t$ is

(a) $\frac{\pi}{s^2(s^2 + \pi^2)}$ (b) $\frac{2\pi}{s^2(s^2 + \pi^2)^2}$

(c) $\frac{2\pi s}{(s^2 + \pi^2)^2}$ (d) $\frac{2\pi}{(s^2 + \pi^2)^2}$

153. The value of $L[t^a]$ is (where 'a' is positive but not necessarily an integer)

(a) $\frac{\Gamma(n+1)}{s^{n+1}}$ (b) $\frac{\Gamma(a+1)}{s^{a+1}}$

(c) $\frac{\Gamma(a+1)}{s^{n+1}}$ (d) $\frac{\Gamma(a+1)}{s^n}$

154. $L^{-1}\left[\frac{p+1}{p^2 + 6p + 25}\right] =$

(a) $e^{-3t}\,[\cos 4t - \frac{1}{4}\sin 4t\,]$

(b) $e^{-3t}\cos 2t$

(c) $e^{-3t}\,[\cos 4t - \frac{1}{2}\sin 4t\,]$

(d) $e^{2t}\sin t$

155. If $f(x) = \begin{cases} 0 & \text{for } x < 3, \\ x - 3 & \text{for } x \geq 3, \end{cases}$ then the Laplace transform of $f(x)$ is

(a) $s^{-2}e^{3s}$ (b) $s^2 e^{-3s}$

(c) s^{-2} (d) $s^{-2}e^{-3s}$

156. Which of the following function is not analytic? (where $z = x + iy$)
(a) $\sin z$
(b) z^3
(c) $e^x(\cos y + i\sin y)$
(d) $\sin x \sin y - i\cos x \cos y$

157. If r is unit circle, $|z| = 1$, then $\oint_r \cot z^2 dz = 0$ because the function $\cot z^2$
(a) is analytic everywhere
(b) is not analytic everywhere but has no singular point inside r
(c) has only one singular point inside r where the residue is zero

(d) has several singular points inside r where the residue is nonzero such that the sum vanishes

158. The principle value of $\ln(i^i)$

(a) $i\pi$ (b) $\pi/2$

(c) $i\pi/2$ (d) $-\pi/2$

159. Let Z_1 and Z_2 be two non – zero complex numbers. If $|Z_1+Z_2| = |Z_1|+|Z_2|$, which of the following is true

(a) $\text{Re}(Z_1\overline{Z}_2)<0,\ \text{Im}(Z_1\overline{Z}_2)=0$

(b) $\text{Re}(Z_1\overline{Z}_2)>0,\ \text{Im}(Z_1\overline{Z}_2)=0$

(c) $\text{Re}(Z_1\overline{Z}_2)>0,\ \text{Im}(Z_1\overline{Z}_2)>0$

(d) $\text{Re}(Z_1\overline{Z}_2)<0,\ \text{Im}(Z_1\overline{Z}_2)<0$

160. The real part of the function $f(z)=e^{z^2}$, where $z = x + iy$, is

(a) $e^{x^2+y^2}$ (b) $e^{x^2-y^2}$

(c) $e^{x^2-y^2}\cos(2xy)$ (d) $e^{x^2+y^2}\sin(2xy)$

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161. The Laurent expansion of a function f(z) having a pole of order 3 at z = 1 is

$$f(z)=\sum_0^{\infty}\frac{(z-1)^n}{2^n}+\frac{1}{z-1}-\frac{1}{(z-1)^2}+\frac{2}{3(z-1)^3}.$$

The residue of $f(z)\ at\ z=1 is$

(a) 1 (b) 2/3

(c) -5 (d) 0

162. The function $f(z)=\dfrac{1}{1+e^z}$

(a) is analytic everywhere

(b) has a branch point singularity at z=0

(c) has poles of order one at $z=\pm i\pi, \pm 3i\pi, \pm 5i\pi, ...$

(d) has poles of order two at $z=\pm\pi, \pm 3\pi, \pm 5\pi, ...$

163. $\displaystyle\int_c \frac{\cos\pi z}{z-1}dz=$ (where c is the circle |z| =3)

(a) $i2\pi$ (b) $-i2\pi$

(c) $i6\pi^2$ (d) $-i6\pi^2$

164. $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx=$

(a) $\dfrac{\pi ab}{a+b}$ (b) $\dfrac{\pi(a+b)}{ab}$

(c) $\dfrac{\pi}{a+b}$ (d) $\dfrac{\pi}{a-b}$

165. Find the nature and location of singular point of $(z+1)\sin\left(\dfrac{1}{z-2}\right)$

(a) Essential singularity at z = 2

(b) Removable singularity at z = 2

(c) Essential singularity at z =- 1

(d) Removable singularity at z = - 1

Common Data (Q.166 - 167)

Consider a function $f(z)=\dfrac{z\sin z}{(z-\pi)^2}$ of a complex variable z.

166. Which of the following statements is TRUE for the function $f(z)$?

(a) $f(z)$ is analytic every where in the complex plane

(b) $f(z)$ has a zero at $z=\pi$

(c) $f(z)$ has a pole of order 2 at $z=\pi$

(d) $f(z)$ has a simple pole at

167. Consider a counterclockwise circular contour |z| = 1 about the origin. The integral over this contour is

(a) -iπ (b) zero

(c) iπ (d) 2iπ

EVALUATION & INTEGRAL

168. The value of $\displaystyle\int_c\frac{dz}{(z^2+a^2)}$ where c is a unit circle (anticlockwise) centered at the origin in the complex plane is

(a) π for $a = 2$ (b) zero for $a = 1/2$

(c) 4π for $a = 2$ (d) $\pi/2$ for $a = 1/2$

169. The value of the integral $\displaystyle\int_0^{\infty}\frac{\cos mx}{(x^2+1)}dx$

(a) $\frac{\pi}{2}e^{-m}$ (b) πe^{-m}

(c) πe^{-im} (d) $\frac{\pi}{2}e^{-2m}$

170. The value of the integral $\int_c \frac{4-3z}{z(z-1)(z-2)}dz$ where c is the circle $|z| = 3/2$

(a) Zero (b) πi

(c) $2\pi i$ (d) $-2\pi i$

171. Let $f(z)$ be continuous in a simply connected domain D and $\int_c f(z)dz = 0$ for every closed path in D. Then $f(z)$ is

(a) a constant in D (b) analytic in D

(c) zero in D (d) a polynomial in D

172. Consider the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ (where $|z| = 3$), which of the following represent the residues at each pole

(a) 7/9, 4/9 (b) 5/9, 7/9

(c) 2/9, 7/9 (d) 5/9, 4/9

173. The value of the integral $\int_c \frac{dz}{z^2-2z}$, $c = |z| = 1$ is

(a) 0 (b) πi

(c) $-\pi i$ (d) $-2\pi i$

174. The value of the integral $\oint \frac{e^z \sin z}{z^2}\frac{dz}{2\pi}$ around the unit circle in the complex plane is

(a) 1 (b) 0

(c) ∞ (d) i

175. If $\bar{z}$ is the complex conjugate of z, then the value of the integral $\int_c \bar{z}dz$ from $z = 0$ to $z = 4 + 2i$ along the curve given by $z = t^2 + it$ is

(a) $10 - 8i/3$ (b) $5 + 4i/3$

(c) 0 (d) $5 - 4i/3$

176. $\int_0^\infty \frac{dx}{(1+x^2)^5} =$

(a) π (b) $\frac{10\pi}{16}$

(c) $\frac{35\pi}{256}$ (d) $\frac{26\pi}{148}$

177. The value of the integral $\int \frac{\cos \pi z}{z^2-1}dz$ around a rectangle with vertices $2 \pm i, -2 \pm i$ is

(a) Zero (b) πi

(c) $2\pi i$ (d) $\frac{i}{\pi}$

178. $\int_c \frac{\sin \pi z^2}{(z-2)(z-1)}dz =$? Where c is the circle $|z| = 3$

(a) $i6\pi$ (b) $i2\pi$

(c) $i4\pi$ (d) 0

179. $\int_c z^2 e^{\frac{1}{Z}} dz = ?$ where c is $|z| = 1$

(a) $i3\pi$ (b) $-i3\pi$

(c) $\frac{i\pi}{3}$ (d) None of these

180. The value of the integral $\oint_c \frac{e^Z \sin(Z)}{z^2} dz$, where the contour C is the unit circle: $|Z-2| = 1$, is

(a) $2\pi i$ (b) $4\pi i$

(c) πi (d) 0

181. The value of the integral $\int_C \frac{e^z}{z^2-3z+2}dz$, where the contour C is the circle $|z| = \frac{3}{2}$ is

(a) $2\pi ie$ (b) πie

(c) $-2\pi ie$ (d) $-\pi ie$

ELEMENTARY PROBABILITY THEORY

182. A card is draw from a pack containing 52 cards with 4 aces and another card is drawn from a pack of 48 cards with 8 aces. What is the probability that both are aces?

(a) $\frac{4}{52}$ (b) $\frac{8}{48}$

(c) $\frac{32}{52\times 48}$ (d) $\frac{1}{78}$

183. In a group of ten people, the probability that at least one person was born on a sunday is

(a) 1/7 (b) $1-(1/10)^7$

(c) $(1/7)^{10}$ (d) $1-(6/7)^{10}$

184. The probability that a man who is x years old will die in a year is p. Then amongst n persons $A_1, A_2, ..., A_n$ each x years old now, the probability that A_1 will die in one year is

(a) $\frac{1}{n^2}$ (b) $1-(1-p)^n$

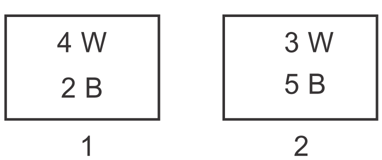
(c) $\frac{1}{n^2}\left[1-(1-p)^n\right]$ (d) $\frac{1}{n}\left[1-(1-p)^n\right]$

185. The odds against a husband who is 45 years old, living till he is 70 are 7: 5 and the odds against his wife who is 36, living till she is 61 are 5 : 3. The probability that at least one of them will be alive 25 years hence, is

(a) 61/96 (b) 5/32

(c) 13/64 (d) 64/13

186. A bag contains 4 white and 2 black balls. Another bag contains 3 white and 5 black balls. If one ball is drawn from each bag, the probability that both are white is



(a) 1/24 (b) 1/4

(c) 5/24 (d) 7/24

187. A man applying for his driver's license estimates that his chances of passing the written test are 2/3, and that his chances of passing the driving test are 1/4. What is the probability that he passes both tests.

(a) 1/7 (b) 1/6

(c) 1/3 (d) 1/4

Common Data (Q.188 - 189)

188. A bag contains 9 marbles, 3 of which are red, 3 of which are blue, and 3 of which are yellow. If three marbles are selected from the bag at random, what is the probability that they are all of different colors.

(a) 9/28 (b) 5/28

(b) 3/28 (c) 11/28

189. The probability of hitting a target is 2/5. A person fires at the target 10 times. What is the probability that he hits the target exactly 6 times?

(a) ${}^{10}C_6\left(\frac{2}{5}\right)^6$ (b) ${}^{10}C_6\left(\frac{3}{5}\right)^4$

(c) ${}^{10}C_6\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^4$ (d) $\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^4$

190. Poisson distribution is given by $P_x = \frac{e^{-m}m^x}{x!}$. Which of the following is correct.

I) $P_{m-1} = P_m$ II) $P_{x-1} = \frac{x}{m}P_m$

III) $P_{x+1} = \frac{m}{x+1}P_x$

(a) I and II only (b) I , II and III

(c) I and III only (d) only I

191. Consider the following statements and Identify the CORRECT ONE

I) Kronecker delta is a mixed tensor of order 2

II) Velocity of a fluid at any point is a contra variant tensor of rank 1

III) A symmetric tensor of second order has $\frac{1}{2}n(n+1)$ different Components

(a) I, II, III (b) I & II only

(c) I & III only (d) II & III only

INTRODUCTORY GROUP THEORY APPENDIX

192. Consider the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the normalized eigenvector corresponding to the eigenvalue 5 is

(a) $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

(c) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (d) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Common Data(Q.193 - 194)

Apply Runge Kutta forth order method to obtain $y(0.2)$, $y(0.6)$ from $dy/dx = 1 + y^2$, with $y = 0$ at $x = 0$. Take step size $h = 0.2$.

193. $y(0.2) =$

(a) 0.2027 (b) 0.4396
(c) 0.3846 (d) 0.9341

194. $y(0.4) =$

(a) 0.1649 (b) 0.8397
(c) 0.4227 (d) 0.1934

NUMERICAL METHODS

195. Back ward Euler method for solving the differential equation $\frac{dy}{dx} = f(x, y)$ is specified by

(a) $y_{n+1} = y_n + h f(x_n, y_n)$

(b) $y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$

(c) $y_{n+1} = y_{n-1} + 2h f(x_n, y_n)$

(d) $y_{n+1} = (1+h)(x_{n+1}, y_{n+1})$

196. The formula used to compute an approximation for the second derivative of a function f at a point x_0 is

(a) $\frac{f(x_0+h)+f(x_0-h)}{2}$

(b) $\frac{f(x_0+h)-f(x_0-h)}{2h}$

(c) $\frac{f(x_0+h)+2f(x_0)+f(x_0-h)}{h^2}$

(d) $\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}$

197. The Newton - Raphson iteration formula for finding $\sqrt[3]{c}$, where $c > 0$ is,

(a) $x_{n+1} = \frac{2x_n^3 + \sqrt[3]{c}}{3x_n^2}$ (b) $x_{n+1} = \frac{2x_n^3 - \sqrt[3]{c}}{3x_n^2}$

(c) $x_{n+1} = \frac{2x_n^3 + c}{3x_n^2}$ (d) $x_{n+1} = \frac{2x_n^3 - c}{3x_n^2}$

198. The Newton – Raphson method is used to find the root of the equation $x^2 - 2$. If the iterations are started from -1, then the iteration will

(a) converge to -1 (b) converge to $\sqrt{2}$
(c) converge to - $\sqrt{2}$ (d) not converge

199. The value of $\int_1^2 \frac{1}{x}dx$ computed using Simpson's rule with a step size of $h = 0.25$ is

(a) 0.69430 (b) 0.69385
(c) 0.69325 (d) 0.69415

200. Given $a > 0$, we wish to calculate its reciprocal value $\frac{1}{a}$ by using Newton – Raphsonmethod $f(x) = 0$. The Newton – Raphson algorithm for the function will be

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$

(b) $x_{k+1} = x_k + \frac{a}{2}x_k^2$

(c) $x_{k+1} = 2x_k - ax_k^2$

(d) $x_{k+1} = x_k + \frac{a}{2}x_k^2$

201. Newton – Raphson formula to find the roots of an equation $f(x)=0$ is given by

(a) $x_{n+1}=x_n-\dfrac{f(x_n)}{f^1(x_n)}$

(b) $x_{n+1}=x_n+\dfrac{f(x_n)}{f^1(x_n)}$

(c) $x_{n+1}=\dfrac{f(x_n)}{x_n f^1(x_n)}$

(d) $x_{n+1}=\dfrac{x_n f(x_n)}{f^1(x_n)}$

202. Match the following and choose the correct combination

Group – I
E) Newton – Raphson method
F) Runge – Kutta method
G) Simpson's Rule
H) Gauss elimination

Group – II
1) Solving non – linear equations
2) Solving linear simultaneous equations
3) Solving ordinary differential equations
4) Numerical integration method
5) Interpolation
6) Calculation of eigen values

(a) E – 6, F – 1, G – 5, H – 3
(b) E – 1, F – 6, G – 4, H – 3
(c) E – 1, F – 3, G – 4, H – 2
(d) E – 5, F – 3, G – 4, H – 1

203. Identify the Newton – Raphson iteration scheme for the finding the square root of 2

(a) $x_{n+1}=\dfrac{1}{2}\left(x_n+\dfrac{2}{x_n}\right)$ (b) $x_{n+1}=\dfrac{1}{2}\left(x_n-\dfrac{2}{x_n}\right)$

(c) $x_{n+1}=\dfrac{2}{x_n}$ (d) $x_{n+1}=\sqrt{2+x_n}$

204. The following equation needs to be numerically solved using the Newton – Raphson method $x^3+4x-9=0$. The iterative equation for this purpose is (k indicates the iteration level)

(a) $x_{k+1}=\dfrac{2x_k^2+9}{3x_k^2+4}$

(b) $x_{k+1}=\dfrac{3x_k^3+9}{2x_k^2+9}$

(c) $x_{k+1}=x_k-3_k^2+4$

(d) $x_{k+1}=\dfrac{4x_k^2+3}{9x_k^2+2}$

205. Matching exercise choose the correct one out of the alternatives A, B, C, D

Group – I
P) 2nd order differential equations
Q) Non – linear algebraic equations
R) Linear algebraic equations
S) Numerical integration

Group – II
1) Runge – Kutta method
2) Newton – Raphson method
3) Gauss Elimination
4) Simpson's Rule

(a) P – 3, Q – 2, R – 4, S – 1
(b) P – 2, Q – 4, R – 3, S – 1
(c) P – 1, Q – 2, R – 3, S – 4
(d) P – 1, Q – 3, R – 2, S – 4

206. A differential equation $\dfrac{dx}{dt}=e^{-2t}u(t)$ has to be solved using trapezoidal rule of integration with a step size $h=0.01$ sec. Function $u(t)$ indicates a unit step function. If $x_0=0$ then the value of x at $t=0.01$ sec will be given by

(a) 0.00099 (b) 0.00495
(c) 0.0099 (d) 0.0198

207. Consider a differential equation $\dfrac{dy(x)}{dx}-y(x)=x$ with initial condition $y_0=0$. Using Euler's first order method with a step size of 0.1 then the value of $y(0.3)$ is

(a) 0.01 (b) 0.031
(c) 0.0631 (d) 0.1

208. The following algorithm computers the integral $J=\int_a^b f(x)dx$ from the given values $f_j=f(x_j)$ at equidistant points $x_0=a,\ x_1=x_0+h,$

$x_2 = x_0 + 2h, ...x_{2m} = x_0 + 2mh = b$ compute

$S_0 = f_0 + f_{2m}$

$S_1 = f_1 + f_3 + ... + f_{2m-1}$

$S_2 = f_2 + f_4 + ... + f_{2m-2}$

$J = \frac{h}{3}[S_0 + 4(S_1) + 2(S_2)]$

The rule of numerical integration, which uses the above algorithm is

(a) Rectangle rule (b) Trapezoidal rule

(c) Four – point rule (d) Simpson's rule

INTEGRATION BY TRAPEZOIDAL & SIMPSON'S RULE

209. By Simpson's rule, the value of $\int_{-3}^{3} x^4 dx$ by taking 6 sub-intervals is

(a) 96 (b) 98

(c) 99 (d) 100

210. By Simpson's rule, the value of $\int_{1}^{2} \frac{dx}{x}$ dividing the interval (1,2) into four equal parts is

(a) 0.6932 (b) 0.6753

(c) 0.6692 (d) 0.6319

211. Using Simpson's $\frac{1}{3}$ rule, the value of

x	1	1.5	2	2.5	3
$f(x)$	2.1	2.4	2.2	2.8	3

$\int_{1}^{3} f(x)dx$ for the following data is

(a) 4.975 (b) 5.05

(c) 11.1 (d) 55.5

212. If $e^0 = 1, e^1 = 2.72, e^2 = 7.39, e^3 = 20.09$ and $e^4 = 54.60$, then by Simpson's rule value of $\int_{0}^{4} e^x dx$ is

(a) 5.387 (b) 52.78

(c) 53.17 (d) 53.87

213. If by Simpson's rule

$$\int_{0}^{1} \frac{1}{1+x^2} dx = \frac{1}{12}\left[3.1 + 4(a+b)\right]$$

When the interval [0, 1] is divided into 4 sub-intervals and a & b are the values of $\frac{1}{1+x^2}$ at two of its division point, then a&b are

(a) $a = \frac{1}{1.0625}, b = \frac{1}{125}$

(b) $a = \frac{1}{1.0625}, b = \frac{1}{1.5625}$

(c) $a = \frac{1}{1.25}, b = 1$

(d) $a = \frac{1}{1.5625}, b = \frac{1}{1.25}$

214. A river is 80 meter wide. Its depth d meter and corresponding distance x meter form one bank is given below in table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

The approximate area of cross-section of river by Trapezoidal rule is

(a) 705 m^2 (b) 710 m^2

(c) 730 m^2 (d) 750 m^2

215. From the following table, using Trapezoidal rule, the area bounded by the curve, the x-axis and the line $x = 7.47, x = 7.52$ is

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

(a) 0.0776 (b) 0.1096

(c) 0.0896 (d) 0.0996

216. Taking four sub-intervals, the value of $\int_{0}^{1} \frac{1}{1+x} dx$ by Simpson's rule, is

(a) 0.6035 (b) 0.6945

(c) 0.6145 (d) 0.5945

217. If $h = 1$ in Simpson's rule, the value of $\int_{1}^{5} \frac{dx}{x}$ is

(a) 1.43 (b) 1.48

(c) 1.56 (d) 1.62

218. A curve passes through the points given by the following table:

x	1	2	3	4	5
y	10	50	70	80	100

By Trapezoidal rule, the area bounded by the curve, the x-axis and the lines $x = 1, x = 5$ is

(a) 255 (b) 275

(c) 305 (d) 310

219. The value of $\int_0^1 x^3 dx$ by Trapezoidal rule taking five sub-intervals is

(a) 0.21 (b) 0.23

(c) 0.24 (d) 0.26

220. Taking the step size $\frac{\pi}{12}$ the value of $\int_0^{\pi/2} \sqrt{1-0.162\sin^2 x}dx$ by Simpson's one third rule is

(a) 1.5058 (b) 1.5759

(c) 2.5056 (d) 1.5056

INTEGRATION BY TRAPEZOIDAL

221. For $dy/dx = 1 + xy$, given that $y = 1$ at $x = 0$. The value of $y(0.1)$ correct to four decimal places using Taylor's series is $(h = 0.1)$

(a) 2.1513

(b) 1.1053

(c) 1.2689

(d) None of the above

222. For $dy/dx = x - y^2$ given that $y = 1$ at $x = 0$. Using Taylor's series the value of $y(0.1)$ correct to four decimal places is

(a) 1.4396 (b) 0.9138

(c) 1.0134 (d) 0.9159

223. For $dy/dx = x^2 + y^2$ given that $y = 0$ at $x = 0$. The solution of differential equation for $x = 0.4$ using Picard's method is

(a) 0.02193 (b) 0.02145

(c) 0.02135 (d) 0.02199

224. For $dy/dx = y - x$ given that $y = 2$ at $x = 0$. Using Picard's method up to third order of approximation the solution of the equation is

(a) $2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{16}$

(b) $2 + 2x + \frac{x^2}{2} + \frac{x^3}{6}$

(c) $2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$

(d) $2 + 2x + \frac{x^2}{2} - \frac{x^3}{6}$

225. For $dy/dx = x + y^2$, given that $y = 0$ at $x = 0$. Using Picard's method up to third order of approximation the solution of the differential equation is

(a) $\frac{x^2}{2} + \frac{x^5}{40} + \frac{x^8}{480} + \frac{x^{11}}{1600}$

(b) $\frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$

(c) $\frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{2400}$

(d) $\frac{x^2}{2} + \frac{x^5}{40} + \frac{x^8}{480} + \frac{x^{11}}{2400}$

226. For $dy/dx = xy$ given that $y = 1$ at $x = 0$. Using Euler method taking the step size 0.1, the y at $x = 0.4$ is

(a) 1.0611 (b) 2.4680

(c) 1.6321 (d) 2.4189

Common Data(Q.227 - 228)

For $dy/dx = x^2 + y^2$ given that $y = 1$ at $x = 0$. Determine the value of y at given x in question using modified method of Euler. Take the step size 0.02.

227. y at $x = 0.02$ is

(a) 1.0468 (b) 1.0204

(c) 1.0346 (d) 1.0348

228. y at $x = 0.04$ is

(a) 1.0316 (b) 1.0301

(c) 1.0408 (d) 1.0416

229. y at $x = 0.06$ is

(a) 1.0348 (b) 1.0539

(c) 1.0638 (d) 1.0796

230. For $dy/dx = x + y$ given that $y = 1$ at $x = 0$. Using modified Euler's method taking step size 0.2, the value of y at $x = 1$ is

(a) 3.401638 (b) 3.405417

(c) 9.164396 (d) 9.168238

231. For the differential equation $dy/dx = x - y^2$ given that

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762

Using Milne predictor-correction method, the y at next value of x is

(a) 0.2498 (b) 0.3046
(c) 0.4648 (d) 0.5114

Common Data(Q.232 - 233)

For $dy/dx = 1 + y^2$ given that

Using Milne's method determine the value of y for x given in question.

x	0	0.2	0.4	0.6
y	0	0.2027	0.4228	0.6841

232. $y(0.8) = ?$

(a) 1.0293 (b) 0.4228
(c) 0.6065 (d) 1.4396

233. $y(1.0) = ?$

(a) 1.9428 (b) 1.3428
(c) 1.5555 (d) 2.168

Common Data(Q.234 - 235)

Apply Runge-Kutta fourth order method to obtain $y(0.2)$, $y(0.4)$ and $y(0.6)$ from $dy/dx = 1 + y^2$, with $y = 0$ at $x = 0$. Take step size $h = 0.2$.

234. $y(0.6) = ?$

(a) 0.9348 (b) 0.2935
(c) 0.6841 (d) 0.563

235. For $dy/dx = x + y^2$, given that $y = 1$ at $x = 0$. Using Runge-Kutta fourth order method the value of y at $x = 0.2$ is ($h = 0.2$)

(a) 1.2735 (b) 2.1635
(c) 1.9356 (d) 2.9468

236. For $dy/dx = x + y$ given that $y = 1$ at $x = 0$. Using Runge-Kutta fourth order method the value of y at $x = 0.2$ is ($h = 0.2$)

(a) 1.1384 (b) 1.9438
(c) 1.2428 (d) 1.6389

237. The second order Runge-Kutta method is applied to the initial value problem

$y' = y, y(0) = y_0$, with step size h then h is

(a) $y_0(h-1)^2$

(b) $\frac{y_0}{2}(h^2 - 2h + 2)$

(c) $\frac{y_0}{6}(h^2 - 2h + 2)$

(d) $y_0\left(1 + h + \frac{h^2}{2} + \frac{h^2}{6}\right)$

238. The Runge-Kutta method of fourth order is used is used to solve the differential equation $dy/dx = f(x), y(0) = 0$ with step size h. The solution at $x = h$ is given by

(a) $y(h) = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right]$

(b) $y(h) = \frac{h}{6}\left[f(0) + 2f\left(\frac{h}{2}\right) + f(h)\right]$

(c) $y(h) = \frac{h}{6}\left[f(0) + f + f(h)\right]$

(d) $y(h) = \frac{h}{6}\left[2f(0) + f\left(\frac{h}{2}\right) + 2f(h)\right]$

239. The method $y_n + 1 = y_n + \frac{1}{4}(k_1 + 3k_2)$ $n = 0,1,...$

$k_1 = hf(x_i\ y_n)$; $k_2 = hf\left(x_n + \frac{2h}{3}, y_n + \frac{2}{3}k_1\right)$

is used to solve the initial value problem $y' = f(x,y) = -10y, y(0) = 1$. The method willproduce stable results if the step size h satisfies

(a) $0.2 < h < 0.5$ (b) $0 < h < 0.2$
(c) $0 < h < 1$ (d) $0 < h < 0.2$

240. The Solving the ordinary differential equation $y' = 2x, y(0) = 0$ using Eular's method, the iterates $y_n, n \in N$ satisfy

(a) $y_n = 2x_n^2$ (b) $y_n = 2x_n$
(c) $y_n = x_n x_{n-1}$ (d) $y_n = 2x_{n-1} + x_n$

ANSWERS

1. (c)	**2.** (b)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (b)	**7.** (d)	**8.** (d)	**9.** (a)	**10.** (a)
11. (b)	**12.** (c)	**13.** (c)	**14.** (a)	**15.** (c)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (c)	**20.** (d)
21. (a,d)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (b)	**29.** (a)	**30.** (b)
31. (c)	**32.** (c)	**33.** (b)	**34.** (c)	**35.** (a)	**36.** (b)	**37.** (a)	**38.** (d)	**39.** (d)	**40.** (c)
41. (d)	**42.** (b)	**43.** (a)	**44.** (a)	**45.** (b)	**46.** (d)	**47.** (b)	**48.** (c)	**49.** (d)	**50.** (c)
51. (a)	**52.** (c)	**53.** (a)	**54.** (c)	**55.** (c)	**56.** (b,c)	**57.** (a)	**58.** (b)	**59.** (d)	**60.** (d)
61. (a)	**62.** (d)	**63.** (c)	**64.** (b)	**65.** (b)	**66.** (c)	**67.** (a)	**68.** (a)	**69.** (b)	**70.** (c)
71. (b)	**72.** (b)	**73.** (a)	**74.** (c)	**75.** (c)	**76.** (c)	**77.** (a)	**78.** (d)	**79.** (d)	**80.** (b)
81. (c)	**82.** (c)	**83.** (a)	**84.** (d)	**85.** (b)	**86.** (b)	**87.** (d)	**88.** (d)	**89.** (a)	**90.** (d)
91. (c)	**92.** (b)	**93.** (c)	**94.** (c)	**95.** (c)	**96.** (b)	**97.** (c)	**98.** (c)	**99.** (a)	**100.** (b)
101. (a)	**102.** (d)	**103.** (a)	**104.** (a)	**105.** (c)	**106.** (c)	**107.** (a)	**108.** (a)	**109.** (a)	**110.** (c)
111. (a)	**112.** (b)	**113.** (b)	**114.** (b)	**115.** (c)	**116.** (c)	**117.** (d)	**118.** (a)	**119.** (c)	**120.** (d)
121. (a)	**122.** (d)	**123.** (c)	**124.** (b)	**125.** (a)	**126.** (b)	**127.** (d)	**128.** (a)	**129.** (d)	**130.** (b)
131. (b)	**132.** (b)	**133.** (c)	**134.** (d)	**135.** (a)	**136.** (b)	**137.** (a)	**138.** (b)	**139.** (d)	**140.** (d)
141. (c)	**142.** (c)	**143.** (c)	**144.** (d)	**145.** (d)	**146.** (b)	**147.** (d)	**148.** (a)	**149.** (c)	**150.** (c)
151. (a)	**152.** (c)	**153.** (b)	**154.** (c)	**155.** (d)	**156.** (d)	**157.** (c)	**158.** (d)	**159.** (a)	**160.** (c)
161. (b)	**162.** (b)	**163.** (b)	**164.** (c)	**165.** (a)	**166.** (d)	**167.** (b)	**168.** (b)	**169.** (b)	**170.** (d)
171. (b)	**172.** (d)	**173.** (c)	**174.** (d)	**175.** (a)	**176.** (c)	**177.** (a)	**178.** (d)	**179.** (c)	**180.** (d)
181. (c)	**182.** (d)	**183.** (d)	**184.** (d)	**185.** (a)	**186.** (b)	**187.** (b)	**188.** (a)	**189.** (c)	**190.** (c)
191. (a)	**192.** (d)	**193.** (a)	**194.** (c)	**195.** (b)	**196.** (d)	**197.** (c)	**198.** (c)	**199.** (c)	**200.** (c)
201. (a)	**202.** (c)	**203.** (a)	**204.** (a)	**205.** (c)	**206.** (c)	**207.** (b)	**208.** (d)	**209.** (b)	**210.** (a)
211. (b)	**212.** (d)	**213.** (b)	**214.** (a)	**215.** (d)	**216.** (b)	**217.** (d)	**218.** (a)	**219.** (d)	**220.** (d)
221. (b)	**222.** (b)	**223.** (c)	**224.** (a)	**225.** (b)	**226.** (a)	**227.** (b)	**228.** (c)	**229.** (c)	**230.** (b)
231. (b)	**232.** (a)	**233.** (c)	**234.** (c)	**235.** (a)	**236.** (c)	**237.** (b)	**238.** (a)	**239.** (d)	**240.** (c)

EXPLANATIONS

1. Vectors are L.D. $\Rightarrow$ determinent = 0

$$\begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{vmatrix} = 0$$

$$\Rightarrow x(x^2 - 1) - 1 \cdot x = 0$$

$$\Rightarrow x^3 - x - x = 0$$

$$\Rightarrow x^3 - 2x = 0$$

$$\Rightarrow x(x^2 - 2) = 0$$

$$x = 0, \quad x^2 = 2$$

$$x = \pm\sqrt{2}$$

2. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 2 & 4 & -1 \end{vmatrix} = 3(4-4) = 0 \Rightarrow$ Vectors are lineary dependent.

3. $f(x) = xe^{-x^2}$

$f(x)$ is maximum $\Rightarrow \dfrac{df}{dx} = 0$

$$e^{-x^2} + xe^{-x^2}(-2x) = 0$$

$$\Rightarrow \quad e^{-x^2}[1 - 2x^2] = 0$$

$$\Rightarrow \quad 2x^2 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{2} \quad \Rightarrow x = \mp\frac{1}{\sqrt{2}}$$

4. $f(x) = x(x-1)$

$$\frac{df}{dx} = x - 1 + x = 2x - 1$$

$$\frac{df}{dx} = 0 \Rightarrow \quad 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

$$\frac{d^2 f}{dx^2} = 2 = +\text{ve}$$

$f(x)$ is minimum at $x = 1/2$.

5.

$$\langle f|g\rangle = (1 - i - 1)\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = 1 - 1$$

$$\langle f|g\rangle = 0$$

$$\langle f|h\rangle = (1 - i - 1)\begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix} = 1 + 1 - 2 = 0$$

$$\langle f|h\rangle = 0$$

$\Rightarrow f$ is orthogonal to g and h.

7. $\begin{vmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & 5 & 7\lambda \end{vmatrix} = 0 \quad \Rightarrow 1(-7\lambda + 25) - 2(-14\lambda + 5\lambda) + 3(-10 + \lambda) = 0 \Rightarrow \lambda = \dfrac{5}{14}$

8. According to D' Alembert's ratio test.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \begin{cases} < 1 & \text{Convergent} \\ > 1 & \text{Divergent} \\ = 1 & \text{Test fails} \end{cases}$$

9.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$$

10.

$$F = (yz, xz, xy)$$

$$F = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \cdot F = 0$$

$$\nabla\times F=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & xz & xy\end{vmatrix}=\hat{i}[x-x]-\hat{j}[y-y]+\hat{k}[z-z]=0$$

11. $\displaystyle\int_{(0,0)}^{\left(\frac{\pi}{2},\frac{\pi}{2}\right)} y\cos x dx+\sin x dy$

Line equation is $x=y$

$$dx=dy \Rightarrow \int_0^{\frac{\pi}{2}}(x\cos x+\sin x)dx=\left[x\sin x-(-\cos x)-\cos x\right]_0^{\pi/2}$$

$$\frac{\pi}{2}\sin\frac{\pi}{2}=\frac{\pi}{2}$$

12. $\vec{V}$ is irrotational.

$\Rightarrow$ $$\nabla\times\vec{V}=0$$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y+az & bx+3y-z & 3x+cy+z\end{vmatrix}=0$$

$$\nabla\times V=0\Rightarrow (c+1)\hat{i}-(3-a)\hat{j}+(b-1)\hat{k}=0$$

$$c+1=0 \qquad a-3=0 \qquad b-1=0$$

$\Rightarrow$ $$c=-1 \qquad a=3 \qquad b=1$$

13. $$(\bar{a}\times\bar{b})\bullet\left\{(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})\right\}=(\bar{a}\times\bar{b})\bullet\left\{[\bar{b}\ \bar{c}\ \bar{a}]\,\bar{c}-[\bar{b}\ \bar{c}\ \bar{c}]\,\bar{a}\right\}$$

$$=(\bar{a}\times\bar{b})\bullet\left\{[\bar{b}\ \bar{c}\ \bar{a}]\,\bar{c}\right\}$$

$$=\left\{(\bar{a}\times\bar{b})\bullet\bar{c}\right\}\left\{[\bar{b}\ \bar{c}\ \bar{a}]\right\}$$

$$=\left[\bar{c}\ \bar{a}\ \bar{b}\right][\bar{b}\ \bar{c}\ \bar{a}]=[\bar{b}\ \bar{c}\ \bar{a}]^2\neq[\bar{a}\ \bar{b}\ \bar{c}\ ]$$

14. $F=(x^2-y^2+x)\hat{i}-(2xy+y)\hat{j}$ (0,0) to (1,1)

$W=\int\vec{F}\cdot\vec{ds}$ $\vec{ds}=dx\hat{i}+dy\hat{j}$

$$\vec{F}\cdot\vec{ds} = (x^2 - y^2 + x)dx - (2xy + y)dy$$

$$y^2 = x$$

$$2dydy = dx$$

$$W = \int_0^1 (x^2 - y^2 + x)2ydy - (2xy + y)dy$$

$$= \int_0^1 (y^4 - y^2 + y^2)2ydy - (2y^3 + y)dy$$

$$= \int_0^1 (2y^5 - 2y^3 - y)dy = \left[2\left(\frac{y^6}{6}\right) - 2\left(\frac{y^4}{4}\right) - \left(\frac{y^2}{2}\right)\right]_0^1 = \frac{1}{3} - \frac{1}{2} - \frac{1}{2} \quad = \frac{1}{3} - 1$$

$$= -\frac{2}{3}$$

$$W = \frac{2}{3}$$

15. $\nabla\phi = \frac{\overline{r}}{r^2} = \frac{\hat{r}}{r} \Rightarrow \phi = \ln|r|$

16. $u = i \qquad v = x\hat{i}$

$\nabla \bullet u = 0 \qquad \nabla \bullet v = 1$

17. $\vec{F} = x\hat{i} + y\hat{j} \qquad \vec{dl} = dx\hat{i} + dy\hat{j}$

$$\int_c \vec{F}\cdot\vec{dl} = \int_c (xdx + ydy) \qquad c : x = y$$

$$= \int_0^1 2xdx = 2\left(\frac{x^2}{2}\right)_0^1 = 1$$

18. $\vec{\nabla}\cdot\vec{A} = 1 + 3 + 1 = 5$

$\vec{\nabla}\cdot\vec{B} = e^x \sin y - e^x \sin y = 0$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + 3z & bx + 3y - z & 3x - y + z \end{vmatrix} = \hat{i}(-1+1) - \hat{j}(3-3) + \hat{k}(1-1) = 0$$

$$\vec{\nabla}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^x\sin y & e^x\cos y & 0\end{vmatrix}=\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(e^x\cos y-e^x\cos y)=0$$

19. Curl Curl Curl Curl $F=\nabla\times\nabla\times\nabla\times\nabla\times F$

F is solenoidal $\Rightarrow\quad \nabla\cdot F=0$

$$=\nabla\times\nabla\times[\nabla(\nabla\cdot F)-\nabla^2 F]$$

$$=-\nabla\times\nabla\times\nabla^2 F$$

$$=-\nabla^2[\nabla\times\nabla\times F]$$

$$=-\nabla^2[\nabla(\nabla\cdot F)-\nabla^2 F]$$

$$=\nabla^4 F$$

20. $f(x)=x^2e^{-x^2}$

$$\frac{df}{dx}=2xe^{-x^2}+x^2e^{-x^2}(-2x)$$

$$\frac{df}{dx}=2xe^{-x^2}(1-x^2)=2e^{-x^2}(x-x^3)$$

$$\frac{d^2f}{dx^2}=2e^{-x^2}(1-3x^2)+2(x-x^3)e^{-x^2}(-2x)=2e^{-x^2}\{1-3x^2-2x^2+2x^4\}$$

$$\frac{d^2f}{dx^2}=2e^{-x^2}[1+2x^4-5x^2]$$

$\frac{df}{dx}=0\quad\Rightarrow\quad 1-x^2=0\quad x=\pm1$

For $x=1$

$$\frac{d^2f}{dx^2}=2e^{-1}[1+2-5]=-4e^{-1}=\text{-ve}$$

For $x=-1$

$$\frac{d^2f}{dx^2}=2e^{-1}[1+2-5]=-4e^{-1}=\text{-ve}$$

21. $V = \frac{r}{r^3}$ $\nabla \cdot V = \nabla \cdot \frac{r}{r^3} = 0 \quad \Rightarrow \nabla \times \left(\frac{\vec{r}}{r^3}\right) = \nabla \times \left(\frac{\hat{r}}{r^2}\right)$

$$\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{1}{r^2} & 0 & 0 \end{vmatrix} = \vec{0}$$

$$\int_s V \cdot ds = \int_s \frac{r}{r^3} \cdot ds = \int_V \nabla \cdot \left(\frac{r}{r^3}\right) dV = 0$$

22. Directional derivative of a function in the direction of $\vec{a}$ is $\nabla f \cdot \frac{\vec{a}}{|a|} \Rightarrow \nabla f = 4x\hat{i} + 6y\hat{j} + 2z\hat{k}$ $\frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} - 2\hat{k}}{\sqrt{5}}$

$$\Rightarrow \nabla f \cdot \frac{\vec{a}}{|a|}\bigg|_{(2,1,3)} = \frac{4x - 4z}{\sqrt{5}} = \frac{4\times 2 - 4\times 3}{\sqrt{5}} = \frac{8-12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

23. $\nabla \cdot \vec{F} = 2\hat{x}$

(a) $\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3z & 5y \end{vmatrix} = \hat{x}(5-3) - \hat{y}(0-2) + \hat{z}(0-0) = 2\hat{x} + 2\hat{y}$

(b) $\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z & 5y \end{vmatrix} = \hat{x}(5-1) - \hat{y}(0-0) + \hat{z}(0-0) = 4\hat{x}$

(c) $\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 3x & 5y \end{vmatrix} = \hat{x}(5-0) - \hat{y}(0-0) + \hat{z}(3-0) = 5\hat{x} + 3\hat{z}$

(d) $\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & 0 & 5y \end{vmatrix} = \hat{x}(5-0) - \hat{y}(0-0) + \hat{z}(0-0) = 5\hat{x}$

24. $\nabla(\log r) = \frac{1}{r}\hat{r} = \frac{\vec{r}}{r^2}$

27. $\vec{A} = e^{xyz}(\hat{i} + \hat{j} + \hat{k})$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix} = \hat{i}e^{xyz}(xz - xy) + \hat{j}e^{xyz}(xy - yz) + \hat{k}e^{xyz}(yz - xz)$$

$$\nabla \times \vec{A}\Big|_{(1,2,3)} = \hat{i}e^6 + \hat{j}e^6(-4) + \hat{k}e^6(3) = e^6(\hat{i} - 4\hat{j} + 3\hat{k})$$

28. $\nabla \times \vec{A} = 0, \nabla \times \vec{B} = 0;\ \nabla \times (\vec{A} \times \vec{B}) = (\vec{B}.\nabla)\vec{A} - (\vec{A}.\nabla)\vec{B} + A\nabla.\vec{B} - B\nabla.\vec{A}$;

$\nabla.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}(\vec{\nabla} \times \vec{B}) = 0$; $\therefore$ If $\vec{A}$ and $\vec{B}$ are irrotational $\vec{A} \times \vec{B}$ is solenoidal.

29. $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$

$$\vec{\nabla}.\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}) = 3(x^2 + y^2 + z^2) = 3R^2;$$

$$\int \vec{A}.d\vec{s} = \int_v \vec{\nabla}.\vec{A}dv$$

$$\therefore \int_s \vec{A}.d\vec{s} = \int 3R^2 dv = \int 3R^2(4\pi R^2)dR = \frac{12}{5}\pi R^5$$

30. $\phi_1 = x^2 + y^2 + z^2 - 1 \quad \Rightarrow \quad \nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\phi_2 = x^2 + y^2 - z - 1 \quad \Rightarrow \quad \nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$$\nabla\phi_1\Big|_{(1,1,-1)} = 2\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\nabla\phi_2\Big|_{(1,1,-1)} = 2\hat{i} + 2\hat{j} - \hat{k}$$

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{4+4+2}{\sqrt{4+4+4}\sqrt{4+4+1}} = \frac{10}{\sqrt{12}\sqrt{9}} = \frac{10}{6\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{10}{6\sqrt{3}}\right)$$

31. V is irrotational

$$\Rightarrow \quad V = \nabla\phi = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

$$\frac{\partial\phi}{\partial x} = y + z;\ \frac{\partial\phi}{\partial y} = z + x;\ \frac{\partial\phi}{\partial z} = x + y$$

32. Let $f(r)=\frac{c}{r^3}$ $\Rightarrow$ $f(r)\vec{r}=\frac{c}{r^3}\vec{r}$

$$\nabla\cdot\left(f(r)\vec{r}\right)=\nabla\cdot\left(\frac{c\vec{r}}{r^3}\right)=c\nabla\cdot\left(\frac{\vec{r}}{r^3}\right)=0$$

33. F is conservative when $\Rightarrow \nabla\times F=0 \Rightarrow F=-\nabla\phi$

34. $u=e^{xyz}$

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial z}\right)=\frac{\partial^2}{\partial x\partial y}(xye^{xyz})=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}[xye^{xyz}]\right\}=\frac{\partial}{\partial x}\left\{xe^{xyz}+(xy)(xz)e^{xyz}\right\}$$

$$=\frac{\partial}{\partial x}\left\{e^{xyz}(x+x^2yz)\right\}=yze^{xyz}(x+x^2yz)+e^{xyz}(1+2xyz)$$

$$=e^{xyz}(xyz+x^2y^2z^2+2xyz+1)=e^{xyz}(1+3xyz+x^2y^2z^2)$$

35.

$$(r.\nabla)r^2=(x\hat{i}+y\hat{j}+z\hat{k})\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)$$

$$=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$=x(2x)+y(2y)+z(2z)=2(x^2+y^2+z^2)=2r^2$$

36.

$$AB=(0,1,0)-(0.5\sqrt{3},0.5,0)=(-0.5\sqrt{3},0.5,0);$$

$$R_{Bc}=\left(\frac{0.5}{\sqrt{3}},0.5,\frac{\sqrt{2}}{3}\right)-(0.5\sqrt{3},0.5,0)=(-0.5773,0,0.4714)$$

$$AB\times BC=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ -0.5\sqrt{3} & 0.5 & 0\\ 0.5773 & 0 & 0.4714\end{vmatrix}=0.2357\hat{x}+0.4082\hat{y}+0.2886\hat{z}$$

$$\left|AB\times BC\right|=0.55269$$

37. $y^2=4x$ and $x^2=4y \Rightarrow y=\sqrt{2x}$ and $y=\frac{x^2}{4}$

$$\frac{x^4}{16}=4x \qquad x(x^3-4^3)=0$$

$$x^3=4^3$$

$$x=4,0$$

$$\iint_A y\,dxdy = \int\left(\frac{y^2}{2}\right)_{\frac{x^2}{4}}^{2\sqrt{x}} dx = \frac{1}{2}\int\left(4x - \frac{x^4}{16}\right)dx = \int\left(2x - \frac{x^4}{32}\right)dx = \left[x^2 - \frac{x^5}{5(32)}\right]_0^4 = 16 - \frac{4^5}{5(32)}$$

$$= 16 - \frac{32}{5} = 10\left(1 - \frac{2}{5}\right) = 16\left(\frac{3}{5}\right) = \frac{48}{5}$$

38. $\phi : x^2 + y^2 - z - 1$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla\phi\big|_{(1,1,1)} = 2\hat{i} + 2\hat{j} - \hat{k}$$

$$\text{Unit vector } = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{4+4+1}} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$$

39. $u_1 = e^{\alpha x}$; $u_2 = e^{-\alpha x}$

$$\frac{d^2}{dx^2}|u_1\rangle = \alpha^2 e^{\alpha x} \quad ; \quad \frac{d^2}{dx^2}|u_2\rangle = \alpha^2 e^{-\alpha x}$$

$$\langle u_1|\frac{d^2}{dx^2}|u_1\rangle = \alpha^2 \quad ; \quad \langle u_2|\frac{d^2}{dx^2}|u_2\rangle = \alpha^2$$

$$\Rightarrow \langle u_1|\frac{d^2}{dx^2}|u_2\rangle = 0 \quad ; \quad \langle u_2|\frac{d^2}{dx^2}|u_1\rangle = 0$$

40. According to D' Alembert's ratio test.

$$\begin{bmatrix} \langle u_1|\frac{d}{dx}|u_1\rangle & \langle u_1|\frac{d}{dx}|u_2\rangle \\ \langle u_2|\frac{d}{dx}|u_1\rangle & \langle u_2|\frac{d}{dx}|u_2\rangle \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$$

41. Rotation matron $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta = 30^\circ \quad \Rightarrow R = \begin{pmatrix} \cos 30^\circ & \sin 30^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{pmatrix} \quad \Rightarrow R = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

43. $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{vmatrix}$

$$|A| = 1(8) - 2(1) + 3(-2) = 8 - 2 - 6 = 0$$

$$\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} = 4 - 12 = -8 \neq 0$$

$\rho(A) = 2$

44. $$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)(2-\lambda) - 6 = 0$

$\Rightarrow$ $2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$

$\Rightarrow$ $\lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow$ $\lambda^2 + \lambda - 4\lambda - 4 = 0$

$\Rightarrow$ $\lambda(\lambda + 1) - 4(\lambda + 1) = 0$

$\Rightarrow$ $(\lambda + 1)(\lambda - 4) = 0$

$\Rightarrow$ $\lambda = -1, 4$

Similar matrix $= \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$

45. $AA' = I \Rightarrow A' = A^{-1}$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\Rightarrow \quad e = \dfrac{d}{D} \Rightarrow \quad D = \dfrac{d}{e}$

46. $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$; $A_\beta = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$

$$A_\alpha^2 = A_\alpha A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha - \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = A_{\alpha+\beta}$$

47. $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 9-4 & -12+4 \\ 3-1 & -4+1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

48. $A = \begin{pmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{pmatrix}$; $A^+ = \begin{pmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{pmatrix}$

$A = A^+$ $\quad\because A$ is unitary $\Rightarrow |A| = 1$

$\Rightarrow \alpha + i\gamma = \alpha - i\gamma$ $\qquad \Rightarrow (\alpha+i\gamma)(\alpha-i\gamma) - (-\beta - i\delta)(\beta - i\delta) = 1$

$$\alpha^2 + \gamma^2 + \beta^2 + \delta^2 = 1$$

49.

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$(I+A)^{-1} = \frac{1}{1+5}\begin{bmatrix} 1 & -(1+2i) \\ 1-2i & 1 \end{bmatrix}$$

$$(I-A)(I+A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1-5 & -1-2i-1-2i \\ 1-2i+1-2i & -5+1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

50.

$$|A| = 1(-4) + 2(2) = 0$$

$\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = 2 \neq 0$ $\qquad \Rightarrow \rho(A) = 2$

51. $$\begin{bmatrix} 1989 & 1990 & 1991 \\ 1992 & 1993 & 1994 \\ 1995 & 1996 & 1997 \end{bmatrix} \underset{\substack{R_2\to R_2-R_1 \\ R_3\to R_3-R_1}}{\simeq} \begin{bmatrix} 1989 & 1990 & 1991 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{bmatrix} \underset{R_3\to\frac{R_3}{2}}{\simeq} \begin{bmatrix} 1989 & 1990 & 1991 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \Rightarrow |A| = 0$$

52. $$A\cdot B = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$A\cdot B = \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos^2\phi\cos\theta\sin\theta + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta + \sin^2\theta\sin^2\phi + \cos\phi\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\theta\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \cos\phi\sin\theta\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

$$A\cdot B = 0 \qquad \Rightarrow \cos(\theta-\phi) = 0$$

$$\theta-\phi = \frac{\pi}{2}$$

53. We have $A(adjA) = |A|\,I_n$, where n is the order. Here, $n = 3$ and $|A| = 2$;

$$\therefore\ A(adj\ \ A) = 2.I_3 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

54. $$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(-1-\lambda)-1 = 0$$

$$\Rightarrow \quad -(1-\lambda)(1+\lambda)-1 = 0$$

$$\Rightarrow \quad 1-\lambda^2+1 = 0$$

$$\Rightarrow \quad 2 = \lambda^2$$

$$\Rightarrow \quad \lambda = \pm\sqrt{2}$$

55. $\text{Tr}\,(ABC) = \text{Tr}\,(CAB) = \text{Tr}\,(BCA)$

56. $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$(-\lambda)(\lambda^2) - 1(-1) = 0$$

$$\Rightarrow \quad -\lambda^3 + 1 = 0$$

$$\Rightarrow \quad \lambda^3 = 1 \quad \Rightarrow \lambda_1 \lambda_2 = 1$$

$$\text{Tr}(A) = 0 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0$$

$$|A| = 1 \Rightarrow \lambda_1 \lambda_2 \lambda_3 = 1$$

57. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\cos^2\theta + \lambda^2 - 2\lambda\cos\theta + \sin^2\theta = 0$$

$$\lambda^2 - 2\cos\theta\lambda + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\Rightarrow \lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

59. $\text{Tr}(A) = 6$

$$|A| = 3(0) + 2(3) = 6$$

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 6, \quad \lambda_1 \lambda_2 \lambda_3 = 6$$

$$\Rightarrow \lambda = 1, 2, 3$$

60. $\text{Tr}(A) = 3 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 3$

$|A| = 0 \Rightarrow \lambda_1 \lambda_2 \lambda_3 = 0 \Rightarrow \lambda = 3, 0, 0$

61. $A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \Rightarrow \quad \text{Tr}(A^2) = 10$

62. $EF - FE = 0 \Rightarrow EF = FE$

$\text{Tr}(EFGH) = \text{Tr}(HEFG) = \text{Tr}(GHEF) = \text{Tr}(GHFE) = \text{Tr}(EGHF)$

63. $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 2, 0$

$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - 4 = 0 \Rightarrow \lambda = 4, 0$

64. Eign values are $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$

Let $\theta = 30^\circ \Rightarrow \frac{\sqrt{3}}{2} \pm i\left(\frac{1}{2}\right) \Rightarrow \frac{1}{2}(\sqrt{3} \pm i)$

65. If A is a real matrix, its eigen values are real or complex conjugate in pairs.

66. $\lambda_1 + \lambda_2 + \lambda_3 = 5$; $\lambda_1\lambda_2\lambda_3 = -5$

$$\lambda_1 = 5 \Rightarrow \lambda_2 + \lambda_3 = 0$$

$$\lambda_2\lambda_3 = -1$$

$$\lambda_2 = 1, \quad \lambda_3 = -1$$

67. $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\begin{vmatrix} -\lambda & 2i \\ 2i & -\lambda \end{vmatrix} = 0 \qquad \Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$

69. $\lambda_1 + \lambda_2 + \lambda_3 = 3$; $\lambda_1\lambda_2\lambda_3 = 0$ $\Rightarrow \lambda = 0, 1, 2$

70. The products of the unitary matrices are also unitary, if A and B are unitary matrices, then AB and BA are also unitary.

71. $\rho(A) = 2 \Rightarrow |A| = 0$

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 7 & \lambda \\ 1 & 4 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(35 - 4\lambda) + 1(20 - \lambda) + 3(16 - 7) = 0$$

$$\Rightarrow \quad 70 - 8\lambda + 20 - \lambda + 48 - 21 = 0$$

$$\Rightarrow \quad -9\lambda + 117 = 0$$

$$\Rightarrow \quad \lambda = \frac{117}{9} = 13$$

73. $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & 12 \end{bmatrix}_{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} \simeq \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & 6 \end{bmatrix}_{R_3 \to R_3 - R_2} \simeq \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & 2 \end{bmatrix}$

If $\lambda = 3$ $\rho(A) = 2$ $\rho(AB) = 3$

$$\rho(A) \neq \rho(AB)$$

$\Rightarrow$ System is inconsistent.

74. $|A| = 8(21-16) + 6(-18+8) + 2(24-14) = 40 - 60 + 20 = 0$

$$\lambda_1 + \lambda_2 + \lambda_3 = 18$$

$$\lambda_1 \lambda_2 \lambda_3 = 0$$

$$\Rightarrow \lambda = 0, 3, 15$$

75. $|adj(adjA)| = |A|^{(n-1)^2} = |A|^{(3-1)^2} = |A|^4$

76. $A = \begin{pmatrix} t^2 & \cos t \\ e^t & \sin t \end{pmatrix} \Rightarrow \dfrac{dA}{dt} = \begin{pmatrix} 2t & -\sin t \\ e^t & \cos t \end{pmatrix}$

78. For the rotation through an angle θ about x axis in the counter clockwise sense.

$$\begin{bmatrix} \cos\theta & i\sin\theta & 0 \\ i\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & i\frac{\sqrt{3}}{2} & 0 \\ i\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \theta = 60^\circ$$

79.

$$A \to \lambda \quad \Rightarrow \quad A^{-1} \to \frac{1}{\lambda}$$

$$\frac{1}{\lambda^3} + \frac{2}{\lambda^2} - \frac{3}{\lambda} + 4 = 0$$

$$1 + 2\lambda - 3\lambda^2 + 4\lambda^3 = 0$$

$$4\lambda^3 - 3\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda^3 - \frac{3\lambda^2}{4} + \frac{\lambda}{2} + \frac{1}{4} = 0$$

80. $$\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \end{bmatrix}_{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_3}} \simeq \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}_{R_4 \to \frac{R_4}{5}}$$

$$\simeq \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2}} \simeq \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

81. $|A| = 0 \Rightarrow \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$

82. $|A - \lambda I| = 0 \quad \Rightarrow \lambda^3 + \lambda(a^2 + b^2 + c^2) = 0$,

83.

$$(D^2 + 2D + 5)x = 0$$

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$m = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$x = e^{-t}[c_1 \cos 2t + c_2 \sin 2t]$$

$$\frac{dx}{dt} = -e^{-t}[c_1 \cos 2t + c_2 \sin 2t] + e^{-t}[-2c_1 \sin 2t + 2c_2 \cos 2t]$$

$$= -e^{-t}[(2c_2 - c_1)\cos 2t - (c_2 + 2c_1)\sin 2t]$$

$$x = 5; \frac{dx}{dt} = -3 \quad \text{at} \quad t = 0$$

$$c_1 = 5 \Rightarrow -3 = 2c_2 - 5 = -3$$

$$2c_2 = 2 \Rightarrow c_2 = 1$$

84. $ydx = (3x + y^4)dy$; $\frac{dx}{dy} = \frac{3x + y^4}{y}$; $\frac{dx}{dy} - \frac{3}{y}x = y^3$

$I.F = e^{-3\int \frac{1}{y}dy} = e^{-3\ln y} = e^{\ln \frac{1}{y^3}} = \frac{1}{y^3}$;

$\therefore x.(I.F) = \int y^3 \frac{1}{y^3} dy + c$; $\frac{x}{y^3} = \int dy + c$; $\frac{x}{y^3} = y + c$

85. Given equator is, $y = y'^2 + xy' - x$ _(1) but $y = P^2 + xP - x$ _(2) ;

Differentiating ern. (2) w. r. t. P,

$\frac{dy}{dP} = 2P + x + P\frac{dx}{dP} - \frac{dx}{dP} \Rightarrow P\frac{dx}{dP} = \alpha P + x + P\frac{dx}{dP}\left(\because \frac{dy}{dP} = \frac{dy}{dx}.\frac{dx}{dP} = P.\frac{dx}{dP} 2P + x - \frac{dx}{dP} = 0\right) \Rightarrow \frac{dx}{dP} = 2P + x$

86. Given that, $xP(x) = 4 + x + x^2 + ... x^2 q(x) = 2 + 3x + 5x^2 + ...$ indical equation is, $k^2 + (P_0 - 1)k + q_0 = 0$;

$\therefore k^2 + (4-1)k + 2 = 0 \;\; k^2 + 3k + 2 = 0; (k+2)(k+1) = 0; k = -1, -2$

87. $y_1 = x; y_2 = ?$ wronskin, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & y_2 \\ 1 & y_2' \end{vmatrix} = xy_2' - y_2$; at $x = \frac{1}{2}, y_2 = 0$

$u; W = \frac{1}{2}y_2' \Rightarrow \frac{dy_2}{dx} = 2W \;\; y_2 = 2Wx + C$;

If $x = \frac{1}{2}; y_2 = 0, 0 = 2W\frac{1}{2} + C;\; C = -W$; $\therefore y_2 = 2Wx - W = W(2x - 1)$

88. $\left(x^3y^3 + xy\right)dx = dy; \frac{dy}{dx} = x^3y^3 + xy;\; \frac{dy}{dx} - xy = x^3y^3; y^{-3}\frac{dy}{dx} - xy^{-2} = x^3$ put $y^{-2} = v$

89. $xdx + ydy + zdz = 0$ By integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} =$ constant;

$x^2 + y^2 + z^2 = C$ which represents the equation of a sphere.

91. $y^2 = 2c(x + \sqrt{c}) \rightarrow (1)$; Differentiating (1) with respect to 'x'

$2yy^1 = 2c \Rightarrow c = yy^1$; $\quad (1) \Rightarrow y^2 = 2yy^1\{x + (yy^1)^{1/2}\}$;

$y^4 - 4xy^3y^1 + 4x^2y^2y^{1^2} = 4y^3y^{1^3}$. Order is '1' degree is 3.

92. $ydx = 2xdx \Rightarrow \int \frac{dx}{x} = 2\int \frac{dx}{x} + c \Rightarrow \ln x = 2\ln y = \ln c \Rightarrow x = cy^2$.

93. $\frac{dy}{dx} + 2xy = 2xy^2$. Dividing throught by y^2. $y^{-2}\frac{dy}{dx} + 2xy^{-1} = 2x$;

out $y^{-1} = v$; $-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$; $-\frac{dv}{dx} + 2xv = 2x$; $\frac{dv}{dx} - 2xv = -2x$

$$I.F = e^{\int(-2x)dx} = e^{-x^2}$$

$$\Rightarrow v(I.F) = \int (-2x)(I.F)dx + C$$

$$ve^{-x^2} = \int -2xe^{-x^2}dx + C = e^{-x^2} + C; v = 1 + ce^{x^2}\ \frac{1}{y} = 1 + ce^{x^2};\ \therefore y = \frac{1}{1+ce^{x^2}}$$

94. $x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 15y = 0$ _(1) which is in the form of a cauchy's homogeneous linear equation

put $x = e^t \Rightarrow t = \log x \Rightarrow x\frac{dy}{dx} = Dy; x^2\frac{d^2y}{dx^2} = D(D-1)y$ where $D = \frac{d}{dt}$,

Then equation (1) becomes, $[D(D-1) - 7D + 15]y = 0 (D^2 - 8D + 15)y = 0$ _(2) A.E.

$$\Rightarrow m^2 - 8m + 15 = 0 \Rightarrow m = 5,3\ y = c_1e^{3t} + c_2e^{5t} = c_1e^{3\log x} + c_2e^{5\log x}\ y = c_1x^3 + c_2x^5$$

95. From Newtons 3^{rd} law, $m\frac{d^2x}{dt^2} = -cx - bv^2$; but $\frac{dx}{dt^2} = \frac{d}{dt}(v) = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

$$\Rightarrow v\frac{dv}{dx} = -cx - bv^2 (\because m = 1) \Rightarrow v\frac{dv}{dx} + bv^2 + cx = 0$$

Put $v^2 = z; 2v\frac{dv}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{2}\frac{dz}{dx} + bz = -cx$

$$\Rightarrow \frac{dz}{dx} + 2bz = -2cx; I.F = e^{\int 2bdx} = e^{2bx}; \therefore e^{2bx} = -2c\int xe^{2bx}dx + c'$$

$$\Rightarrow ze^{2bx} = -2c\left(x\frac{e^{2bx}}{2b} - \frac{e^{2bx}}{4b^2}\right) + c' \Rightarrow z = -\frac{c}{b}x + \frac{c}{2b^2} + c'e^{-2bx};$$

Applying initial condition $V(x=0) = 0\ \therefore z = v^2 = -\frac{c}{b}x + \frac{c}{2b^2} - \frac{c}{2b^2}e^{-2bx}$

96. $(4x+3y+1)dx+(3x+2y+1)dy=0 \Rightarrow Mdx+Ndy=0\frac{2M}{2y}=3=\frac{2N}{2x} \Rightarrow$ the given equation is exact $\therefore$ the solution is, $\int\limits_{y=\text{constant}} Mdx + \int\limits_{\text{terms excluding } x \text{ terms}} = C\int(4x+3y+1)dx+\int(2y+1)dy=C$

$2x^2+3xy+x+y^2+y=C \Rightarrow 2x^2+3xy+y^2+x+y+C=0$. Compare with standard form,

$ax^2+2hxy+by^2+2gx+2fy+c=0;\ h^2-ab=\frac{9}{4}-2=\frac{1}{4}>0$; It is parabola condition

97. Given equation.is, $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\frac{3}{2}(3/2+1)y=0 \Rightarrow \frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{3}{2(1-x^2)}$

$(3/2+1)y=0 \Rightarrow \frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{3}{2(1-x^2)}(3/2+1)=0$

$1-x^2=0 \Rightarrow x=\pm 1$; The solution will diverge at 1 and -1.

98. $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{5x} \Rightarrow (D^2-5D+6)y=e^{5x}\ A.E$

is, $m^2-5m+6=0 \Rightarrow (m-3)(m-2)=0 \Rightarrow m=3,2$

C.F. $= c_1e^{2x}+c_2e^{3x}\ P.I=\frac{1}{D^2-5D+6}e^{5x}=\frac{1}{6}e^{5x}$

$\therefore y=C.F+P.I$; $y=c_1e^{2x}+c_2e^{3x}+\frac{1}{6}e^{5x}$

99. $\frac{d^4x}{dt^4}+4x=\cosh t \Rightarrow (D^4+4)x=\cosh t;$

$$P.I=\frac{1}{(D^4+4)}\left(\frac{e^t+e^{-t}}{2}\right)=\frac{1}{2}\left(\frac{1}{D^4+4}e^t+\frac{1}{D^4+4}e^{-t}\right)=\frac{1}{2}\left(\frac{e^t}{5}+\frac{e^{-t}}{5}\right)=\frac{1}{5}\cosh t$$

100. $(D^3-D)y=e^x+e^{-x}\ P.I=\frac{1}{D^3-D}(e^x+e^{-x})=x\frac{1}{3D^2-1}(e^x+e^{-x})=\frac{x}{2}(e^x+e^{-x})$

101. The given equation is, $y'-\frac{\tan y}{1+x}=(1-x)e^x\sec y$ Dividing the above equation with $\sec y, \cos y\frac{dy}{dx}-\frac{\sin y}{1+x}=(1-x)e^x$. Put $\sin y=v \Rightarrow \cos y\frac{dy}{dx}=\frac{dv}{dx};\ \frac{dv}{dx}-\frac{v}{1+x}=(1-x)e^x$

$I.F=e^{\int -\frac{1}{1+x}dx}=e^{-\log(1+x)}=\frac{1}{1+x}$.

102. The given differential equation is $x^2(y-px)=yp^2$ Let $x^2=u, and\ y^2=\upsilon$, then $2xdx=du$, and $2ydy=d\upsilon$; $\frac{y}{x}\frac{dy}{dx}=\frac{d\upsilon}{du} or, p=\frac{x}{y}\frac{d\upsilon}{du}$ Substituting these in the given equation, we get $x^2\left(y-\frac{x^2}{y}\frac{d\upsilon}{du}\right)=y\frac{x^2}{y^2}\cdot\left(\frac{d\upsilon}{du}\right)^2$; or, $y^2-x^2\frac{d\upsilon}{du}=\left(\frac{d\upsilon}{du}\right)^2$ $or, \upsilon=u\frac{d\upsilon}{du}+\left(\frac{d\upsilon}{du}\right)^2$ Which is of Clairaut's forms. Hence the required solution is $\upsilon=cu+c^2 or, y^2=cx^2+c^2$.

103. $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$, which is a Cauchy's homogeneous linear eqation, put

$x=e^t \Rightarrow t=\log x$ and $\frac{dy}{dx}=Dy$, $x^2\frac{d^2y}{dx^2}=D(D-1)y...$

$\therefore D(D-1)y+Dy-y=0\Rightarrow(D^2-1)y=0$, where $D=\frac{d}{dt}$ $A.E.$ is,

$m^2-1=0; m=\pm1; y=c_1e^t+c_2e^{-t}=c_1x+c_2x^{-1}=c_1x+c_2/x$; $\therefore$ The others solutions is $\frac{1}{x}$.

104. $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0\Rightarrow(4D^2+4D+5)y=0$;

$A.E.$ is, $4m^2+4m+5=0; m=\frac{-4\pm\sqrt{16-80}}{8}=\frac{-4\pm i8}{8}=-\frac{1}{2}\pm i$; $\therefore y=e^{-\frac{x}{2}}(c_1\cos x+c_2\sin x)$

105. $(D^2+4D+4)y=2\sinh 2x$ .A.E. is, $m^2+4m+4=0; m=-2,-2$; $y_{C.F}=(c_1+c_2x)e^{-2x}$.

$$P.I=\frac{1}{D^2+4D+4}2\sinh 2x=\frac{1}{D^2+4D+4}(e^{2x}-e^{-2x})$$

$$=\frac{1}{(D+2)^2}(e^{2x}-e^{-2x})=\frac{1}{16}e^{2x}\frac{x}{2(D+2)}e^{-2x}=\frac{1}{16}e^{2x}-\frac{x^2}{2}e^{-2x};$$

$$\therefore y=y_{C.F}+y_{P.I}=(c_1+c_2x)e^{-2x}+\frac{1}{16}e^{2x}-\frac{x^2}{2}e^{-2x}$$

106. Given D.E. is, $x(1-x^2)dy+(2x^2y-y-ax^3)dx=0$;

$$x(1-x^2)dy=-(2x^2y-y-ax^3)dx;\ \frac{dy}{dx}+\frac{(2x^2-1)}{x(1-x^2)}y=\frac{ax^3}{x(1-x^2)};$$

$$I.F=e^{\int\frac{2x^2-1}{x(1-x^2)}dx};\ \int\frac{2x^2-1}{x(1-x^2)}dx=\int\frac{x^2-1+x^2}{x(1-x^2)}dx=\int\frac{-(1-x^2)}{x(1-x^2)}dx+\int\frac{x^2}{x(1-x^2)}dx$$

$$=-\ln x+\left(-\frac{1}{2}\right)\int\frac{-2x}{1-x^2}dx=-\ln x-\frac{1}{2}\ln(1-x^2)=\ln\frac{1}{x(1-x^2)^{1/2}};\ I.F.=e^{\ln\frac{1}{x(1-x^2)^{1/2}}}$$

$I.F = \frac{1}{x(1-x^2)^{1/2}}$

107. $y'' + (3i+1)y' - 3i = 0\left[D^2 + (3i+1)D - 3i\right]y = 0$; $A.E$ is,

$m^2 + (3i-1)m - 3i = 0(m+3i)(m-1) = 0 \Rightarrow m = 1,-3i$

$y = c_1 e^x + c_2 \cos 3x - c_3 \sin 3x$

108. $\frac{d^2y}{dx^2} + y = 0 \Rightarrow (D^2+1)y; D^2 + 1 = 0 \Rightarrow D = \pm i$;

$\therefore y = c_1 \cos x + c_2 \sin x;\quad y(0) = 1 \Rightarrow 1 = c_1;\; y\left(\frac{\pi}{2}\right) = 2 \Rightarrow c_2 = 2$; $\therefore y = \cos x + 2\sin x$

109. $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0 \Rightarrow (D^4 - D^2) = 0;\; D^2(D^2-1) = 0 \Rightarrow D = 0,0,-1,1;$

$y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$;∴ the set of linearly independent so equations, $\{1, x, e^x, e^{-x}\}$

110. $(2x/y^3)dx + \left[(y^2 - 3x^2)\right]dy = 0\; \frac{dy}{dx} = -\frac{2x/y^3}{(y^2-3x^2)/y^4}$;1

$v + x\frac{dv}{dx} = -\frac{2x/v^3x^3}{(v^2x^2 - 3x^2)/v^4x^4} = -\frac{2x.v^4x^4}{v^3x^3.(v^2x^2-3x^2)} = -\frac{2vx^{21}}{v^2x^2 - 3x^2} = -\frac{2v}{v^2-3}$ the given ern is homogeneous $\frac{dy}{dx} = -\frac{2xy}{(y^2-3x^2)}$.

111. Given $\frac{dy}{dx} = \frac{(y/x)^2 - 1}{2(y/x)} \rightarrow (1);\quad \frac{y}{x} = V \Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx};\quad (1) \Rightarrow \frac{dx}{x} + \frac{2V}{1+V^2}dV = 0 \Rightarrow x^2 + y^2 = cx$ which is circle. The differential equation of the orthogonal trajectories $\frac{dy}{dx} = -\frac{dx}{dy};\quad (1) \Rightarrow \frac{dy}{dx} = \frac{(2xy)}{(x^2-y^2)}$.

112. Let the population be 'x' at time 't' (in years) and 'x_0' be the population at time $t = 0$. $\frac{dx}{dt} \alpha x \Rightarrow \frac{dx}{dt} = kx$; $\frac{dx}{x} = kdt$. By integrating, we get $\log x = kt + \log c \Rightarrow x = ce^{kt}$ _(1) at $t = 0, x = x_0 \Rightarrow x_0 = c$; $\therefore x = x_0 e^{kt} \Rightarrow t = \frac{1}{k}\log\left(\frac{x}{x_0}\right) \Rightarrow k = \frac{1}{t}\log\left(\frac{x}{x_0}\right)$

for $t = 50, x = 2x_0 \Rightarrow k = \frac{\log^2}{50}$. at $t = t'$

$x = 3x_0 \Rightarrow kt' = \log 3, t' = \frac{1}{k}\log 3, t' = \frac{50}{\log 2}\log 3, t' = 50\frac{\log 3}{\log 2}$

113. $(D^3 - D)y = 2\cosh x \Rightarrow D(D^2 - 1)y = 2\cosh x$ $A.E$:

$\Rightarrow m(m^2 - 1) = 0 \Rightarrow m = 0, m = 1, -1$

$$y_{C.E} = c_1 + c_2 e^x + c_3 e^{-x};\ P.I = \frac{1}{(D^3 - D)} 2\cosh x$$

$$= \frac{1}{(D^3 - D)}(e^x + e^{-x}) = x\frac{1}{(3D^2 - 1)}(e^x + e^{-x})$$

$$y_{C.E} = c_1 + c_2 e^x + c_3 e^{-x};\ P.I = \frac{1}{(D^3 - D)} 2\cosh x$$

$$= \frac{1}{(D^3 - D)}(e^x + e^{-x}) = x\frac{1}{(3D^2 - 1)}(e^x + e^{-x})$$

$$= \frac{x}{2}e^x + \frac{x}{2}e^{-x} = \frac{x}{2}(e^x + e^{-x})$$

$\therefore$ The complete solution is, $y = y_{CF} + y_{PI} = c_1 + c_2 e^x + c_3 e^{-x} + \frac{x}{2}(e^x + e^{-x})$

114. $\frac{d^2 s}{dt^2} = g$; Integrating $\Rightarrow \frac{ds}{dt} = gt + c$; Again at $t = 0, \frac{ds}{dt} = u \Rightarrow u = c$; $\therefore \frac{ds}{dt} = gt + u$ _(1);

Again integrating, $s = \frac{1}{2}gt^2 + ut + c' \Rightarrow s = ut + \frac{1}{2}gt^2$ $\left(at\ t = 0, s = 0 \Rightarrow 0 = c' \Rightarrow c' = 0\right)$

$$(x^2 + 1)\frac{dy}{dx} + (y^2 + 1) = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^2 + 1}{x^2 + 1} \Rightarrow \frac{dy}{y^2 + 1} + \frac{dx}{x^2 + 1} = 0.$$

Integrating by applying the method of variables

$$\tan^{-1}(y) + \tan^{-1}(x) = c;\ \tan^{-1}(y) = c - \tan^{-1}(x);\ y = \tan\{c = \tan^{-1}(x)\};$$

$$y = \frac{\tan c - x}{1 + \tan c(x)} [\because \text{expansion of } \tan(A - B)];$$

Given $y(0) = 1, \Rightarrow 1 = \frac{\tan c}{1 + \tan c(0)} = \tan c$;

$$\therefore y = \frac{1 - x}{1 + x} \Rightarrow y = (1 + x) = 1 - x \quad y + 2y = 1 - x \Rightarrow y + x = 1 - xy$$

115. Option (a) order = 4, degree = 1; option (b) = 3, degree = 4; option (c)

$$\left\{1 + \left(\frac{dy}{dx}\right)^3\right\}^{2/3} = 4\left(\frac{d^3 y}{dx^3}\right)\left[1 + \left(\frac{dy}{dx}\right)^3\right]^2 = 4\left(\frac{d^3 y}{dx^3}\right)^3 \Rightarrow \text{order} = 3 = \text{degree}$$

116. $\phi\left(x, y, \frac{dy}{dx}, \frac{d^3y}{dx^3}\right) = 0$; ϕ represent a 3^{rd} order D.E $\Rightarrow 3$ arbitary constants.

117. $(D^2+4)y = y_p = \frac{x}{4\left(1+\frac{D^2}{4}\right)}x = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x = \frac{1}{4}\left(1-\frac{D^2}{4}\right)x = \frac{1}{4}x = \frac{x}{4}$

$\left[\because (1+x)^{-1} = 1-x+x^2 ...\right]$

118. $M(x,y)dx + N(x,y)dy = 0 \; \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$ Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Now we can represent the variables V and x.

119. From option, (c) $(D^3 - 2^3)y = 0 \Rightarrow (D-2)(D^2+2D+4) = 0 \Rightarrow D = 2, D = \frac{-2 \pm \sqrt{4-16}}{2}$

$$= \frac{-2 \pm i2\sqrt{3}}{2} = -1 \pm i\sqrt{3}$$

$$\Rightarrow y = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + c_3 e^{2x}$$

120. $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \left(\frac{1}{Lc}\right)i = 0\left(D^2 + \frac{R}{L}D + \frac{1}{Lc}\right)i = 0$; where $D = \frac{d}{dt}$ A.E. is,

$$m^2 + \frac{R}{L}m + \frac{1}{Lc} = 0 \Rightarrow m = \left[-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{Lc}}\right] / 2$$

$$m = \frac{-R/L \pm \sqrt{\left(\frac{R}{L}\right)^2 - \left(\frac{R}{L}\right)^2}}{2} = -\frac{R}{2L} \; ; \left(\because R^2c = 4L\right)$$

$\therefore$ Solution is, $y = (c_1 + c_2 t)e^{-Rt/2L}$

121. Given D.E. is, $(1+xy)ydx + (1-xy)xdy = 0 \Rightarrow Mdx + Ndy = 0$;

$$Mx - Ny = 0, \therefore I.F = \frac{1}{Mx - Ny} ; \therefore I.F = \frac{1}{xy + x^2y^2 - (xy - x^2y^2)} = \frac{1}{2x^2y^2}$$

123. Given D.E. is, $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$ dividing throught by $\cos^2 y, \sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$ _(1)

Put $V = \tan y \Rightarrow \frac{dV}{dx} = \sec^2 y\frac{dy}{dx}$ then equation (1) can be written as,

$\frac{dV}{dx} + 2xv = x^3$ _(2) Which is of the linear form $\frac{dV}{dx} + Pv = Q$

124. Given equation can be rewritten as $(D^2 + D - 2)y = 0,$

where $D = \frac{d}{dx}$ .A.E. is, $m^2 + m = 2 = 0 \Rightarrow (m+2)(m-1) = 0 \Rightarrow m = -2,1;$

$\therefore y = c_1 e^{-2x} + c_2 e^x;\ y' = -2c_1 e^{-2x} + c_2 e^x;$

$y(0) = 0 \Rightarrow 0 = c_1 + c_2 \Rightarrow c_1 e^x y'(0) = 3; = -2c_1 + c_2 \Rightarrow c_2 = 1$ and $c_1 = -1$; $\therefore y = e^x - e^{-2x}$

125. Given D.E can be rewritten as, $\frac{dx}{dt} + \frac{k_2}{k_1} x = \frac{k_3}{k_1} I.F = e^{\int \frac{k_2}{k_1} dt} = e^{\frac{k_2}{k_1} t}$

$\therefore$ Solution is, $xe^{\frac{k_2}{k_1} t} = \int \frac{k_3}{k_1} e^{\frac{k_3}{k_1} t} dt + c = \frac{k_3}{k_1}, \frac{e^{\frac{k_2}{k_1} t}}{k_2 / k_1} + c = \frac{k_3}{k_2} e^{k_2/k_1 t} + c$

$$x = \frac{k_3}{k_2} + ce^{-k_2/k_1 t}; x(t=0) = 0 \Rightarrow 0 = \frac{k_3}{k_2} + c \Rightarrow c = -\frac{k_3}{k_2}; \therefore x = k_3 / k_2 \left(1 - e^{k_2/k_1 t}\right)$$

126. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0 \Rightarrow (D^2 + 2D + 1)x = 0$

A.E. is, $m^2 + 2m + 1 = 0 (m+1)(m+1) = 0$

$m^2 + 2m + 1 = 0(m+1)(m+1) = 0 \Rightarrow m = -1 -1$

$\Rightarrow x = (c_1 + c_2 t)e^{-t} \frac{dx}{dt} = -c_1 e^{-t} + c_2 e^{-t} - c_2 te^{-t}$

$$\Rightarrow x = (c_1 + c_2 t)e^{-t} \frac{dx}{dt} = -c_1 e^{-t} + c_2 e^{-t} - c_2 te^{-t}$$

$$x(t=0) = 1 \Rightarrow 1 = c_1 \frac{dx}{dt}(t=0) = 0 \Rightarrow 0 = -c_1 + c_2 \Rightarrow c_1 = c_2 = 1$$

$\therefore x(t) = (1+t)e^{-t}; \therefore$ at $t = 1$ $x = 2e^{-1} = 2/e$

127. The function are linearly independent if $W \neq 0$

$$W = \begin{vmatrix} e^{m_1} & e^{m_2 x} & e^{m_3 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} & m_3 e^{m_3 x} \\ m_1^2 e^{m_1 x} & m_2^2 e^{m_2 x} & m_3^2 e^{m_3 x} \end{vmatrix}$$

$= e^{m_1 x} e^{(m_2+m_3)x}(m_2 m_2^2 m_3) + e^{m_2 x} e^{(m_1+m_2)x}(m_1 m_2^2 - m_2 m_1^2)$

$= e^{(m_1+m_2+m_3)x}\{m_2 m_3(m_3 - m_2) + m_1 m_3(m_1 - m_3) + m_1 m_2(m_3 - m_1)\}$

128. $y = ax^3$.....(1) $\Rightarrow a = \dfrac{y}{x^3}$. Differentating equation (1), $\dfrac{dy}{dx} = 3ax^2 = \dfrac{3y}{x}$ for orthogonal tranjectories, $\dfrac{dy}{dx} = -\dfrac{x}{3y} \Rightarrow 3ydy + xdx = 0$ integrating by the method of seperation of variables.

$\therefore$ The curves (1) and (3) form orthogonal trajectories.

129. The equation of the circle with radius r and centre on x-axis is, $(x-a)^2 + y^2 = r^2$....(1)

$(1) \Rightarrow x - a = (r^2 - y^2)^{1/2}\ 2(r^2 - y^2)^{1/2} + 2y\dfrac{dy}{dx} = 0 \Rightarrow (r^2 - y^2)^{1/2} = -y\dfrac{dy}{dx}$;

On squaring, $r^2 - y^2 = y^2\left(\dfrac{dy}{dx}\right)^2 \Rightarrow r^2 = y^2 + y^2\left(\dfrac{dy}{dx}\right)^2$ ù; $r^2 = y^2\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]$

130. The given equation is $4x\left(\dfrac{dy}{dx}\right)^2 + 2x\left(\dfrac{dy}{dx}\right) - y = 0$ ......(1)

Also given primitives is $(y-c)^2 = cx$ ....(2)

The curve passes through (1,2) $(2-c)^2 = c \Rightarrow (c-1)(c-4) = 0 \Rightarrow c = 1,4$. There are two integral curves $(y-1)^2 = x$ *and* $(y-4)^2 = 4x$

131. $ydx - xdy = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x} \Rightarrow \dfrac{dx}{x} - \dfrac{dy}{y} = 0$

By integrating we get $\log x - \log y + \log c = 0\ \log(cx) = \log y \Rightarrow y = cx$ ù;

A family of straight lines.

132. The given equation can be written as, $(D^2+1)y = 0 A.E \Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$;

$$\therefore y = (c_1 \cos x + c_2 \sin x);\ y(0) = 1 \Rightarrow 1 = c_1$$

and $$y(\pi/2) = 1 \Rightarrow 1 = c_2; \therefore y_1 = \cos x + \sin x$$

133. Given, $(D^4 + 8D^2 + 16)y = 0 \Rightarrow (D^2+4)^2 y = 0$

$\Rightarrow (m^2+4)^2 = 0 \Rightarrow m = \pm 2i, \pm 2i;\ \Rightarrow (m^2+4)^2 = 0 \Rightarrow m = \pm 2i, \pm 2i;$

134. The given equations are $xy^1 + 2y = 1 \to (1)$ *and* $xy^1 - 2y = 1 \to (2)$; Both differential equations are linear .I.F of (1) $I_1 = e^{\int \frac{2}{x}dx} = e^{2\log x} = x^2$; (3) and I.F of (2) $I_2 = e^{-\int \frac{2}{x}dx} = e^{-2\log x} = \dfrac{1}{x^2}$ (4) From (3) and (4), $I_1 I_2 = 1$.

135, 136. Given D.E. can be written as, $(D^2-2D+1)y=0\, A.E. \Rightarrow (m^2-2m+1)=0 \Rightarrow m=1,1\,;\therefore$ Solution is, $y=(c_1+c_2x)e^x$.

137. $(1-x^2)y''-2xy'+2y=0 \Rightarrow y''-\dfrac{2x}{1-x^2}y'+\dfrac{2}{1-x^2}y=0$ where $P(x)=-\dfrac{2x}{1-x^2}$ and $Q(x)=\dfrac{2}{1+x^2}\,;\therefore$ The given equation is not analytic at $x=\pm 1$. Ordinary point means the point at which the function is not analytic.

138. $(4D^2+4D+1)y=0 \Rightarrow (D^2+D+1/4)y=0\,;$

$A.E \Rightarrow (m^2+m+1/4)=0 \Rightarrow ,=-1/2,-1/2 \therefore\ y=(A+Bx)e^{-x/2}$

139. Given equation, is $(D^2-1)y=2\cosh t$ where $D=\dfrac{d}{dt}\, A.E. \Rightarrow m^2-1=0 \Rightarrow m=\pm 1\,;$

$$\therefore y_{CF}=c_1e^t+c_2e^{-t};\ P.I=\frac{1}{D^2-1}2\cosh t=\frac{1}{D^2-1}(e^t+e^{-t})=t\frac{1}{2D}(e^t+e^{-t})=\frac{1}{2}(e^t-e^{-t})=t\sinh\ t$$

$$y'=c_1e^t-c_2e^{-t}+t\cosh t+\sinh t\,;\ y(0)=0 \Rightarrow 0=c_1+c_2;\ y'(0)=0 \Rightarrow 0=c_1-c_2$$

Which implies that $c_1=c_2=0$; $\therefore$ The solution of the given D.E, is $y=t\sinh t$

140. $x\dfrac{dy}{dx}+y=x^4 \Rightarrow \dfrac{dy}{dx}+\dfrac{1}{x}y=x^3 I.F.=e^{\int \frac{1}{x}dx}=e^{\log x}=x \Rightarrow yx=\int x^3 x dx+c \Rightarrow xy=\dfrac{x^5}{5}+\dfrac{4}{5} \Rightarrow y=\dfrac{x^4}{5}+\dfrac{4}{5x}\,;$

141. $f(t)=\begin{cases}0 & -2<t<-1\\ k & -1<t<1\\ 0 & 1<t<2\end{cases}$. We have $f(t)=a_0/2+\displaystyle\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi t}{l}$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(t)dt=\frac{1}{2}\int_{-2}^{2}f(t)dt=\frac{1}{2}\int_{-1}^{1}kdt=\frac{k}{2}t\Big|_{-1}^{+1}=(k/2).2=k\ ;$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(t)\sin\frac{n\pi t}{l}dt=\frac{1}{2}\int_{-1}^{1}k\sin\frac{n\pi t}{2}dt=\frac{k}{2}\frac{-\cos\left(\frac{n\pi t}{2}\right)}{\left(\frac{n\pi}{2}\right)}\Bigg|_{-1}^{1}=\frac{k}{n\pi}0=0\,..$$

142. Given, $f(x)=\begin{cases}0 & 0<x<c\\ 1 & c<x<2c\end{cases}$;

By Fourier theorem, $f(x)=\dfrac{A_0}{2}+\displaystyle\sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}$

Where, $A_0 = \frac{2}{2l}\int_0^{2l} f(x)dx,\ A_n = \frac{2}{2l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \ \& \ B_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Here, ; $A_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}dx = \frac{1}{c}\int_c^{2c}\cos\frac{n\pi x}{c}dx$

$$A_n = \frac{2}{2l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \ \& \ B_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Here, $A_0 = \frac{1}{c}\int_0^{2c} f(x)dx = \frac{1}{c}\int_c^{2c} dx = \frac{1}{c}.(2c-c) = 1$

$$A_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}dx = \frac{1}{c}\int_c^{2c}\cos\frac{n\pi x}{c}dx \ ; \ = \frac{1}{c}\left.\frac{\sin\left(\frac{n\pi x}{c}\right)}{\frac{n\pi}{c}}\right|_c^{2c} = \frac{1}{n\pi}(0) = 0$$

$$B_n = \frac{1}{c}\int_0^{2c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx = \frac{1}{c}\int_c^{2c}\sin\left(\frac{n\pi x}{c}\right)dx$$

$$= \frac{1}{c}\left.\frac{-\cos\left(\frac{n\pi x}{c}\right)}{\frac{n\pi}{c}}\right|_c^{2c} = \frac{1}{n\pi}(-1+\cos n\pi) = \begin{cases} 0, \text{if } n \text{ is even} \\ -2/n\pi, \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ The fourier series for given function will be, $f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + ...\right]$

143. The fourier series is a series of sines and cosines in the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

If $f(x)$ is an even function, the corresponding series is a cosine series and if $f(x)$ is as odd function, the corresponding series is a sine series.

144. At $x = 0$; $f(0) = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + ...\right]$ and $f(0) = 1$

145. $f(t) = e^{-at}$ the Fourier sine transform of the given function is, $g_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\omega t dt$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-at}\sin\omega t dt = \sqrt{\frac{2}{\pi}}\frac{\omega}{a^2+\omega^2}\left(\because \int_0^{\infty} e^{-ax}\sin bx dx = \frac{b}{a^2+b^2}\right)$$

146. $L^{-1}\left\{\frac{s+4}{s(s-1)(s^2+4)}\right\}$ $\frac{s+4}{s(s-1)(s^2+4)}=\frac{As+B}{s^2+4}+\frac{c}{s}+\frac{D}{s-1}$

Multiplying throughout by $s(s-1)(s^2+4)$,

We get $s+4=(As+B)s(s-1)+c(s-1)(s^2+4)+D(s^2+4)s$

$s+4 \Rightarrow 4=c(-4) \Rightarrow c=-1\ s=1 \Rightarrow 5=5D \Rightarrow D=1$

Comparing the coefficient of s on both sides, comparing the coefficient of S^3 on both sides.

$0=A+c+D \Rightarrow A=-c-D=1-1=0\ \therefore\ f(s)=\frac{s+4}{s(s-1)(s^2+1)}=\frac{1}{s^2+4}+\frac{-1}{s}+\frac{1}{s-1}$

$\therefore f(t)L^{-1}\{f(s)\}=L^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{2}L^{-1}\left\{\frac{2}{s^2+4}\right\}-L^{-1}\frac{s+4}{s(s-1)(s^2+1)}-L^{-1}\left\{\frac{1}{s}\right\}=e^t-\frac{1}{2}\sin 2t-1$

147. The Fourier transform of a gaussian function is a gaussian in.

148. $f(s)=\int_0^\infty F(t)e^{-st}dt$, we are $L\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}F(s)\ \therefore \int tF(t)e^{-st}dt=L\{tf(t)\}=-\frac{df}{ds}$

149. The Laplace transform of a function $f(x)$ exists only if the function satisfies the following codition:

i) The function $f(t)$ should be as arbitrary continous function in every finite internal and that $f(t)=0$ for all values of t.

ii) The function $f(t)$ should be of expontial order.

150. $L[\sqrt{t}]=L\{t^{1/2}\}=\frac{(\frac{1}{2})!}{S^{3/2}}=\frac{\sqrt{\pi}}{2S^{3/2}}$

151. $L[te^{at}\sin at]$ we are $L\{\sin\ at\}=\frac{a}{(S^2+a^2)}$. Also $L\{t^n f(t)\}=(-1)^n\frac{d^n f(s)}{ds^n}$

where $f(s)=L[F(t)]$ $L\{t\sin\ at\}=(-1)\frac{d}{ds}\frac{a}{(s^2+a^2)}=\frac{2as}{(s^2+a^2)^2}$

We know that $L[e^{at}F(t)]=f(s-a)\ \therefore L\left[te^{at}\sin at\right]=\frac{2as-a}{\left[(s-a)^2+a^2\right]^2}$

152. $f(t)=\sin\pi t$ and $F(s)=L\left[f(t)=\frac{\pi}{S^2+\pi^2}\right];\ \therefore\{t\sin\pi t\}=(-1)\frac{d}{d^s}\left[\frac{\pi}{S^2+\pi^2}\right]=\frac{2\pi S}{(S^2+\pi^2)^2}$

153. Given function is $f(t)=t^a L[f(t)]=\int_0^\infty e^{-st}t^a dt=t^a$

$$e^{-st}\Big|_0^{\infty} - \int_0^{\infty} at^{a-1}\frac{e^{-st}}{-s}dt = \frac{a}{S}\int_0^{\infty} t^{a-1}e^{-st}dt = \frac{a}{s}\frac{(a-1)}{S}\int_0^{\infty} t^{a-2}e^{-st}dt\ ;$$

As proceeding like this we get $L\{t^a\} = \frac{a9a-1)(a-2)(a-3)...3.2.1}{S^a} = \frac{\Gamma a+1}{S^{a+1}}$

154. $\frac{p+1}{(p+3)^2+16} = \frac{p+3}{(p+3)^2+4^2} - \frac{2}{(p+3)^2+16}$

$$L^{-1}\left[\frac{p+1}{p^2+6p+25}\right] = e^{-3t}\left[\frac{p}{p^2+4^2} - \frac{1}{2}\frac{4}{p^2+4^2}\right] = e^{-3t}\left[\cos 4t - \frac{1}{2}\sin 4t\right]$$

155. Given $f(x) = \begin{cases} 0\ , & x<3 \\ x-3, & x>3 \end{cases}$ $\qquad L[f(x)] = \int_0^{\infty} e^{-sx} f(x)dx$

$$= \int_0^3 e^{-sx} f(x)dx + \int_3^{\infty} e^{-sx} f(x)dx = \int_3^{\infty}(x-3)e^{-sx}dx = \int_3^{\infty} xe^{-sx}dx - 3\int_3^{\infty} e^{-sx}dx$$

$$= x\frac{e^{-sx}}{-s}\Big|_3^{\infty} + \frac{1}{s}\int_s^{\infty} e^{-sx}dx + \frac{3}{s}e^{-sx}\Big|_3^{\infty} = \frac{3}{s}e^{-3s} - \frac{1}{s^2}e^{-sx}\Big|_3^{\infty} - \frac{3}{s}e^{-3s} = \frac{1}{S^2}e^{-3s} = s^{-2}e^{-3s}$$

156. A function is said to be analytic is $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

(cauchy's Reimann condition) and the function must be conditions

$f(z) = \sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i\cos x \sinh y = u + iv$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \cos x \cosh y = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= \sin x \sinh y = -\frac{\partial v}{\partial x} \end{aligned}\right\} f(z) = \sin z$$

(a) is analytic.

(b) $f(z) = z^3 = (x+iy)^3 = x^3 - iy^3 + i3x^2y - 3xy^2 = (x^3 - 3xy^2) + i(3x^2y - y^3) = u + iv$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 - 3y^2\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -6xy = -\frac{\partial v}{\partial y} \end{aligned}\right\} f(z) = z^3$$ is analytic function.

(c) $f(z)=z^3 f(z)=e^x(\cos y=i\sin y)=e^x\cos y+ie^x\sin y=u+iv$

$$\left.\begin{array}{l}\dfrac{\partial u}{\partial x}=e^x\cos y=\dfrac{\partial v}{\partial y}\\ \dfrac{\partial u}{\partial y}=-e^x\sin y=-\dfrac{\partial v}{\partial x}\end{array}\right\}\Rightarrow f(z)=e^x(\cos y+i\sin y)\text{ is analytic function.}$$

(d) $f(z)=\sin x\sin y-i\cos x\cos y=u+iv,$ say

$$\frac{\partial u}{\partial x}=\cos x\sin y=\frac{\partial v}{\partial y}\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ \frac{\partial u}{\partial y}=\sin x\cos y=\frac{\partial v}{\partial x};\quad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}\Rightarrow\frac{\partial u}{\partial y}+-\frac{\partial v}{\partial x};$$

$\therefore f(z)=\sin x\sin y-i\cos x\cos y$ is not analytic

157. $|Z|=1\Rightarrow$ unit circle centred at the origin $f(z)=\oint\cot z^2dz=\oint\dfrac{\cos z^2}{\sin z^2}dz$ $z=0$ is a pole.

$\therefore$ The function has only one singular point inside r where the residue is zero

158. $i^i=e^{i\log i}=e^{i(2in\pi+\log i)}=e^{-2n\pi+i\log(e^{i\pi/2})}=e^{-2n\pi+i(i\pi/2)}=e^{-2n\pi-\pi/2}=e^{-(2n+1/2)\pi};$

$\therefore \ln(i^i)=\ln e^{-(2n+1/2)\pi}=-(2n+1/2)\pi$ principal value is obtained by putting $n=0$ $\dot{u}; -\pi/2$

160. Given that $f(z)=e^{z^2}$

$z=x+iy\Rightarrow z^2=x^2-y^2+i2xy\Rightarrow f(z)=e^{x^2-y^2+i2xy}$

$\Rightarrow f(z)=e^{x^2-y^2}e^{i2xy}=e^{x^2-y^2}\left[\cos(2xy)+i\sin(2xy)\right]$

161. $f(z)=\displaystyle\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}+\frac{1}{z-1}-\frac{1}{(z-1)^2}+\frac{2}{3(z-1)^3}$

$\therefore$ The residue of $f(z)$ at $z=1$ is, $2/3$

162. $f(z)=\dfrac{1}{1+e^z}$ Poles of $f(z)=\dfrac{1}{1+e^z}$ are found by equanting to zero, *i.,e.*, $1+e^z=0$

$\Rightarrow e^z=-1=e^{\pm i(2n+1)\pi}\Rightarrow z=\pm i(2n+1)\pi$, Where n = 0, 1, 2, ... $\therefore f(z)$ has poles of order 1 at z=$\pm i\pi,\pm 3i\pi,\pm 5i\pi,.....$

163. $f(z)=\displaystyle\int_c\frac{\cos\pi z}{(z-1)}dz$; Poles of f(z)$\Rightarrow$ z - 1 = 0$\Rightarrow$ $z=1$ is a pole of order 1 and which lies inside the circle whose radius is 3, centred at the origin. Now, by cauchy's integral formula,

$$f(z_0)=\frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}dz \Rightarrow \int \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

Taking $f(z) = \cos(\pi z)$ and $z_0 = 1$, we have ; $\int\limits_{|z|=3} \frac{\cos \pi z}{z-1}dz = 2\pi i\ \cos(\pi z_0)\big|_{z_0=1} = -2\pi i$.

164. Let $\int\limits_c f(z)dz = \int\limits_c \frac{z^2}{(z^2+a^2)(z^2+b^2)}dz,$ where c is the contour consisting of the semicircle C_R of radius R together with the part of the real axis from $-R$ to R as shown in figure? The poles are given by, $(z^2+a^2)(z^2+b^2)=0 \Rightarrow z=\pm ia,\ \pm ib$ of which $z = ia, ib$ only lie inside c.

$\therefore$ By residence theorem, $\int\limits_c f(z)dz = 2\pi i\{\operatorname{Res} f(z)|_{z=ia} + \operatorname{Res} f(z)|_{z=ib}\}$

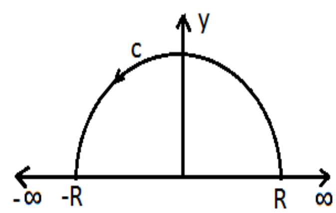
$$= 2\pi i\left\{\lim_{z\to ia}(z-ia)f(z) + \lim_{z\to ib}(z-ib)f(z)\right\}$$

$$= 2\pi i\left\{\lim_{z\to ia}\frac{z^2}{(z+ia)(z^2+b^2)} + \lim_{z\to ib}\frac{z^2}{(z^2+a^2)(z+ib)}\right\}$$

$$= 2\pi i\left\{\frac{-a^2}{2ia.(b^2-a^2)} + \frac{-b^2}{(a^2-b^2)(2ib)}\right\}$$

$$= 2\pi i\left\{\frac{-a^2}{2ia.(a^2-b^2)} - \frac{b^2}{(a^2-b^2)(2ib)}\right\} = \pi\left(\frac{a-b}{a^2-b^2}\right) = \frac{\pi}{a+b}$$

Also, $$\int\limits_c f(z)dz = \int\limits_{-R}^{R} f(x)dx + \int\limits_{C_R} f(z)dz$$



Now, let $R\to\infty$. For any point on CR as $|z|\to\infty$,

$$\left|\int\limits_c f(z)dz\right| = \left|\int\limits_{C_R}\frac{z^2}{(z^2+a^2)(z^2+b^2)}dz\right| < \int\limits_{C_R}\frac{|z|^2|dz|}{|z|^2|z|^2} \text{ as a} > 0, \text{b} > 0 = \int\limits_{C_R}\frac{|dz|}{|z|^2} = \int\limits_{C_R}\frac{dz}{R^2}$$

$\because$ On semicircle C_R , $|z| = R$ $= \frac{\pi R}{R^2} = \frac{\pi}{R} \to 0$ *as* $R\to\infty$ $\therefore \int\limits_{-\infty}^{\infty}\frac{x^2dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$

165. $f(z)=(z+1)\sin\left(\frac{1}{z-2}\right)$

Let $z-2=t \Rightarrow t+3=z+1 \Rightarrow (z+1)\sin\frac{1}{z-2}=$

$$=(t+3)\sin\left(\frac{1}{t}\right)=(t+3)\left[\frac{1}{t}-\frac{1}{3!}\left(\frac{1}{t}\right)^3+\frac{1}{5!}\left(\frac{1}{t}\right)\right]$$

$$=1+\frac{3}{t}-\frac{1}{6}\frac{1}{t^2}-\frac{1}{2}\frac{1}{t^3}+\ldots=1+\frac{3}{t}-\frac{1}{6t^3}-\frac{1}{2t^3}+\ldots$$

$$=1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}-\frac{1}{2(z-2)^3}+..$$

$\therefore f(z)$ has essential singularity at $z=2$

166. $f(z)=\frac{z\sin z}{(z-\pi)^2}=z\frac{\sin(\pi-z)}{(\pi-z)^2}=\frac{z}{(\pi-z)^2}\left\{(\pi-z)-\frac{(\pi-z)^3}{3!}+\frac{(\pi-z)^5}{5!}-\ldots.\right\}$

$=z\left\{\frac{1}{(\pi-z)}-\frac{(\pi-z)}{3!}+\frac{(\pi-z)^3}{5!}-\ldots.\right\}; \Rightarrow f(z)$ has a simple pole at $z=\pi$.

167. $|z|=1 \Rightarrow$ radius of contour = 1, centred at origin. Then the pole $z=\pi$ lies outside the contour. $\therefore \oint f(z)\,dz=0.$

168. Poles $z=\pm ai$ if $a=2$ then $z=\pm 2ai$ lies outside the contour; if $z=\pm\frac{i}{2}$ *i.,e.* $a=1/2$ poles lies inside the contour.

$$\operatorname{Res}_{z=\frac{i}{2}}=2\pi$$

$$\operatorname{Res}_{z=-\frac{i}{2}}=-2\pi$$

169. Let $I=\int_0^\infty \frac{\cos mx}{(x^2+1}dx$ let $\int_0^\infty \frac{\cos mx}{(x^2+1)}dx$ poles of $f(z)$ are $\pm i$ only $z=i$ lies within the contour

$$\operatorname{Re}sf(z)\big|_{z=i}=\lim_{z\to i}(z-i)\frac{\cos mz}{(z+i)(z-i)}=\lim_{z\to i}\frac{(e^{imz})}{z+i}=\frac{e^{-m}}{2i}\ \therefore 2\pi i.\frac{e^{-m}}{2i}=\pi e^{-m,}$$

170. $f(z)=\dfrac{4-3z}{z(z-1)(z-2)}$ *poles* $\Rightarrow z(z-1)(z-2)=0\Rightarrow z=0,1,2$ of which $z=0, z=1$ lie inside the circle of radius 3/2, centred at the origin $\operatorname{Re}sf(z)|_{z=0}=\lim_{z\to 0} z\dfrac{(4-3z)}{Z(Z-1)(Z-2)}=\dfrac{4}{2}=2$

$$\operatorname{Re}sf(z)|_{z=1}=\lim_{z\to 1} z\frac{(4-3z)}{Z(Z-1)(Z-2)}=\frac{1}{-2}=-1 \;\therefore \int_C \frac{(4-3z)}{z(z-1)(z-2)}dz = 2\pi i\Sigma R = 2\pi(1)=2\pi i$$

172. Poles of $f(z)$ are $z=1$ (second order) and $z=-2$

(Simple) Residue of $f(z)$(at $z=1$) $=\underset{z\to 1}{Lt}\dfrac{z}{(z-1)!}\dfrac{d}{dz}\dfrac{(z-1)^2 z^2}{(z-1)^2(z+2)}=5/9;$

Residue of $f(z)$(at $z=-2$) $=\underset{z\to -2}{Lt}\dfrac{(z+2)z^2}{(z-1)^2(z+2)}=4/9$

173. Let $I=\int_C \dfrac{dz}{z^2 2z}$. here $f(z)=\dfrac{1}{z^2-2z}=\dfrac{1}{z(z-2)}$ $z=0$ and z = 2 are poles. But z = 0 is the only pole which lies inside the contour $\therefore \operatorname{Re}f(z)|_{z=0}=\lim_{z\to 0} z\dfrac{1}{z(z-2)}=-1/2;\therefore \int_c \dfrac{dz}{z^2-2z}=2\pi i(-1/2)=-\pi i$

174. Let $I=\oint \dfrac{e^z \sin z}{2\pi z^2}dz \;\; f(z)=\dfrac{e^z\sin z}{2\pi z^2}, z=0$ is a pole of order 2.

$$\therefore \operatorname{Re}sf(z)|_{z=0}=\frac{1}{2}\frac{d}{dz}\left(\lim_{z\to 0} z^2\frac{e^z\sin z}{2\pi z^2}\right)=\frac{1}{4\pi}(e^z\cos z+\cos z e^z)|_{z=0}=\frac{1}{4\pi}(1+1)=\frac{1}{2\pi}$$

$$\therefore \oint \frac{e^z\sin z}{2\pi z^2}=2\pi i\Sigma R=2\pi i.\frac{1}{2\pi}=i$$

175. $\displaystyle\int_0^2 (t^2-it)dt:(2t+i)=\int_0^2\left[(2t^3+t-it)\right]dt=\left[\frac{t^4}{2}+\frac{t^2}{2}-i\frac{t^3}{3}\right]_0^2=10-\frac{8i}{3}$

176. $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^5}$ consider the function, $f(z)=\dfrac{1}{(1+z^2)^5}$ poles, $\Rightarrow z=\pm i$ of order 5. of which only z = i lies inside the contour $\operatorname{Re}sf(z)|_{z=1}=\dfrac{1}{4!}\dfrac{d^4}{dz^4}\left\{(z-i)^5\dfrac{1}{(z+i)^5(z-i)^5}\right\}\Big|_{z=i}$

$$=\frac{1}{4!}\frac{d^4}{dz^4}(z+i)^{-5}\Big|_{z=i}=\frac{(-5)}{4!}\frac{d^3}{dz^3}(z+i)^{-6}\Big|_{z=i}$$

$$= \frac{5\times 6}{4!}\frac{d^2}{dz^2}(z+i)^{-7}\bigg|_{z=i} = \frac{5\times 6\times -7}{4!}\frac{d}{dz}(z+i)^{-8}\bigg|_{z=i}$$

$$= \frac{5\times 6\times 8}{4!}(z+i)^{-9}\bigg|_{z=i} = \frac{5\times 6\times 7\times 8}{4!}\frac{1}{(2i)^9} = \frac{70}{2.2^8 i} = \frac{35}{256i}$$

By cauchy- Residence theorem, $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$

$$\therefore \left|\int_{C_R} f(z)dz\right| = \left|\int_{C_R} \frac{dz}{(1+z^2)^5}\right| \le \int \frac{|dz|}{|1+z^2|^5} \le \int \frac{|dz|}{|z^0|} \text{ (as } 1>0) = \int \frac{|dz|}{R^{10}} (\because 0\ n^{th} \text{ semicircle } |z|=R)$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^5} = 2\pi i.\frac{35}{256\, i} \Rightarrow \int_0^{\infty} \frac{dx}{1+x^2)^5} = \frac{1}{2}\frac{70\,\pi}{256} = \frac{35\,\pi}{256}$$

177. $\frac{\cos \pi z}{z^2-1} = 1/2\left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos \pi z \ \therefore \int \frac{\cos \pi z}{z^2-1}dz = 1/2\left\{\int \frac{\cos \pi z}{z-1}dz - \int \frac{\cos \pi z}{z+1}dz\right\}$

$$= \tfrac{1}{2}\, 2\pi i\left\{\lim_{z\to 1}(z-1)\frac{\cos \pi z}{z-1} - \lim_{z\to -1}(z+1)\frac{\cos \pi z}{(z+1)}\right\}$$

$$= \pi i(-1+1) = 0 \ [\because \cos(-\theta) = \cos\theta]$$

178. Let $f(z) = \frac{\sin \pi z^2}{(z-2)(z-1)}$

z = 2 , z = 1 are simple poles lying within the given circle of radius 3 , centred at the origin.

$$\therefore \int \frac{\sin \pi z^2}{(z-1)(z-2)}dz = 2\pi i\left\{\lim_{z\to 1}(z-1)\frac{\sin \pi z^2}{(z-1)(z-2)} + \lim_{z\to 2}(z-2)\frac{\sin \pi z^2}{(z-1)(z-2)}\right\} = 2\pi i(0+0) = 0.$$

179. $\int_c f(z)dz = \int_c z^2 e^{1/z}dz$; Where c represents a circle of unit radius centred at the origin. $f(z) = z^2 e^{1/z} = z^2\left(1 + 1/z + 1/2!1/z^2 + 1/3!1/z^3 + 1/4!1/z^4 + \ldots\right)$

$$= \left(z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \ldots\right)$$

Res $f(z)\big|_{z=0} = 1/3! = 1/6$; $\therefore \int z^2 e^{1/z}dz = 2\pi i.1/6 = \pi i/3$.

180. Let I = $\oint\limits_{c} \frac{e^Z \, Sin(Z)}{z^2} dz$, where C is c unit circle contour at (2, 0); $Z = 0$, is a pole of order 2, which lies outside the contour. $\therefore \oint \frac{e^z \sin z}{z^2} dz = 0.$

181. Given , $f(z) = \frac{e^z}{z^2 - 3z + 2} = \frac{e^z}{(z-2)(z-1)} \Rightarrow$ Poles are z = 2 , z = 1 , out of which z = 1 is the only pole which lies inside the contour.n

$$\therefore \int \frac{e^z}{z^2 - 3z + 2} dz = 2\pi i \operatorname{Re}s\, f(z)\Big|_{z=1} = 2\pi i \lim_{z \to 1}(z-1)\frac{e^z}{(z-2)(z-1)} = 2\pi i \lim_{z \to 1} \frac{e^z}{z-2} = 2\pi i \frac{e^1}{-1} = -2\pi i e$$

182. Probability that the card, which is drawn from a pack containing 52 cards with 4 aces, become are is, $P_1 = \frac{4c_1}{52c_1} = \frac{4}{52}$ Probability that the card, which is drawn from a pack of 48 cards with 8 aces, become ace is, $P_2 = \frac{8c_1}{48c_1} = \frac{8}{48} = \frac{1}{6}$ $\therefore$ The probability that both are aces is, $P = P_1.P_2 = \frac{4}{52}.\frac{1}{6} = \frac{1}{13}.\frac{1}{6} = \frac{1}{78}$.

184. The probability that no one die, $P = (1-p)(1-p).......(n \text{ times}) = (1-p)^n$.

The probability that at least one dies $= 1-(1-p)^n$ $\therefore$ The probability that A_1 will die in one year is ,

$P(A_1 \text{ dies}) = \frac{1}{n}\left\{1-(1-p)^n\right\}$.

185. Let E = Event that the husband will be alive 25 years hence and F = event that the wife will be alive 25 years . Then $P(E) = \frac{5}{12}$ *and* $P(F) = \frac{3}{8}$; *Thus* $P(\overline{E}) = \frac{7}{12}$ *and* $P(\overline{F}) = \frac{5}{8}$; The probability that at least one of them will be alive 25 years hence $= 1 - P(\overline{E}).P(\overline{F}) = \frac{61}{96}$.

186. Probability that the ball which is drawn from bag 1 become white is, $P_1 = \frac{4c_1}{6c_1} = \frac{4}{6}$

Probability that the ball which is drawn from bag 2 become white is, $P_2 = \frac{3c_1}{8c_1} = \frac{3}{8}$

$\therefore$ The Probability that both are white is, $P = \frac{4}{6}.\frac{3}{8} = \frac{1}{4}$.

187. Since these two events are independent, the probability of passing both is $\frac{2}{3}\times\frac{1}{4}=\frac{1}{6}$.

188. Total marbles of ways of selecting 3 marbles out of 9 is $9c_3 = 84$.

$\therefore$ The probability that all are different $= \frac{3\times3\times3}{84} = \frac{27}{84} = \frac{9}{28}$.

189. If n independent trials constitute one experiment and this experiment be repeated N times, then the frequency of r success is, $N^n c_r P^r q^{n-r}$.

Here, $P = \frac{2}{5}$ $q = 1-\frac{2}{5} = \frac{3}{5}$, $n = 10$, $r = 6$. $\therefore$ The probability $= {}^{10}C_6\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^4$.

190. $P_x = \frac{e^{-m}m^x}{x!}$; $P_{m-1} = \frac{e^{-m}m^{m-1}}{(m-1)!} = \frac{e^{-m}m^m m^{-1}}{m!}m = \frac{e^{-m}m^m}{m!} = P_m$;

$$P_{x-1} = \frac{e^{-m}m^{x-1}}{(x-1)!} = \frac{e^{-m}m^x m^{-1}x}{x!} = \frac{e^{-m}m^x}{x!}\frac{x}{m} = x/mP_x \text{ ; } P_{x+1} = \frac{e^{-m}m^{x+1}}{(x+1)!} = \frac{e^{-m}m^x.m}{x!(x+1)} = P_x.\frac{m}{(x+1)}$$

191.

$$\begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 5$$

$$\Rightarrow \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x+3y = 0 \Rightarrow x = y$$

$$-4z = 0 \Rightarrow z = 0$$

193. $x_0 = 0$ $y_0 = 0$

$$f(x,y) = 1+y^2$$

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = hf(x_0, y_0) = (0.2)(1) = 0.2$$

$$K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right] = (0.2)f(0.1, 0.1) = (0.2)(1.01) = 0.202$$

$$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right] = (0.2)f(0.1, 0.101) = (0.2)(1.010201) = 0.20204$$

$$K_4 = hf\left[x_0 + \frac{h}{2}, y_0 + K_3\right] = (0.2)f(0.1, 0.20204) = (0.2)(1.0408) = 0.20816$$

$$y_1 = 0 + \frac{1}{6}[0.2 + 0.404 + 0.40408 + 0.20816]$$

$$y_{(0.2)} = 0.2027$$

194. $y_{(0.2)} = 0.2027$

Let $x_0 = 0.2$, $y_0 = 0.2027$, $h = 0.2$

$$K_1 = hf(x_0, y_0) = (0.2)(1.041) = 0.2082$$

$$K_2 = hf(0.3, 0.3068) = 0.2188$$

$$K_3 = (0.2)f(0.3, 0.3121) = 0.21948$$

$$K_4 = (0.2)f(0.3, 0.42218) = 0.2356$$

$$y_{(0.4)} = y_{(0.2)} + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y_{(0.4)} = 0.4227$$

195. Back ward Euler method is given by $y_{n+1} = y_n + h\,f\left(x_{n+1}, y_{n+1}\right)$

196. The finite difference approximation for the 2nd derivative of a function f at a point x_0 is

$$f^{11}(x) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

197. Let $x = \sqrt[3]{c}$ then $x^3 - c = 0 \Rightarrow f(x) = x^3 - c = 0$ and $f^1(x) = 3x^2$

Newton – Raphson method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} = x_n - \frac{\left(x_n^3 - c\right)}{3x_n^2} = \frac{3x_n^3 - x_n^3 + c}{3x_n^2};\ x_{n+1} = \frac{2x_n^3 + c}{3x_n^2}$$

198. Let $f(x) = x^2 - 2$ and $x_0 = -1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} = x_n - \frac{\left(x_n^2 - 2\right)}{2x_n} = \frac{\left(x_n^2 + 2\right)}{2x_n};\ x_{n+1} = \frac{(x_n^2 + 2)}{2x_n}$$

$$x_1 = \frac{x_0^2 + 2}{2x_0} = \frac{1+2}{-2} = \frac{-3}{2} = -1.5\ ;\ x_2 = \frac{x_2^2 + 2}{2x_1} = \frac{(-1.5)^2 + 2}{2(-1.5)} = \frac{\frac{9}{4} + 2}{-3} = \frac{17/4}{-3} = -1.4166$$

$$x_3 = \frac{x_2^2 + 2}{2x_2} = \frac{(-1.4166)^2 + 2}{2(-1.4166)} = \frac{2.0067 + 2}{-2.8332} = -1.4141$$

$$x_4 = \frac{x_3^2 + 2}{2x_3} = \frac{(-1.4141)^2 + 2}{2(-1.4141)} = \frac{1.9996 + 2}{-2.8282} = -1.4141$$

$\therefore$ The iteration will converge to $-\sqrt{2} = -1.4141$

199. $f(x) = \frac{1}{x}$, $a = 1$, $b = 2$, $h = 0.25$

x	1	1.25	1.50	1.75	2
$f(x)=\frac{1}{x}$	1	0.8	0.6667	0.5714	0.5

$$\int_1^2 \frac{1}{x}dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right] = 0.69325\,;$$

$$\int_1^2 \frac{1}{x}dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right] = 0.69325$$

200. Let $x = \frac{1}{a}$ (or) $\frac{1}{x} - a = 0$ Then take $f(x) = \frac{1}{x} - a$ and $f^1(x) = -\frac{1}{x^2}$ Now the Newton's formula

gives $x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} = x_n - \frac{\left(\frac{1}{x_n} - a\right)}{-1/x_n^2} = x_n + \left(\frac{1}{x_n - a}\right)x_n^2$ $x_{n+1} = 2x_n - ax_n^2$ for n = 0, 1, 2, 3, ..

201. Newton – Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$ $n = 0, 1, 2, 3, \ldots\ldots\ldots$

202. $E-1, F-3, G-4, H-2$

203. Let $x = \sqrt{2}$ (or) $x^2 - 2 = 0$ Taking $f(x) = x^2 - 2$ and $f^1(x) = 2x$ Newton – Raphson's

formula is $x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} = x_n - \frac{(x_n^2-2)}{2x_n} = \frac{2x_n^2 - x_n^2 + 2}{2x_n}; x_{n+1} = \frac{x_n^2+2}{2x_n}$ $\therefore$ $x_{n+1} = \frac{1}{2}\left[x_n + \frac{2}{x_n}\right]$

204. Given $f(x) = x^3 + 4x - 9 = 0 \Rightarrow f^1(x) = x^2 + 4$ Newton – Raphson formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f^1(x_k)} = x_k - \frac{(x_k^3 + 4x_k - 9)}{(3x_k^2+4)} = \frac{3x_k^3 + 4x_k - x_k^3 - 4x_k + 9}{(3x_k^2+4)} \therefore x_{k+1} = \frac{2x_k^3+9}{3x_k^2+4}$$

205. P – 1, Q – 2, R – 3, S – 4

1. Simpson's Rule is one of the numerical integration technique (method).
2. Gauss – elimination method is used to solve only linear algebraic equations.
3. Runge – Kutta method is used to solve the ordinary differential equations.
4. Newton – Raphson method is used to solve the linear and non – linear algebraic equations.

206. Given $\frac{dx}{dt} = e^{-2t}\, u(t)$, where $u(t)$ is an unit step function.

By the definition of unit step function, we have $u(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases}$

$x(0) = 0$ *and* $x = ?$ *at* $t = 0.01$sec

$$\frac{dx}{dt} = e^{-2t}\, u(t) \Rightarrow dx = e^{-2t} u(t) dt \Rightarrow x = \int_{t=0}^{t=0.01} e^{-2t} u(t) dt \qquad \Rightarrow x = \int_{t=0}^{t=0.01} g(t)\, dt \text{ where } g(t) = e^{-2t}\, u(t)$$

$$= c^{-2t} \begin{cases} 0, & t<o \\ 1, & t\geq 0 \end{cases}$$

$$= \begin{cases} 0, & t<0 \\ e^{-2t}, & t\geq 0 \end{cases}$$

By TrapEzoidal rule, we have $\Rightarrow x = \int_0^{0.1} g(t)\, dt = \frac{h}{2}\left[g(a) + g(b)\right]$

$$= \frac{(0.01)}{2}\left[g(0) + g(0.01)\right] = \frac{0.01}{2}\left[1 + e^{-2(0.01)}\right]$$

$$= \frac{0.01}{2}\left[1 + e^{-0.02}\right]; x = 0.0099$$

207. Given $\frac{dy}{dx} - y = x$............(1) *and* $y(0) = 0$................(2)

Also given $h = 0.1$ $y(0.3) = ?$

From (2), we have

$$x_0 = 0, \quad y_0 = 0 \quad and \quad f(x, y) = \frac{dy}{dx} = y + x$$

$$x_1 = x_0 + 1h = 0 + 0.1 = 0.1$$

$$x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$$

$$x_3 = x_0 + 3h = 0 + 3(0.1) = 0.3$$

Euler's first order method is given by

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h\left(\frac{dy}{dx}\right)_P$$

$$y_1 = 0 + (0.1)(x_0 + y_0) = 0.1[0 + 0] = 0.0$$

$$y_2 = y_1 + hf(x_1,\, y_1) = 0.0 + (0.1)[0.1 + 0.0] = 0.0 + 0.01 = 0.01$$

$$y_3 = y_2 + hf(x_2, y_2) = 0.01 + (0.1)\, [0.2 + 0.01] = 0.01 + 0.021 = 0.031$$

209. The Table of x and corresponding value of $f(x)$ are as follows :

i	x	$y = x^4$
0	-3	$y_0 = 81$
1	-2	$y_1 = 16$
2	-1	$y_2 = 1$
3	0	$y_3 = 0$
4	1	$y_4 = 1$
5	2	$y_5 = 16$
6	3	$y_6 = 81$

Using Simpson rule, we have $\int_{-3}^{3} x^4 dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6\right]$

$$= \frac{1}{3}\left[81 + 4(16 + 0 + 16) + 2(1+1) + 81\right] = \frac{1}{3}\left[81 + 128 + 4 + 81\right] = 98$$

210. Here $h = \frac{2-1}{4} = 0.25$;The table of x and corresponding value of $f(x)$ are as follows

i	x_i	y_i
0	1	1
1	1.25	0.80
2	1.5	0.667
3	1.75	0.571
4	2	0.5

Using Simpson rule we have

$$\int_1^2 \frac{dx}{x} = \frac{h}{3}\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right] = \frac{0.25}{3}\left[1 + 4(0.8 + 0.571) + 2(0.667) + 0.5\right] = 0.6932$$

211. Here $h = 0.5$ Using Simpson's rule

$$\int_1^3 f(x)dx = \frac{0.5}{3}\left[2.1 + 20.8 + 4.4 + 3\right] = \frac{0.5}{3}\left[2.1 + 20.8 + 4.4 + 3\right] = 5.05$$

212. Here $h = 1$;By Simpson rule we have

$$\int_0^4 e^x dx = \frac{1}{3}\left[1 + 4(2.72 + 20.09) + 2(7.39) + 54.60\right] = \frac{1}{3}\left[1 + 91.24 + 14.78 + 54.60\right] = 53.87$$

213. $a = y_1 = \frac{1}{1+x^2} = \frac{1}{1+(0.25)^2} = \frac{1}{1.0625}$ $b = y_3 = \frac{1}{1+x_3^2} = \frac{1}{1+(0.75)^2} = \frac{1}{1.5625}$

214. Here $h = 10$ Using Trapezoidal rule

$$A = \frac{10}{2}\left[0+3+2(4+7+9+12+15+14+8)\right] = 5\left[0+3+130\right] = 705\ m^2$$

215. Using Trapezoidal rule $\int_{7.47}^{7.52} f(x)dx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3+y_4)+y_5\right]$

$$= \frac{0.01}{2}\left[1.93+2(1.95+2.01+2.03)+2.06\right] = 0.005\left[1.93+15.94+2.06\right] = 0.0996$$

216. Taking four equal part between 0 & 1, $h = 0.25$.Using Simpson's formula

i	x_i	$y_i = \frac{1}{x_i}$
0	1	$y_0 = 1$
1	2	$y_1 = 0.5$
2	3	$y_2 = 0.33$
3	4	$y_3 = 0.25$
4	5	$y_4 = 0.2$

$$\int_0^1 \frac{1}{1+x}dx = \frac{h}{3}\left[y_0 + 4(y_1+y_3)+2y_2+y_4\right] = \frac{0.25}{3}\left[1+4(0.8+0.571)+2(0.67)+0.5\right] = 0.6945$$

217. The Table of x and corresponding $f(x)$ are as follows :Using Simpson's rule we have

$$\int_1^5 \frac{dx}{x} = \frac{h}{3}\left[y_0 + 4(y_1+y_3)+2y_2+y_4\right] = \frac{1}{3}\left[1+4(0.5+0.25)+2(0.33)+0.2\right] = 1.62$$

218. Here $h = 1$ $\int_1^5 ydx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3)+y_4\right] = \frac{1}{2}\left[10+2(50+70+80)+100\right] = 255$

219. Here $h = \frac{1-0}{5} = 0.2$ Calculating all values of $f(x)$

x	0	0.2	0.4	0.6	0.8	0.1
$f(x) = x^3$	0	0.008	0.064	0.216	0.512	1

$$\int_0^1 x^3 dx = \frac{h}{2}\left[y_0 + (y_1+y_2+y_3+y_4)+y_5\right] = \frac{0.2}{2}\left[0+(0.008+0.064+0.216+0.512)+1\right] = 0.26$$

220. $f(x)=\sqrt{1-0.162\sin^2 x}=\sqrt{1-0.162\left(\frac{1-\cos 2x}{2}\right)}=\sqrt{0.919+0.081\cos 2x}$ Here Where

$h=\pi/12$

when $x_0=0$, $y_0=1$ when $x_1=\pi/12$, $y_1=\sqrt{0.919+0.081\cos 30^0}==0.9795$

when $x_2=\pi/6$, $y_2=\sqrt{0.919+0.081\cos 60^0}=0.9795$

when $x_3=\pi/4$, $y_3=\sqrt{0.919+0.081\cos 90^0}=0.9586$

when $x_4=\pi/3$, $y_4=0.9372$

when $x_5=5\pi/12$, $y_5=0.9213$

when $x_6=\pi/2$, $y_6=0.9154$

$$\int_0^{\pi/2} f(x)dx=\frac{h}{2}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]=1.5056$$

221. We have $h=0.1,\ x_0=0,\ x_1=x_0+h=0.1$ Now $\frac{dy}{dx}=1+xy$ or $\frac{d^2y}{dx^2}=x\frac{dy}{dx}+y$ or

$\frac{d^3y}{dx^3}=x\frac{d^2y}{dx^2}+2\frac{dy}{dx}$ or $\frac{d^4y}{dx^4}=x\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}$; From $y(0)=1$ we have

$\frac{dy}{dx}=1;\ \frac{d^2y}{dx^2}=1,\ \frac{d^3y}{dx^3}=2,\ \frac{d^4y}{dx^4}=3$ and so on

Taylor series expression gives $y(x+h)=y(x)+h\frac{dy}{dx}+\frac{h^2}{2!}\frac{d^2y}{dx^2}+\frac{h^3}{3!}\frac{d^3y}{dx^3}+\frac{h^4}{4!}\frac{d^4y}{dx^4}+....$

$$y(0.1)=1+01\times1+\frac{(0.1)^2}{2!}1+\frac{(0.1)^3}{3!}2+...=1+0.1+\frac{0.01}{2}+\frac{0.001}{3}+....$$

$$=1+0.1+0.005+0.000033................\qquad=1.1053$$

222. Let $h=0.1$, given $x_0=0$, $y_0=1$; $x_1=x_0+h=0.1$, Now $\frac{dy}{dx}=x-y^2$ and

$\left(\frac{dy}{dx}\right)_{0,1}=-1$; $\frac{d^2y}{dx^2}=1-2y\frac{dy}{dx}$ and $\left(\frac{d^2y}{dx^2}\right)_{0,1}=3$; $\frac{d^3y}{dx^3}=-2\left(\frac{dy}{dx}\right)^2-2y\frac{d^2y}{dx^2}$ and

$\left(\frac{d^3y}{dx^3}\right)_{0,1}=-3$; $\frac{d^4y}{dx^4}=-2\left[3\frac{dy}{dx}\frac{d^2y}{dx^2}+y\frac{d^3y}{dx^3}\right]$ and $\left(\frac{d^4y}{dx^4}\right)_{0,1}=34$

The Taylor series expression gives $y(x+h)=y(x)+h\frac{dy}{dx}+\frac{h^2}{2!}\frac{d^2y}{dx^2}+\frac{h^3}{3!}\frac{d^3y}{dx^3}+\frac{h^4}{4!}\frac{d^4y}{dx^4}+...$

$$y(0.1)=1+0.1(-1)+\frac{(0.1)^2}{2!}3+\frac{(0.1)^3}{3!}(-8)+\frac{(0.1)^4}{4!}34+.....$$

$$=1-0.1+0.015-0.001333+0.0001417 \ =0.9138$$

223. Here $f(x,y)=x^2+y^2, x_0=0 \ y_0=0$. We have, by Picard's method

$$y=y_0+\int_{x_0}^{x} f(x,y)\,dx \qquad \text{...(1)}$$

The first approximation to y is given by $y^{(1)}=y_0+\int_{x_0}^{x} f(x,y_0)\,dx$,where

$$y_0=0+\int_0^x f(x,0)\,dx=\int_0^x x^2 dx \qquad \text{...(2)}$$

The second approximation to y is given by $y^{(2)}=y_0+\int_{x_0}^{x} f\left(x,y^{(1)}\right)dx=0+\int_0^x f\left(x,\frac{x^3}{3}\right)dx$

$$=0+\int_0^x\left(x^2+\frac{x^6}{9}\right)dx=\frac{x^3}{3}+\frac{x^7}{63}$$

Now, $$y(0.4)=\frac{(0.4)^3}{3}+\frac{(0.4)^7}{63}=0.02135$$

224. Here $f(x,y)=y-x; x_0=0, y_0=2$;We have by Picard's method $y=y_0+\int_{x_0}^{x} f(x,y)\,dx$

The first approximation to y is given by

$$y^{(1)}=y_0+\int_{x_0}^{x} f(x,y_0)\,dx=2+\int_0^x f(x,2)\,dx \quad =2+\int_0^x(2-x)\,dx=2+2x-\frac{x^2}{2} \qquad \text{...(1)}$$

The second approximation to y is given by

$$y^{(2)}=y_0+\int_{x_0}^{x} f\left(x,y^{(1)}\right)dx \ =2+\int_{x_0}^{x} f\left(x,2+2x-\frac{x^2}{2}\right)dx$$

$$=2+\int_0^2\left(2+2x-\frac{x^2}{2}-x\right)dx \ = 2+2x+\frac{x^2}{2}-\frac{x^3}{6} \ \text{.........(2)}$$

The third approximation to y is given by

$$y^{(3)}=y_0+\int_{x_0}^{x} f\left(x,y^{(2)}\right)dx \ =2+\int_{x_0}^{x} f\left(x,2+2x+\frac{x^2}{2}-\frac{x^3}{6}\right)dx$$

$$= 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} - \frac{x^3}{6} - \right) dx = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$$

225. Here $f(x, y) = x + y^2,\ x_0 = 0\ y_0 = 0$;

We have, by Picard's method $y = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

The first approximation to y is given by $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = 0 + \int_0^x f(x, 0)\, dx = 0 + \int_0^x x dx = \frac{x^2}{2}$

The second approximation to y is given by

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}\right) dx = 0 + \int_0^x f\left(x, \frac{x^2}{2}\right) dx = \int_0^x \left(x + \frac{x^4}{4}\right) dx = \frac{x^2}{2} + \frac{x^5}{50}$$

The third approximation is given by

$$y^{(3)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(2)}\right) dx = 0 + \int_0^x f\left(x, \frac{x^2}{2} + \frac{x^5}{20}\right) dx$$

$$= \int_0^x \left(x + \frac{x^4}{4} + \frac{x^{10}}{400} + \frac{2x^7}{40}\right) dx = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$$

226. We have x : 0.1 0.2 0.3 0.4
Euler's method gives

$$y_{n+1} = y_n + h(x_n, y_n) \quad \text{.........(1)}$$

By putting $n = 0$ in (1) gives

$$y_1 = y_0 + hf(x_0, y_0)$$

Here $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$$y_1 = 1 + 0.1 f(0, 1) = 1 + 0 = 1$$

$n = 0$ in (1) gives $\quad y_2 = y_1 + hf(x_1, y_1)$

$$= 1 + 0.1 f(0.1, 1) = 1 + 0.1(0.1) = 1 + 0.01$$

Thus $\quad y_2 = y_{(0.2)} = 1.01$

$n = 2$ in (1) gives $\quad y_3 = y_2 + hf(x_2, y_2) = 1.01 + 0.1 f(0.2, 1.01)$

Thus $\quad y_3 = y_{(0.3)} = 1.01 + 0.0202 = 1.0302$

$n = 3$ in (1) gives $\quad y_4 = y_3 + hf(x_3, y_3) = 1.0302 + 0.1 f(0.3, 1.0302)$

$$= 1.0302 + 0.03090$$

Thus $\qquad y_4 = y_{(0.4)} = 1.0611$

Hence $\qquad y_{(0.4)} = 1.0611$

227. The Euler's modified method gives $y_1^* = y_0 + hf(x_0, y_0)$,

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right]$$

Now, here

228. $y_2^* = y_1 + hf(x_1, y_1) = 1.0202 + 0.02[f(0.02, 1.0202)] = 1.0202 + 0.0204 = 1.0406$

Next $\qquad y_2 = y_1 + \frac{h}{2}\left[f(x, y) + f(x_2, y_2^*)\right]$

$$= 1.0202 + \frac{0.02}{2}[f(0.02, 1.0202) + f(0.04, 1.0406)]$$

$= 1.0202 + 0.01[1.0206 + 1.0422] = 1.0408$ Thus

$y_2 = y_{(0.04)} = 1.0408$

229. $y_3^* = y_2 + hf(x_2, y_2) = 1.0416 + 0.02d(0.04, 1.0416) = 1.0416 + 0.0217 = 1.0633$ Next

$y_3 = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^*)\right]$ $y_3 = 1.0416 + \frac{0.02}{2}[f(0.04, 1.0416) + f(0.06, 1.0633)]$

$= 1.0416 + 0.01[1.0865 + 1.1342] = 1.0638$ $y_3 = y_{(0.06)} = 1.0638$

230. We assume that x : 0 0.2 0.4 0.6 0.8 1

Euler's modified method gives $y_1^* = y_0 + hf(x_0, y_0)$ $y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right]$

Here $y_0 = 1, x_0 = 0, h = 0.2$

$y_1^* = 1 + 0.2f(0,1) = 1 + 0.2(1) = 1.2$; $1 + \frac{0.2}{2}[f(0,1) + f(0.2, 1.2)]$

$= 1 + 0.1[1 + 1.4], y_1 = 1.24 = y(0.2)$ Now $y_2^* = y_1 + hf(x_1 y_1) = 1.24 + 0.2f(0.2, 1.24)$

$= 1.24 + 0.2 \times 1.44 = 1.528$ $y_1 = y_1 + \frac{h}{2}\left[f(x_1 y_1) + f(x_2, y_2^*)\right]$

$= 1.24 + \frac{0.2}{2}[f(0.2, 1.24) + f(0.4, 1.528)] = 1.24 + 0.1[1.44 + 1.928] = 1.5768$

$y_2 = y(0.4) = 1.5768$

Now

$y_3^* = y_2 + hf(x_2, y_2) = 1.5768 + 0.2f(0.2f(0.4, 1.5768) = 1.5768 + 0.3953 = 1.9721$

$$y_3 = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^*)\right] = 1.5768 + \frac{0.2}{2}\left[f(0.4, 1.5768) + f(0.6, 1.9721)\right]$$

$$= 1.5768 + 0.1\left[1.9768 + 25721\right] = 2.03169,\ y_3 = 2.03169 \text{ Next}$$

$$y_4^* = y_3 + hf(x_3, y_3) = 2.0317 + 0.5f(0.6, 20317) = 2.0317 + 0.5263 = 2.558$$

$$y_4 = y_3\frac{h}{2}\left[f(x_3 + y_3) + f(x_4, y_4^*)\right]$$

$$= 2.0317 + \frac{0.2}{2}\left[f(0.6, 2.0317) + f(0.8, 2.558)\right] = 2.0317 + 0.1\left[2.6317 + 3.358\right] = 2.63067 \text{ Next}$$

$$y_5^* = y_4 + hf(x_4, y_4) = 2.63067 + 0.2f(0.8, 2.63067) = 2.63067 + 02(3.43067) = 3.3168$$

$$y_5 = y_4 + \frac{h}{2}\left[f(x_4, y_4) + (x_5, y_5^*)\right]\ 2.63067 + \frac{0.02}{2}\left[f(0.8, 2.63067) + f(1, 3.3168)\right]$$

$$= 2.63067 + 0.1\left[3.43067 + 4.3168\right], y_5 = 3.405417$$

231. We have x: 0 0.2 0.4 0.6

$$f(x) = x - y_2$$

On calculation we get $f_1(x) = 0.1996$ $f_2(x) = 0.3937$ $f_3(x) = 0.5689$

Using predictor formula $y_4^{(p)} = y_0 + \frac{4}{3}h(2f_1 - f_2 + 2f_3)$

$$h = 0.2\ = 0 + \frac{0.8}{3}\left[2(0.1996) - (0.3937) + 2(0.5689)\right]$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 - 4f_3 + f_3^*)\quad f_4^* = f(x_4, y_4^{(p)}) = f(0.8, 0.3049) = 0.07070$$

$$y_4^{(c)} = 0.0795 + \frac{2}{30}\left[0.3937 + 4(0.5689) + 0.7070\right] = 0.3046$$

232. We have $f(x) = 1 + y^2$ On calculation we get $f_1 = 1.04108, f_2 = 1.17875$ *and* $f_3 = 1.46778$

By predicated formula $y_4^{(p)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]; 0 + \frac{8}{30}\left[2(1.04108) - 1.17875 + 2(1.46778)\right]$

$$y_4^{(p)} = 1.0237,\ y_4^{(p)} = 1.0237,\ y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^*);$$

$$f_4^* = f(x_4, y_4^{(p)}) = f(0.8, 1.0237) f_4^* = 2.0480\quad y_4^{(c)} = 0.4228 + \frac{2}{3}\left[1.1787 + 4(1.4678) + 2.0480\right]$$

$$y_{(0.8)} = 1.0293$$

233. $y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4) = 0.2027 + \frac{0.8}{3}(2\times1.1787 - 1.4678 + 2(2.0594) = 1.5382$

$f_5^* = f(x_5, y_5^{(p)}) = f(1{,}15382) = 3.3660$; $y_5^{(c)} = y_3 + \frac{0.2}{3}\left[f_3 + 4f_4 + f_5^*\right]$

$= 0.6841 + \frac{2}{30}\left[1.467 + 4\times 2.0594 + 3.3360\right] = 1.5555$

234. We now to find $y_3 = y_{(0.6)}$; $k_1 = hf(x_2, y_2) = (0.2)f(0.4, 0.4228) = 0.2357$

$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.2)f(0.5, 0.5406) = 0.2584$;

$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.2)f(0.5, 0.520) = 0.2609$ $k_4 = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= \frac{1}{6}\left[0.2357 + 2(0.2584) + 2(0.2609) + 0.2935\right] = \frac{1}{6}\left[0.2357 + 0.5168 + 0.5218 + 0.2935\right] = 0.2613$

$y_3 = y_{(0.6)} = y_2 + k = 0.4228 + 0.2613 = 0.6841$

235. Here given $x_0 = 0$ $y_0 = 1, h = 0.2$; $f(x, y) = x + y^2$ To find $y_1 = y_{(02)}$;

$k_1 = hf(x_0, y_0) = (0.2)f(0,1) = (0.2)\times 1 = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)f(0.1, 1.1) = 0.2(1.31) = 0.262$;

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f(0.1, 1.131) = 0.2758$

$k_4 hf(x_0 + h, y_0 + k_3) = (0.2)f(0.2, 1.2758) = 0.3655$; $k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + 2k_4\right]$

$= \frac{1}{6}\left[0.2 + 2(0.262) + 2(0.2758) + 0.3655\right] = 0.2735$ Here $y_1 = y_{(0.2)} = y_0 + k = 1 + 0.2735 \Rightarrow 1.2735$

236. Here $f(x, y) = x + y,$ $h = 0.2$ To find $y_1 = y_{(0.2)}$ $k_1 = hf(x_0, y_0) = 0.2f(0,1) = 0.2$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)(0.2)f(0.1, 1.12) = 0.244$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)f(0.2, 1.244) = 0.2888$ $k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}\left[0.2 + 2(0.24) + 2(0.244) + 0.2888\right] = 0.2428$; $y_1 = y_{(0.2)} = y_0 + k = 1 + 0.2428 = 1.2428$

237. Here $y_1 = y_0 + k \ where \ k = \frac{1}{2}(k_1 + k_2)$; $k_1 = hf(x_0, y_0)$; $k_2 = hf(x_0 + h, k_1)$; $\frac{dy}{dx} = -y = f(x, y)$

Thus $y(0) = y_0 \ k_1 = -hy_0$; $k_2 = hf(h, y_0 + k_1) = h[-1(y_0 + k_1)] = h[-y_0 + hy_0] = -hy_0(1-h)$

$$k = \frac{1}{2}[(-hy_0) + \{-hy_0(1-h)\}] = \frac{1}{2}[-2hy_0 + h^2 y_0] = \frac{h^2}{2} y_0 - hy_0 ;$$

Thus $y_1 = y_0 + \frac{h^2}{2} y_0 - hy_0 = \frac{y_0}{2}[h^2 - 2 + 2]$

238. We have $\frac{dy}{dx} = f(x)$ Step size $= h, y_0 = 0, x_0 = 0$;By Runge-Kutta of the fourth order

$$k_1 = hf(x_0, y_0) = h[f(x_0)] = hf(0) ; k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = hf\left(x_0 = \frac{1}{2}h\right) = hf\left(\frac{h}{2}\right) ;$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = hf\left(\frac{h}{2}\right) ; k_3 = hf(x_0 + h, y_0 + k_3) = hf(h)$$

Hence bu fourth order Runge-Kutta formula the solution at $x = h; \ y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$;

$$y(h) = 0 + \frac{1}{6}\left[hf(0) + 2hf\left(\frac{h}{2}\right) + 2hf\left(\frac{h}{2}\right) + 2hf\left(\frac{h}{2}\right) + hf(h)\right] = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right]$$

239. $y_{n+1} = y_n + \frac{1}{4}(k_1 3k_2) \ n = 0,1,2,3,..... \ k_1 nf(x_n, y_n)$; $k_2 = hf(k_n + \frac{2h}{3}, y_n + \frac{3k_1}{3})$ and

$y' = -10y = f(x, y) \ x_0 = 0, y_0 = 1$. We can take step size, $h = 0.1 \ or \ 0.2$ Thus $0 < h < 0.2$

240. $\frac{dy}{dx} = 2x = f(x, y)$; Initially $x_0 = 0.1, y_0 = 0$ taking $h = 1$;

$y(1) = y_1 = y_0 + hf(x_0 + y_0) = 0$; $y(2) = y_2 = y_1 + hf(x_0 + h, y_1) = 2$;

$y(3) = y_3 = y_2 + hf(x_0 + 2h, y_2) = 6$; $y(4) = y_4 = y_3 + hf(x_0 + 3h, y_3) = 12$;

$y(5) = y_5 = y_4 + hf(x_0 + 4h, y_4) = 20$;Clearly

$y(1) = 0.1, (2) = 1.2, \ y(3) = 2.3$; $y(4) = 3.4 \ y(5) = 4.5$. Thus $y_n = x_{n-1}.x_n$